Lecture 1

Topics to Recall

1- Vector
2- Scalar
3- Vector Addition
4- Scalar Multiplication with a vector.
5- Position vector.
6- Unit vector.
7- Direction Cosine.
8- Scalar Mul/Product OR Dot product. With properties.
9- Vector product OR Cross product with properties.
10- Scalar Triple product With Properties.
11- Vector Triple product with Properties.
12- Vector product of more than two vectors.

Equation Of Plane

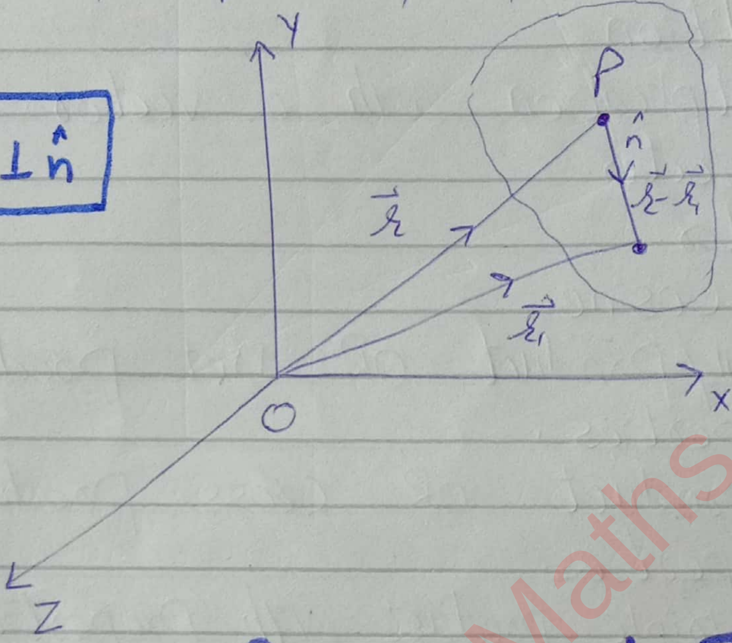
If $\vec{r}$ is the position vector of point P with $\hat{n}$ as the unit normal vector to the plane then the eqn $\vec{r}\cdot\hat{n} = p \longrightarrow (1)$ is the Lar distance of the plane from the origin. This is obviously eqn of such a plane which is at a distance p from origin.

If $\vec{r}_1$ be the position vector of any other point then eqn of plane will be

$$(\vec{r}-\vec{r}_1)\cdot\hat{n}=0 \longrightarrow (2)$$

eq/n (1) and eq/n (2) are different form of eq/n of plane.

$\boxed{r-r_1 \perp \hat{n}}$



where $\vec{r}-\vec{r}_1$ is $\perp$ $\hat{n}$

Curves With Torsion.

Curve A curve is a locus of a point whose position vector $\vec{r}$ relative to a fixed origin may be expressed as a ftn of a single Variable (Single Parameter say 't'). Then its Cartesian co-ordinates are also ftn of the same parameter. Then the eq/n of curve in parametric form can be written as;

$x=f_1(t)$, $y=f_2(t)$, $z=f_3(t)$

f_1, f_2, f_3 are ftns of t.

Space Curve OR Twisted Curve OR Skew Curve OR Tortious Curve.

When all the pts of a curve do not lye in the same plane then it is said to be a space curve OR skew curve OR tortious curve otherwise a plane curve.

Q If $\vec{r}(t) = a\cos t\,\hat{i} + b\sin t\,\hat{j} + 0\,\hat{k}$ is a p.s of a curve then find that curve?

For requirement compare eqn ① with eqn ②

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \rightarrow ②$$

$x = a\cos t$ $\qquad$ $\cos t = \dfrac{x}{a} \rightarrow ③$

$y = b\sin t$ $\qquad$ $\sin t = \dfrac{y}{b} \rightarrow ④$

$z = 0$

By squaring and adding ③, ④

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Therefore required curve is an ellipse.

Q Find the eqn of tangent of a point on a curve?

Solution;

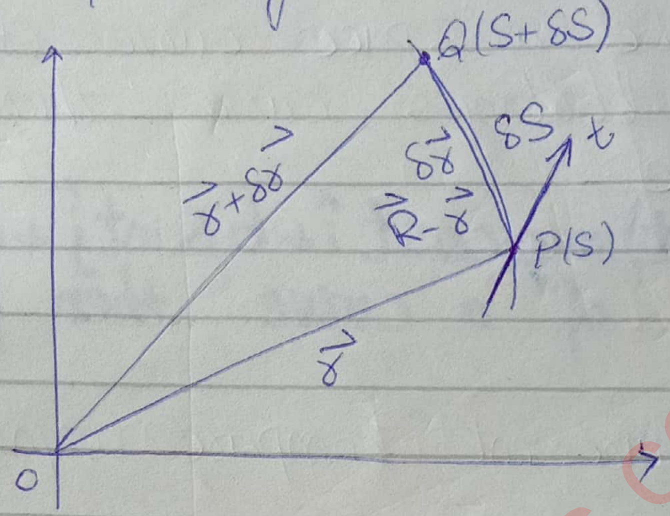
Let P and Q be the pts on the given curve C whose position vectors are $\vec{r}$ and $\vec{r} + \delta\vec{r}$ respectively corresponding

$\frac{1}{\delta S}(\delta \vec{r}) \parallel \delta \vec{r}$

$\vec{r} + \vec{PQ} = \vec{R}$

$\vec{PQ} = \vec{R} - \vec{r}$

to the values of parameter S and $S + \delta S$ respectively.



$\frac{\delta r}{\delta S} \parallel \delta r$

$OP + PQ = OQ$

$r + PQ = r + \delta r$

$PQ = \delta r$

From Figure $\delta \vec{r}$ is the vector PQ and quotient value $\frac{\delta \vec{r}}{\delta S}$ is the vector along $\delta \vec{r}$ $\left(\frac{\delta \vec{r}}{\delta S} \parallel \delta \vec{r}\right)$ and for the limit i.e, $\lim \delta S \to 0$ this becomes the tangent of point P.

If t is the tangent at point P then

$$\boxed{t = \lim_{\delta S \to 0} \frac{\delta \vec{r}}{\delta S} = \frac{d\vec{r}}{ds} = \vec{r}'}$$

If in figure $\vec{r} + \delta \vec{r}$ replace with $\vec{R}$ then the ~~curve~~ vector along PQ will be $\vec{R} - \vec{r}$. But there is another vector along PQ which is tangent t, therefore $\vec{R} - \vec{r}$, is parallel to t. that is written as,

$$\vec{R} - \vec{r} = u\vec{t}$$

$$\boxed{\vec{R} = \vec{r} + u\vec{t}} \longrightarrow (2)$$

u is any real number. Eqn (2) is required eqn of tangent.

Q Find the eq/n of tangent to the curve? whose coordinates are

Solution; $x = a\cos\theta$, $y = a\sin\theta$, $z=0$

$\because \quad \vec{R} - \vec{r} = u\vec{t}$

$$\frac{\vec{R}-\vec{r}}{\vec{t}} = u$$

when we take

$\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$t = \vec{r}' = x'\hat{i} + y'\hat{j} + z'\hat{k}$

$$\frac{X-x}{x'} = \frac{Y-y}{y'} = \frac{Z-z}{z'} = u \longrightarrow (1)$$

$\vec{x}' = -a\sin\theta \qquad \vec{y}' = a\cos\theta \qquad \vec{z}' = 0$

$$\frac{X - a\cos\theta}{-a\sin\theta} = \frac{Y - a\sin\theta}{a\cos\theta} = \frac{Z-0}{0} = u$$

neglect

$$\frac{X - a\cos\theta}{-\cancel{a}\sin\theta} = \frac{Y - a\sin\theta}{\cancel{a}\cos\theta}$$

$\cos\theta\,(X - a\cos\theta) = -\sin\theta\,(Y - a\sin\theta)$

$X\cos\theta - a\cos^2\theta = -Y\sin\theta + a\sin^2\theta$

$$\boxed{X\cos\theta + Y\sin\theta = a}$$

$\frac{X-x}{x'} = \frac{Y-y}{y'} = \frac{Z-z}{0}$

$t = \vec{r}'$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$

$\frac{R-r}{t} = u$

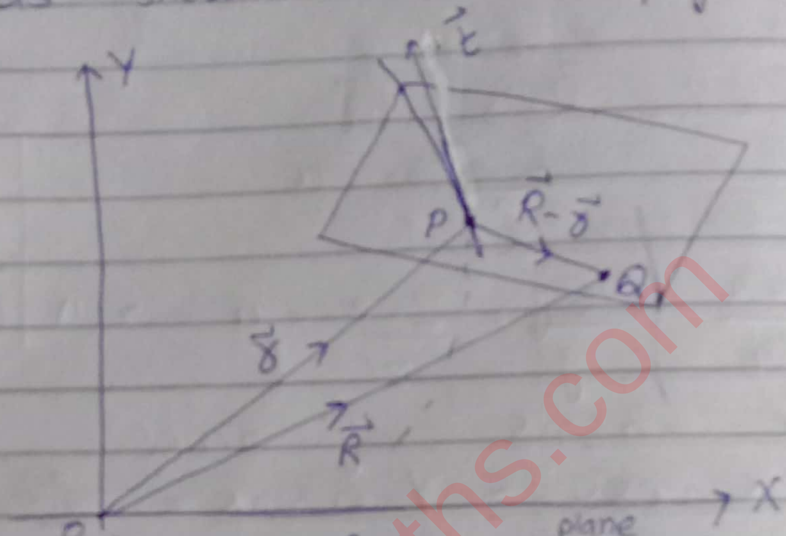
NORMAL PLANE ;

Derive the eq/n

The plane normal to the tangent to a curve at the point of contact is called the normal plane at that point

Let P be a point on the curve with position vector $\vec{r}$ and Q be another point with position vector $\vec{R}$ and $\vec{R}-\vec{r}$ is the position vector of any line in the plane as shown in the fig.

$\vec{R}-\vec{r} \parallel \vec{t}$



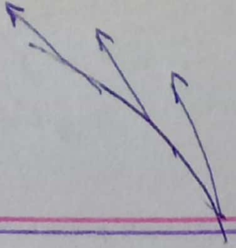
By the definition of Normal plane we can say that $\vec{R}-\vec{r}$ is $\perp$ to the tangent $\vec{t}$ of curve i.e $(\vec{R}-\vec{r})\cdot\vec{t}=0$ which is the eqn of normal plane $\therefore$ every line through P in this plane is normal to the curve.

rate of rotation $\frac{d\theta}{ds}$

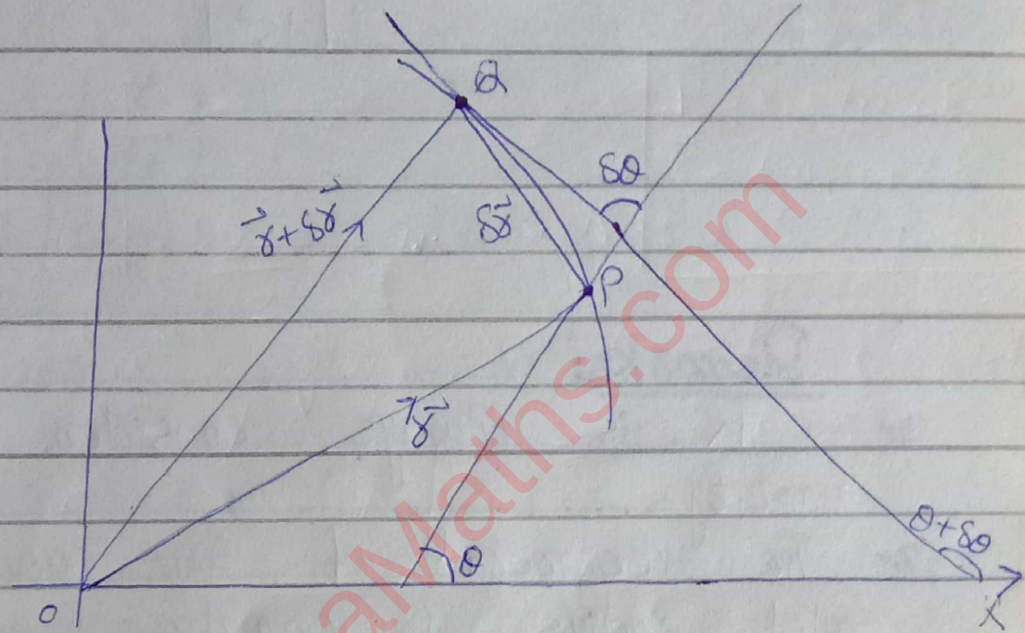
Curvature:

The curvature of a curve at any point is defined as the ~~arc~~ rate of rotation of the tangent. It is called 1st curvature or circular curvature. It is denoted by K (kepa)

Derived an expression of Curvature



Let C be a curve and OX be a fixed ~~position~~ direction/horizontal
Let P and Q be two points with position vector $\vec{r}$ and $\vec{r}+\delta\vec{r}$



Let tangent at P makes an angle θ with x-axis and tangent at Q makes an angle $\theta+\delta\theta$ with x-axis. This means that $\delta\theta$ is the angle b/w tangents at P and Q.

If S is the parameter of the curve and δS is the length for PQ i.e, $\delta S = \vec{PQ}$

By the definition of the curvature $\frac{\delta\theta}{\delta S}$ is the average curvature of arc PQ and for the limiting case i.e, $\lim_{\delta S \to 0} \frac{\delta\theta}{\delta S} = \frac{d\theta}{ds} = \kappa$

Arc rate of rotation $\frac{\delta\theta}{\delta S}$

$r' = t$

$r'' =$ curvature

$$\lim_{\delta s \to 0} \frac{\delta\theta}{\delta s} = \frac{d\theta}{ds} = \theta' = K$$

If $\vec{r}$ is the position vector of point P on the curve then $\left|\frac{d^2 r}{ds^2}\right| = K$

$\frac{dt}{ds} = |t'| = K$

$\left|\frac{d^2 r}{ds^2}\right| = k$

Remarks:

1- K is always consider as +ve quantity.

2- The reciprocal of ~~the~~ curvature is called radius of curvature and it is denoted by ρ i.e, $\rho = \frac{1}{K}$

or $K = \frac{1}{\rho}$

Q Find the curvature of curve $x = a\sin\theta$, $y = a\cos\theta$, & $z = a\theta\cot B$

$\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\Rightarrow \vec{r} = a\sin\theta\,\hat{i} + a\cos\theta\,\hat{j} + a\theta\cot B\,\hat{k}$

$\frac{dr}{ds} = \frac{dr}{d\theta} \cdot \frac{d\theta}{ds}$

$\Rightarrow \frac{dr}{ds} = (a\cos\theta\,\hat{i} - a\sin\theta\,\hat{j} + a\cot B\,\hat{k})\,\theta'$

$|t| = \sqrt{a^2 + a^2\cot^2 B}\;\theta'$

$$1 = a \operatorname{cosec} B \, \theta'$$

$$\theta' = \frac{\sin B}{a} \Rightarrow \theta'' = 0$$

$$\therefore \frac{d\gamma}{ds} = (a\cos\theta\,\hat{i} - a\sin\theta\,\hat{j} + a\cos B\,\hat{k})\cdot\theta'$$

Now, $$\frac{d^2\gamma}{ds^2} = \frac{d}{ds}\left(\frac{d\gamma}{ds}\right)$$

$$= \frac{d}{d\theta}\left[(a\cos\theta\,\hat{i} - a\sin\theta\,\hat{j} + a\cot B\,\hat{k})\cdot\theta'\right]\cdot\frac{d\theta}{ds}$$

$$= \left[(-a\sin\theta\,\hat{i} - a\cos\theta\,\hat{j} + 0)\,\theta' + \theta''(a\cos\theta\,\hat{i} - a\sin\theta\,\hat{j} + a\cot B\,\hat{k})\right]\theta'$$

$$\because \theta'' = 0$$

$$\frac{d^2\gamma}{ds^2} = (-a\sin\theta\,\hat{i} - a\cos\theta\,\hat{j})\cdot(\theta')^2$$

$$\left|\frac{d^2\gamma}{ds^2}\right| = \sqrt{(a^2\sin^2\theta + a^2\cos^2\theta)\left(\frac{\sin^2 B}{a^2}\right)}$$

$$= \sqrt{a^2}\left(\frac{\sin^2 B}{a^2}\right) = a\left(\frac{\sin^2 B}{a^2}\right)$$

$$\left|\frac{d^2\gamma}{ds^2}\right| = \frac{\sin^2 B}{a}$$

$$\boxed{K = \frac{\sin^2 B}{a}}$$

Plane of Curvature or Osculating plane;

The plane through point P and containing the tangents at two cosecutive points is called the osculating plane. if $\bar{\gamma}$ is the

$$\begin{bmatrix} X-x & Y-y & Z-z \\ x' & y' & z' \\ \frac{x''}{k} & \frac{y''}{k} & \frac{z''}{k} \end{bmatrix} = 0$$

position vector of any on the curve and $\vec{R}$ is the position vector of the point very near to P. Then the three vectors $\vec{R}-\vec{r}$, $\vec{t}$, & $\vec{n}$ are coplanar

i.e, $[\vec{R}-\vec{r}, \vec{t}, \hat{n}] = 0 \rightarrow (1)$

This equation is called an eqn of osculating plane.

eqn ① $\Rightarrow$

$$\left[\vec{R}-\vec{r}, \vec{r}', \frac{\vec{r}''}{k}\right] = 0 \rightarrow (2)$$

eqn (2) is another form of osculating plane

$k = r''$

$1 \cdot k = |r''|$

$|\vec{n}| \cdot k = |r''|$

$k = \frac{|r''|}{|\vec{n}|}$

$\Rightarrow k = \frac{r''}{n}$

$\Rightarrow n = \frac{r''}{k}$

Q Find the eqn. of osculating plane, find the curve?

$$\vec{r}(s) = a\cos s\,\hat{i} + a\sin s\,\hat{j} + 0\,\hat{k}$$

$\therefore$ eqn of osculating plane is

$$\left[\vec{R}-\vec{r}, \vec{r}', \frac{\vec{r}''}{k}\right] = 0 \Rightarrow \begin{vmatrix} X-x & Y-y & Z-z \\ x' & y' & z' \\ \frac{x''}{k} & \frac{y''}{k} & \frac{z''}{k} \end{vmatrix} = 0$$

$$\begin{vmatrix} X-a\cos s & Y-a\sin s & Z-0 \\ -a\sin s & a\cos s & 0 \\ -a\cos s & -a\sin s & 0 \end{vmatrix} = 0$$

$$-a\sin s(-a\sin s) + a^2\cos^2 s = 0$$

$$a^2(\cos^2 s + \sin^2 s) = 0$$

$$Z(a^2) = 0$$

$$Z = 0 \qquad a \neq 0$$

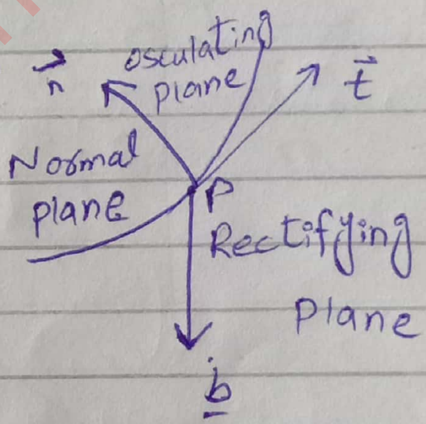
Lecture # 5, 6

Binormal And Principal Normal;

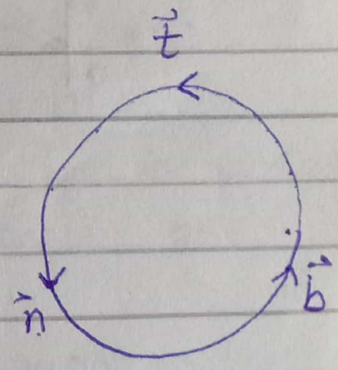
A vector ⊥ar to the tangent at P and line on the osculating plane at that point P is called the <u>principal normal</u> at point P, and it is denoted by $\vec{n}$.

The normal at P which is ⊥ar to the osculating plane is called binormal.
It is denoted by $\vec{b}$.

Note that $\vec{t}$ $\vec{n}$ and $\vec{b}$ are mutually ⊥ar. and these satisfy all the properties of rectangular coordinate system.



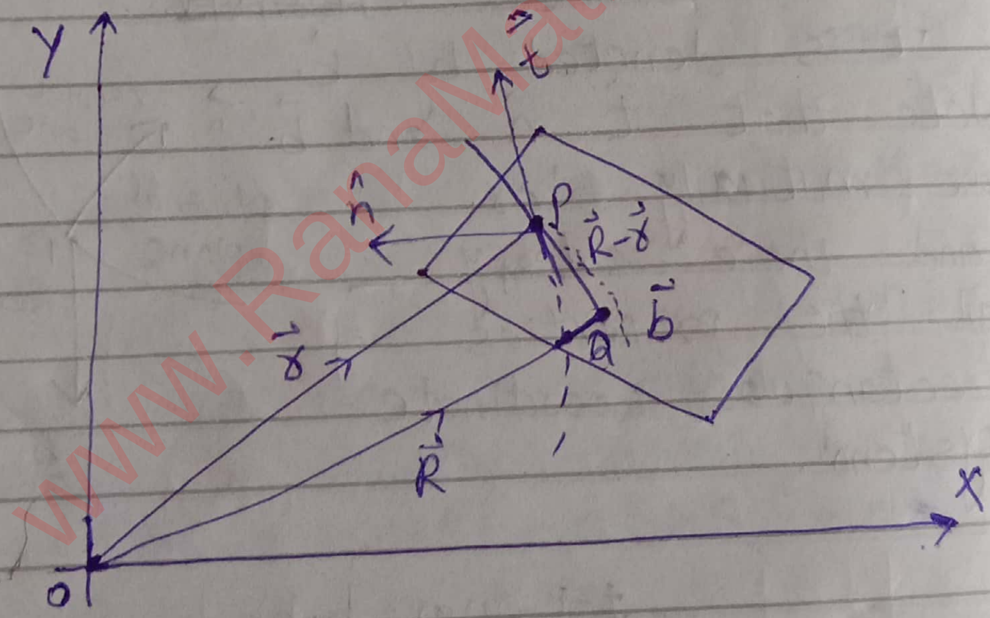
$$t\cdot t = n\cdot n = b\cdot b = 1$$
$$t\cdot n = n\cdot b = b\cdot t = 0$$
$$t \times n = b$$
$$n \times b = t$$
$$b \times t = n$$
$$t\times t = n\times n = b\times b = 0$$



The triplet $(\vec{t}, \vec{n}, \vec{b})$ is unit tangent, unit normal and unit binormal is called the moving trihedron.

Equation of Binormal;

Let P be a point on curve with position vector $\vec{r}$ and take another point Q with position vector R and a unit vector $\vec{b}$ $\perp$ar to the osculating plane is taken as binormal.



Then from fig we can write

$$\vec{R}-\vec{r}=\mu\vec{b}$$

$$\boxed{\vec{R}=\vec{r}+\mu\vec{b}} \longrightarrow (1)$$

The rotation of osculating plane is called torsion. $\frac{r\,\sigma''}{K}$

$$R = r + \mu\,(t \times n)$$

$$R = r + \mu\left(r' \times \frac{r''}{K}\right)$$

$$R = r + \frac{\mu}{K}\,(r' \times r'')$$

$$R = r + \nu\,(r' \times r'') \longrightarrow (2)$$

TORSION

Torsion is defined as the rate of rotation of osculating plane.
It can also be defined as,
The rate of turning of the binormal is called torsion of the curve at any point P.
It is denoted by τ.

Curvature is always +ve but torsion may be +ve or -ve.
Torsion is regarded as +ve if rotation of binormal is in the sense as S (Parameter) increases i.e, is of same sense as of right handed screw

Radius of Torsion;

is defined as the reciprocal of torsion it is denoted by $\sigma = \frac{1}{\tau}$ or $\tau = \frac{1}{\sigma}$

Serret - Frenet Formulas:

1- $\underline{t}' = \frac{d\underline{t}}{ds} = K\underline{n}$

2- $b' = \frac{d\underline{b}}{ds} = -\tau n$

3- $\vec{n}' = \frac{d\underline{n}}{ds} = \tau\vec{b} - K\vec{t}$

Let $\vec{t}$ be the tangent for the point of the curve then we can write $\vec{t}\cdot\vec{t} = 1$

Diff w.r.t "s"

$$2\vec{t}\,\frac{d\vec{t}}{ds} = 0$$

$$\vec{t}\,\frac{d\vec{t}}{ds} = 0$$

$t \neq 0$ $\qquad \frac{d\vec{t}}{ds} = 0$

$$\frac{d\vec{t}}{ds} \parallel d\vec{t}$$

or $\frac{\delta\vec{t}}{\delta s} \parallel \delta t$

$\Rightarrow$ The quotient $\frac{\delta t}{\delta s} \parallel \delta t$ and for the $\lim\limits_{\delta s \to 0} \frac{\delta t}{\delta s}$ its direction is Lar to the tangent.

$\vec{r}'' = kn$ $\quad k = |\vec{r}''|$

$1 \cdot k = |\vec{r}''|$

$|\hat{n}| \cdot k = \vec{r}''$

$k = \frac{\vec{r}''}{\hat{n}}$

$\therefore$ the Limiting value of $\frac{\delta t}{\delta s}$ is same as that of $\lim_{\delta s \to 0} \frac{\delta \theta}{\delta s}$ which is k.

Here we can write

$$\vec{r}'' = \frac{dt}{ds} = \lim_{\delta s \to 0} \frac{\delta t}{\delta s} = kn$$

$\Rightarrow \vec{r}'' = kn \rightarrow (1)$

(ii) Let b be a binormal

$\vec{b} \cdot \vec{b} = 1$

$b^2 = 1$

$2b \frac{db}{ds} = 0$

$b \neq 0 \quad \frac{db}{ds} = 0$

$\vec{b} \cdot \vec{b} = 1$

$2\vec{b} \cdot \frac{d\vec{b}}{ds} = 0$

$\vec{b} \cdot \vec{b}' = 0$

$\Rightarrow \vec{b}$ lar $\vec{b}' \rightarrow (1)$

consider the relation

$$\vec{t} \cdot \vec{b} = 0$$

$$\vec{t} \cdot \vec{b}' + \vec{t}' \cdot \vec{b} = 0$$

$$\vec{t} \cdot \vec{b}' + k\vec{n} \cdot \vec{b} = 0$$

$$\vec{t} \cdot \vec{b}' + k(0) = 0$$

$$\vec{t} \cdot \vec{b}' = 0$$

$$\vec{t} \text{ lar } \vec{b}' \rightarrow (2)$$

From result (1) and (2) b' is the vector which is parallel to $\vec{n}$.

* i.e, $\vec{b}' = -\tau \vec{n}$

$-ve$ sign is choosen to keep the torsion $+ve$

(iii) Consider

$$\vec{n} = \vec{b} \times \vec{t}$$

d

$$\frac{dn}{ds} = \frac{db}{ds} \times t + b \times \frac{dt}{ds}$$

$$= b' \times t + b \times t'$$

$$= b' \times t + b \times r''$$

$$= b' \times t + b \times (kn)$$

$$= b' \times t + k(b \times n)$$

$$= b' \times t - k\vec{t}$$

$$= -\tau n \times t - k\vec{t}$$

$$= -\tau(n \times t) - k\vec{t}$$

$$= \tau\vec{b} - k\vec{t}$$

Lecture #7,8

Q Let r is the position vector of the point on the curve with Parameter s then find $\gamma'(s)$, $\gamma''(s)$, $\gamma'''(s)$ and $\gamma^{iv}(s)$.

As,

$$r = r(s)$$

$$r' = \frac{dr}{ds} = \vec{t}$$

$$r'' = \frac{d^2\vec{r}}{ds^2} = \frac{d}{ds}\left(\frac{dr}{ds}\right)$$

$$= \frac{d}{ds}(\vec{t})$$

$$\gamma'' = k\vec{n}$$

$$r''' = \frac{d}{ds}\left(\frac{d^2 r}{ds^2}\right)$$

$$= \frac{d}{ds}(k\vec{n})$$

$$= k\vec{n}' + k'\vec{n}$$

$$= k(\tau\vec{b} - k\vec{t}) + k'\vec{n}$$

$$r''' = k\tau\vec{b} - k^2\vec{t} + k'\vec{n}$$

$$r^{iv} = \frac{d}{ds}\left(\frac{d^3 r}{ds^3}\right)$$

$$= \frac{d}{ds}\left(k\tau\vec{b} - k^2\vec{t} + k'\vec{n}\right)$$

$$= k\tau\vec{b}' + k'\tau\vec{b} + k\tau'\vec{b} - (2kk'\vec{t}) - k^2 t' + k''\vec{n} + k'\vec{n}'$$

$n' = \tau b - kt \quad \because \quad b' = -\tau n \quad$ and $\quad t' = kn$

$$= k\tau(-\tau n) + k'\tau b + k\tau' b - 2kk't - k^2 \cdot kn + k''n + k'(\tau b - kt)$$

$$= -k\tau^2 n + k'\tau b + k\tau' b - 2kk't - k^3 n + k''n + k'\tau b - kk't$$

$$= (k'' - k^3 - k\tau^2)\vec{n} + (-3kk')\vec{t} + (2k'\tau + k\tau')\vec{b}$$

$\frac{dr}{ds} = t$ $r' = t$

Suppose the path traced by the particle $x = x(t)$ is then prove that the

(i) Acceleration vector lies in the osculating plane.

(ii) Find the tangential and normal components of acceleration.

Since we have $x = x(t)$

Diff w.r.t "t" $$\frac{d\vec{x}}{dt} = \frac{dx}{ds} \cdot \frac{ds}{dt}$$

$$\frac{d\vec{x}}{dt} = \vec{t} \cdot \frac{ds}{dt}$$

$$\frac{d}{ds}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2} \cdot \frac{dt}{ds}$$

$$\frac{d^2\vec{x}}{dt^2} = \frac{d}{ds}\left(\vec{t} \cdot \frac{ds}{dt}\right) \cdot \frac{ds}{dt}$$

$$\frac{d^2\vec{x}}{dt^2} = \left[\vec{t}' \cdot \frac{ds}{dt} + \frac{d^2s}{dt^2}\frac{dt}{ds}\vec{t}\right] \cdot \frac{ds}{dt}$$

$$\frac{d^2\vec{x}}{dt^2} = \vec{t}' \cdot \left(\frac{ds}{dt}\right)^2 + \frac{d^2s}{dt^2} \cdot \vec{t}$$

$t' = r'' = kn$ $$\vec{a} = k \cdot \left(\frac{ds}{dt}\right)^2 \cdot n + \frac{d^2s}{dt^2} \cdot t$$

Tangential component of $\vec{a} = \frac{d^2s}{dt^2}$

Normal component of $\vec{a} = k\left(\frac{ds}{dt}\right)^2$

Assignment#1

Q Prove that the position vector of current point $\vec{r}$ on a curve with parameter s satisfies the following diff eqns.

(i) $$\frac{d}{ds}\left\{\sigma\frac{d}{ds}\left(\rho\frac{d^2\vec{r}}{ds^2}\right)\right\}+\frac{d}{ds}\left(\frac{\sigma}{\rho}\frac{d\vec{r}}{ds}\right)+\frac{\rho}{\sigma}\frac{d^2\vec{r}}{ds^2}=0$$

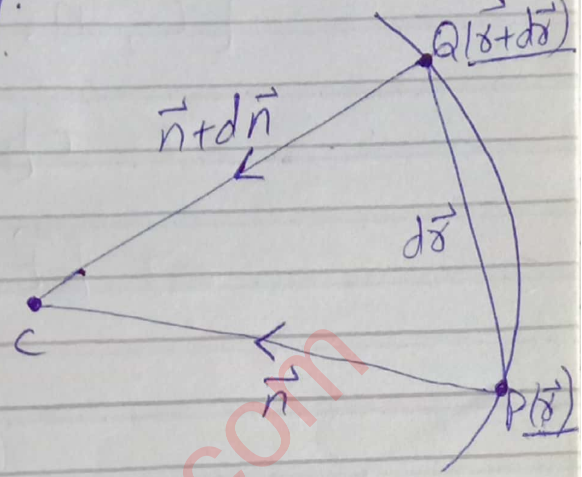
(ii) $$\tau=\frac{1}{\kappa^2}\left[r', r'', r'''\right]$$

Q Show that the principle normals at consecutive pts do not intersect unless $\tau=0$

Note: If the curve is a planar curve then the two principle normals will intersect and no question will arise like above. This question arise only for non-planar curve for which the principle normal at any consecutive pts do not intersect becoz of torsion in the curve.

Proof

Let P and Q are/be two consecutive pt on the curve with position vector $\vec{r}$ and $\vec{r}+d\vec{r}$ and principle normals $\vec{n}$ and $\vec{n}+d\vec{n}$ respectively.



Suppose that these principle normals intersect then the vectors $d\vec{r}$, $\vec{n}$, and $\vec{n}+d\vec{n}$ are coplanar

i.e,
$$\left[d\vec{r}, \vec{n}, \vec{n}+d\vec{n}\right] = 0$$

or
$$\left[d\vec{r}, \vec{n}, \vec{n}\right] + \left[d\vec{r}, \vec{n}, d\vec{n}\right] = 0$$

or
$$0 + \left[d\vec{r}, \vec{n}, d\vec{n}\right] = 0$$

$$(ds)(ds)\left[\frac{d\vec{r}}{ds}, \vec{n}, \frac{d\vec{n}}{ds}\right] = 0$$

$$(ds)^2\left[\vec{r}', \vec{n}, \vec{n}'\right] = 0$$

$$(ds)^2\left[\vec{t}, \vec{n}, \tau\vec{b} - k\vec{t}\right] = 0$$

$$\Rightarrow \left[\vec{t}, \vec{n}, \tau\vec{b}\right] - \left[\vec{t}, \vec{n}, k\vec{t}\right] = 0$$

$R_1 = R_3$

$$\Rightarrow [\vec{t}, \vec{n}, \tau\vec{b}] - k[\vec{t}, \vec{n}, \vec{t}] = 0$$

$$[\vec{t}, \vec{n}, \tau\vec{b}] - 0 = 0$$

$$\tau[\vec{t}, \vec{n}, \vec{b}] = 0$$

$$\Rightarrow [\vec{t}, \vec{n}, \vec{b}] \neq 0$$

$$\Rightarrow \tau = 0$$

Proved.

بڑے Formula imp ہے

Important Formulas; Of Curvature;

Torsion

Case-I When the given curve has a point with position vector $\vec{r}$ which is the ftn of its arc length say s i.e,

$$\vec{r} = \vec{r}(s)$$

Then $k = |\vec{r}''| = \sqrt{(x'')^2 + (y'')^2 + (z'')^2}$

and $\tau = \dfrac{1}{k^2}[r', r'', r''']$

Case-II

When in the given curve the position vector $\vec{r}$ is the ftn of some general parameter say u. Let

$$\vec{r} = \vec{r}(u)$$

$$\frac{d\vec{r}}{du} = \frac{d\vec{r}}{ds} \cdot \frac{ds}{du}$$

$$r' = \frac{d\vec{r}}{du} = t \cdot s' \longrightarrow ①$$

$$\frac{d^2\vec{r}}{du^2} = \frac{d}{du}\left(\frac{dr}{du}\right) = \frac{d}{du}\left(t \cdot s'\right)$$

$$r'' = \frac{d}{du}\left(t \cdot \frac{ds}{du}\right)$$

$\frac{d}{ds} \cdot \frac{ds}{du}$

$$= \frac{d}{ds}\left(t \cdot s'\right) \frac{ds}{dt}$$

$\frac{d}{du}\left(\frac{ds}{du}\right)\frac{du}{dt}$

$$= \left(\frac{dt}{ds} \cdot s' + t \qquad\right) \cdot \frac{ds}{dt}$$

$\frac{d^2s}{du^2}$

$$= \frac{d}{du}(t \cdot s')$$

$$= t \cdot s'' + s' \frac{d}{du}(t)$$

$$= t \cdot s'' + s' \left(\frac{d}{ds} t \cdot \frac{ds}{du} \right)$$

$$= t \cdot s'' + s' (kn \cdot s')$$

$$\vec{\gamma}'' = t \cdot s'' + kn \cdot (s')^2 \longrightarrow (2)$$

$$\vec{\gamma}''' = \frac{d}{du} \left[t \cdot s'' + kn \cdot (s')^2 \right]$$

$$\frac{dt}{ds} \cdot \frac{ds}{du}$$

$$= \left[t \cdot s''' + s's'' \cdot \frac{d}{du}(t) \right] + k'n(s')^2 + kn'(s')^3 + kn \cdot 2s' \cdot s''$$

$$\frac{dn}{du} = \frac{dn}{ds} \cdot \frac{ds}{du}$$

$$= s''' \cdot \underline{t} + s''s' k\underline{n} + k'\underline{n}(s')^2 + k\underline{n}'(s')^3 + 2s's'' k\underline{n}$$

$$=$$

$$\gamma''' = \gamma'\gamma'' k\vec{n} + \gamma''' t + k'(\gamma') n + k(\gamma')^3(\tau b - kt) + 2k\gamma'\gamma''\vec{n} \rightarrow (3)$$

$t \times t = 0$

$t \times k =$

$$\gamma' \times \gamma'' = (\vec{t} \cdot \gamma') \times [\vec{t} \cdot s'' + k\vec{n}(\gamma')^2] \rightarrow$$

$$\gamma' \times \gamma'' = kb(\gamma')^3 \rightarrow (4)$$

$$|\gamma' \times \gamma''| = k(\gamma')^3 \qquad |b| = 1$$

$$k = \frac{|\gamma' \times \gamma''|}{(\gamma')^3} \rightarrow$$

and for τ

$$\vec{\gamma}' \times \vec{\gamma}'' \cdot \vec{\gamma}''' = k^2 (s')^6 \tau \, b \cdot b$$

$$[\vec{\gamma}' \;\; \vec{\gamma}'' \;\; \vec{\gamma}'''] = k^2 (s')^6 \tau$$

$$\Rightarrow \tau = \frac{[\vec{\gamma}' \;\; \vec{\gamma}'' \;\; \vec{\gamma}''']}{k^2 (\vec{\gamma}')^6}$$

Lecture #9,10

$$K = \frac{|\gamma' \times \gamma''|}{(\gamma')^3} \qquad \tau = \frac{[\gamma', \gamma'', \gamma''']}{K^2 (\gamma')^6}$$

If we don't use s as parameter then above formulas become

$$K = \frac{|\gamma' \times \gamma''|}{(\gamma')^3} \qquad \tau = \frac{[\gamma', \gamma'', \gamma''']}{K^2 (\gamma')^6}$$

Q For the curve

$$x = a(3u - u^3)$$

$$y = 3au^2, \qquad z = a(3u + u^3)$$

then prove that $K = \tau$

$$\vec{\gamma} = (x, y, z)$$

$$\vec{\gamma} = (a(3u - u^3), 3au^2, a(3u + u^3))$$

$$\vec{\gamma}' = (a(3 - 3u^2), 6au, a(3 + 3u^2)) \rightarrow (1)$$

$$K = \frac{|\gamma' \times \gamma''|}{(\gamma')^3} \qquad \tau = \frac{1}{K(\gamma')^6} [\gamma', \gamma'', \gamma''']$$

$$\vec{\gamma}'' = (-6au, 6a, 6au) \rightarrow (2)$$

$$\vec{\gamma}''' = (-6a, 0, 6a) \rightarrow (3)$$

$$|\vec{\gamma}'| = \sqrt{a^2(3 - 3u^2)^2 + 36a^2u^2 + a^2(3 + 3u^2)^2}$$

$$= \sqrt{a^2(9 + 9u^4 - 18u^2) + 36a^2u^2 + a^2(9 + 9u^4 + 18u^2)}$$

$$= \sqrt{9a^2 + 9a^2u^4 \cancel{- 18a^2u^2} + 36a^2u^2 + 9a^2 + 9a^2u^4 \cancel{+ 18a^2u^2}}$$

$$= \sqrt{18a^2 + 18a^2u^4 + 36a^2u^2}$$

$$= \sqrt{2(9a^2 + 9a^2u^4 + 18a^2u^2)}$$

$$= \sqrt{2}\sqrt{(3a)^2 + 2(3a)(3au^2) + (3au^2)^2}$$

$$= \sqrt{2}\sqrt{(3a + 3au^2)^2}$$

$$= \sqrt{2}(3a + 3au^2)$$

$$|\vec{r}'| = 3\sqrt{2}\,a\,(1+u^2)$$

$$r' \times r'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a(3-3u^2) & 6au & a(3+3u^2) \\ -6au & 6a & 6au \end{vmatrix}$$

$$= \hat{i}\left[36a^2u^2 - 6a^2(3+3u^2)\right] - \hat{j}\left[6a^2u(3-3u^2) + 6a^2u(3+3u^2)\right] + \hat{k}\left[6a^2(3-3u^2) + 36a^2u^2\right]$$

$$= \hat{i}\left[36a^2u^2 - 18a^2 - 18a^2u^2\right] - \hat{j}\left[18a^2u - \cancel{18a^2u^3} + 18a^2u + \cancel{18a^2u^3}\right] + \hat{k}\left[18a^2 - 18a^2u^2 + 36a^2u^2\right]$$

$$= \hat{i}(18a^2u^2 - 18a^2) - \hat{j}(36a^2u) + \hat{k}(18a^2 + 18a^2u^2)$$

$$= 18a^2\left[(u^2-1)\hat{i} - 2u\hat{j} + (1+u^2)\hat{k}\right]$$

$$|r' \times r''| = \sqrt{(18a^2)^2\left[(u^2-1)^2 + (-2u)^2 + (1+u^2)^2\right]}$$

$$= \sqrt{(18a^2)^2\left[u^4 + 1 - \cancel{2u^2} + 4u^2 + 1 + u^4 + \cancel{2u^2}\right]}$$

$$= 18a^2\sqrt{2u^4 + 2 + 4u^2}$$

$$= \sqrt{2}\, 18a^2 \sqrt{u^4 + 2u^2 + 1}$$

$$|r' \times r''| = 18\sqrt{2}\, a^2 \sqrt{(u^2+1)^2}$$
$$= 18\sqrt{2}\, a^2 (u^2+1)$$

$$k = \frac{|r' \times r''|}{(r')^3} = \frac{18\sqrt{2}\, a^2 (u^2+1)}{\left[3\sqrt{2}\, a\,(u^2+1)\right]^3}$$

$$= \frac{\overset{2}{\cancel{18}}\cancel{\sqrt{2}}\, \cancel{a^2}\, \cancel{(u^2+1)}}{\underset{3}{\cancel{27}}\,(2)\cancel{\sqrt{2}}\, a^{\cancel{3}}\, (u^2+1)^{\cancel{3}}}$$

$$= \frac{\cancel{2}}{3\cancel{6}\, a\,(u^2+1)^2} = \frac{1}{3a(u^2+1)^2}$$

Now

$$[r', r'', r'''] = \begin{vmatrix} a(3-3u^2) & 6au & a(3+3u^2) \\ -6au & 6a & 6au \\ -6a & 0 & 6a \end{vmatrix}$$

$$= a(3-3u^2)\left[36a^2 - 0\right] - 6au\left[-36a^2u + 36a^2u\right] + a(3+3u^2)\left[0 + 36a^2\right]$$

(3) 36
6
216

(4) 18
6
108

$$= \left[36a^3(3-3u^2) + 216a^3u^2 - 108a^3u - 108a^3u^3\right] \times$$

$$= 36a^2\left[a(3-3u^2)\right] - 6au\left[\cancel{-36a^2u} + \cancel{36a^2u}\right]$$
$$+ a(3+3u^2)(36a^2)$$

$$= 36a^2\left[3a - \cancel{3au^2} + 3a + \cancel{3au^2}\right]$$

$$[\gamma', \gamma'', \gamma'''] = 36a^2(6a)$$
$$= 216a^3$$
$$\tau = \frac{[\gamma', \gamma'', \gamma''']}{k^2 (\gamma')^6}$$
$$= \frac{216a^3}{\left[\frac{1}{3a(u^2+1)^2}\right]^2 \left[3\sqrt{2}a(1+u^2)\right]^6}$$
$$= \frac{216a^3 \times \left[3a(u^2+1)^2\right]^2}{\left[3\sqrt{2}a(u^2+1)\right]^6}$$
$$= \frac{216a^3 \left[9a^2 (u^2+1)^4\right]}{729(8)a^6 (u^2+1)^6}$$
$$= \frac{1944}{5832a(u^2+1)^2}$$
$$= \frac{1}{3a(u^2+1)^2}$$
$$k = \tau$$

Q Find k and τ for the following curve
$$x = e^t, \quad y = e^{-t} \quad z = -\sqrt{2}\,t$$

$$\gamma = (x, y, z) = (e^t, e^{-t}, -\sqrt{2}\,t)$$
$$k = \frac{|\gamma' \times \gamma''|}{(\gamma')^3}$$
$$\tau = \frac{[\gamma', \gamma'', \gamma''']}{k^2 (\gamma')^6}$$
$$\gamma' = (e^t, -e^{-t}, -\sqrt{2})$$
$$\gamma'' = (e^t, e^{-t}, 0)$$

$$|\gamma'| = \sqrt{e^{2t} + e^{-2t} + 2}$$

$$\gamma' \times \gamma'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ e^{t} & -e^{-t} & -\sqrt{2} \\ e^{t} & e^{-t} & 0 \end{vmatrix}$$

$$= \hat{i}(0 + \sqrt{2}e^{-t}) - \hat{j}(0 + \sqrt{2}e^{t}) + \hat{k}(1+1)$$

$$= \sqrt{2}e^{-t}\,\hat{i} - \sqrt{2}e^{t}\,\hat{j} + 2\hat{k}.$$

$$|\gamma' \times \gamma''| = \sqrt{2e^{-2t} + 2e^{2t} + 4}$$

$$= \sqrt{2}\sqrt{e^{-2t} + e^{2t} + 2}$$

$$k = \frac{|\gamma' \times \gamma''|}{|\gamma'|^{3}}$$

$$= \frac{\sqrt{2}\sqrt{e^{-2t} + e^{2t} + 2}}{(e^{2t} + e^{-2t} + 2)^{3/2}}$$

$$= \sqrt{2}\,(e^{2t} + e^{-2t} + 2)^{1/2 - 3/2}$$

$$= \sqrt{2}\,(e^{2t} + e^{-2t} + 2)^{-1}$$

$$k = \frac{\sqrt{2}}{(e^{2t} + e^{-2t} + 2)^{1}}$$

$$\tau = \frac{[\gamma', \gamma'', \gamma''']}{k^{2}\,|\gamma'|^{6}}$$

$$[\gamma', \gamma'', \gamma'''] = \begin{vmatrix} e^{t} & -e^{-t} & -\sqrt{2} \\ e^{t} & e^{-t} & 0 \\ e^{t} & -e^{-t} & 0 \end{vmatrix}$$

$$= e^{t}(0-0) + e^{-t}(0-0) - \sqrt{2}(-1-1)$$

$$= -\sqrt{2}(-2)$$

$$= 2\sqrt{2}$$

$$\tau = \frac{2\sqrt{2}}{(e^{2t} + e^{-2t} + 2)^{6 \times 1/2}} \times \frac{(e^{2t} + e^{-2t} + 2)^{2}}{(\sqrt{2})^{2}}$$

$$= \frac{\sqrt{2}\,(e^{2t}+e^{-2t}+2)^2}{(e^{2t}+e^{-2t}+2)^3}$$

$$= \frac{\sqrt{2}}{(e^{2t}+e^{-2t}+2)^1}$$

So

$\tau = k$

prooved.

Q2 For the point of the curve of intersection of the surface

$$x^2 - y^2 = c^2$$

&

$$y = x \tanh \frac{z}{c}$$

Prove that $\rho = \sigma = \frac{2x^2}{c}$

Solution

$$x^2 - y^2 = c^2$$

or $$x^2 - y^2 = c^2 \cdot 1$$

$$x^2 - y^2 = c^2 (\cosh^2\theta - \sinh^2\theta)$$

$x^2 = c^2\cosh^2\theta$ $\qquad$ $y^2 = c^2 \sinh^2\theta$

$x = c\cosh\theta \rightarrow (1)$ $\qquad$ $y = c\sinh\theta \rightarrow (2)$

dividing (2) ~~and~~ by (1)

$$\frac{y}{x} = \frac{c\sinh\theta}{c\cosh\theta} = \tanh\theta$$

$$\frac{y}{x} = \tanh\theta \rightarrow (3)$$

$\because$ $$y = x \tanh\frac{z}{c}$$

$$\Rightarrow \frac{y}{x} = \tanh\frac{z}{c} \rightarrow (4)$$

By comparing (3) & (4)

$$\tanh\theta = \tanh\frac{z}{c}$$

$$\theta = \frac{z}{c} \Rightarrow z = c\theta \rightarrow (5)$$

Let γ is the point of intersection of given two surfaces that can be written as,

$$\vec{\gamma} = (c\cosh\theta, c\sinh\theta, c\theta)$$

$$\gamma' = (c\sinh\theta, c\cosh\theta, c)$$

$$\gamma'' = (c\cosh\theta, c\sinh\theta, 0)$$

$$\gamma''' = (c\sinh\theta, c\cosh\theta, 0)$$

$$|\gamma'| = \sqrt{c^2\sinh^2\theta + c^2\cosh^2\theta + c^2}$$

$$= c\sqrt{\sinh^2\theta + \cosh^2\theta + 1}$$

$$= c\sqrt{\cancel{\sinh^2\theta} + \cosh^2\theta + \cosh^2\theta - \cancel{\sinh^2\theta}}$$

$$|\gamma'| = c\sqrt{2\cosh^2\theta} = \sqrt{2}c\cosh\theta$$

Since $k = \dfrac{|\gamma' \times \gamma''|}{(\gamma')^3} \rightarrow (A)$

$$\gamma' \times \gamma'' = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ c\sinh\theta & c\cosh\theta & c \\ c\cosh\theta & c\sinh\theta & 0 \end{bmatrix}$$

$$= \hat{i}\left[0 - c^2\sinh^2\theta\right] - \hat{j}\left[0 - c^2\cosh^2\theta\right] + \hat{k}\left[c^2\sinh^2\theta - c^2\cosh^2\theta\right]$$

$$= -c^2\sinh^2\theta\,\hat{i} + c^2\cosh^2\theta\,\hat{j} - c^2(\cosh^2\theta - \sinh^2\theta)\hat{k}$$

$$= -c^2\sinh^2\theta\,\hat{i} + c^2\cosh^2\theta\,\hat{j} - c^2\hat{k}$$

$2[\cosh^2\theta(1-\sinh^2\theta)-\sinh^2\theta]$

$(\sinh^2\theta)^2+(\cosh^2\theta)^2+(\cosh^2\theta-\sinh^2\theta)+2(\cosh^2\theta\sinh^2\theta)-2(\cosh^2\theta\sinh^2\theta)$

$(\sinh^2\theta+\cosh^2\theta)^2+\cosh^2\theta-\sinh^2\theta+2\cosh^2\theta\sinh^2\theta$

$2(\cosh^2\theta\sinh^2\theta+\cosh^2\theta\sinh^2\theta)\;1+1$

$2(1+\cosh^2\theta\sinh^2\theta)$

$$|\gamma'\times\gamma''|=\sqrt{c^4\sinh^4\theta+c^4\cosh^4\theta+c^4}$$
$$=c^2\sqrt{\sinh^4\theta+\cosh^4\theta+1}$$

$$\gamma'\times\gamma''=\hat{i}(0-c^2\sinh\theta)-\hat{j}(-c^2\cosh\theta)+\hat{k}(c^2\sinh^2\theta-c^2\cosh^2\theta)$$

$$=c^2(-\sinh\theta\,\hat{i}+\cosh\theta\,\hat{j}-\hat{k})$$

$$|\gamma'\times\gamma''|=c^2\sqrt{\sinh^2\theta+\cosh^2\theta+1}$$
$$=c^2\sqrt{\sinh^2\theta+\cosh^2\theta+\cosh^2\theta-\sinh^2\theta}$$
$$=c^2\sqrt{2}\cosh\theta$$

$$k=\frac{|\gamma'\times\gamma''|}{(\gamma')^3}$$

$$k=\frac{\sqrt{2}c^2\cosh\theta}{(\sqrt{2}c\cosh\theta)^3}=\frac{1}{2c\cosh^2\theta}\qquad\because\cosh\theta=\frac{x}{c}$$

$$=\frac{1}{2c\left(\frac{x}{c}\right)^2}=\frac{1}{2c}\times\frac{c^2}{x^2}=\frac{c}{2x^2}$$

$$\rho=\frac{1}{k}=\frac{2x^2}{c}$$

$$[\gamma',\gamma'',\gamma''']=\begin{vmatrix}c\sinh\theta & c\cosh\theta & c\\ c\cosh\theta & c\sinh\theta & 0\\ c\sinh\theta & c\cosh\theta & 0\end{vmatrix}$$

$$=c\sinh\theta(0)-c\cosh\theta(0)+c(c^2\cosh^2\theta-c^2\sinh^2\theta)$$

$$=c^3$$

$$\tau=\frac{[\gamma',\gamma'',\gamma''']}{k^2(\gamma')^6}$$

$$=\frac{c^3}{\left(\frac{c}{2x^2}\right)^2(\sqrt{2}c\cosh\theta)^6}$$

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$$= \frac{\cancel{c^3}\,(\cancel{4}x^4)}{c^2\,(\cancel{8}_2\, c^6 \cosh^6\theta)} = \frac{x^4}{2c^8\cosh^6\theta}$$

$$\left[\begin{array}{ll} & \sigma = \frac{1}{\tau} = \frac{2c^2\cosh^6\theta}{x^4} \\ \rho = \frac{2x^2}{c} & \sigma = \frac{2c^2\cosh^6\theta}{x^4} \end{array}\right] \times$$

$$\tau = \frac{x^4}{c^5\,(2)\left(\frac{x}{c}\right)^6}$$

$$\tau = \frac{x^4}{2c^5} \times \frac{c^6}{x^6}$$

$$\tau = \frac{c}{2x^2}$$

$$\sigma = \frac{2x^2}{c}$$

Hence proved

$$\rho = \sigma = \frac{2x^2}{c}$$

straight

Results:

(i) If $k = 0$ at all points then prove that curve is a st line.

(ii) If $\tau = 0$ at all points then proved that curve is a plane.

(iii) Prove necessary and sufficient condition for a curve to be plane is

$$[r', r'', r'''] = 0$$

I- Proof

$$\because t' = kn$$

$$\because k = 0 \quad \text{then} \quad t' = 0$$

$$\Rightarrow t = \text{constant}$$

Since tangent is fixed therefore curve is a st line.

II - Proof

$$\because b' = -\tau n$$

$$\text{As} \quad \tau = 0$$

$$\Rightarrow b' = 0$$

$$\Rightarrow b = \text{constant}$$

Since binormal is fixed therefore curve is plane.

III- Proof (Necessary condition)

C curve is plane $\tau = 0$

To prove $[r', r'', r'''] = 0$

$$\tau = \frac{[r', r'', r''']}{k^2 |r'|^6}$$

$$0 = \frac{[\gamma', \gamma'', \gamma''']}{k^2 |\gamma'|^6}$$

$$\Rightarrow [\gamma', \gamma'', \gamma'''] = 0$$

Sufficient Condition;

i.e, $[\gamma', \gamma'', \gamma'''] = 0$

To prove $\tau = 0$

$\because [\gamma', \gamma'', \gamma'''] = 0$

$$\frac{[\gamma', \gamma'', \gamma''']}{k^2 |\gamma'|^6} = 0$$

$$\Rightarrow \tau = 0$$

Q prove that the curve
$x = a\sin^2 u \quad y = a\sin u\cos u$
$z = a\cos u$ lies on a sphere
and also verify that all the normal
planes pass through origin.

Solution

$$\gamma = (x, y, z) = (a\sin^2 u,\ a\sin u\cos u,\ a\cos u)$$

$$x^2+y^2+z^2 = a^2\sin^4 u + a^2\sin^2 u\cos^2 u + a^2\cos^2 u$$

$$= a^2[\sin^2 u(\sin^2 u + \cos^2 u) + \cos^2 u]$$

$$= a^2[\sin^2 u + \cos^2 u]$$

$$x^2+y^2+z^2 = a^2$$

This is equation of sphere centre at origin and with radius a.

Since the equation of normal plane is

$$(R-r)\cdot t = 0$$

or $R\cdot t - r\cdot t = 0 \rightarrow (1)$

If the plane passes through the origin then $R = 0$

curve passes through plane and plane passes through origin That's why $R=0$

$\therefore (1) \Rightarrow$

$$0 - r\cdot t = 0$$

$$r\cdot t = 0 \rightarrow (2)$$

This is the condition all the normal planes passes through the origin which is required to prove

$$r = (a\sin^2 U,\ a\sin U\cos U,\ a\cos U)$$

$$t = r' = \frac{dr}{du}\cdot\frac{du}{ds}$$

$$t = (2a\sin U\cos U,\ a\cos^2 U - a\sin^2 U,\ -a\sin U)U'$$

$\therefore \star \Rightarrow$

$r\cdot t$

$$= \left[2a^2\sin^3 U\cos U + a^2\sin U\cos^3 U - a^2\sin^3 U\cos U - a^2\sin U\cos U\right]U'$$

$$= \left[a^2\sin^3 U\cos U + a^2\sin U\cos^3 U - a^2\sin U\cos U\right]U'$$

$$= \left[a^2 \sin u \cos u (\sin^2 u + \cos^2 u) - a^2 \sin u \cos u\right] u'$$

$$= \left[a^2 \sin u \cos u - a^2 \sin u \cos u\right] u'$$

$$= 0 = R.H.S$$

Hence proved the required result.

Home Task

Q By using $\gamma'' = t$ and Serret-Frenet formulas find γ'', γ''', γ'''' and prove the relations.

(i) $\gamma' \cdot \gamma'' = 0$

(ii) $\gamma'' \cdot \gamma''' = -k^2$

(iii) $\gamma' \cdot \gamma'''' = -3kk'$

(iv) $\gamma'' \cdot \gamma''' = kk'$

(v) $\gamma'' \cdot \gamma'''' = k(k - k^3 - k\tau^2)$

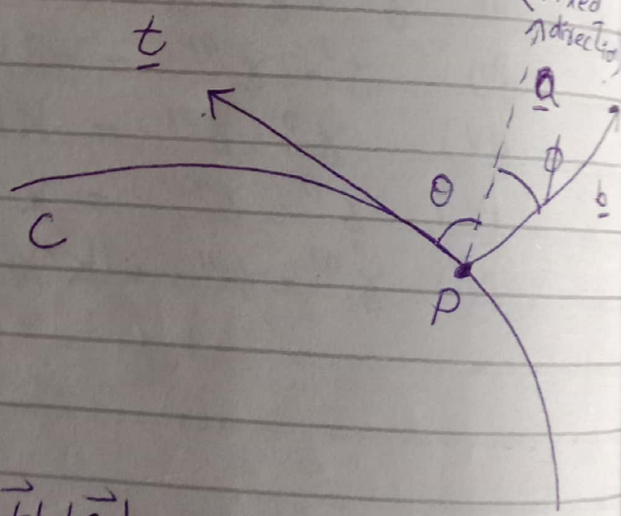
(vi) $\gamma''' \cdot \gamma'''' = k'k'' + 2k^3k' + k'\tau\tau' + kk'\tau^2$

If Tangent And Binormal at a point of curve make angle θ and binormal ϕ respectively with fixed direction. Show that

$$\frac{\sin\theta \, d\theta}{\sin\phi \, d\phi} = \frac{k}{\tau}$$

Solution.

Let C be any curve with a point P on it. Let "a" is any unit vector along a fixed direction making angles θ and ϕ with tangent and binormal as shown in fig.



Now,

$$\vec{t}\cdot\vec{a} = |\vec{t}|\,|\vec{a}|\cos\theta$$

$$t\cdot a = \cos\theta \rightarrow (1)$$

and

$$\vec{b}\cdot\vec{a} = |\vec{b}|\,|\vec{a}|\cos\phi$$

$$\vec{b}\cdot\vec{a} = \cos\phi \rightarrow (2)$$

diff (1) (2) w.r.t "S"

$$(1) \Rightarrow \quad t' \cdot \vec{a} = -\sin\theta \frac{d\theta}{ds} \quad \rightarrow (3)$$

and

$$(2) \Rightarrow \quad \vec{b}' \cdot \vec{a} = -\sin\phi \frac{d\phi}{ds} \quad \rightarrow (4)$$

$\div$ing (3) & (4)

$$\frac{\vec{t}' \cdot \vec{a}}{\vec{b}' \cdot \vec{a}} = \frac{-\sin\theta \frac{d\theta}{ds}}{-\sin\phi \frac{d\phi}{ds}}$$

$$\frac{kn \cdot a}{-\tau n \cdot a} = \frac{-\sin\theta d\theta}{-\sin\phi d\phi}$$

$$\frac{\sin\theta}{\sin\phi} \frac{d\theta}{d\phi} = -\frac{k}{\tau}$$

Q If plane of curvature at any point of the curve passes through fixed point then prove that the curve is plane?

Solution; Let R be the position vector of the ~~fixed~~ current point, lying in osculating plane and let $\vec{r}$ be the p.v of the fixed point then $\vec{R} - \vec{r}$ will also be ~~the~~ lies in osculating plane

then

$R-r$, $\vec{t}, \vec{n}$ are coplanar.

i.e, $[R-r, t, n] = 0$

or $(R-r) \cdot t \times n = 0$

$(R-r) \cdot b = 0 \rightarrow ①$

Let R_1 be the position of an other fixed point on the same curve then this R_1 will satisfy equation ①

(The plane of curvature is passing through R_1)

Then eqn ① can be written as

$(R_1 - r) \cdot b = 0 \rightarrow ②$

diff w.r.t "s"

$$(R_1 - r) \cdot \frac{db}{ds} = 0$$

$$(R_1 - r) \cdot (-\tau n) = 0$$

$$-\tau \, (R_1 - r) \cdot n = 0$$

$$\tau \, (R_1 - r) \cdot n = 0$$

This is possible only when $\tau = 0$

$\Rightarrow$ Curve is a plane.

Q If nth derivative of $\vec{r}$ w.r.t "s" is given by

$$\vec{r}^{\,n} = a_n \vec{t} + b_n \vec{n} + c_n \vec{b} \rightarrow (*)$$

then prove that

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$$a_{n+1} = a_n' - kb_n$$
$$b_{n+1} = b_n' + ka_n - \tau c_n$$
$$c_{n+1} = c_n' + \tau b_n$$

Solution;

diff eqn (*) w.r.t 's'

$$\vec{r}^{\,n+1} = a_n\vec{t}\,' + b_n\vec{n}\,' + c_n\vec{b}\,' + t a_n' + b_n' n + c_n' b$$
$$= a_n(kn) + b_n(\tau b - kt) + c_n(-\tau n)$$
$$+ a_n' t + b_n' n + c_n' b$$

$$r^{n+1} = (a_n' - kb_n)t + (a_n k - \tau c_n + b_n')n + (\tau b_n + c_n')b \rightarrow (1)$$

In general $(n+1)$th derivative can be written as,

eqn (*) $\Rightarrow$

$$\vec{r}^{\,n+1} = a_{n+1}\vec{t} + b_{n+1}\vec{n} + c_{n+1}\vec{b} \rightarrow (2)$$

comparing (1) & (2)

$$a_{n+1} = a_n' - kb_n$$
$$b_{n+1} = a_n k - \tau c_n + b_n'$$
$$c_{n+1} = \tau b_n + c_n'$$

Q(4) If m_1, m_2 and m_3 are the moments about the origin of the unit vectors t, n and b respectively then show that

(i) $m_1' = k m_2$

(ii) $m_2' = b - k m_1 + \tau m_3$

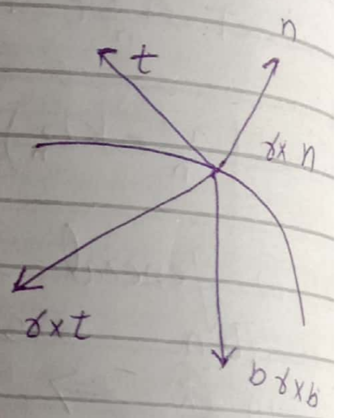
(iii) $m_3' = -n - \tau m_2$

If r is ~~[illegible]~~ is Position vector of current point of moment of forces about that point then by definition we can write

$m_1 = r \times t \rightarrow$ (i)

$m_2 = r \times n \rightarrow$ (ii)

$m_3 = r \times b \rightarrow$ (iii)



To find the req diff eqn ① ② & ③ w.r.t "S"

① $\Rightarrow$ $m_1' = r' \times t + r \times t'$

$= t \times t + r \times kn$

$m_1' = 0 + k(r \times n)$

$m_1' = k m_2$ ① Proved.

② $\Rightarrow$ $m_2' = r' \times n + r \times n'$

$m_2' = t \times n + r \times (\tau b - kt)$

$m_2' = t \times n + \tau(r \times b) - k(r \times t)$

$m_2' = b + \tau m_3 - k(m_1)$

$m_2' = b - k m_1 + \tau m_3$

$$m_3' = \gamma \times b$$
$$m_3' = \gamma' \times b + \gamma \times b'$$
$$m_3' = t \times b + \gamma \times (-\tau n)$$
$$m_3' = t \times b - \tau(\gamma \times n)$$
$$m_3' = -n - \tau m_2$$

Skew Curvature

The arc rate of rotation of normal is called skew curvature and mathematically it can be written as,

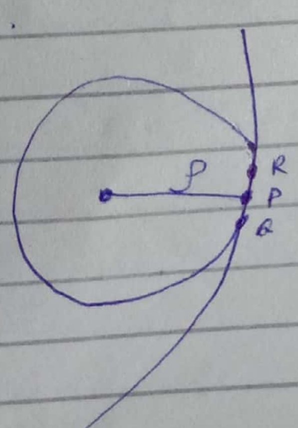
$$\frac{dn}{ds} = n' = \tau \underline{b} - k\underline{t}$$

and $\left|\frac{d\underline{n}}{ds}\right| = |\underline{n}'| = \sqrt{\tau^2 + k^2}$

is called magnitude of skew curvature.

Circle Of Curvature

The circle of curvature at point P is the circle passing through three points ~~is passing~~ ~~through~~ on the circle is called circle of curvature.

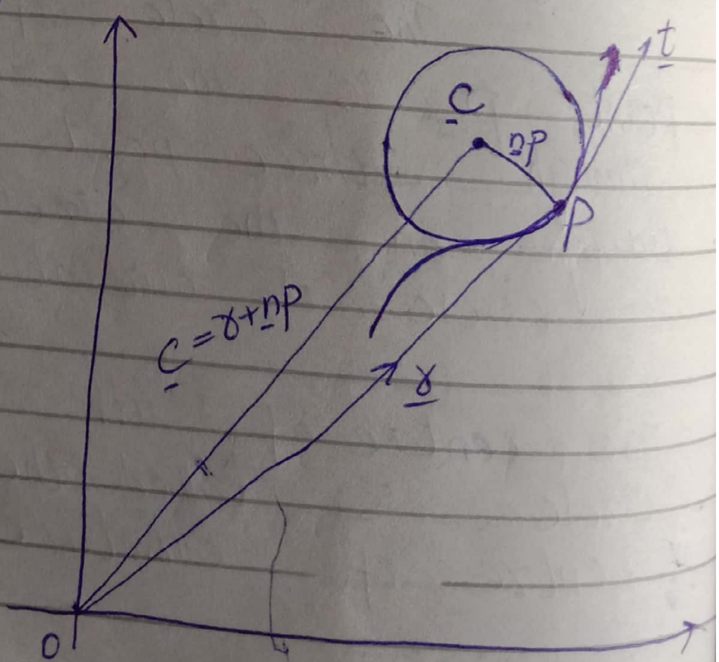


The centre of such circle is called the centre of curvature the radius of such circle is called radius of curvature usually the centre is denoted by 'C' and radius is denoted by ρ. and this circle lies in the osculating plane w.r.t point P and its curvature is same as that of the curvature at P.

That is $$k = \frac{d\theta}{ds} = \frac{1}{\rho}$$

Equation Of Centre of Curvature:

The centre of curvature (C) corresponding to point P on the curve with p.v "$\underline{r}$" is shown in fig



In figure $\underline{c} = \underline{r} + \rho \underline{n}$ is called eqn of centre of curvature

In this eqn ρ is the radius of curvature and its direction is always along the direction of normal to the point

Theorem;

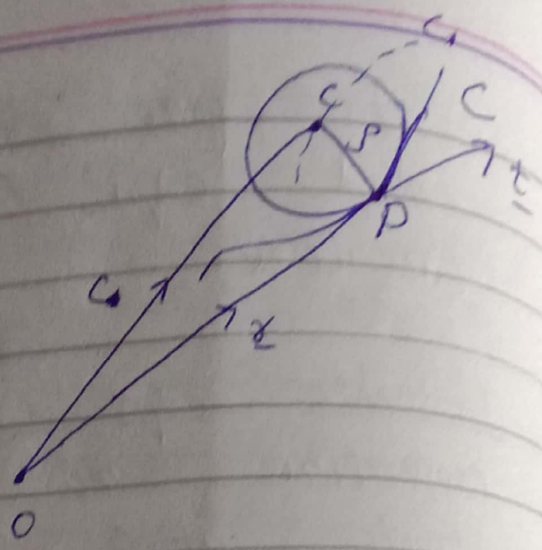
Q Prove that the tangent to the Locus of the centre of Curvature lies in the normal plane at P to the curve C. Also find expression for unit tangent of the locus C_1.

Proof

Let S be the arc length of C and S_1 be the arc length of C_1, and r be the p.v of point "P" then by the definition of centre of curvature the p.v of centre of curvature can be written as

$$\underline{c} = \underline{r} + \underline{n}\rho \rightarrow (1)$$

Shown in the figure.



Diff eqn ① w.r.t "s"

$$\frac{d\bar{c}}{ds} = \frac{d\bar{r}}{ds} + \rho n' + \rho' n$$

$$\frac{d\bar{c}}{ds} = r' + \rho(\tau b - kt) + \rho' n$$

$$\frac{d\bar{c}}{ds} = t + \tau\rho b - \rho k t + \rho' n$$

$$\frac{dc}{ds} = (1-\rho k)t + \rho' n + \tau\rho b$$

$$= \cancel{t} + \rho\tau b - \frac{1}{k}k\cancel{t} + \rho' n$$

$$\frac{d\vec{c}}{ds} = \rho\tau \underline{b} + \rho' \underline{n} \quad \rightarrow ②$$

In eqn ② $\frac{d\bar{c}}{ds}$ is the vector along the tangent of the circle of curvature and it lies in the normal plane of centre of curvature

(b) In eqn ②

$\frac{d\bar{c}}{ds}$ is not the unit tangent of the locus c_1. To find the unit tangent of the locus c_1.

Consider the eqn ①

i.e, $\bar{c} = \bar{r} + \rho\bar{n}$

diff w.r.t "s_1"

$$\cancel{\frac{d\bar{c}}{ds_1} = \frac{d\bar{r}}{ds_1} + \rho\frac{d\bar{n}}{ds_1} + \bar{n}\frac{d\rho}{ds_1}}$$

$$\frac{dc}{ds_1} = \frac{d}{ds}(\underline{r} + \rho\underline{n})\frac{ds}{ds_1}$$

$$t_1 = (\rho\tau b + \rho' n)\frac{ds}{ds_1} \quad \text{③} \quad \text{by ②}$$

For $\frac{dc}{ds_1}$ find the magnitude of ③

$$|\underline{t}_1| = \sqrt{\rho^2\tau^2 + \rho'^2}\,\frac{ds}{ds_1}$$

$$\frac{\rho\tau b + \rho' n}{\sqrt{\rho^2\tau^2 + \rho'^2}}$$

$$\frac{ds}{ds_1} = \frac{1}{\sqrt{\rho^2\tau^2 + \rho'^2}}$$

$$\frac{ds}{ds_1} = \frac{1}{\sqrt{\frac{\tau^2}{k^2} + \frac{1}{(k')^2}}}$$

Theorem; If the radius of curvature is constt for the given curve C then prove that

① The tangent to the locus of the centre of curvature is parallel to the binormal at a point P to the curve "C".

② Curvature of locus C_1 is same as curvature of the given curve C, i.e,

$$K_1 = K$$

③ Torsion of the locus of centre of curvature varies inversely as the torsion of the given curve C.

i.e, $\tau_1 \propto \dfrac{1}{\tau}$

Proof

(i) ∵ the eqn of centre of curvature (P.V of $\bar{c}$)

$$\bar{c} = \bar{r} + \bar{n}\rho$$

diff w.r.t 's'

$$\frac{d\bar{c}}{ds} = t + \rho' n + n'\rho$$

$$\frac{d\bar{c}}{ds} = t + n\rho' + \frac{1}{k}(\tau b - kt)$$

$\rho' = 0$

$$\frac{d\bar{c}}{ds} = \cancel{t} + \rho\tau b - \cancel{t}$$

$$\frac{d\bar{c}}{ds} = \rho\tau \underline{b} \rightarrow ① \quad \because \rho \text{ is constant}$$

eqn ① shows that $\dfrac{d\bar{c}}{ds} \parallel \underline{b}$

(ii) Again consider

$$\bar{c} = \bar{r} + \bar{n}\rho$$

$$\frac{d\bar{c}}{ds_1} = \frac{d}{ds}(r + n\rho)\frac{ds}{ds_1}$$

$$t_1 \quad \frac{dc}{ds_1} = \rho\tau b \cdot \frac{ds}{ds_1} \qquad \text{by } ①$$

$$1 \cdot t_1 = \rho\tau \frac{ds}{ds_1} b$$

By comparing

$$t_1 = b \rightarrow ② \qquad 1 = \rho\tau\frac{ds}{ds_1} \rightarrow$$

$$\frac{ds}{ds_1} = \frac{1}{\rho\tau} \rightarrow ③$$

Again diff eqn ② w.r.t 's_1'

$$\frac{d\bar{t_1}}{ds_1} = \frac{d\bar{b}}{ds} \cdot \frac{ds}{ds_1}$$

$$\underline{t}_1' = \underline{b}' \cdot \frac{ds}{ds_1}$$

$$\underline{t}_1' = \underline{b}' \cdot \frac{1}{\rho\tau}$$

$$k_1\underline{n}_1 = -\cancel{\tau}\underline{n} \cdot \frac{1}{\cancel{\tau}\rho}$$

$$k_1\underline{n}_1 = -k\underline{n}$$

By comparing

$$k_1 = k \qquad \text{and} \qquad \underline{n}_1 = -\underline{n}$$

(iii) using eqn (3) and (4)

$$\bar{t}_1 = \bar{b} \quad \text{and} \quad \bar{n}_1 = -\bar{n}$$

Taking cross product of these two.

$$\bar{t}_1 \times \bar{n}_1 = -\bar{b} \times \bar{n}$$

$$\Rightarrow \bar{b}_1 = \bar{t} \rightarrow (5)$$

diff (5) w.r.t "s_1"

$$\frac{d\bar{b}_1}{ds_1} = \frac{dt}{ds} \cdot \frac{ds}{ds_1}$$

$$\frac{d\bar{b}_1}{ds_1} = t' \cdot \frac{ds}{ds_1}$$

$$\frac{d\bar{b}_1}{ds_1} = kn \cdot \frac{1}{\rho\tau}$$

$$-\tau_1 n_1 = k^2 \cdot \frac{n}{\tau}$$

By comparing

$$\bar{n}_1 = -\bar{n} \quad \& \quad \tau_1 = \frac{k^2}{\tau}$$

$$\Rightarrow \tau_1 \propto \frac{1}{\tau}$$

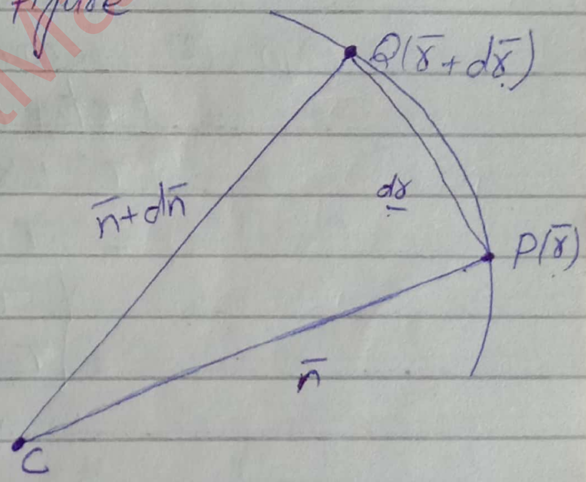
proved.

Lecture

Q Prove that the shortest distance b/w the principle normals at consecutive pts, and distance ds apart is $\frac{\rho}{\sqrt{\rho^2+\sigma^2}}\, ds$.

Proof

Let $\vec{r}$ and $\vec{r}+d\vec{r}$ be the p.v of two consecutive pts P and Q respectively and $\underline{n}$ and $\bar{n}+d\bar{n}$ are the unit principle normals at P and Q respectively, as shown in figure.



we have to find the shortest distance b/w $\bar{n}$ and $\bar{n}+d\bar{n}$ i.e we have to find $\perp$ar distance.

[Cross prod is n]

First we have to find $\perp$ar

unit vector is along that vector having magnitude 1.

distance

~~The Lar vector~~ which is Lar to both n and n+dn.

i.e. $\bar{n}\times(\bar{n}+d\bar{n}) = n\times n + n\times dn$

$= 0 + \bar{n}\times d\bar{n}$

$= \bar{n}\times \frac{d\bar{n}}{ds} ds$

Let $\hat{e}$ be the vector in the direction of vector of ①

$= \bar{n}\times(\tau b - kt)ds$

$= (\tau \underline{t} + k\underline{b})ds \to ①$

∵ Shortest distance = projection of $d\underline{r}$ upon $\hat{e}$

$= \hat{e}\cdot dr$ (say) $\to ②$

∴ eqn ② becomes shortest distance

$$= \frac{(\tau \bar{t} + k\bar{b})}{\sqrt{\tau^2+k^2}}\frac{ds}{ds}\cdot\left(\frac{dr}{ds}\right)ds$$

$$= \frac{\tau\bar{t}+k\bar{b}}{\sqrt{\tau^2+k^2}}\cdot t\, ds$$

$$= \frac{[\tau(t\cdot t)+k(b\cdot t)]}{\sqrt{\tau^2+k^2}}ds$$

$$= \frac{\tau\, ds}{\sqrt{\tau^2+k^2}} = \frac{\frac{1}{\sigma}ds}{\sqrt{\frac{1}{\sigma^2}+\frac{1}{\rho^2}}}$$

$$= \frac{\frac{\cancel{\sigma}\rho}{\sigma}ds}{\sqrt{\rho^2+\sigma^2}}$$

$$= \frac{\rho\, ds}{\sqrt{\rho^2+\sigma^2}}$$

Proved.

Q Prove that the product of curvatures of C and C_1 at the consecutive pts is equal to the product of the torsion of those points

i.e, $\kappa\kappa_1 = \tau\tau_1$

Soln:-

$\because$ We know that

$$t_1 = b$$

$$\frac{dt_1}{ds_1} = \frac{db}{ds} \times \frac{ds}{ds_1}$$

$$k_1 n_1 = b' \cdot \frac{ds}{ds_1}$$

$$\cancel{k_1 n_1 = -\tau n \frac{1}{\rho_1}}$$

$$\cancel{k_1 n_1 = -\tau n \cdot k}$$

$$k_1 n_1 = -\tau n \frac{ds}{ds_1}$$

$n_1 = -n$

$$\boxed{\frac{k_1}{\tau} = \frac{ds}{ds_1}} \rightarrow (1)$$

$$b_1 = t$$

$$\frac{db_1}{ds_1} = \frac{dt}{ds} \cdot \frac{ds}{ds_1}$$

$$-\tau_1 n_1 = t' \cdot \frac{ds}{ds_1}$$

$$-\tau_1 n_1 = (kn) \cdot \frac{k_1}{\tau}$$

$\frac{ds}{ds_1} = \frac{\tau_1}{k} \rightarrow (2)$ $\quad n_1 = -n$

compare (1) & (2)

$n_1 = -n$

$\tau_1 \tau = k k_1$

proved.

After Mids:

Sphere of Curvature oR Osculating Sphere

The sphere which passes through four points on the curve. Ultimately consident with p is called the sphere of curvature or osculating sphere at point p.

Its centre is called the centre of spherical curvature and is denoted by S whereas its radius is called radius of spherical curvature and is denoted by $\bar{R}$

Derive an expression for the radius of spherical curvature and also the p.v of centre of spherical curvature?

Proof

Let r be the p.v of the point p on the curve. And $\vec{S}$ is the p.v of centre of spherical curvature and this centre of curvature is the limiting position of the intersection of three normal plane at adjacent points. Now the eqn of normal plane at point P is given as

$$(\vec{S}-\vec{r})\cdot\vec{t}=0 \rightarrow (1)$$

diff w.r.t "s"

$$(\bar{S}-\bar{r})\cdot\frac{d\bar{t}}{ds}+\left(\frac{d\bar{S}}{ds}-\frac{dr}{ds}\right)\cdot\bar{t}=0$$

$$(S-r)kn+\frac{dS}{ds}\cdot t-\frac{dr}{ds}\cdot t=0$$

$$(S-r)\cdot kn - t\cdot t=0$$

$$(S-r)\cdot kn=1$$

$$(S-r)\cdot n=\rho \rightarrow (2) \quad \because \boxed{\rho=\tfrac{1}{k}}$$

$$\left(\frac{ds}{ds}-\frac{dr}{ds}\right)\cdot\cancel{\frac{dt}{ds}}=0$$

$$(1-t)\cdot\cancel{t'}=0$$

$$\cancel{(1-t)\cdot kn=0}$$

$$\cancel{kn-k(t\cdot n)=0}$$

$$\cancel{kn=0}$$

$+1$

$$(1-t)\cdot t+(R-S)\,t'=0$$

$$(1-t)\cdot t+(R-S)\cdot kn=0$$

$$(t-1)+k(S\cdot n)-k(r\cdot n)=0$$

As $\bar{S}$ is the vector along $\hat{n}$ and hence $\frac{d\bar{S}}{ds}\cdot\bar{t}=0$

Again diff w.r.t "s"

$$(\bar{S}-\bar{\gamma})\cdot\frac{d\bar{n}}{ds}+\left(\frac{d\bar{S}}{ds}-\frac{d\bar{\gamma}}{ds}\right)\cdot\bar{n}=\rho'$$

$$(\vec{S}-\vec{\gamma})(\tau\vec{b}-k\vec{t})+\frac{d\vec{S}}{ds}\cdot\vec{n}-\frac{d\vec{\gamma}}{ds}\cdot\vec{n}=\rho'$$

$$(\vec{S}-\vec{\gamma})(\tau\vec{b}-k\vec{t})+\frac{d\vec{S}}{ds}\cdot\vec{n}=\rho'$$

Since $\frac{d\bar{S}}{ds}$ is ~~the~~ in the vector along $\bar{n}$ hence $\frac{ds}{ds}\cdot n$ is a Scalar quantity and for simplicity so merge this quantity in ρ'.

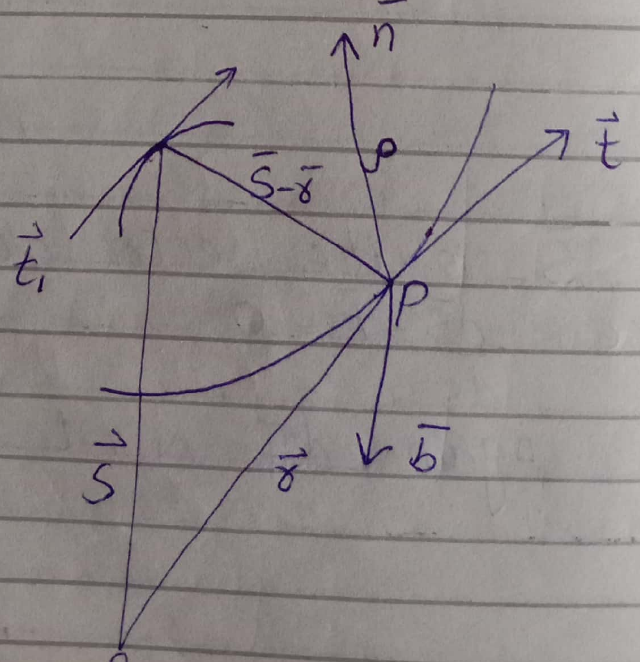
$$(\vec{S}-\vec{\gamma})\cdot(\tau b-kt)=\rho'$$

$$\tau(\vec{S}-\vec{\gamma})\cdot\vec{b}-k(\vec{S}-\vec{\gamma})\cdot\vec{t}=\rho'$$

$$\because (\vec{S}-\vec{\gamma})\cdot t=0$$

$$\tau(\vec{S}-\vec{\gamma})\cdot b=\rho'$$

$$(\vec{S}-\vec{\gamma})\cdot\vec{b}=\sigma\rho' \rightarrow (3)$$



$$(S-\gamma)\cdot t = 0 \rightarrow (1)$$
$$(S-\gamma)\cdot n = \rho \rightarrow (2)$$
$$(S-\gamma)\cdot b = \sigma\rho' \rightarrow (3)$$

Eqn (1) is possible only when $(\vec{S}-\vec{\gamma}) = 0$

Eqn (2) is possible only when

$$(\vec{S}-\vec{\gamma}) = \rho\vec{n}$$

Eqn (3) is possible only when

$$(S-\gamma) = \sigma\rho' b$$

Therefore we can write

$$\vec{S}-\vec{\gamma} = \rho\vec{n} + \sigma\rho'\vec{b} \rightarrow (3A)$$

$$\boxed{\vec{S} = \vec{\gamma} + \rho\vec{n} + \sigma\rho'\vec{b}} \rightarrow (4)$$

eqn (4) is used to determine the p.v S of the centre of spherical curvature.

To find the radius of spherical curvature. Taking the square on both sides of eqn (3A)

$$(\vec{S}-\vec{\gamma})^2 = (\rho\vec{n} + \sigma\rho'\vec{b})^2$$

$$(\bar{R})^2 = (\rho\vec{n})^2 + (\sigma\rho'\vec{b})^2 + 2\sigma\rho\rho'\,\vec{n}\cdot\vec{b}$$

$$R^2 = \rho^2 + \sigma^2\rho'^2$$

$$\boxed{R = \sqrt{\rho^2 + \sigma^2\rho'^2}} \rightarrow (5)$$

This is the radius of osculating Sphere.

Remarks: For the curve of constant curvature i.e. $\rho' = 0$ then eq/n (5) implies that $\boxed{R = \rho}$. It means that ~~radius~~ centre of spherical curvature coincides with the centre of circular curvature.

Theorem Prove that the tangential, principle normal and binormal to the locus of the centre of osculating sphere C_1 are Parallel to the binormal, principle normal and tangent to the curve C

Proof

Let C be the p.v of the centre of spherical curvature i.e. $\bar{C} = \bar{r} + \bar{n}\rho + \sigma\rho'\bar{b}$ →(1)

Differentiate w.r.t "s_1"

$$\frac{d\bar{c}}{ds_1} = \frac{d}{ds}(\bar{r} + \bar{n}\rho + \sigma\rho'\bar{b})\frac{ds}{ds_1}$$

$$t_1 = \left[\frac{d\bar{r}}{ds} + \rho\frac{d\bar{n}}{ds} + \bar{n}\rho' + \sigma'\rho'\bar{b} + \sigma\rho''\bar{b} + \sigma\rho'\frac{d\bar{b}}{ds}\right]\frac{ds}{ds_1}$$

$$t_1 = \left[t + \rho(\tau b - kt) + \rho' n + \sigma'\rho'\bar{b} + \sigma\rho''\bar{b} + \sigma\rho'(-\tau n)\right]\frac{ds}{ds_1}$$

$$t_1 = \left[\cancel{t} + \rho\tau b - \cancel{t} + \rho' n + \sigma'\rho'\bar{b} + \sigma\rho''\bar{b} - \frac{\tau}{\sigma}\sigma\rho' n\right]\frac{ds}{ds_1}$$

$$t_1 = (\rho\tau\vec{b} + \sigma'\rho'\vec{b} + \sigma\rho''\vec{b})\frac{ds}{ds_1}$$

$$t_1 = \bar{b}\,(\rho\tau + \sigma'\rho' + \sigma\rho'')\,\frac{ds}{ds_1}$$

Compare the vectors and scalars.

$$\boxed{\bar{t}_1 = \bar{b}} \rightarrow (2) \qquad 1 = (\rho\tau + \sigma'\rho' + \sigma\rho'')\frac{ds}{ds_1}$$

Now diffe w.r.t s_1 eqn (2)

$$\frac{dt_1}{ds_1} = \frac{db}{ds}\cdot\frac{ds}{ds_1}$$

$$t_1' = (-\tau n)\cdot\frac{1}{(\rho\tau + \sigma'\rho' + \sigma\rho'')}$$

$$k_1 n_1 = -\tau n \cdot \frac{ds}{ds_1}$$

$$\boxed{n_1 = -n} \rightarrow (3) \qquad k_1 = \tau\frac{ds}{ds_1}$$

Taking cross product of (2) & (3)

$$t_1 \times n_1 = -\bar{b} \times \bar{n}$$

$$\boxed{b_1 = t}$$

proved.

$t_1 \parallel t_1$

$n_1 \parallel n$

$b_1 \parallel b$

$t_1 = t$

$b_1 = b$

$n_1 = -n$

$t_1 = \lambda t$

$n_1 = \mu n$

$b_1 = \nu b$

Theorem Prove that the necessary and sufficient condition for a curve to be a spherical curve is

$$\frac{\rho}{\sigma} + \frac{d}{ds}\left(\frac{\rho'}{\tau}\right) = 0$$

For every point of that curve

Proof

Necessary Condition:

Let the curve is spherical curve and

To prove $\frac{\rho}{\sigma} + \frac{d}{ds}\left(\frac{\rho'}{\tau}\right) = 0$

$\because$ the curve is spherical

$\therefore$ it lies in the sphere

Hence the sphere is osculating plane and the value of its radius R is given by

$$R^2 = \rho^2 + (\sigma\rho')^2 \longrightarrow (1)$$

diff w.r.t s

R = constant

$$\cancel{2R\frac{dR}{ds} = 2\rho\frac{d\rho}{ds} + 2(\sigma\rho')\sigma'\rho''}$$

$$0 = 2\rho\rho' + 2\sigma\rho'\frac{d}{ds}(\sigma\rho')$$

$$= 2\rho'\left(\rho + \sigma\frac{d}{ds}(\sigma\rho')\right)$$

hence $2\rho' \neq 0$

$\rho \neq$ constant

$$\rho + \sigma \frac{d}{ds}(\sigma \rho') = 0$$

$$\frac{\rho}{\sigma} + \frac{d}{ds}\left(\frac{\rho'}{\tau}\right) = 0$$

Sufficient Condition:

Let $\frac{\rho}{\sigma} + \frac{d}{ds}\left(\frac{\rho'}{\tau}\right) = 0$ is given

then we have to show that the given curve is spherical curve

For this we have to prove that the radius and the centre of osculating are fixed/unchanged.

Since we have

$$\frac{\rho}{\sigma} + \frac{d}{ds}\left(\frac{\rho'}{\tau}\right) = 0$$

$$\Rightarrow \rho + \sigma \frac{d}{ds}\left(\frac{\rho'}{\tau}\right) = 0$$

Xing both sides by $2\rho'$

$$2\rho\rho' + 2\sigma\rho' \frac{d}{ds}\left(\frac{\rho'}{\tau}\right) = 0$$

Integrating w.r.t "s"

$$\rho^2 + (\sigma\rho')^2 = c^2 \longrightarrow (1)$$

where c is constt of integration

To find the value of C we have the Radius of osculating sphere.

$$R^2 = \rho^2 + (\sigma\rho')^2 \rightarrow (2)$$

By comparing (1) & (2)

$$\Rightarrow R^2 = C^2$$

$$\Rightarrow R = C$$

$\Rightarrow$ Radius of osculating sphere is fixed.

Now ~~th~~ take the p.v of the centre of osculating sphere.

$$\bar{C} = \bar{r} + n\bar{\rho} + \sigma\rho'\bar{b}$$

diff w.r.t "S"

$$= \frac{dr}{ds} + n'\bar{\rho} + n\rho' + \sigma'\rho' b + \sigma'\rho'' b + \sigma\rho' b'$$

$$= t + (\tau b - kt)\rho + n\rho' + \sigma'\rho' b + \sigma'\rho'' b + \sigma\rho'(-\tau n)$$

$$= \cancel{t} + \tau b\rho - \cancel{t} + \cancel{n\rho'} + \sigma'\rho' b + \sigma'\rho'' b - \cancel{\tau n\rho'\sigma}$$

$$\frac{dc}{ds} = \tau b\rho + \sigma'\rho' b + \sigma'\rho'' b$$

$$\frac{dc}{ds} = b\left(\frac{\rho}{\sigma} + \sigma'\rho' + \sigma\rho''\right)$$

$$\frac{dc}{ds} = b\left[\frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho')\right]$$

$\frac{dc}{ds} = 0$ (b) by eq/n ①

$\frac{dc}{ds} = 0 \Rightarrow c =$ constant.

$\Rightarrow$ centre of osculating sphere is fixed

Hence the curve is spherical curve at every point.

$\boxed{\text{Theorem}}$ Fundemental Theorem of Space Curve

A curve is uniquely determined except as to position fixed in space when its curvature and torsion are given ftns of arc length S.

Proof

Consider two curves C_1 and C_2 having equal curvatures $K_1 = K_2$ and equal torsions $\tau_1 = \tau_2$ for the same parameter s. and let $\bar{t}, \bar{n}$ and $\bar{b}$ refers to curve ① and t_1, n_1, b_1 refers to other curve.

For same values of S we will find $\frac{d}{ds}(\bar{t}\cdot\bar{t}_1)$, $\frac{d}{ds}(\bar{n}\cdot\bar{n}_1)$,

$$\frac{d}{ds}(\bar{t}\cdot\bar{t}_1) = \bar{t}'\cdot\bar{t}_1 + \bar{t}\cdot\bar{t}_1'$$

$$= kn\cdot t_1 + t\cdot(k_1 n_1)$$

$$= k(n\cdot t_1) + k_1(t\cdot n_1) \rightarrow ①$$

$$\frac{d}{ds}(\bar{n}\cdot\bar{n}_1) = n'\cdot n_1 + n\cdot n_1'$$

$$= (\tau b - kt)\cdot n_1 + n\cdot(\tau_1 b_1 - k_1 t_1)$$

$$= \tau(b\cdot n_1) - k(t\cdot n_1) + \tau_1(n\cdot b_1) - k_1(n\cdot t_1) \rightarrow ②$$

$$\frac{d}{ds}(b\cdot b_1) = b'\cdot b_1 + b\cdot b_1'$$

$$= (-\tau n)\cdot b_1 + b\cdot(-\tau_1 n_1)$$

$$= -\tau(n\cdot b_1) - \tau_1(b\cdot n_1) \rightarrow ③$$

Adding ① ② and ③

$$= k(n\cdot t_1) + k(t\cdot n_1) + \tau(b\cdot n_1) - k(t\cdot n_1) + \tau_1(n\cdot b_1) - k_1(n\cdot t_1) - \tau(n\cdot b_1) - \tau_1(b\cdot n_1)$$

$$= 0$$

$$\Rightarrow \frac{d}{ds}(t\cdot t_1 + n\cdot n_1 + b\cdot b_1) = 0$$

$$\Rightarrow t\cdot t_1 + n\cdot n_1 + b\cdot b_1 = \text{Constant} \rightarrow ④$$

To find the value of constt of integration in ④

Suppose that the two curves are placed so that their initial points coinsides from which S is measured

i.e, For this $t = t_1$
$n = n_1$
$b = b_1$

Then eqn ④ ⟹

$$t^2 + n^2 + b^2 = \text{constant}$$
$$1 + 1 + 1 = \text{constant}$$
$$3 = \text{constant}$$

∴ eqn ④ can be written as

$$\boxed{t \cdot t_1 + n \cdot n_1 + b \cdot b_1 = 3}$$

but the sum of three angles can be equal to 3 only when each of the three angle vanishes or is an integral multiple of 2π this required that all the pairs of corresponding pts have $t = t_1$, $n = n_1$, $b = b_1$

Now let r be the p.v of any point on the first curve c_1 and r_1 be the p.v of any point on the other curve c_2 and use the relation

$$t = t_1$$

$$\bar{t} - \bar{t}_1 = 0$$

$$\frac{d\bar{r}}{ds} - \frac{d\bar{r}_1}{ds} = 0$$

$$\Rightarrow \frac{d}{ds}(\bar{r} - \bar{r}_1) = 0$$

$$\Rightarrow r - r_1 = \text{constant}$$

This constt difference vanishes at initial point and therefore ~~...~~ it vanishes throughout the curve

Therefore we can write $\bar{r} - \bar{r}_1 = 0$

$$\Rightarrow \bar{r} = \bar{r}_1$$

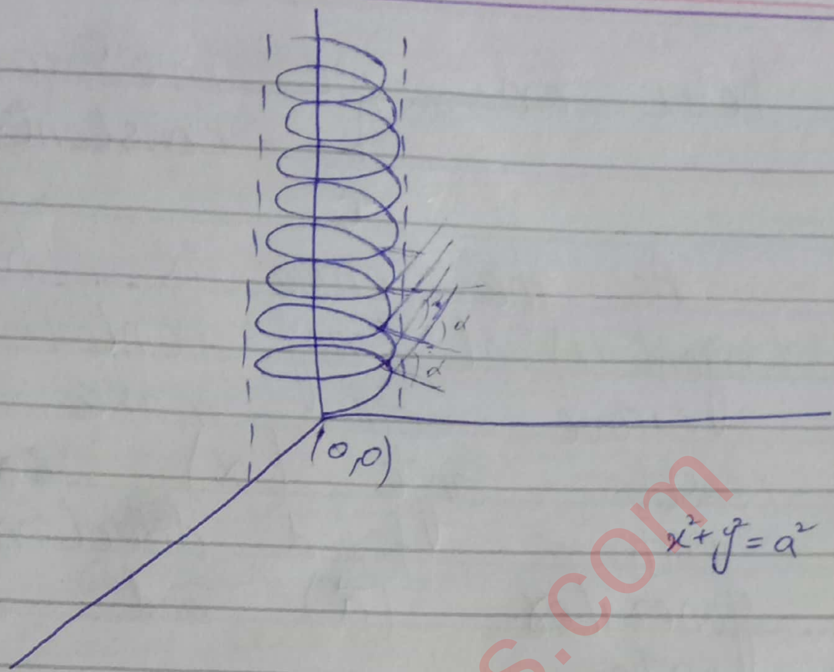
At all the corresponding pts For two curves which consides

Hence proved the theorem.

Helix (Important)

Helix is a curve traced on the surface of cylender and cutting ~~...~~ the generator ~~and~~ at the constt angle say α.

(Helix is a curve which generate a fixed Generator)



If the cylinder is a right cylinder then the curve drawn on that cylinder which is making constt angle with the generator is called a circular Helix.

Theorem; (Property of Helix)

Prove that the necessary and sufficient condition for a curve to be a helix is that the ratio of its curvature and torsion is constt.

Proof;

Necessary Condition;
Let the given curve is

helix and we have to prove

$$\frac{k}{\tau} = \text{constant.}$$

As the curve is helix so tangent (t) at any point to this curve will make a constt angle (α) with fixed direction or generator $(\vec{a})$ and we can write

$$\bar{t}\cdot\bar{a} = |\bar{t}||\bar{a}|\cos\alpha$$

$$\bar{t}\cdot\bar{a} = (1)(1)\cos\alpha$$

$$\bar{t}\cdot\bar{a} = \cos\alpha \quad \rightarrow (1)$$

diff w.r.t "s"

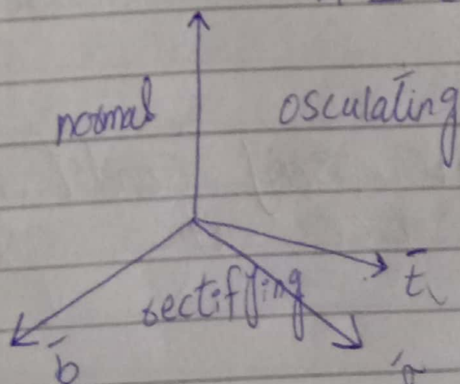
$$t'\cdot a + t\cdot a' = 0$$

$$k\bar{n}\cdot\bar{a} = 0 \quad \rightarrow (2)$$

if $k=0$ then the curve is straight line and there is nothing to prove OR the required condition is proved.

If $k \neq 0$ then $\bar{n}\cdot\bar{a} = 0$

$\bar{n} \perp \bar{a}$



Thus $\bar{a}$ will be in the plane determined by $\underline{t}$ and $\underline{b}$

So $\bar{a}$ can be written as

$$\bar{a} = \cos\alpha\,\bar{t} + \sin\alpha\,\bar{b} \rightarrow (2)$$

diff w.r.t "s"

$$0 = \cos\alpha\,\bar{t}' + \sin\alpha\,\bar{b}'$$

$$0 = \cos\alpha\,(k\bar{n}) + \sin\alpha\,(-\tau\bar{n})$$

$$0 = (k\cos\alpha - \tau\sin\alpha)\,\bar{n}$$

$$\because \bar{n} \neq 0$$

$$\therefore k\cos\alpha - \tau\sin\alpha = 0$$

$$\frac{k}{\tau} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha$$

$$\frac{k}{\tau} = \tan\alpha$$

$\because$ α is constt $\therefore$ $\tan\alpha$ is constt

$$\Rightarrow \frac{k}{\tau} = \text{constt.}$$

Sufficient Condition;

Let $\frac{k}{\tau} = \text{constant}$

We have to prove that the given curve is a helix. For this we have to prove $\bar{t}\cdot\bar{a} = \text{constt}$

$\therefore \quad \frac{k}{\tau} = \text{constant}$

$\frac{k}{\tau} = \frac{1}{c} \rightarrow (1)$

By TNB frame

$t' = kn$

$\frac{dt}{ds} = \frac{\tau}{c} n \rightarrow (2)$ by eqn (1)

Again consider

$\bar{b}' = -\tau \bar{n}$

$\frac{d\bar{b}}{ds} = -\tau \bar{n}$

$\frac{1}{c} \frac{d\bar{b}}{ds} = -\frac{\tau}{c} \bar{n} \rightarrow (3)$

Adding (2) & (3)

$\frac{d\bar{t}}{ds} + \frac{1}{c} \frac{d\bar{b}}{ds} = 0$

$\frac{d}{ds}\left(\bar{t} + \frac{1}{c}\bar{b}\right) = 0$

$\Rightarrow \frac{d}{ds}\left(c\bar{t} + \bar{b}\right) = 0$

Integ w.r.t 's' $\quad c\bar{t} + \bar{b} = \text{constt}$

$c\bar{t} + \bar{b} = \bar{a} \quad (\text{const} = \bar{a})$

Taking dot product of $\bar{t}$ on both sides.

$$\bar{t}\cdot(c\bar{t}+\bar{b}) = \bar{t}\cdot\bar{a}$$

$$c(\bar{t}\cdot\bar{t}) + (\bar{b}\cdot\bar{t}) = \bar{t}\cdot\bar{a}$$

$$c + 0 = \bar{t}\cdot\bar{a}$$

$$\Rightarrow \bar{t}\cdot\bar{a} = c$$

$$\Rightarrow \bar{t}\cdot\bar{a} = \text{constt.}$$

Lecture

Remark

(i) if $\frac{k}{\tau} = 0$ then the curve is a straight line.

(ii) if $\frac{k}{\tau} = \alpha$ then the curve is a plane curve.

Spherical Indicatrix:

The locus of a pt whose p.v is equal to the unit tangent $\bar{t}$ of a given curve C is caled spherical indicatrix of tangent ~~of tangent~~ ~~to th~~ to the curve C and it is denoted by $\vec{r}_1 = \vec{t}$

NOTE: Spherical indicatrix lies on the surface of a unit sphere

Theorem - Prove that the curvature of ~~tangent~~ spherical indicatrix of tangent is the ratio of skew curvature and circular curvature.

$$k_1 = \frac{\sqrt{\tau^2 + k^2}}{k}$$

also prove that the torsion of spherical curvature is $\tau_1 = \dfrac{k\tau' - \tau k'}{k(k^2+\tau^2)}$

Proof

$\because$ spherical indicatrix of tangent is $\vec{r}_1 = \vec{t}$

diff w.r.t s_1

$$\frac{dr_1}{ds_1} = \frac{dt}{ds} \cdot \frac{ds}{ds_1}$$

$$t_1 = t' \cdot \frac{ds}{ds_1}$$

$$t_1 = kn \cdot \frac{ds}{ds_1}$$

$$t_1 = n \rightarrow (1) \qquad \frac{ds_1}{ds} = k \rightarrow (2)$$

$$\frac{dt_1}{ds_1} = \frac{dn}{ds} \cdot \frac{ds}{ds_1}$$

$$k_1 n_1 = (\tau b - kt) \cdot \frac{ds}{ds_1}$$

$$k_1 n_1 = \frac{\tau b}{k_1} - t$$

$$k_1 n_1 = \frac{\tau b - kt}{k}$$

Squaring on both sides

$$K_1 = \frac{\sqrt{\tau^2 + K^2}}{K} \to (A)$$

K_1 = Skew curvature

circular curvature

As the eqn of osculating sphere

$$R^2 = \rho^2 + \sigma^2 \rho'^2 \to (3)$$

As for spherical indicatrix the sphere must be unit sphere

i.e, R will be of 1.

$$R = 1$$

put $R = 1$ in (3)

$$1 = \rho^2 + \sigma^2 \rho'^2$$

and for spherical indicatrix this eqn takes the form.

$$1 = \rho_1^2 + \sigma_1^2 \rho_1'^2 \to$$

$$1 = \frac{1}{K_1^2} + \frac{1}{\tau_1^2} \rho_1'^2 \to (4)$$

$$\because \rho_1'^2 = \frac{K'^2}{K_1^4}$$

$$\rho_1 = \frac{1}{K_1}$$

$$\rho_1 = K_1^{-1}$$

$$\frac{d\rho_1}{ds} = (-1) K_1^{-2} K_1'$$

$$\rho_1'^2 = (K_1^{-2})^2 K_1'^2 = \frac{K_1'^2}{K_1^4}$$

Put in (4)

$$1 = \frac{1}{K_1^2} + \frac{1}{\tau_1^2} \cdot \frac{K_1'^2}{K_1^4}$$

$$1 = \frac{\tau_1^2 K_1^2 + K_1'^2}{K_1^4 \tau_1^2}$$

$$1 = \frac{\tau_1^2 + \rho^2 K_1^2}{K \tau_1^2}$$

$$\tau_1^2 =$$

$$1 = \frac{K_1^4 \tau_1^2 + K_1^2 \cdot K_1'^2}{K_1^4 \tau_1^2}$$

$$= \frac{K_1^2 (\tau_1^2 + K_1'^2)}{K_1^4 \tau_1^2}$$

$$\tau_1^2 = \frac{\tau_1^2 + K_1'^2}{K_1^2}$$

$$\tau_1 = \frac{K'}{K_1 \sqrt{K_1^2 - 1}}$$

$$1 = \frac{k_1^2 \tau_1^2 + k_1'^2}{\tau_1^2 k_1^4}$$

$$1 - \frac{1}{k_1^2} = \frac{k_1'^2}{\tau_1^2 k_1^4}$$

$$\tau_1 =$$

$$\tau_1 = \frac{k_1'^2}{k_1^2 \sqrt{k_1^2 - 1}}$$

$$\frac{k_1^2 - 1}{k_1^2} = \frac{k_1'^2}{\tau_1^2 k_1^4}$$

$$\frac{k_1^2 - 1}{1} = \frac{k_1'^2}{\tau_1^2 k_1^2}$$

$$\frac{k_1^2 - 1}{k_1'^2} = \frac{1}{\tau_1^2 k_1^2}$$

$$k_1 = \frac{\sqrt{\tau^2 + k^2}}{k}$$

diff w.r.t s_1

$$\frac{dk_1}{ds_1} = \frac{d}{ds}\left(\frac{\sqrt{\tau^2 + k^2}}{k}\right)\frac{ds}{ds_1}$$

$$= \frac{\frac{1}{2}(\tau^2 + k^2)^{-1/2}(2\tau\tau' + 2kk')k - \sqrt{\tau^2 + k^2}\,(k')}{k^2} \cdot \frac{ds}{ds_1}$$

$$= \frac{\frac{(\tau\tau' + kk')k}{(\tau^2 + k^2)^{1/2}} - \frac{1}{}\, k'\sqrt{\tau^2 + k^2}}{k^2}$$

$$= \frac{(\tau\tau' + kk')k - k'(\tau^2 + k^2)}{k^2(\tau^2 + k^2)^{1/2}} \cdot \frac{ds}{ds_1}$$

$$= \frac{\tau\tau' k + \cancel{k^2 k'} - k'\tau^2 - \cancel{k^2 k'}}{k^2(\tau^2 + k^2)^{1/2}} \cdot \frac{ds}{ds_1}$$

$$k_1' = \frac{\tau\tau' k - k'\tau^2}{k^2(\tau^2 + k^2)^{1/2}} \cdot \frac{1}{k}$$

$$k_1' = \frac{\tau\tau' k - k'\tau^2}{k^3(\tau^2 + k^2)^{1/2}} \rightarrow (7)$$

Put (A) and (7) in (6)

$$\tau_1 = \frac{1}{k_1\sqrt{k_1^2 - 1}} \cdot k_1'$$

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$$\tau_1 = \frac{1}{\frac{\sqrt{\tau^2+k^2}}{k}\sqrt{\frac{\tau^2+k^2}{k^2}-1}} \cdot \frac{k\tau\tau' - k'\tau^2}{k^3(\tau^2+k^2)^{1/2}}$$

$$= \frac{1}{\frac{\sqrt{\tau^2+k^2}\sqrt{\tau^2+\cancel{k^2}-\cancel{k^2}}}{k^2}} \cdot \frac{k\tau\tau' - k'\tau^2}{k^3(\tau^2+k^2)^{1/2}}$$

$$= \frac{\cancel{k^2}}{\tau\sqrt{\tau^2+k^2}} \cdot \frac{k\tau\tau' - k'\tau^2}{k^{\cancel{3}}(\tau^2+k^2)^{1/2}}$$

$$= \frac{k\cancel{\tau}\tau' - k'\tau^{\cancel{2}}}{k\cancel{\tau}(k^2+\tau^2)}$$

$$\tau_1 = \frac{k\tau' - k'\tau}{k(k^2+\tau^2)}$$

Assignment

Q Prove that the curvature and torsion of the spherical indicatrix of the binormal is given by

$\bar{\gamma}_1 \not\parallel \bar{n}$

$$k_1 = \frac{\sqrt{k^2+\tau^2}}{\tau}$$

Thats why ($\bar{\gamma}_1$ is not along to $\bar{n}$)

&

$$\tau_1 = \frac{k'\tau - \tau' k}{\tau(k^2+\tau^2)}$$

Working Rule to Find out the Spherical Indicatrix (Images)

normal

1- Give a space curve

$$\vec{r} = \vec{r}(\theta) + \vec{r}(t) \rightarrow ①$$

Find out the unit tangent i.e. t by differentiating eq/n ①

Similarly, find out the unit normal by using the result of "t" and also also find the binormal by using unit tangent and unit normal.

2- Equate $\vec{t} = \vec{r}$

$\vec{b} = \vec{r}$

$\vec{n} = \vec{r}$

to find the required images.

Find out the required spherical images (indicatrix) of the circular helix

$$\vec{r} = (a\cos\theta, a\sin\theta, c\theta) \qquad c \neq 0$$

$$\frac{dr}{ds} = (-a\sin\theta, a\cos\theta, c)\frac{d\theta}{ds}$$

$$t = (-a\sin\theta, a\cos\theta, c) \cdot \theta' \rightarrow ①$$

$$\frac{d\theta}{ds} = \frac{1}{\sqrt{a^2+c^2}}$$

$$\Rightarrow \frac{ds}{d\theta} = \sqrt{a^2+c^2} = \lambda \; ② \text{ (any scalar)}$$

① $\Rightarrow$

$$t = (-a\sin\theta, a\cos\theta, c) \cdot \frac{1}{\lambda} \rightarrow ③$$

$$\frac{dt}{ds} = \frac{1}{\lambda}(-a\cos\theta, -a\sin\theta, 0)\cdot\frac{d\theta}{ds}$$

$$kn = (-a\cos\theta, -a\sin\theta, 0)\cdot\frac{1}{\lambda^2} \rightarrow (4)$$

~~and~~ $k^2 = \frac{a^2}{\lambda^4} \Rightarrow k = \frac{a}{\lambda^2}$ Put in (4)

$$\frac{\cancel{a}}{\cancel{\lambda^2}}\cdot n = (-\cos\theta, -\sin\theta, 0)\frac{\cancel{a}}{\cancel{\lambda^2}}$$

$$n = (-\cos\theta, -\sin\theta, 0)$$

$\because t \times n = b$

$$t \times n = \frac{1}{\lambda}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin\theta/\cancel{\lambda} & a\cos\theta/\cancel{\lambda} & c/\cancel{\lambda} \\ -\cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= \hat{i}\left[a\cos\theta(0) + c\sin\theta\right] - \hat{j}\left[0 + c\cos\theta\right] + \hat{k}\left(a \cancel{\quad}\right)$$

$$\underline{b} = \frac{1}{\lambda}\left[c\sin\theta\,\hat{i} - c\cos\theta\,\hat{j} + a\hat{k}\right]$$

$$b = \frac{1}{\lambda}(c\sin\theta, -c\cos\theta, a)$$

To find the spherical images equate $\bar{r}_1$ and $\bar{t}$

say $\vec{r}_1 = x\hat{i} + y\hat{j} + z\hat{k}$

$$x\hat{i} + y\hat{j} + z\hat{k} = \left(-\frac{a\sin\theta}{\lambda}, \frac{a\cos\theta}{\lambda}, \frac{c}{\lambda}\right)$$

$x = \frac{-a\sin\theta}{\lambda}$ $y = \frac{a\cos\theta}{\lambda}$ $z = \frac{c}{\lambda}$

are spherical indicatrix of tangent

Now equate $\bar{r}_1$ and $\bar{n}$ say

$r_1 = x\hat{i} + y\hat{j} + z\hat{k}$

$x = -\cos\theta$ $y = -\sin\theta$ $z = 0$

are spherical images/indicatrix of normal

Equate $\bar{r}_1$ and $\bar{b}$ say $r_1 = x\hat{i} + y\hat{j} + z\hat{k}$

$\Rightarrow$ $x = \frac{c\sin\theta}{\lambda}$ $y = \frac{-c\cos\theta}{\lambda}$ $z = \frac{a}{\lambda}$

are spherical indicatrix of binormal

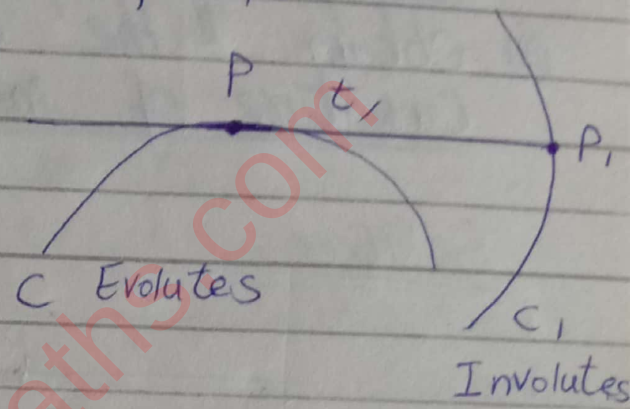
Home Task

Taking any r from previous lect and find indicatrix.

Lecture

Involutes And Evolutes :

When the tangents to the curve C are ~~normals~~ Iar normals to the another curve C_1 then C_1 is called involutes of C and C is called an evolutes of C_1.



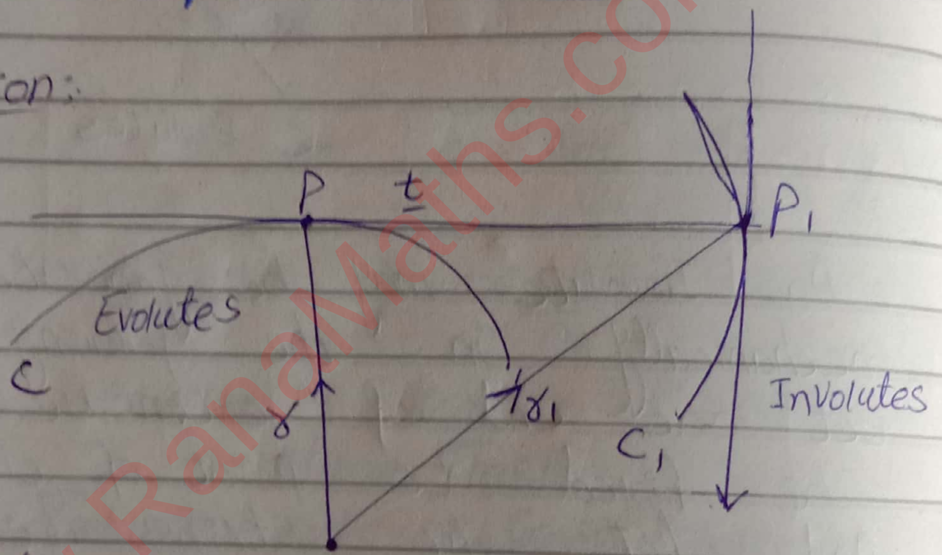
An Involute may be generated mechanically in the following manner. Let one end of an extensible string be fixed to point of curve C. And let the string be kept tight while it is wrapped around the curve on its convex sides. Then any particle of the string describes an ~~envelope~~ involute since at each instant the free part of the string is the tangent to the curve while the direction of the motion of particle is at right angle to this tangent.

Theorem; (i) Prove that there exist infinite many involutes to the given space curve C.

(ii) prove that the tangent at any point P_1 of the involutes C_1 is parallel to the normal of a corresponding point to the curve C.

(iii) Obtain the expression for the curvature of involutes C_1.

Solution:



Let the given space curve C of parameter s with p.v $\vec{\gamma}$

i.e, $\vec{\gamma} = \vec{\gamma}(s)$

and the p.v of the pt P_1 of the curve C_1 is γ_1

i.e, $\gamma_1 = \gamma + \lambda \vec{t} \rightarrow (1)$

$PP_1 \parallel t$

$PP_1 = \lambda t$

diff (1) w.r.t s_1

$$\frac{d\gamma_1}{ds_1} = \frac{d\gamma}{ds} \cdot \frac{ds}{ds_1} + \left(\lambda' t + \lambda \frac{dt}{ds}\right) \cdot \frac{ds}{ds_1}$$

$\gamma = \gamma(s)$

$\gamma_1 = \gamma + \lambda t$

dot product with t

$1+\lambda' = 0$

$\lambda = c - s$ (b) $\lambda' = -1$

(c) $t_1 = n$

dif w.r.t s

$$\left[t_1 = t \cdot \frac{ds}{ds_1} + \lambda' t + \lambda t' \cdot \frac{ds}{ds_1} \right] \times$$

$$t_1 = (t + \lambda' t + \lambda t') \frac{ds}{ds_1}$$

$$t_1 = (t + \lambda' t + \lambda (kn)) \frac{ds}{ds_1}$$

$$t_1 = (t(1+\lambda') + k\lambda n) \frac{ds}{ds_1}$$

$$\frac{ds}{ds_1} = \frac{1}{\sqrt{(1+\lambda')^2 + k^2\lambda^2}}$$

$$t_1 = \frac{t(1+\lambda') + \lambda kn \cdot 1}{\sqrt{(1+\lambda')^2 + k^2\lambda^2}}$$

Taking dot product with 't'

$$0 = (t \cdot t + \lambda' t \cdot t + \lambda k n \cdot t) \frac{ds}{ds_1}$$

$$0 = (1 + \lambda' + 0) \frac{ds}{ds_1}$$

$\frac{ds}{ds_1} \neq 0$ $\qquad$ $1 + \lambda' = 0$

$$\lambda' = -1$$

$$\boxed{\lambda = -s + c}$$

Substitute λ in eqn ①

$$\vec{\gamma}_1 = \vec{\gamma} + (-s + c)\vec{t}$$

This eqn shows that there are different values of γ_1 and arbitrary constant c. Hence proved that there ~~are~~ exist infinite many involutes for each evolute c.

(ii) review (i) in this part

$$t_1 = (\gamma' + \lambda' t + \lambda k n)\frac{ds}{ds_1}$$

where $\lambda' = -1$

and $\lambda = c - s$

$$t_1 = (\cancel{t} + (-1)t + (c-s)kn)\frac{ds}{ds_1}$$

$$t_1 = \left[(c-s)\,k\frac{ds}{ds_1}\right]\vec{n}$$

$$\Rightarrow \vec{t}_1 \parallel \vec{n}$$

and $$\frac{ds}{ds_1} = \frac{1}{(c-s)k}$$

(iii) $\because$ by (ii)

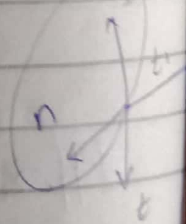
$$\vec{t}_1 = \vec{n}$$

Diff w.r.t s_1

$$\frac{dt'}{ds_1} = \frac{dn}{ds}\cdot\frac{ds}{ds_1}$$

$$k_1 n_1 = (\tau b - kt)\cdot\frac{ds}{ds_1}$$

$$k_1 n_1 = (\tau b - kt)\frac{1}{(c-s)k}$$



$$k_1^2 = \frac{\tau^2 + k^2}{k^2(c-s)^2}$$

$$k_1 = \frac{\sqrt{\tau^2 + k^2}}{k(c-s)}$$

which is the required expression of curvature of curve C_1.

can be

$\gamma = \gamma(s)$

$\gamma_1 = \gamma + PP_1$

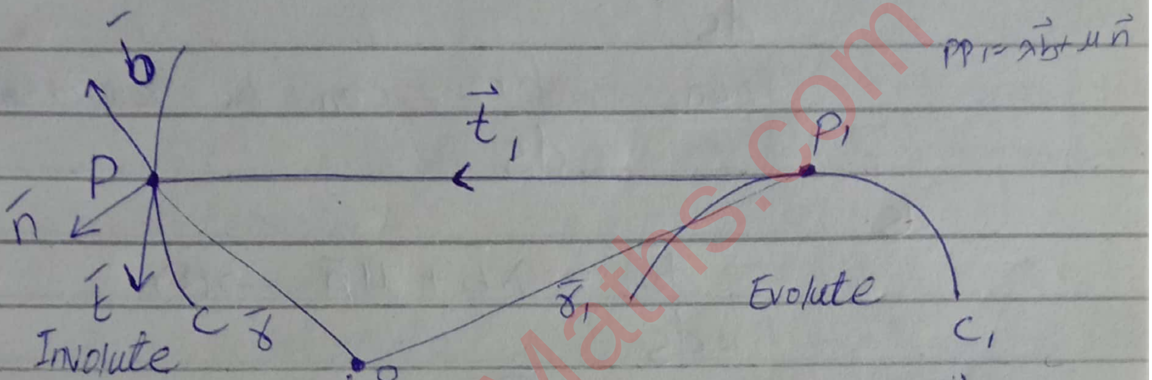
$PP_1 = \lambda b + \mu n$

Theorem: Prove that there are infinite family of evolutes for the space curve C (Involutes)

proof:

Let γ be the p.v of the space curve C with parameter s

i.e, $\vec{\gamma} = \vec{\gamma}(s)$



From figure the tangent $\vec{t}_1$ at P_1 of C_1 is normal at corresponding point P of C that is the tangent t_1 at P_1 lies in the normal plane.

$\therefore$ the vector PP_1 can be expressed as the linear combination of $\underline{b}$ and $\underline{n}$

i.e, $\vec{PP_1} = \lambda\vec{b} + \mu\vec{n}$

and $\vec{\gamma}_1 = \vec{\gamma} + \vec{PP_1}$

$$\vec{\gamma}_1 = \vec{\gamma} + \lambda\vec{b} + \mu\vec{n} \rightarrow (1)$$

diff w.r.t "s"

$$\frac{d\bar{r}_1}{ds} \cdot \cancel{\frac{ds}{ds_1}} = r' + \lambda' b + \lambda b' + \mu' n + \mu n'$$

$$\frac{d\bar{r}_1}{ds} = t + \lambda' b + \lambda(-\tau n) + \mu' n + \mu(\tau b - kt)$$

$$\bar{t}_1 = (1-\mu k)\bar{t} + (\lambda' + \mu\tau)\bar{b} + (\mu' - \lambda\tau)\bar{n} \rightarrow (2)$$

As $\frac{d\bar{r}_1}{ds}$ lies in normal plane then $\frac{d\bar{r}_1}{ds}$ can be expressed as

$$\frac{d\bar{r}_1}{ds} = \lambda\bar{b} + \mu\bar{n} \rightarrow (3)$$

$\therefore$ eqn (2) $\Rightarrow$

$$\lambda\bar{b} + \mu\bar{n} = (1-\mu k)t + (\lambda' + \mu\tau)\bar{b} + (\mu' - \tau\lambda)\bar{n}$$

Comparing vectors

$(1-\mu k) = 0 \rightarrow (4)$ $\quad \lambda' + \mu\tau = \lambda \rightarrow (6)$ $\quad \mu' - \lambda\tau = \mu \rightarrow (7)$

$1 = \mu k$ $\quad \frac{\lambda' + \mu\tau}{\lambda} = 1 \quad (6)$ $\quad \frac{\mu' - \tau\lambda}{\mu} = 1 \quad (7)$

$\boxed{\mu = \rho} \rightarrow (5)$

Comparing (6) & (7)

$$\frac{\lambda' + \mu\tau}{\lambda} = \frac{\mu' - \tau\lambda}{\mu}$$

$$\mu(\lambda' + \mu\tau) = \lambda(\mu' - \tau\lambda)$$

$$\mu\lambda' - \mu'\lambda = -(\tau\lambda^2 + \mu^2\tau)$$

$$\underline{\mu\lambda' - \mu'\lambda} = -\tau(\lambda^2 + \mu^2)$$

$x^2\left(1+\frac{\mu^2}{x^2}\right)\quad(\mu' x - \mu x'$

$\tan^{-1}\left(\frac{\mu}{\lambda}\right)$

$\mu = \rho$

$$\tau = \frac{1}{\mu^2+\lambda^2}(\lambda\mu' - \lambda'\mu)$$

$$\tau = \frac{1}{\frac{\lambda^2}{\lambda^2}(\mu^2+\lambda^2)}(\lambda\mu' - \mu\lambda')$$

$$\tau = \frac{1}{\left(1+\frac{\mu^2}{\lambda^2}\right)}\frac{(\lambda\mu' - \mu\lambda')}{\lambda^2}$$

$$\tau = \frac{d}{ds}\left[\tan^{-1}\left(\frac{\mu}{\lambda}\right)\right]$$

$\cot(\phi(s)-c')$

$\cot(90+\phi(s)-c')$

Integrating w.r.t "s"

$$\int \tau\, ds = \tan^{-1}\left(\frac{\mu}{\lambda}\right) + c'$$

$$\phi(s) = \tan^{-1}\left(\frac{\rho}{\lambda}\right) + c' \quad \text{by } (5) \because \mu = \rho$$

$$\phi(s) - c' = \tan^{-1}\left(\frac{\rho}{\lambda}\right)$$

$$\lambda \tan(\phi(s) - c) = \rho$$

$$\Rightarrow \quad \lambda = \rho \cot(\phi(s) - c')$$

$\because$ the rotation of the curve is at right angle

$$\Rightarrow \lambda = \rho \cot(90 + \phi(s) - c')$$

$$\Rightarrow \lambda = -\rho \tan(\phi(s) - c')$$

$\cot(90+\alpha) = -\tan\alpha$

$$\Rightarrow \lambda = -\rho \tan(\phi(s) + \alpha) \rightarrow (6)$$

$\boxed{-c' = \alpha}$

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Substituting eqn (5) & (8) in (1)

$$r_1 = r + \lambda \bar{b} + \mu \bar{n}$$

$$\bar{r}_1 = r + \rho n - \rho \tan(\phi(s) + \alpha)\, \bar{b}$$

This is the eqn of evolutes for C_1 and for different values of α. We can obtain different evolutes and hence infinite many evolutes C_1 for the curve C exist.

Theorem:
Prove that the locus of the centre of curvature is an evolute only when the curve is a plane curve.

Proof

Since the equation of evolute can be written as

$$\bar{r}_1 = \bar{r} + \rho\bar{n} - \rho \tan\{\phi(s) + \alpha\}\bar{b} \rightarrow ①$$

In equation ① we can obtain different evolutes for different values of α, also the locus of centre of curvature can be written as,

$$\bar{\tau} = \bar{r} + \rho\bar{n} \rightarrow ②$$

Eqn ① & ② identical only when

$$\rho \tan\{\phi(s) + \alpha\} \underline{b} = 0$$

$\rho \neq 0$ $\quad \underline{b} \neq 0 \Rightarrow \tan\{\phi(s) + \alpha\}\underline{b} = 0$

$$\tan\{\phi(s) + \alpha\} = \tan n\pi$$

(n is an integer)

$$\phi(s) + \alpha = n\pi$$

$$\phi(s) = n\pi - \alpha$$

$$\Rightarrow \phi'(s) = 0 \rightarrow ③$$

$\because$ we know that

$$\int \tau\, ds = \phi(s)$$

$$\tau = \phi'(s) \rightarrow ④$$

$\because$

comparing ③ & ④

$\tau = 0$

Hence proved.

Prove that the ratio of torsion and curvature of an evolute of a space curve (involute) is given by

$$\frac{\tau_1}{k_1} = -\tan(\phi+\theta)$$

where $\phi = \int \tau \, ds$

Proof

Since the equation of an evolute is given by

$$\bar{r}_1 = r + \bar{n}\rho - \rho\tan\{\phi(s)+\alpha\}\underline{b} \to ①$$

Diff w.r.t "s_1"

$$\frac{dr_1}{ds_1} = \frac{dr}{ds}\cdot\frac{ds}{ds_1} + (\rho'\bar{n} + n'\rho)\frac{ds}{ds_1} - \Big\{ \rho'\tan\{\phi(s)+\alpha\}b + \underline{\rho \sec^2\{\phi(s)+\alpha\}(\phi'(s))\, b} + \rho\tan\{\phi(s)+\alpha\}b' \Big\}\frac{ds}{ds_1}$$

$$t_1 = \Big[t + (\rho' n + \rho(\tau b - kt)) - \rho'\tan\{\phi(s)+\alpha\}b - \rho\sec^2\{\phi(s)+\alpha\}\phi'(s)\, b - \rho\tan\{\phi(s)+\alpha\}(-\tau n)\Big]\frac{ds}{ds_1}$$

$$t_1 = \left[\cancel{t} + n\rho' + \rho\tau b - \cancel{t} - \rho'\tan\{\phi(s)+\alpha\}b - \rho\sec^2\{\phi(s)+\alpha\}\tau b + \rho\tan\{\phi(s)+\alpha\}\tau n\right]\frac{ds}{ds_1}$$

t_1

$c = r + n\rho$

$\frac{dc}{ds_1}\cdot\frac{ds_1}{ds}$

$\sec^2\alpha = 1+\tan^2\alpha$

$1-\tan^2\phi(s)+\alpha$

$$t_1 = \bar{n}\rho' + \rho\tau b - \rho' b \tan\{\phi(s)+\alpha\} -$$

$$\rho\sec^2\{\phi(s)+\alpha\}\tau b + \rho\tau n^* \tan\{\phi(s)+\alpha\}$$

$$\cancel{= n\rho' + \rho\tau\tan\{\phi(s)+\alpha\} n}$$

$$= n\rho' + \cancel{\rho\tau b} - \rho'\tan(\phi+\alpha) b + \rho\tau n\tan(\phi+\alpha)$$

$$- \cancel{\rho\tau b} - \rho\tau\tan^2(\phi+\alpha) b \Big] \frac{ds}{ds_1}$$

$$= \Big[\big(n\rho' - \rho'\tan(\phi+\alpha) b\big) + \big(\rho\tau\tan(\phi+\alpha) n$$

$$- \rho\tau\tan^2(\phi+\alpha) b\big)\Big] \frac{ds}{ds_1}$$

$$= \big(\rho' + \rho\tau\tan(\phi+\alpha)\big) n \big($$

$$t_1 = \frac{d\bar{r}_1}{ds_1} = \big(\rho' + \rho\tau\tan(\phi+\alpha)\big)\big(\bar{n} - \tan(\phi+\alpha) b\big) \frac{ds}{ds_1} \longrightarrow (2)$$

$$\frac{ds}{ds_1} = \frac{1}{\big(\rho' + \rho\tau\tan(\phi+\alpha)\big)\sec(\phi+\alpha)} \longrightarrow (3)$$

$$\bar{t}_1 = \frac{\rho' + \rho\tau\tan(\phi+\alpha)}{\rho' + \rho\tau\tan(\phi+\alpha)} \frac{\big(\bar{n} - \tan(\phi+\alpha) b\big)}{\sec(\phi+\alpha)}$$

$$\bar{t}_1 = \frac{\bar{n} - \tan(\phi+\alpha) b}{\sec(\phi+\alpha)}$$

$t = \bar{n}\rho$

$\frac{dt}{ds} \frac{ds_1}{ds}$

~~Substituting~~

$$t_1 = \left(n - \frac{\sin(\phi+\alpha)\, b}{\cos(\phi+\alpha)}\right)\cos(\phi+\alpha)$$

$$t_1 = \cos(\phi+\alpha)\, n - \sin(\phi+\alpha)\, b \rightarrow (4)$$

differentiate (4) w.r.t "s_1"

$$\frac{dt_1}{ds_1} = n'\cos(\phi(s)+\alpha) + n(-\sin(\phi(s)+\alpha))\phi'(s)$$
$$- \left\{ b'\sin(\phi(s)+\alpha) + b\cos(\phi(s)+\alpha)\phi'(s)\right] \frac{ds}{ds_1}$$

$$= \Big[n'\cos(\phi+\alpha) - n\sin(\phi+\alpha)\tau + \tau n \sin(\phi+\alpha) - b\tau\cos(\phi+\alpha)\Big] \frac{ds}{ds_1}$$

$$= \Big[(\cancel{\tau b} - kt)\cos(\phi+\alpha) - \cancel{n\tau\sin(\phi+\alpha)} + \cancel{\tau n\sin(\phi+\alpha)} - \cancel{b\tau\cos(\phi+\alpha)}\Big] \frac{ds}{ds_1}$$

$$k_1 n_1 = t_1' = -kt\cos(\phi(s)+\alpha)\frac{ds}{ds_1}$$

$$n_1 = -t \rightarrow (5)$$

$$k_1 = k\cos(\phi(s)+\alpha)\frac{ds}{ds_1} \rightarrow (6)$$

$$k_1 = k\cos\{\phi(s)+\alpha\}\frac{ds}{ds_1} \rightarrow (A)$$

$-(n \times t)$

$$\because \quad \bar{b}_1 = \bar{t}_1 \times \bar{n}_1$$

$$b_1 = \cos(\phi+\alpha)\underline{b} + \sin(\phi+\alpha)\underline{n} \rightarrow (7)$$

Diff (7) W.r.t s_1

$$b_1' = -\tau_1 n_1 = \Big[b'\cos(\phi(s)+\alpha) + b(-\sin(\phi(s)+\alpha)\phi'(s))$$

$$+ n'\sin(\phi(s)+\alpha) + n\cos(\phi(s)+\alpha)\phi'(s)\Big]\frac{ds}{ds_1}$$

$$-\tau_1 n_1 = \Big[\cancel{\cos(\phi+\alpha)(-\tau n)} - \cancel{b\tau\sin(\phi+\alpha)}$$

$$+ \cancel{\sin(\phi+\alpha)\tau b} - kt\sin(\phi+\alpha) + \cancel{n\tau\cos(\phi+\alpha)}\Big]\frac{ds}{ds_1}$$

$$-\tau_1 n_1 = -kt\sin(\phi+\alpha)\frac{ds}{ds_1}$$

$n_1 = -t$

$$\tau_1 = -k\sin(\phi+\alpha)\frac{ds}{ds_1}$$

$$\frac{\tau_1}{k} = -\sin(\phi+\alpha)\frac{ds}{ds_1} \rightarrow (B)$$

$\div$ing (A) & (B)

$$\frac{\tau_1}{k_1} = \frac{-\sin(\phi+\alpha)}{\cos(\phi+\alpha)} = -\tan(\phi+\alpha)$$

Proved.

$t_1 \times n_1$

$b_1 =$

Lecture 9/1/2020

Next chapter

Differential Geometry of Surfaces

A surface is the locus of a point whose coordinates are ftns of two independent parameters. For example if u and v are two independent parameters and x, y, z depends on these parameters i.e,

$$x = f_1(u,v) \quad , \quad y = f_2(u,v) \; , \; z = f_3(u,v)$$

These are the parametric eqns of a surface.

Remarks:

(i) Elemination of u and v from parametric equations will give rise to the equation

$$f(x,y,z) = c$$

which is called the implicit form of the surface.

(ii) An equation of the form

$$z = f(x,y)$$

or $\quad x_3 = f(x_1, x_2)$

is called monge's form.

Example Of Surfaces:

(1) If the parametric equations of the sphere with centre at the origin and radius a are given as

$$x = a\cos\theta\cos\phi \rightarrow (1)$$
$$y = a\cos\theta\sin\phi \rightarrow (2)$$
$$z = a\sin\theta \rightarrow (3)$$

Eliminating the parameters θ, ϕ from above equations, we will get

$$x^2 + y^2 + z^2 = a^2$$

$$x^2 = a^2\cos^2\theta\cos^2\phi \qquad y^2 = a^2\cos^2\theta\sin^2\phi$$
$$z^2 = a^2\sin^2\theta$$

$$x^2 + y^2 + z^2 = a^2\cos^2\theta\cos^2\phi + a^2\cos^2\theta\sin^2\phi + a^2\sin^2\theta$$
$$= a^2\cos^2\theta(\cos^2\phi + \sin^2\phi) + a^2\sin^2\theta$$
$$= a^2(\cos^2\theta + \sin^2\theta)$$

$$\boxed{x^2 + y^2 + z^2 = a^2}$$

which is the equation of sphere with centre at origin and radius a.

(ii) If the parametric equations of an ellipsoid are given below

$$x = a\cos\theta\cos\phi$$
$$y = b\cos\theta\sin\phi$$
$$z = c\sin\theta$$

Eliminat θ and ϕ from these equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$\frac{x}{a} = \cos\theta\cos\phi$ $\quad \frac{y}{b} = \cos\theta\sin\phi$ $\quad \frac{z}{c} = \sin\theta$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

(iii) If the parametric equations of cone are

$$x = \mu\sin\phi\cos\psi$$
$$y = \mu\sin\phi\sin\psi$$
$$z = \mu\cos\phi$$

Eliminating ϕ & ψ from these above equations.

$$x^2+y^2 = \mu^2 \sin^2\phi$$

$$x^2+y^2 = \mu^2 \frac{\sin^2\phi}{\cos^2\phi} \cos^2\phi$$

$$x^2+y^2 = (\mu\cos\phi)^2 \tan^2\phi$$

$$x^2+y^2 = z^2 \tan^2\phi$$

Tangent Line To The Surfaces;

Tangent to any curve drawn at a surface is called a tangent line to that surface.

Tangent Plane:

Tangent plane to a surface at a point P is the plane containing all the tangent lines to the surface at this point.

Theorem: Derive the equation of tangent plane and the equation of normal plane at a point P to the surface $F(x,y,z)=0$

Proof:

Consider any curve drawn on the given surface

$$F(x,y,z)=0 \longrightarrow (1)$$

Let S be the arc length of the curve measured from a fixed point A upto the current point (x, y, z)

Differentiate eqn ① w.r.t "S"

$$F(x, y, z) = 0$$

$$\frac{dF}{ds} = \frac{\partial F}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial F}{\partial y} \cdot \frac{dy}{ds} + \frac{\partial F}{\partial z} \cdot \frac{dz}{ds} = 0$$

OR

$$\frac{dF}{dS} = F_x \cdot x' + F_y \cdot y' + F_z \cdot z' = 0$$

OR

$$(F_x, F_y, F_z) \cdot (x', y', z') = 0 \rightarrow ②$$

where vector (x', y', z') is the unit tangent $\bar{t}$ to the curve at the point (x, y, z)

Eqn ② also shows that the unit tangent t is $\perp$ar to the vector (F_x, F_y, F_z)

$\therefore$ it is clear that all the tangent lines to the surface at the point (x, y, z) $\perp$ar to the vector (F_x, F_y, F_z) and hence lie in the plane through (x, y, z) $\perp$ar to (F_x, F_y, F_z)

And this plane is called the tangent plane.

$\therefore$ the line joining any point (X, Y, Z) on the tangent plane can be written as

$$F_x(X-x)+F_y(Y-y)+F_z(Z-z)=0 \rightarrow (3)$$

This is the equation of tangent plane.

$R-r=ut$

$X-x=ut$

$(X-x)\hat{i}+(Y-y)\hat{j}+(Z-z)\hat{k}$
$=u(x'\hat{i}+y'\hat{j}+z'\hat{k})$

$(R-r)\cdot t=0$

Similarly (X, Y, Z) is a current point on the normal. Then we have

$$\frac{X-x}{F_x}=\frac{Y-y}{F_y}=\frac{Z-z}{F_z} \rightarrow (A)$$

This is the equation of normal plane.

Theorem: Prove that the sum of intercept made by the tangent plane to the surface

$$x^{2/3}+y^{\frac{2}{3}}+z^{2/3}=a^{2/3}$$

Proof

Since the given surface is

$$x^{2/3}+y^{2/3}+z^{2/3}=a^{2/3}$$

$$x^{2/3}+y^{2/3}+z^{2/3}-a^{2/3}=0 \rightarrow (1)$$

And the equation of tangent plane is

$$(X-x)\frac{\partial F}{\partial x}+(Y-y)\frac{\partial F}{\partial y}+(Z-z)\frac{\partial F}{\partial z}=0 \rightarrow (2)$$

Now,

find $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ From eqn (1) and Substitute in (2)

$$(X-x)\frac{2}{3}x^{-1/3}+(Y-y)\cdot\frac{2}{3}y^{-1/3}+(Z-z)\cdot\frac{2}{3}z^{-1/3}=0$$

$$\frac{X-x}{x^{1/3}}+\frac{Y-y}{y^{1/3}}+\frac{Z-z}{z^{1/3}}=0$$

$$\frac{X}{x^{1/3}}-x^{2/3}+\frac{Y}{y^{1/3}}-y^{2/3}+\frac{Z}{z^{1/3}}-z^{2/3}=0$$

$$\frac{X}{x^{1/3}}+\frac{Y}{y^{1/3}}+\frac{Z}{z^{1/3}}=a^{2/3}$$

$$\frac{X}{a^{2/3}x^{1/3}}+\frac{Y}{a^{2/3}y^{1/3}}+\frac{Z}{a^{2/3}z^{1/3}}=1$$

$\therefore$ the intercept with the coordinate axis of tangent plane are

$a^{2/3}x^{1/3},\ a^{2/3}y^{1/3},\ a^{2/3}z^{1/3}$

and to prove the required.
Sum the squares of these intercept

$(a^{2/3}x^{1/3})^2 + (a^{2/3}y^{1/3})^2 + (a^{2/3}z^{1/3})^2$

$a^{4/3}(x^{2/3}+y^{2/3}+z^{2/3})$

$a^{4/3}\cdot a^{2/3} = a^{6/3} = a^2 = \text{constant}$

Theorem: Prove that the tangent plane at any point to the surface $xyz = a^3$ and the coordinate plane form a tetrahedron of constant volume

As we know that the volume of tetrahedron

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \rightarrow (1)$$

$(0,0,0)$

and the equation of tangent plane.

$(X-x)F_x + (Y-y)F_y + (Z-z)F_z = 0 \rightarrow (2)$

Now find the value of F_x, F_y, F_z
the equation of surface is

$$F(x,y,z) = xyz - a^3 = 0 \rightarrow (3)$$

$$(X-x)yz + (Y-y)xz + (Z-z)xy = 0$$

$$Xyz + Yxz + Zxy - 3xyz = 0$$

$$\frac{X}{3x} + \frac{Y}{3y} + \frac{Z}{3z} = 1$$

$\therefore$ $(0,0,0,0)$, $(3x,0,0)$, $(0,3y,0)$, $(0,0,3z)$

are the points of intersection.

Substitute these in eqn (1)

$$V = \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ 3x & 0 & 0 & 1 \\ 0 & 3y & 0 & 1 \\ 0 & 0 & 3z & 1 \end{vmatrix}$$

Expanding R_1

$$V = \frac{1}{6}\begin{vmatrix} 3x & 0 & 0 \\ 0 & 3y & 0 \\ 0 & 0 & 3z \end{vmatrix}$$

$$V = \frac{1}{\cancel{6}\,2}(3x)(3y)(3z)$$

$$V = \frac{9}{2}xyz = \frac{9a^3}{2}$$

Volume is constant.

Assignment

Question-1 Find the eqn of tangent plane to the surface $z = x^2 + y^2$ at the point $(1, -1, 2)$

Question-2 Show that the tangent plane at the point common to the surface $a(xy + yz + xz) = xyz$ and a sphere whose centre is at origin $x^2 + y^2 + z^2 = b^2$ makes intercepts on the axis whose sum is constt.

Question-3

Lecture

One Parameter Family of Surfaces

An equation of the form
$$F(x, y, z, a) = 0$$
where (a is any constant)
This eqn represents a surface.
If the value of "a" is changed then we obtain different surfaces and the set of all these surfaces corresponding to different values of "a" is called one parameter family of surfaces with parameter "a".

Example

For example
$$F(x, y, z, a) = x^2 + y^2 + z^2 - a^2 = 0$$
This eqn represent the family of sphere (For different values of "a"), with centre at origin.

Characteristics Of A Surface

The curve of intersection of the ~~family~~ two surfaces of the family corresponding to the parameter having value "a" and "$a+\delta a$" is determined by the equation

$$F(x,y,z,a) = 0 \rightarrow (1)$$

and $F(x,y,z,a+\delta a) = 0 \rightarrow (2)$

and these equations can also be written as, $F(a)=0 \rightarrow (3)$ $\frac{F(a+\delta a)-F(a)}{\delta a} = 0 \rightarrow (4)$

$\therefore$ for the sack of simplicity we will write $F(a)$ instead of $F(x,y,z,a)$ and so on

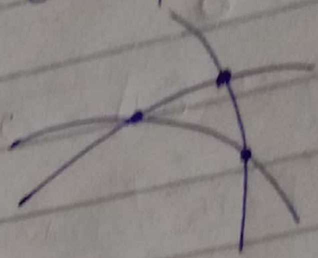
If the eqn (4) $\delta a \rightarrow 0$ then the curve becomes the curve of intersection and the above eqns can be written as,

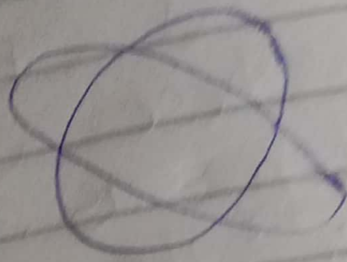
$$F(a)=0 \rightarrow * \quad \text{and} \quad \frac{\partial F(a)}{\partial a} = 0 \rightarrow *'$$

This equation curve is called the characteristic of the surface for parametric value "a".

Envelope of the Family Of Surfaces

The locus of characteristic of $F(a)=0$ (where "a" has different values) is called the envel... of the family of the surfaces.





Find the envelope of a family of paraboloid

$$x^2+y^2=4a(z-a)$$

$$F(x,y,z,a)=x^2+y^2-4az+4a^2=0 \rightarrow (1)$$

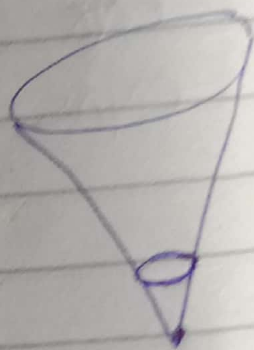
$$\frac{\partial F}{\partial a}=-4z+8a \qquad \frac{\partial F}{\partial a}=0$$

$$-4z+8a=0$$

$$+\cancel{4}z=\cancel{8}a$$

$$z=+2a$$

$$a=\frac{z}{2}$$



$$\therefore \quad x^2+y^2-\cancel{4}z\left(\frac{z}{\cancel{2}}\right)+4\left(\frac{z}{2}\right)^2=0$$

$$x^2+y^2-2z^2+z^2=0$$

$$\Rightarrow \quad x^2+y^2=z^2$$

This is the eqn of circular cone.

$(X-x)^2+(Y-y)^2+(Z-z)^2=b^2$

Q: Sphere of constant radii "b" have their centre on the fixed circle $x^2+y^2=a^2$ & $z=0$. Prove that their envelope in the surface $(x^2+y^2+z^2+a^2+b^2)^2=4a^2(x^2+y^2)$

Solution :-

To prove the required

Let

$P(a\cos\theta, a\sin\theta, 0)$ be the point on the circle

$$x^2+y^2=a^2 \quad \text{and} \quad z=0$$

Then the equation of the sphere of radius b with centre on the given circle

$$(x-a\cos\theta)^2+(y-a\sin\theta)^2+z^2=b^2 \longrightarrow (1)$$

(1) can also be written as,

$$F(x,y,z,\theta)=x^2+y^2+z^2+a^2-2a(x\cos\theta+y\sin\theta)-b^2=0 \rightarrow (2)$$

Diff eqn (2) w.r.t "θ"

$$\frac{\partial F}{\partial\theta} = 2a(x(-\sin\theta)+y\cos\theta)$$

$$\frac{\partial F}{\partial\theta} = 0$$

$y\cos\theta = x\sin\theta$

$\sin\theta = \frac{y}{x}\cos\theta$

$$\Rightarrow \sin\theta = \frac{y}{x}\cos\theta.$$

$$\sin^2\theta = \frac{y^2}{x^2}\cos^2\theta = \frac{y^2}{x^2}(1-\sin^2\theta)$$

$$x^2\sin^2\theta = y^2 - y^2\sin^2\theta$$
$$x^2\sin^2\theta + y^2\sin^2\theta = y^2$$

$$\sin^2\theta = \frac{y^2}{x^2+y^2}$$

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}} \rightarrow (3)$$

By putting (3) and (4) in eq/n (2)

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \rightarrow (4)$$

$$x^2+y^2+z^2+a^2-2a\left[\frac{x^2}{\sqrt{x^2+y^2}}+\frac{y^2}{\sqrt{x^2+y^2}}\right]-b^2=0$$

$$x^2+y^2+z^2+a^2-2a\left[\frac{x^2+y^2}{\sqrt{x^2+y^2}}\right]-b^2=0$$

$$x^2+y^2+z^2+a^2-2a\left(\sqrt{x^2+y^2}\right)-b^2=0$$

$$x^2+y^2+z^2+a^2-2a^2-b^2=0$$
$$x^2+y^2+z^2=a^2+b^2$$

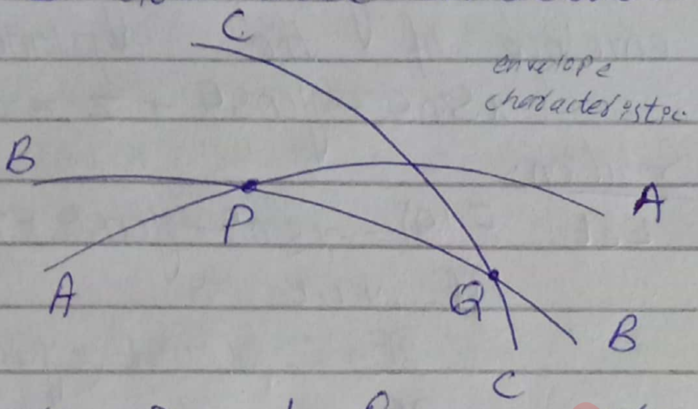
Edge of Regression:-

intersection of characteristic

The locus of the ultimate intersection of consecutive characteristic of one parametric family of surfaces is called the edge of regression

If A, B and C are three consecutive

P and Q



From above P and Q are consecutive pts on the characteristic B as well as called the edge of regression

$x^2+y^2+z^2+a^2+b^2 = a^2+b^2+a^2+b^2$

$= 2a^2+2b^2$

$= 2(a^2+b^2)$

$= 2($

Q Find the edge of regression of the envelope of the planes

$$x\sin\theta - y\cos\theta + z = a\theta$$

Solution:

Let $F(\theta) = x\sin\theta - y\cos\theta + z - a\theta = 0 \rightarrow (1)$

diff w.r.t θ

$$\frac{\partial F}{\partial \theta} = x\cos\theta + y\sin\theta - a = 0 \rightarrow (2)$$

For edges of regression again diff (2) w.r.t θ

$$\frac{\partial^2 F}{\partial \theta^2} = -x\sin\theta + y\cos\theta = 0 \rightarrow (3)$$

and put it equal to zero.

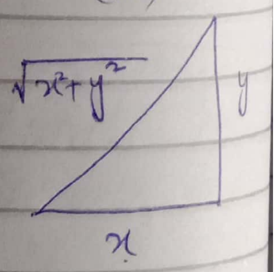
$$-x\sin\theta + y\cos\theta = 0 \rightarrow (3)$$

$$y\cos\theta = x\sin\theta$$

$$\frac{y}{x} = \tan\theta$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right) \rightarrow (4)$$

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \rightarrow (5)$$



$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}} \rightarrow (6)$$

$$F(\theta) = x \sin \tan^{-1}\left(\frac{y}{x}\right) - y \cos \tan^{-1}\left(\frac{y}{x}\right) + \cancel{z}\, a \tan^{-1}\left(\frac{y}{x}\right) = 0$$

$$F(\theta) = \cancel{\frac{xy}{\sqrt{x^2+y^2}}} - \cancel{\frac{xy}{\sqrt{x^2+y^2}}} + z - a\tan^{-1}\left(\frac{y}{x}\right) = 0$$

$$z = a\tan^{-1}\left(\frac{y}{x}\right) \rightarrow (7)$$

putting in (2)

$$\frac{\partial F}{\partial \theta} = \frac{x \cdot x}{\sqrt{x^2+y^2}} + \frac{y \cdot y}{\sqrt{x^2+y^2}} - a = 0$$

$$x^2+y^2 = a\sqrt{x^2+y^2}$$

$$\sqrt{x^2+y^2} = a$$

$$\boxed{x^2+y^2 = a^2}$$

Q Show that the envelope of the plane $\frac{x}{a}\cos\theta\sin\phi + \frac{y}{b}\sin\theta\sin\phi + \frac{z}{c}\cos\phi = 1 \rightarrow (1)$

is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Solution

$$F(\theta,\phi) = \frac{x}{a}\cos\theta\sin\phi + \frac{y}{b}\sin\theta\sin\phi + \frac{z}{c}\cos\phi - 1 = 0 \rightarrow (2)$$

$$\frac{\partial F(\theta,\phi)}{\partial \theta} = -\frac{x}{a}\sin\theta\sin\phi + \frac{y}{b}\cos\theta\sin\phi = 0$$

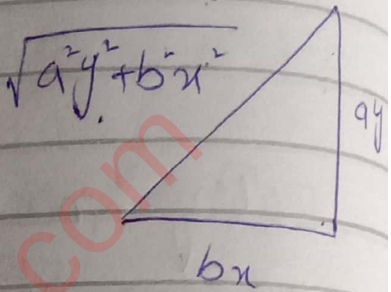
$$\left(\frac{y}{b}\cos\theta = \frac{x}{a}\sin\theta\right)$$

$$\frac{y}{b} = \frac{x}{a}\,\frac{\sin\theta}{\cos\theta}$$

$$\tan\theta = \frac{ay}{bx}$$

$$\theta = \tan^{-1}\left(\frac{ay}{bx}\right)$$

$$\sin\theta = \frac{\cancel{ay}\ \cancel{bx}\ ay}{\sqrt{a^2y^2+b^2x^2}}$$



$$\cos\theta = \frac{\cancel{ay}\ bx}{\sqrt{a^2y^2+b^2x^2}}$$

$$\boxed{\begin{gathered} F(\theta,\phi) = \frac{x}{\cancel{a}}\cdot\frac{\cancel{a}y}{\sqrt{a^2y^2+b^2x^2}}\sin\phi + \frac{y}{\cancel{b}}\cdot\frac{\cancel{b}x}{\sqrt{a^2y^2+b^2x^2}}\sin\phi + \frac{z}{c}\cos\phi - 1 = 0 \\ \frac{xy\sin\phi}{\sqrt{a^2y^2+b^2x^2}} + \frac{xy\sin\phi}{\sqrt{a^2y^2+b^2x^2}} + \frac{z}{c}\cos\phi - 1 = 0 \\ 2xy\sin\phi + \frac{z}{c}\cos\phi\sqrt{a^2y^2+b^2x^2} - \sqrt{a^2y^2+b^2x^2} = 0 \\ \text{and} \\ \frac{\partial F(\theta,\phi)}{\partial\theta} = \frac{-x}{a}\sin\phi\cdot\frac{bx}{\sqrt{a^2y^2+b^2x^2}} \end{gathered}}$$

$$\frac{\partial F(\theta,\phi)}{\partial \phi} = +\frac{x}{a}\cos\theta\cos\phi + \frac{y}{b}\sin\theta\cos\phi - \frac{z}{c}\sin\phi = 0$$

$$\cos\phi\left(\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta\right) = \frac{z}{c}\sin\phi$$

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = \frac{z}{c}\tan\phi$$

$$\tan\phi = \frac{c}{z}\left[\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta\right]$$

$$= \frac{c}{z}\left[\frac{bx\cos\theta + ay\sin\theta}{ab}\right]$$

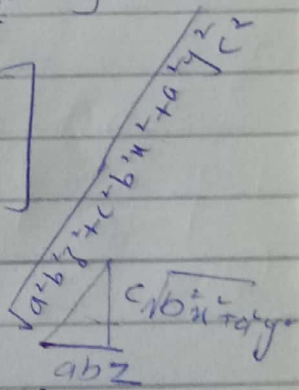
$$= \frac{c}{z}\left[\frac{\dfrac{bx\,\cancel{ay}\,bx}{\sqrt{a^2y^2+b^2x^2}} + \dfrac{ay\cdot\cancel{bx}\,ay}{\sqrt{a^2y^2+b^2x^2}}}{ab}\right]$$

$$= \frac{c}{abz}\left[\frac{axby + abxy}{\sqrt{a^2y^2+b^2x^2}}\right]$$

$$= \frac{c}{abz}\left[\frac{2axby}{\sqrt{a^2y^2+b^2x^2}}\right]$$

$$\sin\phi = \frac{c\sqrt{a^2y^2+b^2x^2}}{\sqrt{a^2b^2z^2+b^2c^2x^2+a^2c^2y^2}}$$

$$\cos\phi =$$



$b^2 = a^2 + c^2$

$\Rightarrow$

$$= \frac{c}{z}\left[\frac{\dfrac{b^2x^2+a^2y^2}{\sqrt{a^2y^2+b^2x^2}}}{ab}\right]$$

$b^2 = a^2 + c^2$

$b = \sqrt{a^2+c^2}$

$$= \frac{c}{abz}\sqrt{a^2y^2+b^2x^2}$$

$= \sqrt{c^2b^2x^2+a^2c^2y^2+a^2b^2z^2}$

$$\tan\phi = \frac{c\sqrt{a^2y^2+b^2x^2}}{abz}$$

$$\sin\phi = \frac{c\sqrt{a^2y^2+b^2x^2}}{\sqrt{a^2b^2z^2+b^2c^2x^2+a^2c^2y^2}}$$

$$\cos\phi = \frac{abz}{\sqrt{a^2b^2z^2+b^2c^2x^2+a^2c^2y^2}}$$

putting in eqn (2)

$$F(\theta,\phi) = \frac{x}{a}\,\frac{bx}{\cancel{\sqrt{a^2y^2+b^2x^2}}}\cdot\frac{c\cancel{\sqrt{a^2y^2+b^2x^2}}}{\sqrt{a^2b^2z^2+b^2c^2x^2+a^2c^2y^2}} + \frac{acy^2}{b\sqrt{\;}} + \frac{abz^2}{c\sqrt{\;}} - 1 = 0$$

$$= \frac{bx^2c}{a\sqrt{a^2b^2z^2+b^2c^2x^2+a^2c^2y^2}}$$

$$= \frac{b^2x^2c^2+a^2c^2y^2+a^2b^2z^2}{abc\sqrt{a^2c^2y^2+a^2b^2z^2+b^2x^2c^2}} - 1 = 0$$

$$\frac{\sqrt{a^2b^2z^2+b^2c^2x^2+a^2c^2y^2}}{abc} = 1$$

$$a^2b^2z^2+b^2c^2x^2+a^2c^2y^2 = a^2b^2c^2$$

$$\frac{z^2}{c^2}+\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

is required.

Q Prove that the envelope of the plane $lx+my+nz=p \rightarrow (A)$ the

is an ellipsoid.

where $p=\sqrt{a^2l^2+b^2m^2+c^2n^2}$.

$$F(l,m,n)=lx+my+nz-p=0 \rightarrow (1)$$

$$F(l,m,n)=lx+my+nz-\sqrt{a^2l^2+b^2m^2+c^2n^2}=0 \rightarrow (2)$$

$$\frac{\partial F(l,m,n)}{\partial l}=x-\frac{(2a^2l)}{2\sqrt{a^2l^2+b^2m^2+c^2n^2}}=0$$

$$x=\frac{a^2l}{\sqrt{a^2l^2+b^2m^2+c^2n^2}} \Rightarrow \frac{px}{a^2}=l$$

$$y=\frac{b^2m}{\sqrt{a^2l^2+b^2m^2+c^2n^2}} \Rightarrow \frac{py}{b^2}=m$$

$$z=\frac{c^2n}{\sqrt{a^2l^2+b^2m^2+c^2n^2}} \Rightarrow \frac{pz}{c^2}=n$$

Putting l,m,n in $*$

$$\frac{px}{a^2}\cdot x+\frac{py}{b^2}\cdot y+\frac{pz}{c^2}\cdot z=p$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is required.

Home Task

Q#3 Find the envelope of family of plane

$$3a^2x - 3ay + z = a^3$$

and show that its edge of regression is the curve of intersection of the surfaces

$$xz = y^2 \quad \text{and} \quad xy = z$$

Q#4 Find the envelope of a family of cone

$$(ax + x + y + z - 1)(ay + z) = ax(x + y + z - 1)$$

Here 'a' being a parameter.

Solution: Given Surface

$$F(a) = 3a^2x - 3ay + z - a^3 = 0 \rightarrow (1)$$

$$\frac{\partial F}{\partial a} = 6ax - 3y - 3a^2 = 0 \rightarrow (2)$$

$$\frac{\partial^2 F}{\partial a^2} = 6x - 6a = 0 \rightarrow (3)$$

$$x - a = 0$$

X'ing (1) by 3 and eqn (3) by $-a^2$

$$3(3a^2x - 3ay + z - a^3) = 0$$

$$9a^2x - 9ay + 3z - 3a^3 = 0 \rightarrow *$$

and

$$-6a^2x + 3ay + 3a^3 = 0 \rightarrow *'$$

Adding * and *'

$$9a^2x - 9ay + 3z - \cancel{3a^3} = 0$$
$$-6a^2x + 3ay \qquad + \cancel{3a^3} \quad 0$$

$$3a^2x - 6ay + 3z = 0 \rightarrow (4)$$

$$a^2x - 2ay + z = 0 \longrightarrow (4)$$

Eqn (2) is

$$6ax - 3y - 3a^2 = 0$$

$$\Rightarrow 3a^2 - 6ax + 3y = 0$$

$$\Rightarrow a^2 - 2ax + y = 0 \longrightarrow (5)$$

Now solving (4) and (5)

$$\frac{a^2}{-2y^2 + 2xz} = \frac{-a}{xy - z} = \frac{1}{-2x^2 + 2y}$$

$$a^2 = \frac{-2y^2 + 2xz}{-2x^2 + 2y} = \frac{+2(xz - y^2)}{2(y - x^2)}$$

$$a = \frac{z - xy}{-2x^2 + 2y} \qquad a^2 = \frac{y^2 - xz}{x^2 - y}$$

$$\Rightarrow$$

$$a^2 = \frac{(z - xy)^2}{(-2x^2 + 2y)^2} = \frac{(z - xy)^2}{4(x^2 - y)^2}$$

$$\frac{(z - xy)^2}{4(x^2 - y)^{\cancel{2}}} = \frac{y^2 - xz}{\cancel{x^2 - y}}$$

$$(z - xy)^2 = 4(y^2 - xz)(x^2 - y)$$

is required equation of Envelope

For edge of regression.

$$6x - 6a = 0$$

$$x = a$$

Putting in ① & ②

$F(a) = 3x^3 - 3xy + z - x^3 = 0$

$$\boxed{2x^3 - 3xy + z = 0} \rightarrow ⑥$$

and

$6x^2 - 3y - 3x^2 = 0$

$3x^2 - 3y = 0$

$x^2 = y \rightarrow$

$$\boxed{y = x^2}$$

Putting $y = x^2$ in ⑥

$2x^3 - 3xy + z = 0$

$2x^3 - 3x^3 + z = 0$

$-x^3 + z = 0$

$$\Rightarrow \boxed{z = x^3} \rightarrow ⑦$$

$x^2 = y$ in ⑦

$z = x^2 \cdot x$

$z = y \cdot x$

$$\boxed{xy = z}$$

Now $x^2 - y = 0$ Xing by y

$x^2y - y^2 = 0$

$x \cdot (xy) - y^2 = 0$

$xz - y^2 = 0$

$$\Rightarrow \boxed{y^2 = xz}$$

Thus req. eqns of edge of regression.

Solution :> 2

$$F(a) = \cancel{y(ax+x+y+z-1)+(ay+z)x - ax(x+y+z-1) = 0}$$

$$\cancel{\frac{\partial F(a)}{\partial a} = (a)x(y) - x(x+y+z-1) + xy}$$

$$F(a) = (ax+x+y+z-1)(ay+z) - ax(x+y+z-1) = 0$$

$$= ay(ax+x+y+z-1) + z(ax+x+y+z-1) - ax(x+y+z-1) = 0$$

$$= y(a^2x+ax+ay+az-a) + z(ax+x+y+z-1) - ax(x+y+z-1) = 0$$

$$\frac{\partial F(a)}{\partial a} = y(2ax+x+y+z-1) + xz - x(x+y+z-1) = 0$$

$$= 2axy + \cancel{xy} + y^2 + yz - y + \cancel{xz} - x^2 - \cancel{xy} - \cancel{xz} + x = 0$$

$$= 2axy + \cancel{xy} - \cancel{xy} + yz - y + x - x^2 + y^2$$

$$2axy = x^2 - y^2 - yz + y - x$$

$$\Rightarrow a = \frac{x^2-y^2-yz+y-x}{2xy} \longrightarrow (2)$$

putting in (1)

$$F(a) = \left[\left(\frac{x^2-y^2-yz+y-x}{2xy}\right)x+y+z-1\right]\left[\left(\frac{x^2-y^2-yz+y-x}{2xy}\right)y+z\right]$$

$$-\left(\frac{x^2-y^2-yz+y-x}{2xy}\right)x(x+y+z-1) = 0$$

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$$=\left((x^2-y^2-yz+y-x)+2y^2+2yz-2y\right)\left((x^2-y^2-yz+y-x)+2xz\right)$$

$$-(x^2-y^2-yz+y-x)(2xy+2y^2+2yz-2y)=0$$

$$F(a)=(x^2+y^2+yz-y-x)(x^2-y^2-yz+2xz+y-x)$$
$$-2(x^2-y^2-yz+y-x)(y^2+xy+yz-y)=0$$

is required equation of envelope.

Assignment # 11

1- Prove

$$\frac{d}{ds}\left\{\sigma \frac{d}{ds}\left(\rho \frac{d^2 r}{ds^2}\right)\right\} + \frac{d}{ds}\left(\frac{\sigma}{\rho}\frac{dr}{ds}\right) + \frac{\rho}{\sigma}\frac{d^2 r}{ds^2} = 0$$

Taking L.H.S

$$\frac{d}{ds}\left\{\sigma \frac{d}{ds}\left(\rho \frac{d^2 r}{ds^2}\right)\right\} + \frac{d}{ds}\left(\frac{\sigma}{\rho}\frac{dr}{ds}\right) + \frac{\rho}{\sigma}\frac{d^2 r}{ds^2}$$

As we know $\rho = \frac{1}{k}$ $\quad \sigma = \frac{1}{\tau}$

$$\frac{dr}{ds} = t \qquad \frac{d^2 r}{dt^2} = t' = kn$$

$$\Rightarrow \frac{d}{ds}\left\{\frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{k}(kn)\right)\right\} + \frac{d}{ds}\left(\frac{1/\tau}{1/k}t\right) + \frac{1/k}{1/\tau}(kn)$$

$$\frac{d}{ds}\left\{\frac{1}{\tau}\frac{dn}{ds}\right\} + \frac{d}{ds}\left(\frac{k}{\tau}t\right) + \frac{\tau}{k}(kn)$$

$$\frac{d}{ds}\left\{\frac{1}{\tau}(\tau b - kt)\right\} + \frac{d}{ds}\left(\frac{k}{\tau}t\right) + \tau n$$

$$\frac{d}{ds}\left\{b - \frac{k}{\tau}t\right\} + \frac{d}{ds}\left(\frac{k}{\tau}t\right) + \tau n$$

$$\frac{db}{ds} - \cancel{\frac{d}{ds}\left(\frac{k}{\tau}t\right)} + \cancel{\frac{d}{ds}\left(\frac{k}{\tau}t\right)} + \tau n$$

$$-\tau \cancel{n} + \tau \cancel{n}$$

$$= 0$$

proved.

2- $\tau = \frac{1}{k^2}\left[\vec{r}', \vec{r}'', \vec{r}'''\right]$

As we know that

$r' = t \qquad r'' = t' = kn \qquad r''' = k'n + kn'$

$= k'n + k(\tau b - kt)$

$= k'\underline{n} + k\tau \underline{b} - k^2 \underline{t}$

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$[\gamma', \gamma'', \gamma'''] = [t, kn, k'n + k\tau b - k^2 t]$

$= [t, kn, k'n] + [t, kn, k\tau b] + [t, kn, -k^2 t]$

$= kk'[t, n, n] + k \cdot k \cdot \tau [t, n, b] + k(-k^2)[t, n, t]$

$= kk'(0) + k^2\tau(1) + 0$

$[\gamma', \gamma'', \gamma'''] = k^2\tau$

$\tau = \frac{1}{k^2}[\gamma', \gamma'', \gamma''']$

proved.

Home Task

By using $\gamma' = t$ and serret-Frenet formula find, γ'', γ''', γ'''' and prove the relations.

(i) $\gamma' \cdot \gamma'' = 0$

$\because \gamma' = t$ and $\gamma'' = kn$

$\gamma''' = k\tau b - k^2 t + k'n$

$\gamma'''' = (k'' - k^3 - k\tau^2)n - 3kk't + (k\tau' + 2k'\tau)b$

L.H.S $= \gamma' \cdot \gamma''$

$= t \cdot kn$

$= k(t \cdot n)$

$= k(0)$

$= 0 =$ R.H.S

(ii) $\gamma' \cdot \gamma''' = -k^2$

$L.H.S = \gamma' . \gamma'''$

$= t . (k\tau b - k^2 t + k' n)$

$= k\tau (t.b) - k^2 (t.t) + k'(t.n)$

$= k\tau (0) - k^2 (1) + k'(0)$

$= 0 - k^2 + 0$

$= -k^2$

$= R.H.S$

(iii) $\gamma' . \gamma'''' = -3kk'$

$L.H.S = \gamma' . \gamma''''$

$= t . [(k'' - k^3 - k\tau^2) n - 3kk' t + (k\tau' + 2k'\tau) b]$

$= (k'' - k^3 - k\tau^2)(t.n) - 3kk'(t.t) + (k\tau' + 2k'\tau)(t.b)$

$= 0 - 3kk'(1) + 0$

$= -3kk'$

$= R.H.S$

(iv) $\gamma'' . \gamma''' = kk'$

$L.H.S = \gamma'' . \gamma'''$

$= kn . (k\tau b - k^2 t + k' n)$

$= k^2\tau (n.b) - k^3 (n.t) + kk'(n.n)$

$= k^2 \tau (0) - k^3 (0) + kk'(1)$

$= 0 - 0 + kk'$

$= kk'$

$= R.H.S$

(v) $\gamma'' . \gamma'''' = k(k'' - k^3 - k\tau^2)$

$L.H.S = \gamma'' . \gamma''''$

$= kn . \{(k'' - k^3 + k\tau^2) n - 3kk' t + (k\tau' + 2k'\tau) b\}$

$= k(k'' - k^3 + k\tau^2)(n \cdot n) - 3k^2k'(n \cdot t) + k(k\tau' + 2k'\tau)(n \cdot b)$

$= k(k'' - k^3 + k\tau^2)(1) - 3k^2k'(0) + k(k\tau' + 2k'\tau)(0)$

$= k(k'' - k^3 + k\tau^2) - 0 + 0$

$= k(k'' - k^3 + k\tau^2)$

(vi) $\gamma''' \cdot \gamma'''' = k'k'' + 2k^3k' + k^2\tau\tau' + kk'\tau^2$

L.H.S $= \gamma''' \cdot \gamma''''$

$\gamma''' = k\tau b - k^2 t + k'n$

$\gamma'''' = (k'' - k^3 + k\tau^2)n - 3kk't + (k\tau' + 2k'\tau)b$

$\gamma''' \cdot \gamma'''' = (k\tau b - k^2 t + k'n) \cdot \left[(k'' - k^3 + k\tau^2)n - 3kk't + (k\tau' + 2k'\tau)b\right]$

$= k\tau(k\tau' + 2k'\tau) + 3k^3k' + k'(k'' - k^3 + k\tau^2)$

$= k^2\tau\tau' + 2kk'\tau^2 + 3k^3k' + k'k'' - k^3k' + kk'\tau^2$

$= k'k'' + 2k^3k' + k^2\tau\tau' + kk'\tau^2$

Proved.

Solution:

As we know that

$$\vec{b}' = -\tau \vec{n}$$

$$b'' = -(\tau' \underline{n} + \tau \underline{n}')$$

$$b'' = -[\tau' \underline{n} + \tau(\tau \underline{b} - k \underline{t})]$$

$$b'' = -[\tau' \underline{n} + (-\tau)(k \underline{t} - \tau \underline{b})]$$

$$b'' = -[\tau' \underline{n} - \tau(k \underline{t} - \tau \underline{b})]$$

$$b'' = -\tau' n + \tau k \underline{t} - \tau \underline{b}$$

$$b'' = \tau(k \underline{t} - \tau \underline{b}) - \tau' \underline{n}$$

Also we know that

$$n' = \tau b - kt$$

$$n'' = \tau' b + \tau b' - (k' t + k t')$$

$$= \tau' b + \tau(-\tau n) - (k' t + k(kn))$$

$$= \tau' b - \tau^2 n - k' t - k^2 n$$

$$= \tau' b - k' t - n(\tau^2 + k^2)$$

Now to find n''' and b'''

$$b'' = \tau(kt - \tau b) - \tau' n$$

$$b''' = \tau'(kt - \tau b) + \tau(k' t + k t' - \tau' b - \tau b') - \tau'' n - \tau n'$$

$$= \tau'(kt - \tau b) + \tau[k' t + k(kn) - \tau' b - \tau(-\tau n)] - \tau'' n - \tau(\tau b - kt)$$

$$= \tau'(kt - \tau b) + \tau(k' \underline{t} + k^2 n - \tau' b + \tau^2 n) - \tau'' n - \tau(\tau b - k \underline{t})$$

$b''' = (2\tau' k + k' t)\underline{t} - 3\tau\tau'\underline{b} + (k^2\tau + \tau^3 - \tau'')\underline{n}$

As $n'' = \tau'\underline{b} - (k^2+\tau^2)\underline{n} - k'\underline{t}$

$n''' = \tau''\underline{b} + \tau' b' - n'(k^2+\tau^2) - \underline{n}(2kk' + 2\tau\tau')$
$- k''\underline{t} - k' t'$

$= \tau''\underline{b} + \tau'(-\tau\underline{n}) - (k^2+\tau^2)(\tau\underline{b} - k\underline{t})$
$- \underline{n}(2kk' + 2\tau\tau') - k''\underline{t} - k'(k\underline{n})$

$= \tau''\underline{b} - \tau\tau'\underline{n} - \tau(k^2+\tau^2)\underline{b} + k(k^2+\tau^2)\underline{t}$
$- 2kk'\underline{n} - 2\tau\tau'\underline{n} - k''\underline{t} - kk'\underline{n}$

$= \tau''\underline{b} - \tau\tau'\underline{n} - k^2\tau\underline{b} - \tau^3\underline{b} + k^3\underline{t} + k\tau^2\underline{t}$
$- 2kk'\underline{n} - 2\tau\tau'\underline{n} - k''\underline{t} - kk'\underline{n}$

$= (\tau'' - k^2\tau - \tau^3)\underline{b} + (-\tau\tau' - 2kk' - 2\tau\tau'$
$- kk')\underline{n} + (k^3 + k\tau^2 - k'')\underline{t}$

$= (\tau'' - k^2\tau - \tau^3)\underline{b} - (3kk' + 3\tau\tau')\underline{n} +$
$(k^3 + k\tau^2 - k'')\underline{t}$

Equation of the locus of centre of curvature.

$$c = \gamma + n\rho$$

diff w.r.t "s"

$$\frac{dc}{ds} = \frac{d}{ds}(\gamma + n\rho)\frac{ds}{ds_1}$$

$$= \gamma' + \rho n' + n\rho' \quad \frac{ds}{ds_1}$$

$$= t + \rho(\tau b - kt) + n\rho' \quad \frac{ds}{ds_1}$$

$$\frac{dc}{ds} = (\rho\tau b + n\rho')\frac{ds}{ds_1}$$

$$\frac{dc}{ds_1} \cdot \frac{ds_1}{ds} = (\rho\tau b + n\rho')$$

$$t_1 \cdot \frac{ds_1}{ds} = \rho\tau b + n\rho'$$

$$\left(\frac{ds_1}{ds}\right)^2 = \rho^2\tau^2 + \rho'^2$$

$$\frac{ds_1}{ds} = \sqrt{\rho^2\tau^2 + \rho'^2}$$

$$\frac{ds_1}{ds} = \sqrt{\left(\frac{\rho}{\sigma}\right)^2 + \rho'^2} \rightarrow (A)$$

As we know that

$$\rho = \frac{1}{K}$$

$$\frac{d\rho}{ds} = \frac{d}{ds}\left(\frac{1}{K}\right)$$

$$\rho' = -\frac{1}{K^2} K'$$

$$(\rho')^2 = +\frac{1}{K^4}(K')^2$$

put in (1)

$$\frac{ds_1}{ds} = \sqrt{\frac{\tau^2}{K^2} + \frac{(K')^2}{K^4}}$$

$$= \sqrt{\frac{K^2\tau^2 + (K')^2}{K^4}}$$

$$\frac{ds_1}{ds} = \frac{1}{K^2}\sqrt{K^2\tau^2 + (K')^2} \rightarrow (B)$$

From (A) and (B)

$$\frac{ds_1}{ds} = \sqrt{\left(\frac{\rho}{\sigma}\right)^2 + \rho'^2} = \frac{1}{K^2}\sqrt{K^2\tau^2 + (K')^2}$$

proved.

$\gamma' = t \qquad \gamma'' = t' = kn \qquad \gamma''' = k'n + kn'$

$\gamma^{iv} = (k'' - k^3 - k\tau^2)n - 3kk't + (k\tau' + 2k'\tau)b \qquad = k'n + k(\tau b - kt)$

$= k'n + k\tau b - k^2 t$

$$[\gamma'', \gamma''', \gamma''''] = \begin{vmatrix} 0 & k & 0 \\ -k^2 & k' & k\tau \\ -3kk' & k'' - k^3 - k\tau^2 & k\tau' + 2k'\tau \end{vmatrix}$$

$$= 0 - k\left[-k^2(k\tau' + 2k'\tau) + 3kk'(k\tau)\right] + 0$$

$$= -k\left[-k^3\tau' - 2k'k^2\tau + 3k^2k'\tau\right]$$

$$= -k\left[-k^3\tau' + k'k^2\tau\right]$$

$$= k^4\tau' - k'k^3\tau$$

$$[\gamma'', \gamma''', \gamma''''] = k^3(k\tau' - k'\tau)\frac{k^2}{k^2}$$

$$= k^5 \frac{d}{ds}\left(\frac{\tau}{k}\right)$$

$\gamma' = t \qquad \gamma'' = kn \qquad \gamma''' = t'' \qquad \gamma'''' = t'''$

$$[t', t'', t'''] = [\gamma'', \gamma''', \gamma''''] = k^3(k\tau' - k'\tau)$$

$$= k^5 \frac{d}{ds}\left(\frac{\tau}{k}\right)$$

(iii) $b' = -\tau n$

$b'' = -(\tau' n + \tau n')$

$= -[\tau' n + \tau(\tau b - kt)]$

$= -\tau' n - \tau(\tau b - kt)$

$= -\tau' n - \tau^2 b + k\tau t$

$b''' = -[\tau'' n + \tau' n'] - [2\tau\tau' b + \tau^2 b']$

$+ k'\tau t + k\tau' t + k\tau t'$

$= -\tau'' n - \tau'(\tau b - kt) - 2\tau\tau' b - \tau^2(-\tau n) + k'\tau t + k\tau' t + k\tau(kn)$

$= -\tau'' n - \tau'\tau b + k\tau' t - 2\tau\tau' b + \tau^3 n + k'\tau t + k\tau' t + k^2\tau n$

$= (k\tau' + k'\tau + k\tau')t + (-\tau'' + \tau^3 + k^2\tau)n + (-\tau\tau' - 2\tau\tau')b$

$= (k'\tau + 2k\tau')t + (k^2\tau + \tau^3 - \tau'')n - 3\tau\tau' b$

$$[b', b'', b'''] = \begin{vmatrix} 0 & -\tau & 0 \\ k\tau & -\tau' & -\tau^2 \\ k'\tau + 2k\tau' & k^2\tau + \tau^3 - \tau'' & -3\tau\tau' \end{vmatrix}$$

$= 0 + \tau\left[-3\tau\tau'(k\tau) + \tau^2(k'\tau + 2k\tau')\right]$

$$-3k\tau^2\tau'$$

$$= \tau\left(-3k^2 k'\tau + k'\tau^3 + 2k\tau'\tau^2\right)$$

$$\cancel{= -3k^2k'\tau^2 + k'\tau^4 + 2k\tau'\tau^3}$$

$$= \tau^3\left(-3k\tau' + 2k\tau' + k'\tau\right)$$

$$= \tau^3\left(k'\tau - k\tau'\right) \times \frac{\tau^2}{\tau^2}$$

$$= \tau^5 \frac{d}{ds}\left(\frac{k}{\tau}\right)$$

$$[b', b'', b'''] = \tau^5 \frac{d}{ds}\left(\frac{k}{\tau}\right) = \tau^3\left[k'\tau - k\tau'\right]$$

Proved.

$\gamma_1 = b$

Differentiate w.r.t s_1

$$\frac{d\gamma_1}{ds_1} = \frac{db}{ds} \cdot \frac{ds}{ds_1}$$

$$t_1 = b' \cdot \frac{ds}{ds_1}$$

$$t_1 = -\tau n \frac{ds}{ds_1}$$

$$t_1 = -n \rightarrow (1) \qquad 1 = \tau \frac{ds}{ds_1}$$

$$\frac{1}{\tau} = \frac{ds}{ds_1} \rightarrow (2)$$

Diff (1) w.r.t s_1

$$\frac{dt_1}{ds_1} = -\frac{dn}{ds} \cdot \frac{ds}{ds_1}$$

$$t_1' = k_1 n_1 = -n' \cdot \frac{ds}{ds_1}$$

$$k_1 n_1 = -(\tau b - kt) \cdot \frac{1}{\tau}$$

Squaring on b.s

$$k_1 = \frac{\sqrt{k^2+\tau^2}}{\tau} \rightarrow (A)$$

Since the eqn of osculating sphere is

$$R^2 = \rho^2 + \sigma^2 \rho'^2 \rightarrow \star$$

$$R = 1$$

$$1 = \rho_1^2 + \sigma^2 \rho_1'^2$$

$$1 = \frac{1}{k_1^2} + \frac{1}{\tau_1^2} \frac{k_1'^2}{k_1^4}$$

$$1 - \frac{1}{k_1^2} = \frac{1}{\tau_1^2} \frac{k_1'^2}{k_1^4}$$

$$\frac{k_1^2 - 1}{k_1^2} = \frac{k_1'^2}{\tau_1^2 k_1^4}$$

$$\Rightarrow \tau_1 = \frac{k_1'}{k_1\sqrt{k_1^2-1}} \rightarrow (B)$$

$$\because \quad k_1 = \frac{\sqrt{k^2+\tau^2}}{\tau}$$

Differentiate w.r.t s_1

$$k_1' = \frac{1}{2}(k^2+\tau^2)^{-1/2}(2kk' + 2\tau\tau')\tau$$

$$- \frac{(k^2+\tau^2)^{1/2}\tau'}{\tau^2} \quad \frac{ds}{ds_1}$$

$$= \frac{(kk' + \tau\tau')\tau - (k^2+\tau^2)\tau'}{(k^2+\tau^2)^{1/2}\quad \tau^2} \cdot \frac{1}{\tau}$$

$$= \frac{k\tau k' + \tau'\cancel{\tau}^2 - k^2\tau' - \tau^2\cancel{\tau}'}{(k^2+\tau^2)^{1/2}\,\tau^3}$$

$$k_1' = \frac{kk'\tau - k^2\tau'}{(k^2+\tau^2)^{1/2}\,\tau^3}$$

$$\Rightarrow \quad \tau_1 = \frac{kk'\tau - k^2\tau'}{\tau^3(k^2+\tau^2)^{1/2}} \times \frac{1}{\sqrt{\frac{k^2+\tau^2}{\tau}} \times \sqrt{\frac{k^2+\tau^2}{\tau^2} - 1}}$$

$$= \frac{kk'\tau - k^2\tau'}{\tau^3(k^2+\tau^2)^{1/2}} \times \frac{1}{\frac{\sqrt{k^2+\tau^2}}{\cancel{\tau}^2}} \times \frac{1}{\cancel{k}}$$

$$\tau_1 = \frac{k'\tau - k\tau'}{\tau^2(k^2+\tau^2)} \quad \checkmark$$