Continuity:

A function f is said to be continuous at a point $a \in \text{Dom} f$ if

i) The point a lies in an open interval contained in Dom f

ii) $\lim_{x \to a} f(x) = f(a)$.

or

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a).$$

Continuity seems to be almost equivalent to that of the existence of $\lim_{x \to a} f(x)$ with the exception that for the limit the point a may not belong to Dom f, but for continuity it is essential that the function must be defined at a.

A function which is not continuous is called discontinuous fn.

Example:

Discuss the continuity of f defined by

$$f(x) = \begin{cases} \dfrac{x-4}{\sqrt{x}-2} & \text{if } x \geq 0 \text{ and } x \neq 4 \\ 4 & \text{if } x = 4 \end{cases}$$

Sol

$$\lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x \to 4} \frac{(\sqrt{x})^2 - (2)^2}{\sqrt{x}-2}$$

$$= \lim_{x \to 4} (\sqrt{x} + 2)$$

$$= 2 + 2$$

$$= 4$$

Thus f, is continuous at $x = 4$.

Limit of Piecewise Function

Q $f(x) = \begin{cases} 3 & \text{if } x \leq -2 \\ -\frac{1}{2}x^2 & \text{if } -2 < x < 2 \\ 3 & \text{if } x \geq 2 \end{cases}$

Find $\lim_{x \to \pm 2^+} f(x)$ and $\lim_{x \to \pm 2^-} f(x)$.

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2} (3) = 3$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2} \left(-\frac{1}{2}x^2\right) = -\frac{1}{2}(2)^2 = -2$

$\lim_{x \to -2^+} f(x) = \lim_{x \to -2} \left(-\frac{1}{2}x^2\right) = -2$

$\lim_{x \to -2^-} f(x) = \lim_{x \to -2} 3 = 3$

Q $f(x) = \begin{cases} x+2 & \text{if } x \leq -1 \\ ax^2 & \text{if } x > -1 \end{cases}$

Find a so that $\lim_{x \to -1} f(x)$ exist.

Because given that $\lim_{x \to -1} f(x)$ exist

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^-} f(x)$

$\lim_{x \to -1} (ax^2) = \lim_{x \to -1} (x+2)$

$a(-1)^2 = (-1+2)$

$+a = 1$

$a = 1$

Examples

Let $f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Examine the continuity of f at $x=0$.

Sol:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1-1/e^{1/x}}{1+1/e^{1/x}} = \frac{1-0}{1+0} = 1$$

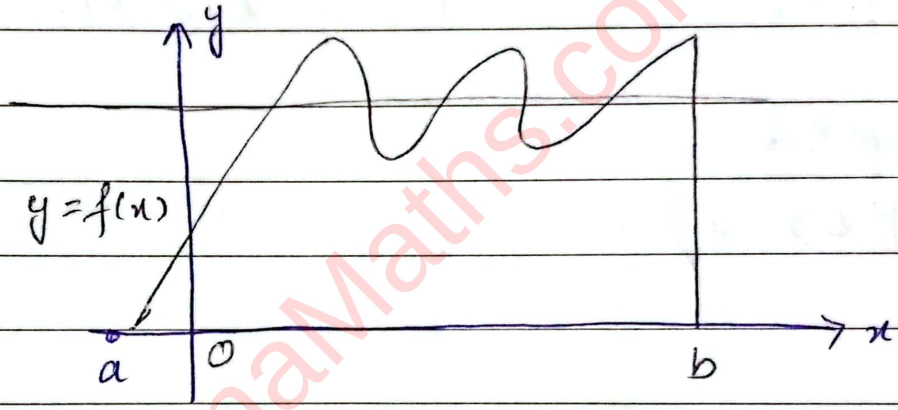
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{e^{1/x}-1}{e^{1/x}+1} = \frac{0-1}{0+1} = -1$$

Thus $\lim_{x\to 0} f(x)$ does not exist and so f is discontinuous at $x=0$.

Intermediate Value Theorem:

Let f be continuous on $[a, b]$ and $c \in R$ such that $f(a) < c$ and $f(b) > c$. Then there is atleast one point $x_0 \in [a, b]$ such that $f(x_0) = c$

There may be more than one such points x_0 as is clear from the figure.



Theorem:

If f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ are of opposite signs, then $f(c_0) = 0$ for at least one $c_0 \in]a, b[$.

Boundedness Theorem;

If f is continuous on $[a, b]$, then it is bounded theorem, (i.e the range of f is bounded.

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Extreme Value Theorem:

If f is continuous on $[a,b]$ and M and m are supremum and infimum of f respectively on this interval, then f assumes each of the values M and m at least once in $[a,b]$ i.e,
there exist $c, d \in [a,b]$ such that
$f(c) = m$ and $f(d) = M$.

Example:

$$g(x) = \begin{cases} x^3 & \text{if } x < 1 \\ -4-x^2 & \text{if } 1 \leq x \leq 10 \\ 6x^2+46 & \text{if } x > 10 \end{cases}$$

Find discontinuity of fn.

Solution:

At $x = 1$

$$g(1) = -4-(1)^2 = -4-1 = -5$$

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} (x^3) = (1)^3 = 1$$

$$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} -4-x^2 = -4-1 = -5$$

$$\lim_{x \to 1^-} g(x) \neq \lim_{x \to 1^+} g(x).$$

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$g(x)$ is discont at $x = 1$.

At $x = 10$

$$g(10) = -4 - (10)^2 = -4 - 100$$
$$= -104$$

$$\lim_{x \to 10^-} g(x) = \lim_{x \to 10} -4 - x^2$$

$$= -4 - (10)^2 = -104$$

$$\lim_{x \to 10^+} g(x) = \lim_{x \to 10^+} 6(10)^2 + 46$$

$$= 6(100) + 46$$
$$= 646$$

$$\lim_{x \to 10^+} g(x) \neq \lim_{x \to 10^-} g(x).$$

$g(x)$ is also discontinuous at $x = 10$.