Limit : (Finite Limit at finite point).

Let a be any real number and let f be a function from R to R which is defined for all values of x near a with the possible exception of the point $x=a$. The function f is said to have limit l as x approaches a if for every $\varepsilon > 0$, there exist a positive real number δ (which usually depends on ε) such that

$$|f(x) - l| < \varepsilon \quad \text{whenever} \quad 0 < |x-a| < \delta$$

In this case, we write

$$\lim_{x \to a} f(x) = l.$$

and say that the function f has the limit l (or $f(x)$ converges to l) as x approaches a. or $f(x) \to l$ as $x \to a$.

Right Hand Limit:

A function f is said to be right hand limit l_1 as x tends to a through values greater than a (i.e $x \to a^+$) if for every $\varepsilon > 0$, there exist a $\delta > 0$ such that

$$|f(x) - l_1| < \varepsilon \quad \text{whenever} \quad 0 < x \leq a + \delta$$

in this case we write $\lim\limits_{x \to a^+} f(x) = l_1$

Left Hand Limit:

A function f is said to have the left hand limit l_2 as x tends to a through values less than a (i.e $x \to a^-$) if for every $\varepsilon > 0$, there exist a $\delta > 0$ such that

$$|f(x) - l_2| < \varepsilon \quad \text{whenever} \quad a - \delta \leq x < a$$

and we write $\lim\limits_{x \to a^-} f(x) = l_2$

Prove that $\lim_{x \to 4} \frac{1}{2}(3x-1) = \frac{11}{2}$

Sol// Let $f(x) = \frac{1}{2}(3x-1)$, $a = 4$ and $L = \frac{11}{2}$

According to definition, we must show that for every $\varepsilon > 0$, there exist a $\delta > 0$ such that

if $0 < |x-4| < \delta$, then $\left|\frac{1}{2}(3x-1) - \frac{11}{2}\right| < \varepsilon$

δ can be found by examining last inequality involving ε. The following is a list of equivalent inequalities.

$$\left|\frac{1}{2}(3x-1) - \frac{11}{2}\right| < \varepsilon$$

$$\frac{1}{2}\left|(3x-1) - 11\right| < \varepsilon$$

$$|3x-1-11| < 2\varepsilon$$

$$|3x-12| < 2\varepsilon$$

$$3|x-4| < 2\varepsilon$$

$$|x-4| < \frac{2}{3}\varepsilon$$

If we let $\delta = \frac{2}{3}\varepsilon$, then if $0 < |x-4| < \delta$. the last inequality in the list is true.

So by definition $\lim_{x \to 4} \frac{1}{2}(3x-1) = \frac{11}{2}$.

Prove by definition that $\lim_{x \to 3} 2x = 6$

Sol// For each $\varepsilon > 0$, we want to find a $\delta > 0$. such that $|2x-6| < \varepsilon$ whenever. $0 < |x-3| < \delta$.

we have $|2x-6| = 2|x-3|$ which leads us to choose $\delta = \frac{\varepsilon}{2}$.

thus for every $\varepsilon > 0$. if we chose $\delta = \varepsilon/2$ then

$|2x-6| = 2|x-3| < 2\delta = \varepsilon$

whenever $0 < |x-3| < \delta$.

Example: $\lim_{x \to 3} \dfrac{x^3-27}{x^2-9}$

write L.H.L and R.H.L.

Sol// L.H.L

$$\lim_{x \to 3-0} \frac{x^3-27}{x^2-9} = \lim_{h \to 0} \frac{(3-h)^3-27}{(3-h)^2-9}$$

$$= \lim_{h \to 0} \frac{(\cancel{3-h-3})((3-h)^2+3(3-h)+9)}{(\cancel{3-h-3})(3-h+3)}$$

$$= \lim_{h \to 0} \frac{9+h^2-6h+9-3h+9}{6-h} = \lim_{h \to 0} \frac{27+h^2-9h}{6-h}$$

$$= \frac{27}{6} = \frac{9}{2} \longrightarrow (1)$$

R.H.L

$$\lim_{x \to 3+0} \frac{x^3-27}{x^2-9} = \lim_{h \to 0} \frac{(3+h)^3-27}{(3+h)^2-9}$$

$$= \lim_{h \to 0} \frac{(\cancel{3+h-3})((3+h)^2+3(3+h)+(3)^2)}{(\cancel{3+h-3})(3+h+3)}$$

$$= \lim_{h \to 0} \frac{9+h^2+6h+9+3h+9}{6+h} \quad \lim_{h \to 0} \frac{27+h^2+9h}{6+h} = \frac{27}{6}$$

$$= \frac{9}{2} \longrightarrow (2)$$

From (1) and (2)

L.H.L = R.H.L

$$\lim_{x \to 3} \frac{x^3-27}{x^2-9} = \frac{9}{2} \quad \underline{\text{Ans}}$$

Theorems on Limits:

Limit of constant function:

If $f(x) = c$ for all $x \in R$, where c is a fixed real number, then

$$\lim_{x \to a} f(x) = c$$ for every real number a.

Identity Function:

$$\lim_{x \to a} f(x) = a.$$ for every $a \in R$.

Sum (Difference).

Let $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$

then

$$\lim_{x \to a} [f(x) \pm g(x)] = l \pm m$$
$$= \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

Product:

Let $\lim_{x \to a} f(x) = l$, $\lim_{x \to a} g(x) = m$

Then $$\lim_{x \to a} (f(x) \cdot g(x)) = lm$$
$$= \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

Quotient:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{l}{m} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}.$$

Inequalities:

If $f(x) \le g(x)$ $\forall x$ in some interval a, then $l \le m$.

Sandwitch Theorem:

Suppose that f, g and h are functions defined on $0 < |x - a| < K$. If $f(x) \le g(x) \le h(x)$ on this domain. and if $\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x)$

then $\lim_{x \to a} g(x) = l$

Some important Formulas:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$$

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a$$

$$\lim_{x\to 0} \frac{x^n - a^n}{x-a} = n a^{n-1}$$

Q Find the limits:-

$$\lim_{x\to 1} \frac{x^3-1}{x-1}$$

$$\lim_{x\to 1} \frac{x^3-1}{x-1} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{x-1}$$

$$= \lim_{x\to 1} (x^2+x+1)$$

$$= 1^2+1+1$$

$$= 3.$$

$$\lim_{y\to x} \frac{y^{2/3} - x^{2/3}}{y-x}$$

$$= \lim_{y\to x} \frac{(y^{1/3})^2 - (x^{1/3})^2}{(y^{1/3})^3 - (x^{1/3})^3}$$

$$= \lim_{y\to x} \frac{(y^{1/3} + x^{1/3})(y^{1/3} - x^{1/3})}{(y^{1/3} - x^{1/3})\left((y^{1/3})^2 + y^{1/3}x^{1/3} + (x^{1/3})^2\right)}$$

$$= \lim_{y\to x} \frac{y^{1/3} + x^{1/3}}{y^{2/3} + y^{1/3}x^{1/3} + x^{2/3}}$$

$$= \frac{x^{1/3} + x^{1/3}}{x^{2/3} + x^{1/3}x^{1/3} + x^{2/3}}$$

$$= \frac{2x^{1/3}}{x^{2/3} + x^{2/3} + x^{2/3}}$$

$$= \frac{2}{3} \frac{x^{1/3}}{x^{2/3}}$$

$$= \frac{2}{3} \frac{1}{x^{2/3 - 1/3}}$$

$$= \frac{2}{3x^{1/3}} \quad \text{Ans}$$

Q $\lim_{x\to 0} \frac{\csc x - \cot x}{x}$

$$= \lim_{x\to 0} \frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{x}$$

$$= \lim_{x\to 0} \frac{1}{x}\left[\frac{1-\cos x}{\sin x}\right]$$

$$= \lim_{x\to 0} \frac{1}{x}\left[\frac{1-\cos x}{\sin x} \times \frac{1+\cos x}{1+\cos x}\right]$$

$$= \lim_{x\to 0} \frac{1}{x}\left[\frac{1-\cos^2 x}{\sin x(1+\cos x)}\right]$$

$$= \lim_{x\to 0} \frac{1}{x}\left[\frac{\sin^2 x}{\sin x(1+\cos x)}\right]$$

$$= \lim_{x\to 0} \frac{1}{x}\left[\frac{\sin x}{1+\cos x}\right]$$

$$= \lim_{x\to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{1+\cos x}\right)$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{1+\cos x}$$

$$= (1) \cdot \frac{1}{1+1} = \frac{1}{2}. \quad \text{Ans}$$

$$\lim_{x \to \pi} \frac{\tan(\sin x)}{\sin x}$$

Let $\sin x = \theta$

When $x \to \pi$, $\theta \to 0$

$$= \lim_{\theta \to 0} \frac{\tan\theta}{\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{1}{\cos\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta \to 0} \frac{1}{\cos\theta}$$

$$= (1) \frac{1}{\cos\theta} = 1 \times \frac{1}{1}$$

$= 1$ Ans

$$\lim_{x \to 1^-} \frac{\sqrt{1-x^2}}{1-x}$$

$$= \lim_{x \to 1^-} \frac{\sqrt{(1-x)(1+x)}}{1-x}$$

$$= \lim_{x \to 1^-} \frac{\sqrt{1-x}\,\sqrt{1+x}}{1-x}$$

$$= \lim_{x \to 1^-} \frac{\sqrt{1+x}}{\sqrt{1-x}}$$

Put $x = 1-h$

$h \to 0$, $x \to 1$

$$= \lim_{h \to 0} \frac{\sqrt{1+1-h}}{\sqrt{1-1+h}}$$

$$= \lim_{h \to 0} \frac{\sqrt{2-h}}{\sqrt{h}}$$

$$= \frac{2}{0} = \infty$$

$$\lim_{x \to -2^-} \frac{x^2+2x-8}{x^2-4}$$

$$= \lim_{x \to -2^-} \frac{x^2+4x-2x-8}{(x+2)(x-2)}$$

$$= \lim_{x \to -2^-} \frac{x(x+4)-2(x+4)}{(x+2)(x-2)}$$

$$= \lim_{x \to -2^-} \frac{(x-2)(x+4)}{(x+2)(x-2)}$$

$$= \lim_{x \to -2^-} \frac{(x+4)}{(x+2)}$$

Put $x = -2-h$

$h \to 0, x \to -2$

$$= \lim_{h \to 0} \frac{-2-h+4}{-2-h+2}$$

$$= \lim_{h \to 0} \frac{2-h}{-h}$$

$$= \lim_{h \to 0} \frac{2}{-h} + 1$$

$$= \infty + 1 = \infty.$$

Theorem:

Limit of polynomials can be found by substitution.

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$

then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \dots + a_0$$

Theorem:

If $f(x) \le g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \le \lim_{x \to c} g(x).$$

Limit as x approaches ∞ or $-\infty$

We say that $f(x)$ has the limit L as x approaches infinity. and we write.

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exist a corresponding number M such that for all x.

$$x > M \Rightarrow |f(x) - L| < \epsilon$$

We say that $f(x)$ has the limit L as x approaches minus infinity and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exist a corresponding number N such that for all x.

$$x < N \Rightarrow |f(x) - L| < \epsilon.$$