INTRODUCTION

NUMERICAL ANALYSIS:

Numerical Analysis is the branch of mathematics that provides tools and methods for solving mathematical problems in numerical form. In numerical analysis we are mainly interested in implementation and analysis of numerical algorithms for finding an approximate solution to a mathematical problem.

It is the study of algorithms that use numerical approximations (as opposed to general symbolic manipulation) for the problem of mathematical analysis (as distinguished from discrete mathematics)

IMPORTANCE: Numerical analysis naturally finds applications in all fields of engineering and the physical sciences, but in the 21^{st} century, the life science and even the arts have adopted elements of scientific computations. Ordinary differential equations appear in the movements of heavenly bodies (planets, stars, galaxies); optimization occurs in portfolio management; numerical linear algebra is important for data analysis; stochastic differential equations and Markov chains are essential in simulating living cells for medicine and biology.

Before the advent of modern computers numerical methods often depended on hand interpolation in large printed tables. Since the mid-20^{th} century, computers calculate the required functions instead. These same interpolation formulas nevertheless continues to be used a part of the software algorithms for solving differential equations.

The overall goal of the field of numerical analysis is the design and analysis of techniques to give approximate but accurate solutions to hard problems, the variety of which is suggested by the following;

- Advanced numerical methods are essential in making numerical weather prediction feasible.
- Computing the trajectory of a space craft requires the accurate numerical solutions of a system of ordinary differential equations.
- Car companies can improve the crash safety of their vehicle by using computer simulations of car crashes. Such simulations essentially consist of solving partial differential equations numerically.
- Hedge funds (Private investment funds) use tools from all fields of numerical analysis to calculate the value of stocks and derivatives more precisely than other market participants.
- Airlines use sophisticated optimization algorithms to decide ticket prices, airplane and crew assignments and fuel needs. Historically, such algorithms were developed within the overlapping fields of operations research.
- Insurance companies use numerical programs for actuarial analysis.

HISTORY:

The field of numerical analysis predates the invention of modern computers by many centuries. Linear interpolation was already in use more than 2000 year ago. Many great mathematicians of the past were preoccupied by numerical analysis, as is obvious from the names of important algorithms like Newton's method, Lagrange's interpolation polynomial, Gaussian elimination or Euler's method.To facilitate computations by hand, large books were produced with formulas and tables of data such as interpolation points and function coefficients. Using these tables, often calculated out to 16 decimal places or more for some functions, one could look up values to plug into the formulas given and achieve very good numerical estimates of some functions. The canonical work in the field is the NIST publications edited by Abramovitz and Stegun, a 1000 plus page book of a very large number of commonly used formulas and functions and their values at many points. The function values are no longer very useful when a computer is available, but the large listing of formulas can still be very handy.

The mechanical calculator was also developed as a tool for hand computation. These calculators evolved into electronic computers in the 1940's and it was then found that these computers were also useful for administrative purposes. But the invention of the computer also influenced the field of numerical analysis, since now longer and more complicated calculations could be done.

Direct methods compute the solution to a problem in a finite number of steps. These methods would give the precise answer if they were performed in infinite precision arithmetic. Examples include Gaussian elimination, the QR factorization method for solving the system of linear equations and the simplex method of linear programming. In practice, finite precision is used and the result is an approximation of the true solution (assuming stability)

In contrast to direct methods, iterative methods are not expected to terminate in a number of steps. Starting from an initial guess, iterative methods from successive approximations that converge to the exact solution only in the limit. A convergence test is specified in order to decide when a sufficiently accurate solution has (hopefully) been found. Even using infinite precision arithmetic these methods would not reach the solution within a finite number of steps (in general). Examples include Newton's method, the Bisection method and Jacobi iteration. In computational matrix algebra, iterative methods are generally needed for large problems. Iterative methods are more common than direct methods in numerical analysis. Some methods are direct in principle but are usually use as though they were not, e.g. GMRES and the conjugate gradient method. For these methods the number of steps needed to obtain the exact solution is so large that an approximation is accepted in the same manner as for an iterative method.

NUMERICAL METHOD (NUMERICAL ITERATION METHOD):

A complete set of rules for solving a problem or problems of a particular type involving only the operation of arithmetic.

OR: A mathematical procedure that generates a sequence of improving approximate solution for a class of problems i.e. the process of finding successive approximations.

NUMERICAL ALGORITHM (ALGORITHM OF ITERATION METHOD):

A complete set of procedures which gives an approximate solution to a mathematical problem. A specific way of implementation of an iteration method, including to termination criteria is called algorithm of an iteration method. In the problem of finding the solution of an equation, an iteration method uses as initial guess to generate successive approximation to the solution.

CRITERIA FOR A GOOD METHOD

1) Number of computations i.e. Addition, Subtraction, Multiplication and Division.
2) Applicable to a class of problems.
3) Speed of convergence.
4) Error management.
5) Stability.

STABLE ALGORITHM:

Algorithm for which the cumulative effect of errors is limited, so that a useful result is generated is called stable algorithm. Otherwise **Unstable**.

NUMERICAL STABILITY:

Numerical stability is about how a numerical scheme propagates error.

Why we use numerical iterative methods for solving equations?

As analytic solutions are often either too tiresome or simply do not exist, we need to find an approximate method of solution. This is where numerical analysis comes into picture.

CONVERGENCE CRITERIA FOR A NUMERICAL COMPUTATION:

If the method leads to the value close to the exact solution, then we say that the method is convergent otherwise the method is divergent. i.e. $\lim_{n\to\infty} x_n = r$

LOCAL CONVERGENCE:

An iterative method is called locally convergent to a root, if the method converges to root for initial guesses sufficiently close to root.

RATE OF CONVERGENCE OF AN ITERATIVE METHOD:

Suppose that the sequence (x_n) converges to "r" then the sequence (x_n) is said to converge to "r" with order of convergence "a" if there exist a positive constant "p" such that $\lim_{n\to\infty}\frac{|x_{n+1}-r|}{|x_n-r|^a}=\lim_{n\to\infty}\frac{\epsilon_{n+1}}{\epsilon_n^{\alpha}}=p\,(error\ constant\,)$ Thus if $a=1$, the convergence is linear. If $a=2$, the convergence is quadratic and so on. Where the number "a" is called convergence factor.

REMARK

- Rate of convergence for fixed point iteration method is linear with $a=1$
- Rate of convergence for Newton Raphson method is quadratic with $a=2$
- Rate of convergence for Secant method is super linear with $a>1$

ORDER OF CONVERGENCE OF THE SEQUENCE

Let (x_0, x_1,x_2,.......)be a sequence that converges to a number "a" and set $\mathcal{E}_n$=a - x_n

If there exist a number "k" and a positive constant "c" such that $\boldsymbol{\lim_{n\to\infty}\frac{|\epsilon_n+1|}{|\epsilon_n|^k}=c}$ Then "k" is called order of convergence of the sequence and "c" the asymptotic error constant.

CONSISTENT METHOD:

a multi – step method is consistent if it has order at least one.

Let $\boldsymbol{x\epsilon[a,b]}$, $\boldsymbol{y\epsilon R^d}$ and the function f:[a,b]×R^d×R_+→R^d may be thought of as the approximate increment per unit step, Or the approximate difference quotient and it defines the method and consider T(x,y:h) is truncation error then the method "f" is called consistent if T(x,y:h)→0 as h→0 uniformly for (x,y) $\boldsymbol{\epsilon}$ [a,b]×R^d

PRECISION:

Precision mean how close are the measurements obtained from successive iterations.

ACCURACY:

Accuracy means how close are our approximations from exact value.

DEGREE OF ACCURACY OF A QUADRATURE FORMULA:

It is the largest positive integer "n" such that the formula is exact for "x^k " for each (k=0,1,2,……n). i.e. Polynomial integrated exactly by method.

CONDITION OF A NUMERICAL PROBLEM:

A problem is well conditioned if small change in the input information causes small change in the output. Otherwise it is **ill conditioned**.

ZERO STABILITY:

A numerical IVP solver is Zero Stable if small perturbation in the initial conditions do not cause the numerical approximation to diverge away from the true solution provided the true solution of the IVP is bounded.

A_STABILITY:

A numerical IVP solver is A- Stable if its region of absolute stability includes the entire complete half plane with negative real part, $\mathbb{C}^-$

STEP SIZE, STEP COUNT, INTERVAL GAP

The common difference between the points i.e. $h = \frac{b-a}{n} = t_{i+1} - t_i$ is called step-size.

ERROR ANALYSIS

ERROR:

Error is a term used to denote the amount by which an approximation fails to equal the exact solution. $Error = Exact\ solution - Approximation$

For example: $\pi = 3.14159265 \ldots\ldots$ is irrational and $\frac{22}{7} = 3.142857 \ldots\ldots$and a rational. And $\pi \approx \frac{22}{7}$ then $Error = 3.14159265 \ldots - 3.142857 \ldots = -0.00126449 \ldots.$ is accurate.

SOURCE OF ERRORS:

Numerically computed solutions are subject to certain errors. Mainly there are three types of errors

1. Inherent errors
2. Truncation errors
3. Round Off errors

INHERENT (EXPERIMENTAL) ERRORS (1st Type of error)

Errors arise due to assumptions made in the mathematical modeling of problems. Also arise when the data is obtained from certain physical measurements of the parameters of the problem i.e. errors arising from measurements.

TRUNCATION ERRORS (2nd Type of error)

Errors arise when approximations are used to estimate some quantity.

These errors corresponding to the facts that a finite (infinite) sequence of computational steps necessary to produce an exact result is "truncated" prematurely after a certain number of steps.

How Truncation error can be removed?:

Use exact solution. OR , Error can be reduced by applying the same approximation to a larger number of smaller intervals or by switching to a better approximation.

ROUND OFF ERRORS (3rd Type of error):

Errors arising from the process of rounding off during computations. These are also called "***chopping***" i.e. discarding all decimals from some decimals on.

CHOPPING:

discarding all decimals from some decimals on. This is cutting of process.

(Rounding off): Cutting of process increasing one digit in last one.

ROUNDING:

For $x \epsilon R$; f(x) is an element of "F" nearest to "x" and the transformation x→f(x) is called Rounding (to nearest).

For Example: let $\frac{22}{7} = 3.142857\ldots$ then $chopping\ off = 3.142$ and $Rounding\ off = 3.143$

There are two common ways to express the size of error.

- Absolute error
- Relative error

RELATIVE ERRORS

If "ā" is an approximate value of a quantity whose exact value is "a" then relative error ($\boldsymbol{\epsilon_r}$) of "ā" is defined by $|\boldsymbol{\epsilon_r}| = \frac{|\boldsymbol{error}|}{|\boldsymbol{true\ value}|} = \frac{|\boldsymbol{\epsilon}|}{|\boldsymbol{a}|}$

EXAMPLE:

Consider $\sqrt{\mathbf{2}}$=1.414213……. upto four decimal places then $\sqrt{\mathbf{2}}$=1.4142+ errors

|error|=|1.4142-1.41421|=0.00001 taking 1.4142 as true or exact value. Hence $\boldsymbol{\epsilon_r} = \frac{\mathbf{0.00001}}{\mathbf{1.4142}}$

OR $Relative\ Error = \frac{True\ value - Approximation}{True\ value}$

EXAMPLE:

If ā = 2.34 is an approximation to a = 2.3456 then the absolute error is 0.00239

REMARK

i. Relative error is useful when two values are so small.
ii. $\boldsymbol{\epsilon_r} \approx \frac{\boldsymbol{\epsilon}}{\bar{a}}$ if | $\boldsymbol{\epsilon}$ | is much less than |ā|
iii. We may also introduce the quantity " $\boldsymbol{\gamma}$= a-ā = - $\boldsymbol{\epsilon}$" and called it the "correction"
iv. $\boldsymbol{True\ value\ =\ Approximate\ value\ +\ Correction}$

ABSOLUTE ERROR

If "ā" is an approximate value of a quantity whose exact value is "a" then the difference

" $\boldsymbol{\epsilon}$ =a-ā" is called absolute error of "a". i.e

$\boldsymbol{Absolute\ Error = |True\ value - Approximation|}$

- ā =a + $\boldsymbol{\epsilon}$

EXAMPLE:

If ā = 2.34 is an approximation to a = 2.3456 then the absolute error is ϵ = 0.00562

PRACTICE: Find the Absolute error and Relative errors of the followings;

i. $y = 10,00000$ and $\bar{y} = 999996$
ii. $p = e \quad and \quad p^* = 2.718$
iii. $p = \sqrt{2}\ and\ p^* = 1.414$
iv. $z = 0.000023$ and $\bar{z} = 0.000009$
v. $p = 0.3000 \times 10^1\ and\ p^* = 0.3100 \times 10^1$
vi. $p = 0.3000 \times 10^{-3}\ and\ p^* = 0.3100 \times 10^{-3}$
vii. $p = 0.3000 \times 10^4\ and\ p^* = 0.3100 \times 10^4$

ANALYTICAL ERROR:

Error arising due to instrument malfunction and operator error**.**

Or Error arising due to wrong patient in test. When patient samples are incorrectly collected, mislabeled, are not delivered in time or lost.

PERCENTAGE ERROR

It is calculated by the formula: $\boldsymbol{percentage\ error} = \left(\frac{\boldsymbol{actual\ error - calculated\ error}}{\boldsymbol{actual\ error}}\right) \times \mathbf{100\%}$

ERROR BOUND: It is a number "β" for "ā" such that |ā-a|≤β i.e. | ϵ |≤β

PROBABLE ERROR:

This is an error estimate such that the actual error will exceed the estimate with probability one – half.

In other words, the actual error is as likely to be greater than the estimate as less. Since this depends upon the error distribution, it is not an easy target and a rough substitute is often used $\sqrt{\boldsymbol{n}}\,\boldsymbol{\epsilon}$ with "$\boldsymbol{\epsilon}$" the maximum possible error.

INPUT ERROR

Error arises when the given values ($y_0 = f(x_0)$, y_1, y_2,..........y_n) are inexact as experimental or computed values usually are.

LOCAL ERROR:

This is the error after first step. Given by $\epsilon_{i+1} = x(t_0+h) - x_1$

The Local Error is the error introduced during one operation of the iterative process.

GLOBAL ERROR: :

This is the error at n-step. Given by $\epsilon_n = x(t_n) - x_n$

The Global Error is the accumulation error over many iterations.

Note that the Global Error is not simply the sum of the Local Errors due to the non-linear nature of many problems although often it is assumed to be so, because of the difficulties in measuring the global error

PROPAGATED ERRORS:

An error in the succeeding steps of a process due to an accurance of an earlier error, such error is an addition to the local error.

LOCAL TRUNCATION ERROR: It is the ratio of local error by step size.i.e. LTE = $\frac{\textit{LOCAL ERROR}}{\textit{STEP SIZE}}$

REMARK : Floating point numbers are not equally spaced.

STIFFNESS:

The phenomenon of Stiffness is not precisely defined in the literature. Some attempts of describing a stiff problem are;

- A problem is stiff if it contains widely varying time scales i.e. some components of solution decay much more rapidly than others.
- A problem is stiff if the step size is dictated by stability requirements rather than by accuracy requirements.
- A problem is stiff if explicit methods don't work, or work only extremely slowly.
- A linear problem is stiff if all of its eigenvalues have negative real part and the ***Stiffness Ratio*** (the ratio of the magnitudes of the real parts of the largest and smallest eigenvalues) is large.
- More generally, a problem is stiff if the eigenvalues of Jacobians of '***f***' differ greatly in magnitude.
- A system is called Stiff if the solution consist of components that vary with very different speed/frequency.

Definition:

A solution is **correct within P decimal places** if the error is less than 0.5×10^{-p}

Question:-

To how many decimal places is $\frac{22}{7}$ accurate as an approximation to π?

Sol:-

$$\left|\pi - \frac{22}{7}\right| = |3.14159265\ldots - 3.142857\ldots|$$

$$= 0.00126449\ldots$$

$$\frac{1}{2} \times 10^{-3} = 0.0005 < 0.00126449\ldots < 0.005 = \frac{1}{2} \times 10^{-2}$$

So, the approximation is accurate to two decimal places.

Example:-

Another approximation to π is $\frac{355}{113}$. To how many decimal places is it accurate?

$$\left|\pi - \frac{355}{113}\right| = |3.14159265 - 3.14159292\ldots|$$

$$= 0.00000027$$

$$\frac{1}{2} \times 10^{-7} < \left|\pi - \frac{355}{113}\right| < \frac{1}{2} \times 10^{-6}$$

This approximation is accurate to 6 decimal places.

SOLUTION OF NON-LINEAR EQUATIONS

ROOTS (SOLUTION) OF AN EQUATION OR ZEROES OF A FUNCTION

Those values of "x" for which f(x) = 0 is satisfied are called root of an equation or zero of a function. Thus "a" is root of f(x) = 0 iff f(a) = 0

DEFLATION: It is a technique to compute the other roots of f(x) = 0

ZERO OF MULTIPLICITY :

A solution "p" of f(x) = 0 is a zero of multiplicity "m" of "f" if for "x ≠ p" we can write
$f(x) = (x-p)^m q(x)$ where "$\lim_{k\to p} q(x) \neq 0$"

Theorem 2.11 The function $f \in C^1[a,b]$ has a simple zero at p in (a,b) if and only if $f(p) = 0$, but $f'(p) \neq 0$. ■

Proof If f has a simple zero at p, then $f(p) = 0$ and $f(x) = (x-p)q(x)$, where $\lim_{x\to p} q(x) \neq 0$. Since $f \in C^1[a,b]$,

$$f'(p) = \lim_{x\to p} f'(x) = \lim_{x\to p}[q(x) + (x-p)q'(x)] = \lim_{x\to p} q(x) \neq 0.$$

Conversely, if $f(p) = 0$, but $f'(p) \neq 0$, expand f in a zeroth Taylor polynomial about p. Then

$$f(x) = f(p) + f'(\xi(x))(x-p) = (x-p)f'(\xi(x)),$$

where $\xi(x)$ is between x and p. Since $f \in C^1[a,b]$,

$$\lim_{x\to p} f'(\xi(x)) = f'\left(\lim_{x\to p}\xi(x)\right) = f'(p) \neq 0.$$

Letting $q = f' \circ \xi$ gives $f(x) = (x-p)q(x)$, where $\lim_{x\to p} q(x) \neq 0$. Thus f has a simple zero at p. ■ ■ ■

ALGEBRAIC EQUATION:

The equation f(X) = 0 is called an algebraic equation if it is purely a polynomial in "x". e.g. $x^3+5x^2-6x+3 = 0$

TRANSCENDENTAL EQUATION:

The equation f(x) = 0 is called transcendental equation if it contains Trigonometric, Inverse trigonometric, Exponential, Hyperbolic or Logarithmic functions.e.g.

i. M = e-esinx

ii. ax^2+ log(x-3) + exsinx = 0

PROPERTIES OF ALGEBRAIC EQUATIONS

1. Every algebraic equation of degree "n" has "n" and only "n" roots.e.g.

   $x^2 - 1=0$ has distinct roots i.e. 1, -1

   $x^2+2x+1 = 0$ has repeated roots i.e. -1, -1

   $x^2+1 = 0$ has complex roots i.e. +i, -i

2. Complex roots occur in pair. i.e. (a+bi) and (a-bi) are roots of $f(x) = 0$
3. If x = a is a root f(x)=0, a polynomial of degree "n" then (x-a) is factor of f(x)=0 on dividing f(x) by $(x - a)$ we obtain polynomial of degree (n-1).

DIFFERENCE BETWEEN ALGEBRAIC EQUATIONS AND TRANSCENDENTAL EQUATION:

An equation f(x) = 0 is called transcendental equation if it contains Trigonometric, Inverse trigonometric, Exponential, Hyperbolic or Logarithmic functions while An equation f(x) = 0 is called algebraic equation if it contains a polynomial.

DISCARTS RULES OF SIGNS

The number of positive roots of an algebraic equation f(x)=0 with the real coefficient cannot exceed the number of changes in sign of the coefficient in f(x)=0.

e.g. $x^3-3x^2+4x-5=0$ changes its sign 3-time, so it has 3 roots.

Similarly, the number of negative roots of f(x)=0 cannot exceed the number of changes in sign of the coefficient of f(-x) =0

e.g. $-x^3-3x^2-4x-5=0$ does not changes its sign, so it has no negative roots.

REMARK: There are two types of methods to find the roots of Algebraic and Transcendental equations.

(i) DIRECT METHODS (ii) INDIRECT (ITERATIVE) METHODS

DIRECT METHODS

1. Methods which calculate the required solution without any initial approximation in finite number of steps.
2. Direct methods give the exact value of the roots in a finite number of steps.
3. These methods determine all the roots at the same time assuming no round off errors.
4. In the category of direct methods; Elimination Methods are advantageous because they can be applied when the system is large.
5. **Examples;** Gauss's Elimination method, Gauss's Jordan method, Cholesky's method.

INDIRECT (ITERATIVE) METHODS

1. Methods based on the concept of successive approximations. The general procedure is to start with one or more approximation to the root and obtain a sequence of iterates "x" which in the limit converges to the actual or true solution to the root.
2. Indirect Methods determine one or two roots at a time.
3. Rounding error have less effect
4. These are self-correcting methods.
5. Easier to program and can be implemented on the computer.
6. **Examples;** Gauss's Jacobi method, Gauss's Seidal method, Relaxation method.

REMEMBER: Indirect Methods are further divided into two categories

I. BRACKETING METHODS II. OPEN METHODS

BRACKETING METHODS

These methods require the limits between which the root lies. e.g. Bisection method, False position method.

OPEN METHODS

These methods require the initial estimation of the solution. e.g. Newton Raphson method.

ADVANTAGES AND DISADVANTAGES OF BRACKETING METHODS

Bracket methods always converge.

The main disadvantage is, if it is not possible to bracket the root, the method cannot applicable.

GEOMETRICAL ILLUSTRATION OF BRACKET FUNCTIONS

In these methods we choose two points "x_n" and "x_{n-1}" such that $f(x_n)$ and $f(x_{n-1})$ are of opposite signs. Intermediate value property suggests that the graph of "y=f(x)" crosses the x-axis between these two points, therefore a root (say) "$x=x_r$" lies between these two points.

REMARK: Always set your calculator at radian mod while solving Transcendental or Trigonometric equations.

How to get first approximation? :

We can find the approximate value of the root of f(x)=0 by "Graphical method" or by "Analytical method".

INTERMEDIATE VALUE THEOREM:

According to this theorem;

If the end points of straight line lies in positive and negative , then the line cuts the x – axis.

OR: If the one value of function is constant and the other is negative and function is continuous then the curve will cuts the x - axis.

OR: if $f(x) \in C[a, b]$ and $f(a).f(b)<0$ then there exist a root "$c \in (a,b)$" such that "$f(c)=0$"

OR: For f(x) being a real valued function defined on [a,b] if

f(x) is continuous at [a,b] and differentiable at (a,b) then there exists "$c \in (a,b)$" such that "$f(c)=0$"

BISECTION METHOD

Bisection method is one of the bracketing methods. It is based on the "Intermediate value theorem"

The idea behind the method is that if f(x) ∈ C [a, b] and f(a).f(b)<0 then there exist a root "c ∈ (a,b)" such that "f(c)=0"
This method also known as **BOLZANO METHOD (or) BINARY SEARCH METHOD.**

ALGORITHM: For a given continuous function f(x)

1. Find a ,b such that $f(a).f(b) < 0$ (this means there is a root "r ∈ (a,b)" such that f(r)=0
2. Let c = $\frac{a+b}{2}$ (mid-point)
3. If f(c)=0; done (lucky!)
4. Else; check if $\boldsymbol{f(c).f(a) < 0}$ or $\boldsymbol{f(c).f(b) < 0}$
5. Pick that interval [a, c] or [c, b] and repeat the procedure until stop criteria satisfied.

STOP CRITERIA

1. Interval small enough.
2. $|f(c_n)|$ almost zero
3. Maximum number of iteration reached
4. Any combination of previous ones
5. $|f(c_n)| < \epsilon$

Remark:

To determine which subinterval of $[a_n, b_n]$ contains a root of 'f' it is better to make use of **signum** function, which is defined as

$$sgn(x) = \begin{cases} -1 & ; if\ x < 0 \\ 0 & ; if\ x = 0 \\ 1 & ; if\ x > 0 \end{cases}$$

Then test $sgn(f(a_n))sgn(f(p_n)) < 0$ instead of $f(a_n)f(p_n) < 0$

Theorem:-

Assume that $f \in C[a,b]$ (f belongs to continuous functions on interval $[a,b]$) and that there exists a number $r \in [a,b]$ such that $f(r)=0$, if $f(a)$ and $f(b)$ have opposite signs and $\{c_n\}_{n=0}^{\infty}$ represent the sequence of midpoints, generated by the bisection process (A) and (B), then

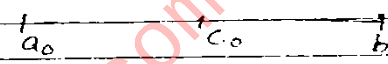
$$|r - c_n| \le \frac{b-a}{2^{n+1}} \; ; \; n = 0, 1, 2, 3, \ldots$$

and therefore the sequence $\{c_n\}_{n=0}^{\infty}$ converges to the root $x = r$

i.e. $\lim_{n\to\infty} c_n = r$

Proof:-

Since both the root "r" and the mid point c_n lies in the interval $[a_n, b_n]$. So, the distance from c_n and r cannot be greater than half the width of this interval



$|r - c_n| \le b_n - a_n$

$|b_1 - a_1| = \frac{|b-a|}{2}$

$|b_2 - a_2| = \frac{|b-a|}{2^2}$

$|b_n - a_n| = \frac{|b-a|}{2^n}$

so $|r - c_n| \le |b-a|/2^{n+1}$

$$|r - c_n| \le \frac{|b_n - a_n|}{2} \; ; \; \forall n \to (1)$$

The successive interval width are

$$|b_1 - a_1| = \frac{|b_0 - a_0|}{2}$$

$$|b_2 - a_2| = \frac{|b_1 - a_1|}{2}$$

$$|b_2 - a_2| = \frac{|b_0 - a_0|}{2^2}$$

Similarly, $|b_n - a_n| = \frac{|b_0 - a_0|}{2^n} \to (2)$

From (1) and (2)

It can be written as

$$|r - c_n| \le \frac{|b_0 - a_0|}{2^{n+1}} \; ; \; \forall n \to (3)$$

Thus, c_n converges to r. Since, for $n \to \infty$

$$\frac{|b_0 - a_0|}{2^{n+1}} \to 0$$

$$\Rightarrow \lim_{n\to\infty} c_n = r$$

Theorem 2.1 Suppose that $f \in C[a, b]$ and $f(a) \cdot f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with

$$|p_n - p| \le \frac{b-a}{2^n}, \quad \text{when} \quad n \ge 1. \qquad \blacksquare$$

Proof For each $n \ge 1$, we have

$$b_n - a_n = \frac{1}{2^{n-1}}(b-a) \quad \text{and} \quad p \in (a_n, b_n).$$

Since $p_n = \frac{1}{2}(a_n + b_n)$ for all $n \ge 1$, it follows that

$$|p_n - p| \le \frac{1}{2}(b_n - a_n) = \frac{b-a}{2^n}. \qquad \blacksquare \blacksquare \blacksquare$$

Because

$$|p_n - p| \le (b-a)\frac{1}{2^n},$$

the sequence $\{p_n\}_{n=1}^{\infty}$ converges to p with rate of convergence $O\left(\frac{1}{2^n}\right)$; that is,

$$p_n = p + O\left(\frac{1}{2^n}\right).$$

It is important to realize that Theorem 2.1 gives only a bound for approximation error and that this bound might be quite conservative. For example, this bound applied to the

CONVERGENCE CRITERIA

Number of iterations needed in the bisection method to achieve certain accuracy

Consider the interval $[a_0, b_0]$,, $c_0 = \frac{a_0+b_0}{2}$ and let $r \in (a_0,b_0)$ be a root then the error is $\epsilon_0 = |r-c_0| \le \frac{b_0-a_0}{2}$

Denote the further intervals as $[a_n,b_n]$ for iteration number "n" then $\epsilon_n = |r-c_n| \le \frac{b_n-a_n}{2} \le \frac{b_0-a_0}{2^{n+1}} = \frac{\epsilon_0}{2^n}$

If the error tolerance is "ϵ" we require "$\epsilon_n \le \epsilon$" then $\frac{b_0-a_0}{2^{n+1}} \le \epsilon$

After taking logarithm $\Longrightarrow$ $\log(b_0-a_0) - n\log 2 \le \log(2\epsilon)$

$$\Longrightarrow \frac{\log(b_0-a_0) - \log(2\epsilon)}{\log 2} \le n \quad \Longrightarrow \frac{\log(b-a) - \log 2\epsilon}{\log 2} \le n \quad \text{(which is required)}$$

MERITS OF BISECTION METHOD

1. The iteration using bisection method always produces a root, since the method brackets the root between two values.
2. **(method always converges)** As iterations are conducted, the length of the interval gets halved. So one can guarantee the convergence in case of the solution of the equation.
3. Bisection method is simple to program in a computer.

DEMERITS OF BISECTION METHOD

1. The convergence of bisection method is slow as it is simply based on halving the interval.
2. Cannot be applied over an interval where there is discontinuity.
3. Cannot be applied over an interval where the function takes always value of the same sign.
4. Method fails to determine complex roots (give only real roots)
5. If one of the initial guesses "a_0" or "b_0" is closer to the exact solution, it will take larger number of iterations to reach the root.

EXAMPLE: Solve $x^3-9x+1 = 0$ for roots between x=2 and x=4 using Bisection method.

SOLUTION

x	2	4
f(x)	-9	29

Since f (2). f (4) <0 therefore root lies between 2 and 4

(1) $x_r = \frac{2+4}{2} = 3$ so f(3) = 1 (+ve)

(2) For interval [2,3] ; $x_r = \frac{2+3}{2} = 2.5$
f (2.5) = -5.875 (-ve)

(3) For interval [2.5,3]; x_r = (2.5+3)/2 = 2.75
f (2.75) = -2.9534 (-ve)

(4) For interval [2.75,3]; x_r = (2.75+3)/2 = 2.875
f (2.875) = -1.1113 (-ve)

(5) For interval [2.875,3]; x_r = (2.875+3)/2 = 2.9375
f (2.9375) = -0.0901 (-ve)

(6) For interval [2.9375,3]; x_r = (2.9375+3)/2 = 2.9688
f (2.9688) = +0.4471 (+ve)

(7) For interval [2.9375,2.9688]; x_r = (2.9375+2.9688)/2 = 2.9532
f (2.9532) = +0.1772 (+ve)

(8) For interval [2.9375,2.9532]; x_r = (2.9375+2.9532)/2 = 2.9453
f (2.9453) = 0.1772
Hence root is 2. 9453 because roots are repeated.

EXAMPLE: Use bisection method to find out the roots of the function describing to drag coefficient of parachutist given by

$f(c) = \frac{667.38}{c}[1-\exp(-0.146843c)]-40$ Where "c =12" to "c =16" perform at least two iterations.

SOLUTION: Given that $f(c) = \frac{667.38}{c}[1-\exp(-0.146843c)]-40$

X	12	13	14	15
f(x)	6.16	3.64	1.47	-2.46

Since f (14). f (15) <0 therefore root lie between 14 and 15

$X_r = \frac{14+15}{2} = 14.5$ So f(14.5) = -4.77

Again f (14). f (14.5) <0 therefore root lie between 14 and 14.5

$X_r = \frac{14+14.5}{2} = 14.25$ So f(14.25) = 1.05 These are the required iterations

EXAMPLE:

Explain why the equation $e^{-x} = x$ has a solution on the interval [0,1]. Use bisection to find the root to 4 decimal places. Can you prove that there are no other roots?

SOLUTION:

If $f(x) = e^{-x} - x$, then $f(0) = 1$, $f(1) = 1/e - 1 < 0$, and hence a root is guaranteed by the Intermediate Value Theorem. Using Bisection, the value of the root is $x^? = .5671$. Since $f^0(x) = -e^{-x} - 1 < 0$ for all x, the function is strictly decreasing, and so its graph can only cross the x axis at a single point, which is the root.

EXAMPLE:

Using Bisection Method Solve x - Cosx = 0

SOLUTION

X	0	1
f(x)	-1	0.46

Since f(0).f(1)<0 therefore root lies between 0 and 1 and let $\epsilon = \frac{1}{2} \times 10^{-3} = 0.0005$

(1) $x_r = \frac{0+1}{2} = 0.5$ so f(0.5) = -0.3775 (-ve)

(2) For interval [0.5,1] ; $x_r = \frac{0.5+1}{2} = 0.75$ so f(0.75) = 0.0183 (+ve)

Similarly, other terms are given below

Interval	x_r	$f(x_r)$
[0.5,0.75]	0.625	-0.1859
[0.625,0.75]	0.6875	-0.0853
[0.6875,0.75]	0.71875	-0.0338
[0.71875,0.75]	0.73	-0.0152
[0.73,0.75]	0.74	-0.0015
[0.73,0.74]	0.735	-0.0068
[0.735,0.74]	0.7375	-0.0026
[0.7375,0.74]	0.73875	-0.00056

Since $f(0.73875) = |-0.0005| \leq \epsilon$ so the root is 0.73875 = 0.739

EXERCISE: :Solve by using Bisection method

i. $x^3 - 3x - 5 = 0$
ii. $x^3 - 4x - 9 = 0$
iii. $e^x - x^2 + 3x - 2 = 0 \quad for \quad 0 \leq x \leq 1$
iv. $Cosx = \sqrt{x}$
v. $3x = \sqrt{1 + Sinx}$

FALSE POSITION METHOD

This method also known as **REGULA FALSI METHOD„ CHORD METHOD „ LINEAR INTERPOLATION** and method is one of the bracketing methods and **based** on intermediate value theorem.

This method is **different** from bisection method. Like the bisection method we are not taking the mid-point of the given interval to determine the next interval and converge faster than bisection method.

ALGORITHM: Given a function f(x) continuous on an interval [a,b]

i. $f(a).f(b) < 0$ for all n = 0,1,2,3…………..

ii. Use following formula to next root $\boldsymbol{x_r} = \boldsymbol{x_f} - \frac{x_f - x_i}{f(x_f) - f(x_i)}$ f(x_f)

iii. $\boldsymbol{f(x_r)} = 0$; done! Otherwise use interval again

We can also use $x_r = x_{n+1}$,,, $x_f = x_n$,,, $x_i = x_{n-1}$

STOPING CRITERIA

1. Interval small enough.
2. |f(c_n)| almost zero
3. Maximum number of iteration reached
4. Same answer.
5. Any combination of previous ones

EXAMPLE:

Using Regula Falsi method Solve x^3-9x+1 = 0 for roots between x=2 and x=4

SOLUTION

X	2	4
f(x)	-9	29

Since f(2).f(4)<0 therefore root lies between 2 and 4

Using formula $\boldsymbol{x_r}$= x_f - $\frac{x_f - x_i}{f(x_f) - f(x_i)}$ f(x_f)

For interval [2,4] we have $x_r = \mathbf{4} - \frac{\mathbf{4-2}}{\mathbf{29-(-9)}} \times 29 = \mathbf{2.4737}$

Which implies $\boldsymbol{f(2.4737) = -6.1263}$ (-ve)

Similarly, other terms are given below

Interval	x_r	$f(x_r)$
[2.4737,4]	2.7399	-3.0905
[2.7399,4]	2.8613	-1.326
[2.8613,4]	2.9111	-0.5298
[2.9111,4]	2.9306	-0.2062
[2.9306,4]	2.9382	-0.0783
[2.9382,4]	2.9412	-0.0275
[2.9412,4]	2.9422	-0.0105
[2.9422,4]	2.9426	-0.0037
[2.9426,4]	2.9439	0.0183
[2.9426,2.9439]	2.9428	-0.0003
[2.9426,2.9439]	2.9428	-0.0003

EXAMPLE: Using Regula Falsi method to find root of equation "$\boldsymbol{logx - cosx = 0}$" upto four decimal places, after 3 successive approximations.

SOLUTION

X	0	1	2
F(X)	-∞	-0.5403	1.1093

Since f(1).f(2)<0 therefore root lies between 1 and 2

Using formula $X_r = x_f - \frac{x_f - x_i}{f(x_f) - f(x_i)} f(x_f)$

For interval [1,2] we have $x_r = 2 - \frac{2-1}{1.1093-(-0.5403)} \times 1.1093 = 1.3275$

Which implies f(2.4737)=0.0424(+ve)

Similarly, other terms are given below

Interval	x_r	$f(x_r)$
[1,1.3275]	1.3037	0.0013
[1,1.3037]	1.3030	0.0001

Hence the root is 1.3030

KEEP IN MIND

- Calculate this equation in Radian mod
- If you have "log" then use "natural log". If you have "$\boldsymbol{log_{10}}$" then use "simple log".

EXAMPLE: Using Regula Falsi method to find root of equation "$\boldsymbol{xlog_{10}x = 1.2}$" upto three decimal places.

SOLUTION: Given that $f(x) = xlog_{10}x - 1.2$

X	2	3
F(X)	-0.5979	0.2314

Since f(2).f(3)<0 therefore root lies between 2 and 3

Using formula $X_r = x_f - \frac{x_f - x_i}{f(x_f) - f(x_i)} f(x_f)$

For interval [2,3] we have $x_r = 3 - \frac{3-2}{0.2314-(-0.5979)} \times 0.2314 = 2.72097$

Which implies f(2.72097)=-0.01713(-ve)

Similarly, other terms are given below

Interval	x_r	$f(x_r)$
[3, 2.72097]	2.7402	-3.8905×10^{-4}

Thus the root of the given equation correct to three decimal places is 2.740

GENERAL FORMULA FOR REGULA FALSI USING LINE EQUATION

Equation of line is $\frac{y-f(x_n)}{x-x_n} = \frac{f(x_{n-1})-f(x_n)}{x_{n-1}-x_n}$

Put (x,0) i.e. y=0 $\frac{-f(x_n)}{x-x_n} = \frac{f(x_{n-1})-f(x_n)}{x_{n-1}-x_n}$

$$\Rightarrow \frac{-f(x_n)}{f(x_{n-1})-f(x_n)} = \frac{x-x_n}{x_{n-1}-x_n} \Rightarrow \frac{-(x_{n-1}-x_n)f(x_n)}{f(x_{n-1})-f(x_n)} = x - x_n \Rightarrow x = x_n - \frac{(x_{n-1}-x_n)f(x_n)}{f(x_{n-1})-f(x_n)}$$

Hence first approximation to the root of f(x) =0 is given by

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})f(x_n)}{f(x_n)-f(x_{n-1})}$$

We observe that $f(x_{n-1})$, $f(x_{n+1})$ are of opposite sign so, we can apply the above procedure to successive approximations.

EXAMPLE: Using False Position method Solve x Tanx = -1 in (2.5,3)

SOLUTION:

Here $f(x) = xTanx + 1$

X	2.5	3
f(x)	-0.8675	0.5723

Since f(0).f(1)<0 therefore root lies between 2.5 and 3 and let $\epsilon = \frac{1}{2} \times 10^{-3} = 0.0005$

Using formula $X_r = x_f - \frac{x_f - x_i}{f(x_f) - f(x_i)} f(x_f)$

For interval [2.5,3] we have $x_r = 3 - \frac{3-2.5}{0.5723-(-0.8675)} \times 0.5723 = 2.426$

Which implies $f(2.426) = 1.0953(-ve)$

Similarly, other terms are given below

Interval	x_r	$f(x_r)$
[2.426,3]	2.8	0.0045
[2.426,2.8]	2.798	-0.0010
[2.798,2.8]	2.7983	-0.0002

Since $f(2.7983) = |-0.0002| \leq \epsilon$ so the root is 2.7983

EXERCISE: :Solve by using Regula Falsi method

i. $x^3 + x - 3 = 0$ correct to four decimal places.
ii. $x^6 - x^4 - x^3 - 1 = 0 \quad for \quad 1.4 \leq x \leq 1.5$ correct to four decimal places after three successive approximations.
iii. $x^3 - sinx + 1 = 0 \quad for \quad -1 \leq x \leq -2$ correct to four decimal places after three successive approximations.
iv. $Cosx - xe^x = 0$
v. $x^3 - 4x - 9 = 0$

SECANT METHOD

The secant method is a simple variant of the method of false position which it is no longer required that the function "f" has opposite signs at the end points of each interval generated, not even the initial interval.

In other words, one starts with two arbitrary initial approximations $\boldsymbol{x_0 \neq x_1}$ and continues with

$$\boldsymbol{x_{n+1} = x_n - \frac{(x_n - x_{n-1}) f(x_n)}{f(x_n) - f(x_{n-1})}} \qquad ; n=1,2,3,4\ldots\ldots\ldots\ldots$$

This method also known as **<u>QUASI NEWTON'S METHOD</u>.**

ADVANTAGES

1. No computations of derivatives
2. One f(x) computation each step
3. Also rapid convergence than Falsi method

EXAMPLE: Using Secant method Solve $x log_{10} x = 1.2$

SOLUTION:

Here $f(x) = x log_{10} x - 1.2$

X	0	1	2	3
f(x)	-1.2	-1.2	-0.598	0.23

Since f(2).f(3)<0 therefore root lies between 2and 3 and let $\epsilon = \frac{1}{2} \times 10^{-3} = 0.0005$

Using formula $\boldsymbol{x_{n+1}} = \boldsymbol{x_n} - \frac{(x_n - x_{n-1}) f(x_n)}{f(x_n) - f(x_{n-1})}$

For n = 1 ⇒ $\boldsymbol{x_2} = \boldsymbol{x_1} - \frac{(x_1 - x_0) f(x_1)}{f(x_1) - f(x_0)} = 2.73 \Rightarrow f(\boldsymbol{x_2}) = -0.00928$

For n = 2 ⇒ $\boldsymbol{x_3} = \boldsymbol{x_2} - \frac{(x_2 - x_1) f(x_2)}{f(x_2) - f(x_1)} = 2.74 \Rightarrow f(\boldsymbol{x_2}) = -0.0005$

Since $f(\boldsymbol{x_3}) = |-0.0002| \leq \epsilon$ so the root is 2.74

EXAMPLE: Using Secant method Solve $x = Cosx$

SOLUTION:

Here $f(x) = x - Cosx$

X	0	1
f(x)	-1	0.4597

Since f(0).f(1)<0 therefore root lies between 0and 1 and let $\epsilon = \frac{1}{2} \times 10^{-3} = 0.0005$

Using formula $\boldsymbol{x_{n+1}} = \boldsymbol{x_n} - \frac{(x_n - x_{n-1}) f(x_n)}{f(x_n) - f(x_{n-1})}$

For n = 1 ⇒ $\boldsymbol{x_2} = \boldsymbol{x_1} - \frac{(x_1 - x_0) f(x_1)}{f(x_1) - f(x_0)} = 0.779 \Rightarrow f(\boldsymbol{x_2}) = 0.0674$

For n = 2 ⇒ $\boldsymbol{x_3} = \boldsymbol{x_2} - \frac{(x_2 - x_1) f(x_2)}{f(x_2) - f(x_1)} = 0.73681 \Rightarrow f(\boldsymbol{x_3}) = -0.0038$

For n = 3 ⇒ $\boldsymbol{x_4} = \boldsymbol{x_3} - \frac{(x_3 - x_2) f(x_3)}{f(x_3) - f(x_2)} = 0.74 \Rightarrow f(\boldsymbol{x_4}) = 0.0015$

For n = 4 ⇒ $\boldsymbol{x_5} = \boldsymbol{x_4} - \frac{(x_4 - x_3) f(x_4)}{f(x_4) - f(x_3)} = 0.738 \Rightarrow f(\boldsymbol{x_5}) = -0.0018$

For n = 5 ⇒ $\boldsymbol{x_6} = \boldsymbol{x_5} - \frac{(x_5 - x_4) f(x_5)}{f(x_5) - f(x_4)} = 0.73 \Rightarrow f(\boldsymbol{x_5}) = -0.0016$

Since $f(\boldsymbol{x_5}) = |-0.0016| \leq \epsilon$ so the root is 0.73

FIXED POINT:

The real number "x" is a fixed point of the function "f" if f(x) =x

The number x=0.7390851332 is an approximate fixed point of f(x) = cosx

REMARK : Fixed point are roughly divided into three classes

ASYMPTOTICALLY STABLE: with the property that all nearby solutions converge to it.

STABLE: All nearby solutions stay nearby.

UNSTABLE: Almost all of whose nearby solutions diverge away from the fixed point

FIXED POINT ITERATION METHOD

ALGORITHM

1. Consider f(x) =0 and transform it to the form x= φ (x)
2. Choose an arbitrary x_0
3. Do the iterations $x_{k+1}=\varphi(x_k)$; k=0,1,2,3……….

STOPING CRITERIA: Let "ϵ" be the tolerance value

1. $|x_k - x_{k-1}| \le \epsilon$
2. $|x_k - f(x_k)| \le \epsilon$
3. Maximum number of iterations reached.
4. Any combination of above.

CONVERGENCE CRITERIA: Let "x" be exact root such that r=f(x) out iteration is $x_{n+1} = f(x_n)$

Define the error $\epsilon_n = x_n - r$ Then

$$\epsilon_{n+1} = x_{n+1} - r = f(x_n) - r = f(x_n) - f(r) = f'(\S)(x_n - r)$$

(Where $\S \in (x_n, r)$; since f is continuous)

$$\epsilon_{n+1} = f'(\S)\epsilon_n \Rightarrow \epsilon_{n+1} \le |f'(\S)||\epsilon_n|$$

OBSERVATIONS:

If $|f'(\S)| < 1$, error decreases, the iteration converges (linear convergence)

If $|f'(\S)| \ge 1$, error increases, the iteration diverges.

REMEMBER: If $|\varphi'(x)| < 1$ in questions then take that point as initial guess.

EXAMPLE:

Find the root of equation $\mathbf{2x = cosx + 3}$ correct to three decimal points using fixed point iteration method.

SOLUTION:

Given that $f(x) = 2x - cosx - 3$

X	0	1	2
F(X)	-4	-1.5403	1.4161

Root lies between "1" and "2"

Now $2x - cosx - 3 = 0 \Rightarrow x = \frac{cosx+3}{2} = \varphi(x)$

$\Rightarrow \varphi'(x) = \frac{1}{2}(-\sin x) \Rightarrow |\varphi'(x)| = |\frac{1}{2}(-\sin x)|$

Now $x_{n+1} = \varphi(x_n) \Rightarrow x_{n+1} = \frac{1}{2}(coxx_n + 3)$

Here we will take "x_0" as mid-point. So

$X_0 = \frac{1+2}{2} = 1.5$

> If by putting 1 we get $|\varphi'(x)|$< 1 then take it as "x_0" if not then check for 2 rather take their mid-point

$X_1 = \frac{1}{2}(cosx_0 + 3) = 1.5354$	$F(x_1) = 0.0354$
$X_2 = \frac{1}{2}(cosx_1 + 3) = 1.5177$	$F(x_2) = -0.0177$
$X_3 = 1.5265$	$F(x_3) = 0.0087$
$x_4 = 1.5221$	$F(x_4) = -0.0045$
$X_5 = 1.5243$	$F(x_5) = 0.0021$
$X_6 = 1.5232$	$F(x_6) = -0.0012$
$X_7 = 1.5238$	$F(x_7) = 0.0006$
$X_8 = 1.5235$	$F(x_8) = -0.0003$
$X_9 = 1.5236$	$F(x_9) = 0.0000$

Hence the real root is 1.5236

EXAMPLE: Find the root of equation $\boldsymbol{e^{-x} = 10x}$ correct to four decimal points using fixed point iteration method.

SOLUTION: Given that $f(x) = e^{-x} - 10x$

X	0	1
F(X)	1	-9.6321

Root lies between "0" and "1"

Now $e^{-x} - 10x = 0 \Rightarrow x = \frac{e^{-x}}{10} = \varphi(x) \Rightarrow \varphi'(x) = -\frac{e^{-x}}{10}$

Now since $|\varphi'(0)| = 0.1$ is less than "1" therefore $x_0 = 0$ then $x_{n+1} = \varphi(x_n) \Rightarrow x_{n+1} = \frac{e^{-x_n}}{10}$

$X_1 = \frac{e^{-x_0}}{10} = \frac{e^{-0}}{10} = 0.1000$	$f(x_1) = -0.0952$
$X_2 = 0.0905$	$f(x_2) = 0.0085$
$X_3 = 0.0913$	$f(x_3) = -0.0003$
$x_4 = 0.0913$	$f(x_4) = -0.0003$

Hence the real root is 0.0913

EXAMPLE: Find the root of equation $\boldsymbol{x^3 + x^2 - 1 = 0}$ by iteration method.

SOLUTION: Given that $f(x) = x^3 + x^2 - 1$

x	0	1
f(x)	-1	1

Root lies between "0" and "1"

Now rewrite the equation $x^2(x+1) = 1 \Rightarrow x = \frac{1}{\sqrt{x+1}} = \varphi(x)$

$\Rightarrow \varphi'(x) = -\frac{1}{2(x+1)^{3/2}} \Rightarrow |\varphi'(x)| < 1$ then $x_{n+1} = \varphi(x_n) \Rightarrow x_{n+1} = \frac{1}{\sqrt{x_n+1}}$

Here we will take initial value "$x_0 = 1$" So

$x_1 = \frac{1}{\sqrt{x_0+1}} = \frac{1}{\sqrt{2}} = 0.70711$	$f(x_1) = -0.14644$
$x_2 = 0.76537$	$f(x_2) = 0.03414$
$x_3 = 0.75263$	$f(x_3) = 7.2213 \times 10^{-3}$
$x_4 = 0.75536$	$f(x_4) = 1.55658 \times 10^{-3}$
$x_5 = 0.75477$	$f(x_5) = -3.44323 \times 10^{-4}$
$X_6 = 0.7549$	$f(x_6) = 7.38295 \times 10^{-5}$

Hence the real root is 0.7549

REMARK:

The given equation can be written in many ways. Suppose we rewrite $x^2 = 1 - x^3$ or $x = \sqrt{1 - x^3} = \varphi(x) \Rightarrow \varphi'(x) = -\frac{3x^2}{2\sqrt{1-x^3}} \Rightarrow |\varphi'(1)| = \infty$ in the interval (0,1) then the condition $|\varphi'(x)| < 1$ violated.

Example 1. $f(x) = x - cosx.$

Choose $g(x) = \cos x$, we have $x = \cos x$.

Choose $x_0 = 1$, and do the iteration $x_{k+1} = \cos(x_k)$:

$$\begin{aligned} x_1 &= \cos x_0 = 0.5403 \\ x_2 &= \cos x_1 = 0.8576 \\ x_3 &= \cos x_2 = 0.6543 \\ &\vdots \\ x_{23} &= \cos x_{22} = 0.7390 \\ x_{24} &= \cos x_{23} = 0.7391 \\ x_{25} &= \cos x_{24} = 0.7391 \quad \text{stop here} \end{aligned}$$

Our approximation to the root is 0.7391.

Example 2. Consider $f(x) = e^{-2x}(x-1) = 0$. We see that $r = 1$ is a root. Rewrite as

$$x = g(x) = e^{-2x}(x-1) + x$$

Choose an initial guess $x_0 = 0.99$, very close to the real root. Iterations:

$$\begin{aligned} x_1 &= \cos x_0 = 0.9886 \\ x_2 &= \cos x_1 = 0.9870 \\ x_3 &= \cos x_2 = 0.9852 \\ &\vdots \\ x_{27} &= \cos x_{26} = 0.1655 \\ x_{28} &= \cos x_{27} = -0.4338 \\ x_{29} &= \cos x_{28} = -3.8477 \quad \text{Diverges. It does not work.} \end{aligned}$$

Convergence depends on x_0 and $g(x)$!

EXERCISE: :Solve by using iteration method

i. $2x = Cosx + 3$ correct to three decimal places.
ii. $Cosx - xe^x = 0$

Theorem:- Assume that g is continuous function and that $\{P_n\}_{n=0}^{\infty}$ is a sequence generated by fixed point Iteration. If $\lim_{n\to\infty} P_n = P$, then P is a fixed point of $g(x)$.

Proof:-

Suppose that

$$\lim_{n\to\infty} P_n = P \rightarrow (1)$$

then

$$\lim_{n\to\infty} P_{n+1} = P \rightarrow (2)$$

Consider the fixed point Iteration

$$P_{n+1} = g(P_n)$$

Taking $\lim_{n\to\infty}$ on both sides:

$$\lim_{n\to\infty} P_{n+1} = \lim_{n\to\infty} g(P_n)$$

As g is continuous. So

$$P = g\left[\lim_{n\to\infty} P_n\right]$$

$P = g(P)$ and hence P is fixed point of g(P).

Note:-
Given a root finding problem $f(p)=0$, we can define a function g with a fixed point at p in a number of ways, for example as

$$g(x) = x - f(x)$$

Conversely, if the function g has a fixed point at p, then the function defined by

$$f(x) = x - g(x)$$

has a zero at p.

Theorem:

(a) If $g \in C[a,b]$ and $g(x) \in [a,b]$, then g has a fixed point in $[a,b]$

(b) If in addition, $g'(x)$ exists on $]a,b[$ and a positive constant $k<1$ exists with

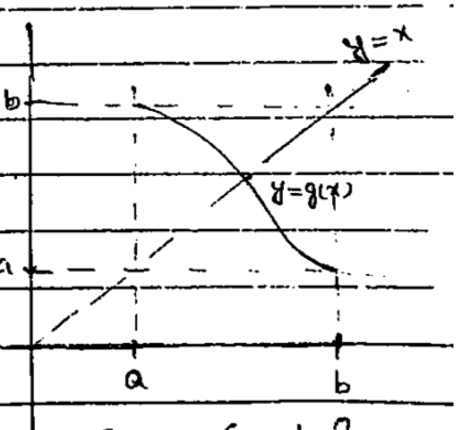
$$|g'(x)| \le k \quad \forall x \in]a,b[$$

then the fixed point in $[a,b]$ is unique.

Proof:-

(a) If $g(a)=a$ or $g(b)=b$, then g has a fixed point at an end point.

Suppose not, then it must true that $g(a)>a$, $g(b)<b$



$g(x) \in [a,b]$

The function $h(x) = g(x) - x$ is continuous on $[a,b]$ and

$h(a) = g(a) - a > 0$

$h(b) = g(b) - b < 0$

The Intermediate value theorem implies that there exists $r \in (a,b)$ for which $h(r) = 0$

For this value of r

$h(r) = g(r) - r = 0$

$\Rightarrow g(r) = r$

or $r = g(r)$

which shows that r is a fixed point of g.

(b)

Suppose in addition that $|g'(x)| \le k < 1$ and that p and q are both fixed point in $[a,b]$ with $p \neq q$.

By Mean value theorem, there exists a number r exists between p and q and hence in interval $[a,b]$ with

$$g'(r) = \frac{g(p) - g(q)}{p - q} \rightarrow (1)$$

$$|p - q| = |g(p) - g(q)| = |p - q||g'(r)| \text{ (by 1)}$$

$$\le |p - q| \cdot k$$

$$< |p - q| \cdot 1$$

$$|p - q| < |p - q|$$

$\Rightarrow |p - q| < |p - q|$ impossible.

which is a contradiction.

This contradiction must come from

The only supposition $p \neq q$. Hence $p \neq q$ and the fixed point in [a,b] is unique.

The following theorem gives sufficient conditions for the existence and uniqueness of a fixed point.

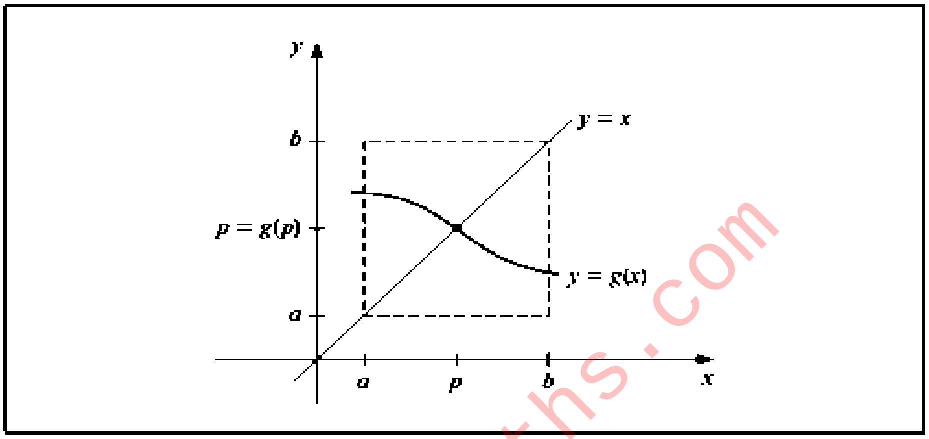
Theorem 2.3 (i) If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has at least one fixed point in $[a,b]$.

(ii) If, in addition, $g'(x)$ exists on (a,b) and a positive constant $k < 1$ exists with

$$|g'(x)| \le k, \quad \text{for all } x \in (a,b),$$

then there is exactly one fixed point in $[a,b]$. (See Figure 2.4.) ■

Figure 2.4



Proof

(i) If $g(a) = a$ or $g(b) = b$, then g has a fixed point at an endpoint. If not, then $g(a) > a$ and $g(b) < b$. The function $h(x) = g(x) - x$ is continuous on $[a,b]$, with

$$h(a) = g(a) - a > 0 \quad \text{and} \quad h(b) = g(b) - b < 0.$$

The Intermediate Value Theorem implies that there exists $p \in (a,b)$ for which $h(p) = 0$. This number p is a fixed point for g because

$$0 = h(p) = g(p) - p \quad \text{implies that} \quad g(p) = p.$$

(ii) Suppose, in addition, that $|g'(x)| \le k < 1$ and that p and q are both fixed points in $[a, b]$. If $p \ne q$, then the Mean Value Theorem implies that a number ξ exists between p and q, and hence in $[a, b]$, with

$$\frac{g(p) - g(q)}{p - q} = g'(\xi).$$

Thus

$$|p - q| = |g(p) - g(q)| = |g'(\xi)||p - q| \le k|p - q| < |p - q|,$$

which is a contradiction. This contradiction must come from the only supposition, $p \ne q$. Hence, $p = q$ and the fixed point in $[a, b]$ is unique. ■ ■ ■

Theorem 2.4 **(Fixed-Point Theorem)**

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all x in $[a, b]$. Suppose, in addition, that g' exists on (a, b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \le k, \quad \text{for all } x \in (a, b).$$

Then for any number p_0 in $[a, b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad n \ge 1,$$

converges to the unique fixed point p in $[a, b]$. ■

Proof Theorem 2.3 implies that a unique point p exists in $[a, b]$ with $g(p) = p$. Since g maps $[a, b]$ into itself, the sequence $\{p_n\}_{n=0}^{\infty}$ is defined for all $n \ge 0$, and $p_n \in [a, b]$ for all n. Using the fact that $|g'(x)| \le k$ and the Mean Value Theorem 1.8, we have, for each n,

$$|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(\xi_n)||p_{n-1} - p| \le k|p_{n-1} - p|,$$

where $\xi_n \in (a, b)$. Applying this inequality inductively gives

$$|p_n - p| \le k|p_{n-1} - p| \le k^2|p_{n-2} - p| \le \cdots \le k^n|p_0 - p|. \tag{2.4}$$

Since $0 < k < 1$, we have $\lim_{n\to\infty} k^n = 0$ and

$$\lim_{n\to\infty} |p_n - p| \le \lim_{n\to\infty} k^n|p_0 - p| = 0.$$

Hence $\{p_n\}_{n=0}^{\infty}$ converges to p. ■ ■ ■

NEWTON RAPHSON METHOD

Nature and Nature's laws lay hid in night:
God said, Let Newton be! And all was light.
Alexander Pope, 1727

The Newton Raphson method is a powerful technique for solving equations numerically. It is based on the idea of linear approximation. Usually converges much faster than the linearly convergent methods.

ALGORITHM: The steps of Newton Raphson method to find the root of an equation "f(x) =0" are **;** Evaluate $f'(x)$

Use an initial guess (value on which f(x) and $f''(x)$ becomes (+ve) of the roots "x_n" to estimate the new value of the root "x_{n+1}" as $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ ……… ***this value is known as Newton's iteration***

STOPING CRITERIA

1. Find the absolute relative approximate error as $|\epsilon_\propto| = |\frac{x_{n+1}-x_n}{x_{n+1}}| \times 100$
2. Compare the absolute error with the pre-specified relative error tolerance "ϵ_s".
3. If $|\epsilon_a| > \epsilon_s$ then go to next approximation. Else stop the algorithm.
4. Maximum number of iterations reached.
5. Repeated answer.

<u>When the Generalized Newton Raphson method for solving equations is helpful?</u>

To find the root of "f(x)=0" with multiplicity "p" the Generalized Newton formula is required.

<u>What is the importance of Secant method over Newton Raphson method?</u>

Newton Raphson method requires the evaluation of derivatives of the function and this is not always possible, particularly in the case of functions arising in practical problems.

In such situations Secant method helps to solve the equation with an approximation to the derivatives.

<u>Why Newton Raphson method is called Method of Tangent?</u>

In this method we draw tangent line to the point" $P_0(x_0,f(x_0))$". The (x,0) where this tangent line meets x-axis is 1^{st} approximation to the root**.** **Similarly,** we obtained other approximations by tangent line. So, method also called Tangent method.

Difference between Newton Raphson method and Secant method.

Secant method needs two approximations x_0, x_1 to start, whereas Newton Raphson method just needs one approximation i.e. x_0. Also Newton Raphson method converges faster than Secant method.

Newton Raphson method is an Open method, how?

Newton Raphson method is an open method because initial guess of the root that is needed to get the iterative method started is a single point. While other open methods use two initial guesses of the root but they do not have to bracket the root.

INFLECTION POINT

For a function "f(x)" the point where the concavity changes from up-to-down

or down-to-up is called its Inflection point.

e.g. $f(x) = (x-1)^3$ changes concavity at x=1,, Hence (1,0) is an Inflection point.

DRAWBACKS OF NEWTON'S RAPHSON METHOD

- Method diverges at inflection point.
- For f(x)=0 Newton Raphson method reduce. So one must be avoid division by zero. Rather method not converges.
- Root jumping is another drawback.
- Results obtained from Newton Raphson method may oscillate about the Local Maximum or Minimum without converging on a root but converging on the Local Maximum or minimum. Eventually, it may lead to division by a number close to zero and may diverge.
- The requirement of finding the value of the derivatives of f(x) at each approximation is either extremely difficult (if not possible) or time consuming.

(GEOMETRICALLY EXPLAIN NR METHOD TO FIND A ROOT OF THE EQUATION $f(x) = 0$ AND HENCE DERIVE THE GENERAL FORMULA)

GEOMETRICAL INTERPRETATION (GRAPHICS) OF NEWTON RAPHSON FORMULA

Suppose the graph of function "y=f(x)" crosses x-axis at " $\propto$ " then "$x = \propto$ " is the root of equation"$f(x) = 0$".

CONDITION

Choose "x_0" such that "$f(x)$" and f"(x) have same sign. If "$(x_0, f(x_0))$" is a point then slope of tangent at "$(x_0, f(x_0)) = m = \frac{dy}{dx}|_{(x_0,f(x_0))} = f'(x_0)$"

Now equation of tangent is

$$y - y_0 = m(x - x_0)$$

$$\Rightarrow y - f(x_0) = f'(x_0)(x - x_0) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Since $(x_1, f(x_1) = y_1 = 0)$ as we take x_1 as exact root

(i)$\Longrightarrow$ $\quad 0 - f(x_0) = f'(x_0)(x - x_0)$

$$\Rightarrow -\frac{f(x_0)}{f\prime(x0)} = x_1 - x_0 \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f\prime(x_0)}$$

Which is first approximation to the root " $\propto$ " . If "P_1" is a point on the curve corresponding to "x_1" then tangent at "P_1" cuts x-axis at $P_1(x_2, 0)$ which is still closer to "$\propto$" than "x_1". Therefore "x_2" is a 2nd approximation to the root.

Continuing this process, we arrive at the root "$\propto$".

FORMULA DARIVATION FOR NR-METHOD

Given an equation "f(x) = 0" suppose "x_0" is an approximate root of "f(x) = 0"

Let $x_1 = x_0 + h \ldots\ldots\ldots\ldots\ldots\ldots (1)$ $\quad$ ***since*** $x_1 - x_0 = h$

Where "h" is the small; exact root of f(x)=0 Then $f(x_1) = 0 = f(x_0 + h) \quad \therefore \quad x_1 = x_0 + h$

By Taylor theorem $f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) \ldots\ldots\ldots\ldots\ldots\ldots. = 0$

Since "h" is small therefore neglecting higher terms we get

$$f(x_0 + h) = f(x_0) + hf'(x_0) = 0 \Longrightarrow h = -\frac{f(x_0)}{f\prime(x_0)}$$

$$(1) \Longrightarrow \boldsymbol{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

Similarly $\boldsymbol{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$

$$\boldsymbol{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$$

$$\vdots = \vdots - \vdots$$

$\boldsymbol{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$ This is required Newton's Raphson Formula.

EXAMPLE:

Apply Newton's Raphson method for $\boldsymbol{cosx = xe^x \ at \ x_0 = 1}$ correct to three decimal places.

SOLUTION

$f(x) = cosx - xe^x$

$f'(x) = -sinx - e^x - xe^x$

Using formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

at $x_0 = 1$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.653 \quad (after\ solving)$

$f(x_1) = -0.460$; $f'(x_1) = -3.783$

Similarly

n	$\boldsymbol{x_n}$	$\boldsymbol{f(x_n)}$	$\boldsymbol{f'(x_n)}$
2	0.531	-0.041	-3.110
3	0.518	-0.001	-3.043
4	0.518	-0.001	-3.043

Hence root is "0.518"

REMARK

1. If two are more roots are nearly equal, then method is not fastly convergent.
2. If root is very near to maximum or minimum value of the function at the point, NR-method fails.

EXAMPLE:

Apply Newton's Raphson method for $\boldsymbol{xlog_{10}x = 4.77}$ correct to two decimal places.

SOLUTION

$f(x) = xlog_{10}x - 4.77$

$f'(x) = log_{10}x + x\frac{1}{x}log_{10}e$

$f'(x) = log_{10}x + log_{10}e$

$f'(x) = log_{10}x + 0.4343 \qquad since\ e = 2.71828$

$f''(x) = \frac{1}{x}log_{10}e = \frac{0.4343}{x}$

For interval

X	0	1	2	3	4	5	6	7
f(x)	-4.77	-4.77	-4.17	-3.34	-2.36	-1.28	-0.10	1.15

Root lies between 6 and 7 and let x_0 =7

Using formula $\boldsymbol{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$

Thus $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 6.10\ \ after\ solving$

$f(x_1) = 0.02\ ;\ f'(x_1) = 1.22$

Similarly

n	$\boldsymbol{x_n}$	$\boldsymbol{f(x_n)}$	$\boldsymbol{f'(x_n)}$
2	6.08	0.00	0.00

Hence root is "6.08"

CONDITION FOR CONVERGENCE OF NR-METHOD

Since by Newton Raphson method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ ……………..(1)

And by General Iterative formula $x_{n+1} = \varphi(x_n)$ ………………(2)

Comparing (1) and (2) $\varphi(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$ and $\varphi(x) = x - \frac{f(x)}{f'(x)}$ $(simply)$

Since by iterative method condition for convergence $|\varphi'(x)| < 1$ ……………(3)

So $\varphi'(x) = 1 - [\frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2}] \Rightarrow \varphi'(x) = 1 - \frac{(f'(x))^2}{(f'(x))^2} + \frac{f(x)f''(x)}{(f'(x))^2}$

$\varphi'(x) = \frac{f(x)f''(x)}{(f'(x))^2}$ Using in (3) we get

$\Rightarrow \left|\frac{f(x)f''(x)}{(f'(x))^2}\right| < 1 \qquad \Rightarrow |f(x)f''(x)| < (f'(x))^2$

Which is required condition for convergence of Newton Raphson method, provided that initial approximation "x_0" is choose sufficiently close to the root and $\boldsymbol{f(x)}, \boldsymbol{f'(x)}, \boldsymbol{f''(x)}$ are continuous and bounded in any small interval containing the root.

NEWTON RAPHSON METHOD IS QUADRATICALLY CONVERGENT

(OR) NEWTON RAPHSON METHOD HAS SECOND ORDER CONVERGENCE

(OR) ERROR FOR NEWTON RAPHSON METHOD

Let " $\propto$ " be the root of f(x) =0 and $\begin{pmatrix} x_n - \propto = \epsilon_n \\ x_{n+1} - \propto = \epsilon_{n+1} \end{pmatrix}$ ………………………..(1)

If we can prove that $\epsilon_{n+1} = k\, \epsilon_n^p$ where "k" is constant then "p" is called order of convergence of iterative method then we are done.

Since by Newton Raphson formula we have $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ Then using (1) in it

$\Rightarrow \propto + \epsilon_{n+1} = \propto + \epsilon_n - \frac{f(\propto + \epsilon_n)}{f'(\propto + \epsilon_n)} \Rightarrow \epsilon_{n+1} = \epsilon_n - \frac{f(\propto + \epsilon_n)}{f'(\propto + \epsilon_n)}$

Since by Taylor expansion we have $\epsilon_{n+1} = \epsilon_n - \frac{f(\propto) + \epsilon_n f'(\propto) + \frac{\epsilon_n^2}{2!} f''(\propto)}{f'(\propto) + \epsilon_n f''(\propto) + \frac{\epsilon_n^2}{2!} f'''(\propto)}$

Since " $\propto$ " is root of f(x) therefore "$f(\propto) = 0$"

$$\epsilon_{n+1} = \epsilon_n - \frac{\epsilon_n f'(\alpha) + \frac{\epsilon_n^2}{2!} f''(\alpha)}{f'(\alpha) + \epsilon_n f''(\alpha) + \frac{\epsilon_n^2}{2!} f'''(\alpha)} = \frac{\epsilon_n [f'(\alpha) + \epsilon_n f''(\alpha) + \frac{\epsilon_n^2}{2!} f'''(\alpha)] - \epsilon_n f'(\alpha) - \frac{\epsilon_n^2}{2!} f''(\alpha)}{f'(\alpha) + \epsilon_n f''(\alpha) + \frac{\epsilon_n^2}{2!} f'''(\alpha)}$$

$$\Rightarrow \epsilon_{n+1} = \frac{\frac{\epsilon_n^2}{2} f''(\alpha) + \frac{\epsilon_n^3}{2!} f'''(\alpha)}{f'(\alpha) + \epsilon_n f''(\alpha) + \frac{\epsilon_n^2}{2!} f'''(\alpha)}$$

After neglecting higher terms $\epsilon_{n+1} = \frac{\frac{\epsilon_n^2}{2} f''(\alpha)}{f'(\alpha) + \epsilon_n f''(\alpha)}$

$$\epsilon_{n+1} = \frac{\epsilon_n^2 f''(\alpha)}{2f'(\alpha)[1 + \frac{\epsilon_n f''(\alpha)}{f'(\alpha)}]} = \frac{\epsilon_n^2 f''(\alpha)}{2f'(\alpha)} \left[1 + \frac{\epsilon_n f''(\alpha)}{f'(\alpha)}\right]^{-1}$$

$$\epsilon_{n+1} = \frac{\epsilon_n^2 f''(\alpha)}{2f'(\alpha)} \left[1 + (-1)\frac{\epsilon_n f''(\alpha)}{f'(\alpha)} + neglected\right] = \frac{\epsilon_n^2 f''(\alpha)}{2f'(\alpha)} - \frac{\epsilon_n^3 f''^2(\alpha)}{2f'^2(\alpha)}$$

$$\epsilon_{n+1} = \frac{\epsilon_n^2 f''(\alpha)}{2f'(\alpha)} = k\,\epsilon_n^2 \qquad where\ \ k = \frac{f''(\alpha)}{2f'(\alpha)}$$

It shows that Newton Raphson method has second order convergence
Or converges quadratically.

EXAMPLE 2.10 Newton's Method for a Problem with a Root of Multiplicity > 1

Consider the function $f(x) = x(1 - \cos x)$, which has a root of multiplicity three at $x = 0$. The following table shows the results of ten iterations of Newton's method applied to this problem with a starting value of $p_0 = 1$. For comparison, the results of the bisection method, starting from the interval $[-2, 1]$ are shown in the third column.

	Newton's Method	Bisection Method
1	0.6467039965	−0.5000000000
2	0.4259712109	0.2500000000
3	0.2825304410	−0.1250000000
4	0.1879335654	0.0625000000
5	0.1251658102	−0.0312500000
6	0.0834075192	0.0156250000
7	0.0555942620	−0.0078125000
8	0.0370596587	0.0039062500
9	0.0247054965	−0.0019531250
10	0.0164700517	0.0009765625

Convergence of Newton-Raphson Method

- Usually converges quadratically

Example: $f(x) = e^{-x} - x$ (true solution = 0.567143290409784)

Solved with 2 methods:

Newton-Raphson with x_0=0

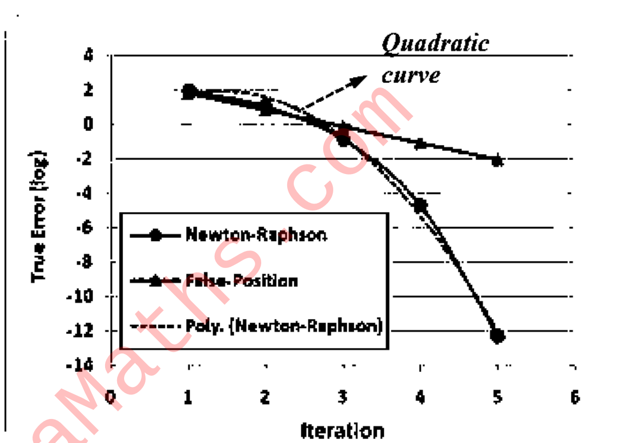
False-Position Method with x_l=0 and x_u=20

Newton-Raphson

	Iterations	true error
x_0 =	0	100.000000000%
x_1 =	0.500000000000000	11.838858282%
x_2 =	0.566311003197218	0.146750782%
x_3 =	0.567143165034862	0.000022106%
x_4 =	0.567143290409781	0.000000000%

False-Position

	Iterations	true error
x_0 =	0.952380952	67.925984240%
x_1 =	0.607944265065116	7.194121018%
x_2 =	0.571658116501746	0.796064446%
x_3 =	0.567645088312370	0.088478152%
x_4 =	0.567199089558233	0.009838633%



NEWTON RAPHSON EXTENDED FORMULA

(CHEBYSHEVES FORMULA OF 3RD ORDER)

Consider f(x) =0. Expand f(x) by Taylor series in the neighborhood of "x_0". We obtain after retaining the first term only.

$$0 = f(x) = f(x_0) + (x - x_0)\, f'(x_0) + neglected \Rightarrow 0 = f(x_0) + (x - x_0)\, f'(x_0)$$

$$\Rightarrow x - x_0 = -\frac{f(x_0)}{f'(x_0)} \quad \Rightarrow x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

This the first approximation to the root therefore

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots..(1)$$

Again expanding f(x) by Taylor Series and retaining the second order term only

$$0 = f(x) = f(x_0) + (x - x_0)\, f'(x_0) + \frac{(x-x_0)^2}{2!} f"(x_0)$$

$$0 = f(x_1) = f(x_0) + (x_1 - x_0)\, f'(x_0) + \frac{(x_1-x_0)^2}{2!} f"(x_0) \qquad \therefore f(x) = f(x_1)$$

$$0 = f(x_0) + (x_1 - x_0)\, f'(x_0) + \frac{(x_1-x_0)^2}{2!} f"(x_0) \quad \dots\dots\dots\dots\dots\dots\dots..(2)$$

Using eq. (1) in (2) we get

$$f(x_0) + (x_1 - x_0)\, f'(x_0) + \frac{1}{2}[-\frac{f(x_0)}{f'(x_0)}]^2\, f"(x_0) = 0$$

$$f(x_0) + (x_1 - x_0)\, f'(x_0) = -\frac{1}{2}\left[\frac{f(x_0)}{f'(x_0)}\right]^2\, f"(x_0)$$

$$x_1\, f'(x_0) = x_0\, f'(x_0) - f(x_0) - \frac{1}{2}[\frac{f(x_0)}{f'(x_0)}]^2\, f"(x_0)$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\frac{[f(x_0)]^2}{[f'(x_0)]^3}\, f"(x_0)$$

This is Newton Raphson Extended formula. Also known as "Chebysheves formula of third order"

NEWTON SCHEME OF ITERATION FOR FINDING THE SQUARE ROOT OF POSITION NUMBER

The square root of "N" can be carried out as a root of the equation

$$x = \sqrt{N} \implies x^2 = N \implies x^2 - N = 0$$

Here $f(x) = x^2 - N$; $f(x_n) = x_n^2 - N$

$f'(x) = 2x$; $f'(x_n) = 2x_n$

Using Newton Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$\implies \boldsymbol{x_{n+1} = x_n - \frac{(x_n^2 - N)}{2x_n} = \frac{2x_n^2 - x_n^2 + N}{2x_n}} = \frac{1}{2}\left[\frac{\boldsymbol{x_n^2 + N}}{\boldsymbol{x_n}}\right]$$

$$\implies \boldsymbol{x_{n+1} = \frac{1}{2}[x_n + \frac{N}{x_n}]}$$ This is required formula.

QUESTION: Evaluate $\sqrt{12}$ by Newton Raphson formula.

SOLUTION: Let x=$\sqrt{12} \Rightarrow x^2 = 12 \Rightarrow x^2 - 12 = 0$

Here $f(x) = x^2 - 12$; $f'(x) = 2x$; $f''(x) = 2$

X	0	1	2	3	4
F(x)	-12	-11	-8	-3	4

Root lies between 3 and 4 and x_0 =4

Now using formula $x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right] \Rightarrow x_{n+1} = \frac{1}{2}\left[x_n + \frac{12}{x_n}\right]$

For n=0 $x_1 = \frac{1}{2}\left[x_0 + \frac{12}{x_0}\right] \Rightarrow x_1 = \frac{1}{2}\left[4 + \frac{12}{4}\right]$=3.5

For n=2 $x_2 = \frac{1}{2}\left[x_1 + \frac{12}{x_1}\right] \Rightarrow x_2 = \frac{1}{2}\left[3.5 + \frac{12}{3.5}\right] = 3.4643$

Similarly $x_3 = 3.4641$ *and* $x_4 = 3.4641$ Hence $\sqrt{12} = 3.4641$

NEWTON SCHEME OF ITERATION FOR FINDING THE "pth" ROOT OF POSITION NUMBER "N"

Consider $x = N^{\frac{1}{p}} \Rightarrow x^p = N \Rightarrow x^p - N = 0$

Here $f(x) = x^p - N$; $f(x_n) = x_n^p - N$

$f'(x) = px^{p-1}$; $f'(x_n) = px_n^{p-1}$

Since by Newton Raphson formula

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \Rightarrow x_{n+1} = x_n - \frac{(x_n^p - N)}{(px_n^{p-1})} \Rightarrow x_{n+1} = \frac{1}{px_n^{p-1}}\left[px_n^{p-1+1} - x_n^p + N\right]$

$x_{n+1} = \frac{1}{px_n^{p-1}}\left[(p-1)x_n^p + N\right] \Rightarrow x_{n+1} = \frac{1}{p}\left[\frac{(p-1)x_n^p + N}{x_n^{p-1}}\right]$ Required formula for pth root.

QUESTION: Obtain the cube root of 12 using Newton Raphson iteration.

SOLUTION: Consider $x = 12^{\frac{1}{3}} \Rightarrow x^3 = 12 \Rightarrow x^3 - 12 = 0$

Here $f(x) = x^3 - 12$ and $f'(x) = 3x^2$; $f''(x) = 6x$

x	0	1	2	3
f(x)	-12	-11	-4	15

Root lies between 2 and 3 and x_0=3

Since by Newton Raphson formula for pth root.

$$x_{n+1} = \frac{1}{p}\left[\frac{(p-1)x_n^p + N}{x_n^{p-1}}\right] \quad \Rightarrow x_{n+1} = \frac{1}{3}\left[\frac{(3-1)x_n^3 + 12}{x_n^{3-1}}\right] = \frac{1}{3}\left[\frac{2x_n^3 + 12}{x_n^2}\right]$$

Put n= 0 $\quad x_1 = \frac{1}{3}\left[\frac{2x_0^3 + 12}{x_0^2}\right] = \frac{1}{3}\left[\frac{2(3)^3 + 12}{(3)^2}\right] = 2.4444$

Similarly $\quad x_2 = 2.2990$, $x_3 = 2.2895$, $x_4 = 2.2894 \quad x_5 = 2.2894$
Hence $\quad \sqrt[3]{12} = 2.2894$

DARIVATION OF NEWTON RAPHSON METHOD FROM TAYLOR SERIES

Newton Raphson method can also be derived from Taylor series.

For the general function "f(x)" Taylor series is

$$f(x_{n+1}) = f(x_n) + f'(x_{n+1} - x_n) + \frac{f''(x_n)}{2!}(x_{n+1} - x_n)^2 + \cdots \ldots \ldots \ldots \ldots \ldots$$

As an approximation, taking only the first two terms of the R.H.S.

$$f(x_{n+1}) = f(x_n) + f'(x_{n+1} - x_n)$$

And we are seeking a point where f(x) =0

That is If we assume f (x_{n+1}) =0 $\Longrightarrow \quad f(x_n) + f'(x_n)(x_{n+1} - x_n) = 0$

This gives $\quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ This is the formula for Newton Raphson Method.

EXERCISE: :Solve by using Newton Raphson method

i. $x^3 - x - 1 = 0$ correct to four decimal places.
ii. $xe^x - 2 = 0$ correct to two decimal places.
iii. Obtain the real root of the equation $x^3 - 3x - 5 = 0$ after third iteration.
iv. $x^4 - x - 10 = 0$ correct to four decimal places.
v. $e^x Sinx = 1$
vi. x=Cosx

THE SOLUTION OF LINEAR SYSTEM OF EQUATIONS

A system of "m" linear equations in "n" unknowns "$\boldsymbol{x_1, x_2, x_3, \dots\dots\dots.., x_n}$" is a set of the equations of the form

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots\dots\dots\dots\dots a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots\dots\dots\dots\dots\dots a_{2n}x_n = b_2$$

……………………………………………………………

……………………………………………………………

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots\dots\dots\dots\dots\dots a_{mn}x_n = b_m$$

Where the coefficients "$\boldsymbol{a_{ik}}$" and "$\boldsymbol{b_i}$" are given numbers.

The system is said to be homogeneous if all the "$\boldsymbol{b_i}$" are zero. Otherwise it is said to be non-homogeneous.

SOLUTION OF LINEAR SYSTEM EQUATIONS: A solution of system is a set of numbers "$\boldsymbol{x_1, x_2, \dots., x_n}$" which satisfy all the "m" equations.

PIVOTING: Changing the order of equations is called pivoting.

We are interested in following types of Pivoting

1. PARTIAL PIVOTING
2. TOTAL PIVOTING

PARTIAL PIVOTING: In partial pivoting we interchange rows where pivotal element is zero.

In Partial Pivoting if the pivotal coefficient "$\boldsymbol{a_{ii}}$" happens to be zero or near to zero, the i^{th} column elements are searched for the numerically largest element. Let the j^{th} row (j>i) contains this element, then we interchange the "i^{th}" equation with the "j^{th}" equation and proceed for elimination. This process is continued whenever pivotal coefficients become zero during elimination.

TOTAL PIVOTING: In Full (complete, total) pivoting we interchange rows as well as column.

In Total Pivoting we look for an absolutely largest coefficient in the entire system and start the elimination with the corresponding variable, using this coefficient as the pivotal coefficient (may change row and column). Similarly, in the further steps. It is more complicated than Partial Pivoting. Partial Pivoting is preferred for hand calculation.

Why is Pivoting important?: Because Pivoting made the difference between non-sense and a perfect result.

PIVOTAL COEFFICIENT: For elimination methods (Guass's Elimination, Guass's Jordan) the coefficient of the first unknown in the first equation is called Pivotal Coefficient.

BACK SUBSTITUTION

The analogous algorithm for upper triangular system "Ax=b" of the form

$$\begin{pmatrix} a_{11} & a_{12} \dots\dots\dots a_{1n} \\ 0 & a_{22} \dots\dots\dots a_{2n} \\ \vdots & \vdots \qquad\qquad \vdots \\ 0 & 0 \quad \cdots\cdots \quad a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$ Is called **Back Substitution**.

The solution "x_i" is computed by $\boldsymbol{x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}}}$; $\boldsymbol{i = 1, 2, 3, \dots\dots\dots n}$

FORWARD SUBSTITUTION

The analogous algorithm for lower triangular system "Lx=b" of the form

$$\begin{pmatrix} l_{11} & 0 \quad \dots\dots 0 \\ l_{21} & l_{22} \dots\dots\dots 0 \\ \vdots & \vdots \qquad\qquad \vdots \\ l_{n1} & l_{n2} \cdots\cdots \quad l_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$ Is called **Forward Substitution**.

The solution "x_i" is computed by $\boldsymbol{x_i = \frac{b_i - \sum_{j=1}^{i-1} l_{ij}x_j}{l_{ii}}}$; $\boldsymbol{i = 1, 2, 3, \dots\dots\dots n}$

THINGS TO REMEMBER

Let the system $\boldsymbol{AX = B}$ is given

- If $\boldsymbol{B \neq 0}$ then system is called non homogenous system of linear equation.
- If $\boldsymbol{B = 0}$ then $\boldsymbol{AX = 0}$ then system is called homogenous system of linear equation.
- If the system $\boldsymbol{AX = B}$ has solution then this system is called consistent.
- If the system $\boldsymbol{AX = B}$ has no solution then this system is called inconsistent.

RANK OF A MATRIX

The rank of a matrix 'A' is equal to the number of non – zero rows in its echelon form or the order of $\boldsymbol{I_r}$ in the conical form of A.

KEEP IN MIND

- TYPE I: when number of equations is equal to the number of variables and the system $AX = B$ is non – homogeneous then unique solution of the system exists if matrix 'A' is non- singular after applying row operation.
- TYPE II: when number of equations is not equal (may be equal) to the number of variables and the system $AX = B$ is non – homogeneous then system has a solution if $rank\ A = rank A_b$
- TYPE III: a system of 'm' homogeneous linear equations $AX = 0$ in 'n' unknown has a non- trivial solution if $rank\ A < n$ where 'n' is number of columns of A.
- TYPE IV: if $rank\ A = rank A_b < number\ of\ unknown$ then infinite solution exists
- TYPE V: if $rank\ A \neq rank A_b$ then no solution exists

GAUSS ELIMINATION METHOD

ALGORITHM

- In the first stage, the given system of equations is reduced to an equivalent upper triangular form using elementary transformation.
- In the second stage, the upper triangular system is solved using back substitution procedure by which we obtain the solution in the order "$x_n, x_{n-1}, \dots \dots \dots \dots x_2, x_1$"

REMARK:

Gauss's Elimination method fails if any one of the Pivotal coefficient become zero. In such a situation, we rewrite the equation in a different order to avoid zero Pivotal coefficients.

QUESTION Solve the following system of equations using Elimination Method.

$$\mathbf{2x + 3y - z = 5}$$

$$\mathbf{4x + 4y - 3z = 3}$$

$$\mathbf{-2x + 3y - z = 1}$$

SOLUTION We can solve it by elimination of variables by making coefficients same.

$$2x + 3y - z = 5 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots (i)$$

$$4x + 4y - 3z = 3 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots (ii)$$

$$-2x + 3y - z = 1 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots (iii)$$

Multiply (i) by 2 and subtracted by (ii) $\quad 2y + z = 7 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (iv)$

Adding (i) and (iii) $\quad 6y - 2z = 6 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (v)$

Now eliminating "y" Multiply (iv) by 3 then subtract from (v) $\quad z = 3$

Using "z" in (iv) we get $\quad y = 2 \quad$ and Using "y", "z" in (i) we get $\quad x = 1$

Hence solution is $\quad x = 1, y = 2, z = 3$

QUESTION: Solve the following system of equations by Gauss's Elimination method with partial pivoting.

$$\boldsymbol{x + y + z = 7}$$

$$\boldsymbol{3x + 3y + 4z = 24}$$

$$\boldsymbol{2x + y + 3z = 16}$$

SOLUTION

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 24 \\ 16 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 3 & 3 & 4 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 24 \\ 7 \\ 16 \end{bmatrix} \quad \sim R_{12} \quad \Longrightarrow \begin{bmatrix} 1 & 1 & \frac{4}{3} \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 16 \end{bmatrix} \quad \sim \frac{1}{3}R_1$$

$$\Longrightarrow \begin{bmatrix} 1 & 1 & \frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 16 \end{bmatrix} \quad \sim R_2 - R_1 \quad \Longrightarrow \begin{bmatrix} 1 & 1 & \frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \\ 0 & -1 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 0 \end{bmatrix} \quad \sim R_3 - 2R_1$$

2nd row cannot be used as pivot row as a_{22} =0, So interchanging the 2nd and 3rd row we get

$$\begin{bmatrix} 1 & 1 & \frac{4}{3} \\ 0 & -1 & \frac{1}{3} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} \quad \sim R_{23}$$

Using back substitution

$$-\frac{1}{3}z = -1 \quad \Longrightarrow z = 3$$

$$-y + \frac{1}{3}z = 0 \quad \Longrightarrow y = 3 \quad \therefore z = 3$$

$$x + y + \frac{4}{3}z = 8 \quad \Longrightarrow x = 3 \quad \therefore y = 3 \;, \; z = 3$$

QUESTION: Solve the following system of equations using Gauss's Elimination Method with partial pivoting.

$$0x_1 + 4x_2 + 2x_3 + 8x_4 = 24$$

$$4x_1 + 10x_2 + 5x_3 + 4x_4 = 32$$

$$4x_1 + 5x_2 + 65x_3 + 2x_4 = 26$$

$$9x_1 + 4x_2 + 4x_3 + 0x_4 = 21$$

SOLUTION

$$\begin{bmatrix} 0 & 4 & 2 & 8 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 65 & 2 \\ 9 & 4 & 4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 24 \\ 32 \\ 26 \\ 21 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 4 & 4 & 0 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 65 & 2 \\ 0 & 4 & 2 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 21 \\ 32 \\ 26 \\ 24 \end{bmatrix} \quad \sim R_{14} \quad \Rightarrow \begin{bmatrix} 1 & 4/9 & 4/9 & 0 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 65 & 2 \\ 0 & 4 & 2 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 21 \\ 32 \\ 26 \\ 24 \end{bmatrix} \quad \sim \frac{1}{9}R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 4/9 & 4/9 & 0 \\ 0 & 4 & 2 & 8 \\ 4 & 5 & 65 & 2 \\ 4 & 10 & 5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 24 \\ 26 \\ 32 \end{bmatrix} \sim R_{24}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & 4 & 2 & 8 \\ 0 & \frac{29}{9} & \frac{85}{18} & 2 \\ 0 & \frac{74}{9} & \frac{29}{9} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 24 \\ 16.6668 \\ 22.668 \end{bmatrix} \quad \sim R_3 - 4R_1 \quad \text{and} \quad \sim R_4 - 4R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 4/9 & 4/9 & 0 \\ 0 & 1 & 1/2 & 2 \\ 0 & 29/9 & 85/18 & 2 \\ 0 & 74/9 & 29/9 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 6 \\ 16.6668 \\ 22.668 \end{bmatrix} \quad \sim \frac{1}{4}R_2$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & 1 & ½ & 2 \\ 0 & 0 & 3.111 & -4.444 \\ 0 & 0 & -0.889 & 16.444 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 6 \\ -2.6665 \\ -26.665 \end{bmatrix} \sim R_3 - \frac{29}{9}R_2 \; and \sim R_4 - \frac{74}{9}R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 4/9 & 4/9 & 0 \\ 0 & 1 & 1/2 & 2 \\ 0 & 0 & 1 & -1.428 \\ 0 & 0 & 0 & 15.175 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 6 \\ -0.857 \\ -27.427 \end{bmatrix} \qquad \sim \frac{R_3}{3.111} \; and \sim R_4 + 0.889R_3$$

$$\Rightarrow \quad 15.175x_4 = -27.427 \qquad \Rightarrow x_4 = -1.8074$$

$$\Rightarrow x_3 - 1.428x_4 = -0.857 \qquad \Rightarrow x_3 = -3.438 \qquad \therefore x_4 = -1.8074$$

$$\Rightarrow x_2 + \frac{1}{2}x_3 + 2x_4 = 6 \Rightarrow \; x_2 = 11.3338 \qquad \therefore x_4 = -1.8074 \;,\; x_3 = -3.438$$

$$\Rightarrow x_1 + \frac{4}{9}x_2 + \frac{4}{9}x_3 = 2.333 \Rightarrow \; x_1 = -1.1762 \quad \therefore x_2 = 11.3338 \;,\; x_3 = -3.438$$

Hence required $x_1 = -1.1762$, $x_2 = 11.3338$, $x_3 = -3.438$, $x_4 = -1.8074$

EXERCISE:

1) Solve by using Gauss's Elimination method

i. $2x + 3y - z = 5$
$4x + 4y - 3z = 3$
$-2x + 3y - z = 1$

ii. $x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 = 1$
$\frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3 = 0$
$\frac{1}{3}x_1 + \frac{1}{4}x_2 + \frac{1}{5}x_3 = 0$

iii. $4x_1 + x_2 + x_3 = 4$
$x_1 + 4x_2 - 2x_3 = 4$
$3x_1 + 2x_2 - 4x_3 = 6$

iv. $10x - 7y + 3z + 5w = 6$
$-6x + 8y - z - 4w = 5$
$3x + y + 4z + 11w = 2$
$5x - 9y - 2z + 4w = 7$

2) Solve by using Gauss's Elimination method with partial pivoting

i. $2.5x - 3y + 4.6z = -1.05$
$-3.5x + 2.6y + 1.5z = -14.46$
$-6.5x - 3.5y + 7.3z = -17.735$

ii. $x_1 + x_2 - 2x_3 = 3$
$4x_1 - 2x_2 + x_3 = 5$
$3x_1 - 1x_2 + 3x_3 = 8$

GAUSS JORDAN ELIMINATION METHOD

The method is based on the idea of reducing the given system of equations "$\boldsymbol{Ax = b}$" to a diagonal system of equations "$\boldsymbol{Ix = b}$" where "$\boldsymbol{I}$" is the identity matrix, using row operation. It is the verification of Gauss's Elimination Method.

ALGORITHM

1) Make the elements below the first pivot in the augmented matrix as zeros, using the elementary row transformation.
2) Secondly make the elements below and above the pivot as zeros using elementary row transformation.
3) Lastly divide each row by its pivot so that the final matrix is of the form

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \boldsymbol{d_1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \boldsymbol{d_2} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \boldsymbol{d_3} \end{bmatrix}$$

Then it is easy to get the solution of the system as $\boldsymbol{x_1 = d_1, x_2 = d_2, x_3 = d_3}$

Partial Pivoting can also be used in the solution. We may also make the pivot as "1" before performing the elimination.

ADVANTAGE/DISADVANTAGE

The Gauss's Jordan method looks very elegant as the solution is obtained directly. However, it is computationally more expensive than Gauss's Elimination. Hence we do not normally use this method for the solution of the system of equations.

The most important application of this method is to find inverse of a non-singular matrix.

<u>What is Gauss Jordan variation?</u>

In this method Zeroes are generated both below and above each pivot, by further subtractions. The final matrix is thus diagonal rather than triangular and back substitution is eliminated. The idea is attractive but it involves more computing than the original algorithm, so it is little used.

QUESTION: Solve the system of equations using Elimination method

$$\boldsymbol{x + 2y + z = 8}$$

$$\boldsymbol{2x + 3y + 4z = 20}$$

$$\boldsymbol{4x + 3y + 2z = 16}$$

ANSWER

$$\begin{bmatrix} 1 & 2 & 1 & \vdots & 8 \\ 2 & 3 & 4 & \vdots & 20 \\ 4 & 3 & 2 & & 16 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & & 8 \\ 0 & -1 & -2 & \vdots & 4 \\ 0 & -5 & -2 & & -16 \end{bmatrix} R_2 - R_1 \text{ and } R_3 - 4R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & & 8 \\ 0 & 1 & -2 & \vdots & -4 \\ 0 & 1 & -2/5 & & 16/5 \end{bmatrix} (-1)R_2 \text{ and } (-1/5)R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & & 8 \\ 0 & 1 & -2 & \vdots & -4 \\ 0 & 0 & 12/5 & & 36/5 \end{bmatrix} \quad R_3 - R_2 \Rightarrow \begin{bmatrix} 1 & 2 & 1 & & 8 \\ 0 & 1 & -2 & \vdots & -4 \\ 0 & 0 & 1 & & 3 \end{bmatrix} \quad (5/12)R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 & & 5 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & & 3 \end{bmatrix} \quad R_1 - R_3 \text{ and } R_2 + 2R_3 \Rightarrow \begin{bmatrix} 1 & 0 & 0 & & 1 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & & 3 \end{bmatrix} \quad R_1 - 2R_2$$

Hence solutions are $\quad x = 1, y = 2, z = 3$

EXERCISE: Solve by using Gauss's Jordan method.

i. $x + y + z = 7$
$3x + 3y + 4z = 24$
$2x + y + 3z = 16$

ii. $10x_1 + x_2 + x_3 = 12$
$x_1 + 10x_2 + x_3 = 12$
$x_1 + x_2 + 10x_3 = 12$

MATRIX INVERTION

A "$n \times n$" matrix "M" is said to be non-singular (or Invertible) if a "$n \times n$" matrix "M^{-1}" exists with "MM^{-1}" = "$M^{-1}M$" $= I$ then matrix "M^{-1}" is called the inverse of"M". A matrix without an inverse is called Singular (or Non-invertible)

MATRIX INVERSION THROUGH GUASS ELIMINATION

1. Place an identity matrix, whose order is same as given matrix.
2. Convert matrix in upper triangular form.
3. **Take largest value as Pivot.**
4. Using back substitution get the result.

NOTE: In order to increase the accuracy of the result, it is essential to employ Partial Pivoting. In the first column use absolutely largest coefficient as the pivotal coefficient (for this we have to interchange rows if necessary). Similarly, for the second column and vice versa.

MATRIX INVERSION THROUGH GUASS JORDAN ELIMINATION

1. Place an identity matrix, whose order is same as given matrix.
2. Convert matrix in upper triangular form.
3. **No need to take largest value as Pivot.**
4. Using back substitution get the result.

QUESTION : Find inverse using Guass Elimination Method $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{4} & \mathbf{3} & \mathbf{-1} \\ \mathbf{3} & \mathbf{5} & \mathbf{3} \end{bmatrix}$

ANSWER

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 \vdots & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 3 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 \vdots & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{bmatrix} \quad R_{12} \Rightarrow \begin{bmatrix} 1 & \frac{3}{4} & \frac{-1}{4} & 0 & \frac{1}{4} & 0 \\ 1 & 1 & 1 \vdots & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{bmatrix} \quad \frac{1}{4}R_1$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{3}{4} & \frac{-1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{5}{4} \vdots & 1 & -\frac{1}{4} & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{bmatrix} \quad R_2 - R_1 \Rightarrow \begin{bmatrix} 1 & \frac{3}{4} & \frac{-1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{5}{4} \vdots & 1 & -\frac{1}{4} & 0 \\ 0 & \frac{11}{4} & \frac{15}{4} & 0 & -\frac{3}{4} & 1 \end{bmatrix} \quad R_3 - 3R_1$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{3}{4} & \frac{-1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{11}{4} & \frac{15}{4} \vdots & 0 & -\frac{3}{4} & 1 \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 \end{bmatrix} \quad R_{23} \Rightarrow \begin{bmatrix} 1 & \frac{3}{4} & \frac{-1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{15}{11} \vdots & 0 & -\frac{3}{11} & \frac{4}{11} \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 \end{bmatrix} \quad \frac{4}{11}R_2$$

$$\Rightarrow\begin{bmatrix}1 & \frac{3}{4} & \frac{-1}{4} & \vdots & 0 & \frac{1}{4} & 0\\ 0 & 1 & \frac{15}{11} & \vdots & 0 & -\frac{3}{11} & \frac{4}{11}\\ 0 & 0 & \frac{10}{11} & \vdots & 1 & -\frac{2}{4} & -\frac{1}{11}\end{bmatrix} \quad R_3-\frac{1}{4}R_2 \Rightarrow \begin{bmatrix}1 & \frac{3}{4} & \frac{-1}{4} & \vdots & 0 & \frac{1}{4} & 0\\ 0 & 1 & \frac{15}{11} & \vdots & 0 & -\frac{3}{11} & \frac{4}{11}\\ 0 & 0 & 1 & \vdots & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10}\end{bmatrix} \qquad \frac{11}{10}R_3$$

$$\Rightarrow\begin{bmatrix}1 & \frac{3}{4} & 0 & \vdots & \frac{11}{40} & \frac{1}{5} & -\frac{1}{40}\\ 0 & 1 & 0 & \vdots & -\frac{3}{2} & 0 & \frac{1}{2}\\ 0 & 0 & 1 & \vdots & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10}\end{bmatrix} \quad R_1+\frac{1}{4}R_3 \text{ and } R_2-\frac{15}{11}R_3$$

$$\Rightarrow\begin{bmatrix}1 & 0 & 0 & \vdots & \frac{7}{5} & \frac{1}{5} & -\frac{2}{5}\\ 0 & 1 & 0 & \vdots & -\frac{3}{2} & 0 & \frac{1}{2}\\ 0 & 0 & 1 & \vdots & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10}\end{bmatrix} \quad R_1-\frac{3}{4}R_2 \quad \text{Hence} \quad A^{-1}=\begin{bmatrix}\frac{7}{5} & \frac{1}{5} & -\frac{2}{5}\\ -\frac{3}{2} & 0 & \frac{1}{2}\\ \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10}\end{bmatrix}$$

QUESTION: Find inverse using Gauss's Jordan Elimination Method $\boldsymbol{A}=\begin{bmatrix}1 & \mathbf{1} & \mathbf{1}\\ \mathbf{4} & \mathbf{3} & \mathbf{-1}\\ \mathbf{3} & \mathbf{5} & \mathbf{3}\end{bmatrix}$

ANSWER

$$\begin{bmatrix}1 & 1 & 1 & \vdots & 1 & 0 & 0\\ 4 & 3 & -1 & \vdots & 0 & 1 & 0\\ 3 & 5 & 3 & \vdots & 0 & 0 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 1 & 1 & \vdots & 1 & 0 & 0\\ 0 & -1 & -5 & \vdots & -4 & 1 & 0\\ 0 & 2 & 0 & \vdots & -3 & 0 & 1\end{bmatrix} \quad R_2-4R_1 \text{ and } R_3-3R_1$$

$$\Rightarrow\begin{bmatrix}1 & 1 & 1 & \vdots & 1 & 0 & 0\\ 0 & 1 & 5 & \vdots & 4 & -1 & 0\\ 0 & 2 & 0 & \vdots & -3 & 0 & 1\end{bmatrix} \quad -1R_2 \Rightarrow \begin{bmatrix}1 & 1 & 1 & \vdots & 1 & 0 & 0\\ 0 & 1 & 5 & \vdots & 4 & -1 & 0\\ 0 & 0 & -10 & \vdots & -11 & 2 & 1\end{bmatrix} \quad R_3-2R_2$$

$$\Rightarrow\begin{bmatrix}1 & 1 & 1 & \vdots & 1 & 0 & 0\\ 0 & 1 & 5 & \vdots & 4 & -1 & 0\\ 0 & 0 & 1 & \vdots & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10}\end{bmatrix} \quad -\frac{1}{10}R_3 \Rightarrow \begin{bmatrix}1 & 1 & 0 & \vdots & \frac{-1}{10} & \frac{1}{5} & \frac{1}{10}\\ 0 & 1 & 0 & \vdots & \frac{-3}{2} & 0 & \frac{1}{2}\\ 0 & 0 & 1 & \vdots & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10}\end{bmatrix} \quad R_1-R_3\,, R_2-5R_3$$

$$\Rightarrow\begin{bmatrix}1 & 0 & 0 & \vdots & \frac{7}{5} & \frac{1}{5} & -\frac{2}{5}\\ 0 & 1 & 0 & \vdots & -\frac{3}{2} & 0 & \frac{1}{2}\\ 0 & 0 & 1 & \vdots & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10}\end{bmatrix} \qquad R_1-R_2 \qquad \text{Hence} \qquad A^{-1}=\begin{bmatrix}\frac{7}{5} & \frac{1}{5} & -\frac{2}{5}\\ -\frac{3}{2} & 0 & \frac{1}{2}\\ \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10}\end{bmatrix}$$

QUESTION: Find A^{-1} if $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

SOLUTION: we first find the co-factor of the elements of A

$a_{11}= (-1)^2(2+1)= 3$

$a_{12}= (-1)^3(-1)= 1$

$a_{13}= (-1)^4(0-2) = -2$

$a_{21}= (-1)^3(0+2)= -2$

$a_{22}=(-1)^4(1-2)= -1$

$a_{23}= (-1)^5(-1-0) = 1$

$a_{31}= (-1)^4(0-4) = -4$

$a_{32}= (-1)^5(1-0) = -1$

$a_{33}= (-1)^6(2-0) = 2$

Thus $[A_{ij}]_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ -2 & -1 & 1 \\ -4 & -1 & 2 \end{bmatrix}$

$adjA = [A'_{ij}]_{3\times3} = \begin{bmatrix} 3 & -2 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix}$ $\quad$ *and* $\quad |A| = -1$

So $A^{-1} = \frac{1}{|A|}\ adjA = \begin{bmatrix} -3 & 2 & 4 \\ -1 & 1 & 1 \\ 2 & -1 & -2 \end{bmatrix}$ after putting the values.

EXERCISE

1) Find inverse using Gauss's Elimination Method $\quad A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$
2) Find inverse of the matrix $\quad A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 2 & 5 \\ 1 & -1 & 0 \end{bmatrix}$
3) Find inverse using Gauss's Jordan Method
   $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 2 & 4 & 7 \end{bmatrix}$

POSITIVE DEFINITE MATIRX: A matrix is positive definite matrix if it is symmetric and if $X^t AX > 0$ for every n – dimensional vector $X \neq 0$

QUESTION: Show that matrix $\begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}$ is positive definite.

SOLUTION: to show the matrix is positive definite we have to show that $X^t AX > 0$ for any matrix X

$$X^t AX = [x_1, x_2, x_3]\begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow X^t AX = [x_1, x_2, x_3]\begin{bmatrix} 4x_1 - x_2 + x_3 \\ -x_1 + 4.25x_2 + 2.75x_3 \\ x_1 + 2.75x_2 + 3.5x_3 \end{bmatrix}$$

$$\mathrm{X^tAX} = 4\mathrm{x_1}^2 - \mathrm{x_1x_2} + \mathrm{x_1x_3} - \mathrm{x_1x_2} + 4.25\mathrm{x_2}^2 + 2.75\mathrm{x_2x_3} + \mathrm{x_1x_3} + 2.75\mathrm{x_2x_3} + 3.5\mathrm{x_3}^2$$

$$\mathrm{X^tAX} = 4\mathrm{x_1}^2 - 2\mathrm{x_1x_2} + 2\mathrm{x_1x_3} + 4.25\mathrm{x_2}^2 + 7.56\mathrm{x_2x_3} + 3.5\mathrm{x_3}^2 > 0$$

This show that A is positive definite.

TRIDIAGONAL MATRIX: Those matrices in which mostly elements are zero while diagonal and adjacent elements are non-zero.

HESSENBERG MATRIX: Matrix in which either the upper or lower triangle is zero except for the elements adjacent to the main diagonal.
If the upper triangle has the zeroes, the matrix is the **Lower Heisenberg** and vice versa.

SPARSE: A coefficient matrix is said to be sparse if many of the matrix entries are known to be zero.

ORTHOGONAL MATRIX: A "$\boldsymbol{n \times n}$" matrix "M" is called orthogonal if

$$\boldsymbol{MM^t = I \text{ i.e. } M^t = M^{-1}}$$

PERMUTATION MATRIX: A "$n \times n$" matrix $P = [P_{ij}]$ is a permutation matrix obtained by rearranging the rows of the identity matrix "I_n". This gives a matrix with precisely one non-zero entry in each row and in each column and each non-zero entry is "1"

For example $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

CONVERGENT MATRIX: We call a "$n \times n$" matrix "M" convergent if $\lim_{k\to\infty}(M^k)_{ij} = 0$ for each $i, j = 0, 1, 2 \ldots n$; Consider $M = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \Longrightarrow M^k = \begin{bmatrix} (\frac{1}{2})^k & 0 \\ \frac{k}{2^{k+1}} & (\frac{1}{2})^k \end{bmatrix}$

Then $\lim_{k\to\infty}(\frac{1}{2})^k = 0$ and $\lim_{k\to\infty}\frac{k}{2^{k+1}} = 0 \Longrightarrow M$ is convergent.

LOWER TRIANGULATION MATRIX: A matrix having only zeros above the diagonal is called Lower Triangular matrix.

(OR) A "$n \times n$" matrix "L" is lower triangular if its entries satisfy $l_{ij} = 0 \;\; for\; i < j$

i.e. $\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$

UPPER TRIANGULATION MATRIX: A matrix having only zeros below the diagonal is called Upper Triangular matrix.

(OR) A "$n \times n$" matrix "U" is upper triangular if its entries satisfy $u_{ij} = 0 \;\; for\; i > j$

i.e. $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

CROUTS REDUCTION METHOD

In linear Algebra this method factorizes a matrix as the product of a Lower Triangular matrix and an Upper Triangular matrix.

Method also named as **Cholesky's reduction method, triangulation method, or LU-decomposition (Factorization)**

ALGORITHM: For a given system of equations $\sum_1^n \boldsymbol{x_i} = \boldsymbol{m}\,;\boldsymbol{m} \in \boldsymbol{Z}$

1. Construct the matrix "A"
2. Use "A=LU" (without pivoting) and "PA=LU" (with pivoting) where "P" is the pivoting matrix and find "$\boldsymbol{u_{ij}}, \boldsymbol{l_{ij}}$"
3. Use formula "AX=B" where "X" is the matrix of variables and "B" is the matrix of solution of equations.
4. Replace "AX=B" by "LUX=B" and then put "UX=Z" i.e. "LZ=B"
5. Find the values of "$\boldsymbol{Z_{i's}}$" then use "Z=UX" find "$\boldsymbol{X_{i's}}$" ; i=1, 2, 3, …….n

ADVANTAGE/LIMITATION (FAILURE)

1. Cholesky's method widely used in Numerical Solution of Partial Differential Equation.
2. Popular for Computer Programming.
3. This method fails if $\boldsymbol{a_{ii}} = \boldsymbol{0}$ in that case the system is Singular.

QUESTION: Solve the following system of equations using Crout's Reduction Method

$$\mathbf{5x_1 - 2x_2 + x_3 = 4}$$

$$\mathbf{7x_1 + x_2 - 5x_3 = 8}$$

$$\mathbf{3x_1 + 7x_2 + 4x_3 = 10}$$

ANSWER

Let $$A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

Step I….

$$[A] = [L][U]$$

$$\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

After multiplication on R.H.S $\Longrightarrow \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$

$\Longrightarrow l_{11} = 5,\ l_{21} = 7,\quad l_{31} = 3 \ \Longrightarrow l_{11}u_{12} = -2 \ \Longrightarrow 5u_{12} = -2 \ \Longrightarrow u_{12} = -2/5$

$\Longrightarrow l_{11}u_{13} = 1 \ \Longrightarrow 5u_{13} = 1 \ \Longrightarrow u_{13} = 1/5$

$\Longrightarrow l_{21}u_{12} + l_{22} = 1 \ \Longrightarrow 7(-2/5) + l_{22} = 1 \ \Longrightarrow l_{22} = 19/5$

$\Longrightarrow l_{31}u_{12} + l_{32} = 7 \ \Longrightarrow 3(-2/5) + l_{32} = 7 \ \Longrightarrow l_{32} = 41/5$

$\Longrightarrow l_{21}u_{13} + l_{22}u_{23} = -5 \ \Longrightarrow 7\left(\frac{1}{5}\right) + \left(\frac{19}{5}\right)u_{23} = -5 \ \Longrightarrow u_{23} = -32/19$

$\Longrightarrow l_{31}u_{13} + l_{32}u_{23} + l_{33} = 4 \ \Longrightarrow 3\left(\frac{1}{5}\right) + \left(\frac{41}{5}\right)\left(\frac{-32}{5}\right) + l_{33} = 4 \ \Longrightarrow l_{33} = 327/19$

Step II…. Put $[A][X] = [B] \ \Longrightarrow \ [L][U][X] = [B]$

Put [U] [X] = [Z] $[L][Z] = [B]$

$$\Longrightarrow \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix} \Longrightarrow \begin{bmatrix} 5 & 0 & 0 \\ 7 & 19/5 & 0 \\ 3 & 41/5 & 327/19 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

$\Longrightarrow 5z_1 = 4 \ \Longrightarrow z_1 = 4/5 \Longrightarrow 7z_1 + \frac{19}{5}z_2 = 8 \ \Longrightarrow 7\left(\frac{4}{5}\right) + \frac{19}{5}z_2 = 8 \Longrightarrow z_2 = 12/19$

$\Rightarrow 3z_1 + \frac{41}{5}z_2 + \frac{327}{19}z_3 = 10 \quad \Rightarrow \quad 3\left(\frac{4}{5}\right) + \frac{41}{5}\left(\frac{12}{19}\right) + \frac{327}{19}z_3 = 10 \quad \Rightarrow z_3 = 46/327$

Step III…. Since [U][X] = [Z]

$$\Rightarrow \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2/5 & 1/5 \\ 0 & 1 & -32/19 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 12/19 \\ 46/327 \end{bmatrix}$$

$\Rightarrow x_3 = 46/327 \Rightarrow x_2 - \frac{32}{19}x_3 = \frac{12}{19} \Rightarrow x_2 - \frac{32}{19}\left(\frac{46}{327}\right) = \frac{12}{19} \qquad \Rightarrow x_2 = 284/327$

$\Rightarrow x_1 - \left(\frac{2}{5}\right)x_2 + \frac{1}{5}x_3 = \frac{4}{5} \Rightarrow x_1 - \left(\frac{2}{5}\right)\left(\frac{284}{327}\right) + \frac{1}{5}\left(\frac{46}{327}\right) = \frac{4}{5} \qquad \Rightarrow x_1 = 366/327$

Hence required solutions are $\Rightarrow x_1 = 366/327, \qquad x_2 = 284/327, \qquad x_3 = 46/327$

1) Solve using Crout's Reduction method.
   i. $6x_1 - x_2 = 3$
      $-x_1 + 6x_2 - x_3 = 4$
      $-x_2 + 6x_3 = 3$
   ii. $x + y + z = 3$
      $2x - y + 3z = 16$
      $3x + y - z = -3$

2) Solve using Cholesky's Reduction method.
   $4x_1 - 3x_2 + 2x_3 = 11\ ,, 2x_1 + x_2 + 7x_3 = 2\ ,, 3x_1 - x_2 + 5x_3 = 8$

3) Using Crout's Reduction method decompose the matrix $A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$ in LU form and hence solve the system of equations
   $5x - 2y + z = 4\ ,, \quad 7x + y - 5z = 8\ ,, \quad 3x + 7y + 4z = 10$

► **EXAMPLE 2.9** Prove that $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ does not have an LU factorization.

The factorization must have the form

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} b & c \\ 0 & d \end{bmatrix} = \begin{bmatrix} b & c \\ ab & ac + d \end{bmatrix}.$$

Equating coefficients yields $b = 0$ and $ab = 1$, a contradiction. ◄

The fact that not all matrices have an LU factorization means that more work is required before we can declare the LU factorization a general Gaussian elimination algorithm. The related problem of swamping is described in the next section. In Section 2.4, the PA = LU factorization is introduced, which will overcome both problems.

DIAGONALLY DOMINANT SYSTEM:

Consider a square matrix "$\boldsymbol{A} = \{\boldsymbol{a_{ij}}\}$" then system is said to be Diagonally Dominant if $|\boldsymbol{a_{ii}}| \geq \sum_{j=1, j\neq i}^{n} |\boldsymbol{a_{ij}}| \quad ; \boldsymbol{i = 1, 2, 3, \ldots \ldots \ldots \ldots n}$

If we remove equality sign, then "A" is called strictly diagonally dominant and 'A' has the following properties

- 'A' is regular, invertible, its inverse exist and Ax = b has a unique solution.
- Ax = b can be solved by Gaussian Elimination without Pivoting.

For example $A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{bmatrix}$ *and* $B = \begin{bmatrix} 6 & 4 & -3 \\ 4 & -2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

Then non- symmetric matrix 'A' is strictly diagonally dominant because

$|7| > |2| + |0| \quad ; \quad |5| > |3| + |-1| \quad ; \quad |-6| = |6| > |0| + |5|$

But 'B' and 'A^t' are not strictly diagonally dominant (Check!)

NORM:

A norm measures the size of a matrix.

Let "$x \in R^n$ *or* $x \in R^{n\times n}$" *then* $\|x\|$ *satisfies*

- $\|x\| \geq 0$
- Iff x =0 then $\|x\| = 0$
- $\|ax\| = |a|\,\|x\|$ Where 'a' is constant.
- $\|x + y\| \leq \|x\| + \|y\|$ i.e. Triangular inequality

INFINITY NORM $\|\boldsymbol{X}\|_\infty$:

The infinity (maximum) norm of a matrix 'X' is

$\|X\|_\infty = maximum\ of\ absolut\ values\ of\ components\ of "X" = \max_{1\leq i\leq n} |x_i|$

Consider $X = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix}$

$\|X\|_\infty = maximum\ of\ absolute\ row\ sum = max \begin{bmatrix} 1+2+3 \\ 4+5+6 \\ 7+1+2 \end{bmatrix} = 15$

EUCLIDEAN NORM $\|X\|_2$:

The Euclidean norm for the matrix 'X' is

$$\|X\|_2 = [\sum_{i=1}^{n} x_i^2]^{\frac{1}{2}}$$

We name it Euclidean norm because it represents the usual notation of distance from the origin in case x is in $R = R^1, R^2 \; or \; R^3$

Take square of each element, add &then square root

Consider

$$X = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix}$$

$$\|X\|_2 = (1+4+9+16+25+36+49+1+4)^{1/2} = 12$$

Theorem:- If $\|\cdot\|$ is a vector norm on R^n, then

$$\|A\| = \max_{\|\vec{x}\|=1} \|A\vec{x}\| \rightarrow (1)$$

is a matrix norm.

Corollary:- For any vector $\vec{x} \neq 0$, matrix A, and any natural norm $\|\cdot\|$,

$$\|A\vec{x}\| \leq \|A\| \|\vec{x}\|$$

Proof:- For any vector $\vec{x} \neq 0$, the vector $\vec{x}/\|\vec{x}\|$ has length 1. Using the result (1), it can be written as

$$\|A(\frac{\vec{x}}{\|x\|})\| \leq \|A\| \rightarrow (2)$$

But $\|\vec{x}\|$ is a non-zero real number, which implies that

$$A(\frac{\vec{x}}{\|\vec{x}\|}) = \frac{1}{\|\vec{x}\|} A\vec{x}$$

$$\frac{1}{\|\vec{x}\|} \|A\vec{x}\| = \|\frac{1}{\|x\|} A\vec{x}\| = \|A(\frac{\vec{x}}{\|x\|})\|$$

$$\frac{1}{\|\vec{x}\|} \|A\vec{x}\| \leq \|A\| \quad (\text{from } (2))$$

$$\Rightarrow \|A\vec{x}\| \leq \|A\| \|\vec{x}\|$$

Similarly, $\|A\|_\infty = \max_{\|\vec{x}\|_\infty = 1} \|A\vec{x}\|_\infty$

$$\|A\|_2 = \max_{\|\vec{x}\|_2 = 1} \|A\vec{x}\|_2$$

$\|A\|_1 = \mathbf{max}_{\|x\|_1} |A\vec{x}|$

$\|x\|_1 = 1$

Note:-

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$$

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$$

$$\|A\|_2 = \left[\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2\right]^{1/2}$$

Example:- Find all three norms $\|\cdot\|_\infty$, $\|\cdot\|_1$ and $\|\cdot\|_2$ of the following matrix

Sol:-

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 5 & -1 & +1 \end{bmatrix}$$

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$$

$$= \max_{1\le i\le n} [1+2+1 \quad 3+1 \quad 5+1+1]$$

$$= \max_{1\le i\le n} [4 \quad 4 \quad 7]$$

$$\|A\|_\infty = 7$$

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$$

$$= \max_{1\le j\le n} [1+5 \quad 2+3+1 \quad 1+1+1]$$

$$= \max_{1\le j\le n} [6 \quad 6 \quad 3]$$

USEFUL DEFINATIONS

Let “x_a” be an approximate solution of the linear system “Ax =b” then

The residual is the vector $r = b - Ax_a$

The backward error is the norm of residual $\|r = b - Ax_a\|$

The forward error is $\|x - x_a\|_\infty$ and The relative backward error is $\frac{\|r\|_\infty}{\|b\|_\infty}$

The relative forward error is $\frac{\|x - x_a\|_\infty}{\|x\|_\infty}$

And error magnification factor is equals to $\frac{Relative\ forward\ error}{Relative\ backward\ error}$

CONDITION NUMBER: For a square matrix ‘A’ condition number is the maximum possible error magnification factor for solving Ax=b

Or The condition number of the "$\boldsymbol{n} \times \boldsymbol{n}$" matrix is defined as

$$\boldsymbol{k}(\boldsymbol{A}) = \boldsymbol{cond}\ (\boldsymbol{A}) = \|\boldsymbol{A}\|.\|\boldsymbol{A}^{-1}\|$$

Remember: Identity matrix has the lowest condition number.

THEOREM: Condition number is always at least one.

PROOF: Since $AA^{-1} = I \Rightarrow \|AA^{-1}\| = \|I\| \Rightarrow \|I\| \leq \|A\|\|A^{-1}\| \Rightarrow 1 \leq cond\ (A)$

Remember: $\|I\| = \max_{\|x\|=1}|Ix^{-1}| = 1 \Rightarrow \|I\| = 1$

UNSTABLE OR ILL CONDITION LINEAR SYSTEM

In practical application small change in the coefficient of the system of equations sometime gives the large change in the solutions of system. This type of system is called ill-condition linear system otherwise it is well-condition.

PROCEDURE (TEST, MEASURE OF CONDITION NUMBER)

- Find determinant. If system is ill condition, then determinant will be very small.
- Find condition number.
- If condition number is very large then system of condition is ill-condition rather it is well-condition. Also determinant will be small.

EXAMPLE: Consider $A = \begin{bmatrix} 2 & 1 \\ 2 & 0.1 \end{bmatrix} \Rightarrow |A| = 0.02 \; and \; \Rightarrow \|A\|_2 = 3.165$

$$\Rightarrow A^{-1} = \frac{adjA}{|A|} = \frac{1}{0.02}\begin{bmatrix} 1.01 & -1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 50.5 & -50 \\ -100 & 100 \end{bmatrix}$$

$$and \; \|A^{-1}\|_2 = ((50.5)^2 + (-50)^2 + (-100)^2 + (100)^2)^{\frac{1}{2}} = 158.273$$

Now condition number = $\|\boldsymbol{A}\|.\left\|\boldsymbol{A^{-1}}\right\|$= 500.93 (very large)

Since condition Number is very large therefore system will be ill-condition.

Example :- The solution vector corresponding to the system $x - y = 1$, $x - 1.0001y = 0$ is (100001, 1000,000) while the solution vector corresponding to the system $x - y = 1$, $x - 0.99999y = 0$ is (−99,999, −100,000)

So This system is unstable (ill-conditioned)

Example:-

Determine whether the system corresponding to the following matrix is ill-conditioned or well-conditioned.

$$A = \begin{bmatrix} 1 & 0.5 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/2 & 1/4 & 1/5 \end{bmatrix}$$

Sol: $\text{Cond}(A) = \|A\|_1 \cdot \|A^{-1}\|_1 \rightarrow (1)$

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1} |a_{ij}| \quad ; \quad n = 3$$

$$= \max_{1\le j\le n} [\,2 \quad 1.083 \quad 0.783\,]$$

$$\|A\|_1 = 2$$

Now find A^{-1} by Gauss Elimination method

$$A|I = \left[\begin{array}{ccc|ccc} 1 & 0.5 & 0.333 & 1 & 0 & 0 \\ 0.5 & 0.333 & 0.25 & 0 & 1 & 0 \\ 0.5 & 0.25 & 0.2 & 0 & 0 & 1 \end{array}\right]$$

$m_{2,1} = \frac{0.5}{1} = 0.5$, $m_{3,1} = \frac{0.5}{1} = 0.5$

$$\begin{array}{l} R_2 - m_{2,1}R_1 \\ R_3 - m_{3,1}R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0.5 & 0.333 & 1 & 0 & 0 \\ 0 & 0.083 & 0.0835 & -0.5 & 1 & 0 \\ 0 & 0 & 0.0335 & -0.5 & 0 & 1 \end{array}\right]$$

Now

$$\begin{bmatrix} 1 & 0.5 & 0.333 \\ 0 & 0.083 & 0.0835 \\ 0 & 0 & 0.0335 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -0.5 \\ -0.5 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} x_{11} + 0.5x_{21} + 0.333x_{31} &= 1 \rightarrow (1) \\ 0.083x_{21} + 0.0835x_{31} &= -0.5 \rightarrow (2) \\ 0.0335x_{31} &= -0.5 \rightarrow (3) \end{aligned}$$

from (3) $x_{31} = -14.93$

Put in (2) $0.083x_{21} + 0.0835(-14.93) = -0.5$

$$x_{21} = 8.996$$

put these in (1)

$$x_{11} + 0.5(8.996) + 0.333(-14.93) = 1$$

$$x_{11} = 1.47$$

$$\begin{bmatrix} 1 & 0.5 & 0.333 \\ 0 & 0.083 & 0.0835 \\ 0 & 0 & 0.0335 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_{12} + 0.5x_{22} + 0.333x_{32} = 0 \rightarrow (4)$$

$$0.083x_{22} + 0.0835x_{32} = 1 \rightarrow (5)$$

$$0.0335x_{32} = 0 \rightarrow (6)$$

from (6) $x_{32} = 0$

put in (5)

$$0.083x_{22} + 0 = 1$$

$$x_{22} = 12.05$$

put these in (4)

$$x_{12} + 0.5(12.05) + 0.333(0) = 0$$

$$x_{12} = -6.025$$

Now

$$\begin{bmatrix} 1 & 0.5 & 0.333 \\ 0 & 0.083 & 0.0835 \\ 0 & 0 & 0.0335 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad x_{13} + 0.5x_{23} + 0.333x_{33} = 0 \rightarrow (7)$$

$$0.083x_{23} + 0.0835x_{33} = 0 \rightarrow (8)$$

$$0.0335x_{33} = 1 \rightarrow (9)$$

From (9)

$$x_{33} = 29.85$$

put in (8)

$$0.083x_{23} + 0.0835(29.85) = 0$$

$$x_{23} = -30.03$$

put these in (7)

$-x_{13} + 0.5(-30.03) + 0.333(29.85) = 0$

$x_{13} = 5.075$

So,

$$A^{-1} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1.47 & -6.025 & 5.075 \\ 8.996 & 12.05 & -30.03 \\ -14.93 & 0 & 29.85 \end{bmatrix}$$

$$\|A^{-1}\|_1 = \max_{1\le j\le n} \sum_{i=1}^{3=n} |x_{ij}| \quad ; n=3$$

$$= \max_{1\le j\le n} [25.396 \quad 18.075 \quad 64.96]$$

$\|A^{-1}\|_1 = 64.96$

put in (1)

$Cond(A) = 2 \times 64.96$

$Cond(A) = 129.92$

So, the system is ill-conditioned.

Example:-

Find $Cond(A)$ and

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 5 & -1 & 1 \end{bmatrix}$$

Sol:-

$Cond(A) = \|A\|_\infty \|A^{-1}\|_\infty \rightarrow (1)$

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$$

$$= \max_{1\le i\le n} [4 \quad 4 \quad 7]$$

$\|A\|_\infty = 7$

Now, find A^{-1} and

$A^{-1} = adj.A / |A| \quad ; |A| \neq 0.$

$$|A| = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 5 & -1 & 1 \end{vmatrix}$$

$$= 1(3-1) - 2(0+5) - 1(0-15)$$

$$= 2 - 10 + 15$$

$$|A| = 7 \neq 0$$

$$\text{adj}\, A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\text{adj}A = \begin{bmatrix} 1\begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix} & -1\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} & 1\begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix} \\ -1\begin{vmatrix} 0 & -1 \\ 5 & 1 \end{vmatrix} & 1\begin{vmatrix} 1 & -1 \\ 5 & 1 \end{vmatrix} & -1\begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} \\ 1\begin{vmatrix} 0 & 3 \\ 5 & -1 \end{vmatrix} & -1\begin{vmatrix} 1 & 2 \\ 5 & -1 \end{vmatrix} & 1\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} \end{bmatrix}$$

$$\text{adj}A = \begin{bmatrix} 1(3-1) & -1(2-1) & 1(-2+3) \\ -1(0+5) & 1(1+5) & -1(-1+0) \\ 1(0-15) & -1(-1-10) & 1(3-0) \end{bmatrix}$$

$$\text{adj}A = \begin{bmatrix} 2 & -1 & 1 \\ -5 & 6 & 1 \\ -15 & 11 & 3 \end{bmatrix}$$

Now,

$$A^{-1} = \frac{\text{adj}A}{|A|} \quad \text{if } |A| \neq 0$$

$$A^{-1} = \begin{bmatrix} 2 & -1 & 1 \\ -5 & 6 & 1 \\ -15 & 11 & 3 \end{bmatrix} \frac{1}{7}$$

$$A^{-1} = \begin{bmatrix} 2/7 & -1/7 & 1/7 \\ -5/7 & 6/7 & 1/7 \\ -15/7 & 11/7 & 3/7 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0.286 & -0.143 & 0.143 \\ -0.714 & 0.857 & 0.143 \\ -2.143 & 1.572 & 0.429 \end{bmatrix}$$

$$\|A^{-1}\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |x_{ij}| \quad ; n=3$$

$$= \max_{1\le i\le n} [0.572 \quad 1.714 \quad 4.144]$$

$$\|A^{-1}\|_\infty = 4.144$$

Put in (1)

$$Cond(A) = 7 \times 4.114$$

$$Cond(A) = 29.01$$

So, the system is ill-conditioned or unstable because Cond (A) is greater than '1'.

Condition number:

The condition number of the non-singular matrix A relative to a norm $\|.\|$ is $K(A) = Cond(A) = \|A\|\|A^{-1}\|$

Note:- Ill-conditioning of a matrix can usually be expected when $|A|$ is small in the system $A\vec{x} = \vec{b}$

Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 1.01 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 \\ 2 & 1.01 \end{vmatrix} = 2.02 - 2$$

$$|A| = 0.02 \neq 0$$

$$A^{-1} = \text{adj}A / |A|$$

$$\text{adj }A = \begin{bmatrix} 1.01 & -1 \\ -2 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 50.5 & -50 \\ -100 & 100 \end{bmatrix}$$

Now $\|A^{-1}\|_2 = \left[\sum_{i=1}^{n}\sum_{j=1}^{n} |x_{ij}|^2\right]^{1/2}$

for A^{-1} $= [25050.25]^{1/2}$

$$\|A^{-1}\|_2 = 158.3$$

$$\|A\|_2 = [10.0201]^{1/2}$$

$$\|A\|_2 = 3.165$$

Now

$$\text{Cond}(A) = \|A\|_2 \|A^{-1}\|_2$$

$$= 3.165 \times 158.3$$

$$\text{Cond}(A) = 501.0$$

Since, the condition number is very large, so the system corresponding to the matrix is ill-conditioned.

Ill Conditioning Example (Just Read)

Here is a simple example of ill conditioning. Suppose that $Ax = b$ is supposed to be

$2x+6y= 8$ *and* $2x+6.00001y = 8.00001$

The actual solution is $x = 1$, $y = 1$. Suppose further that due to representation error, the system on the machine is changed slightly to

$2x+6y= 8$ and $2x +5.99999y= 8.00002$

The solution to this system is $x = 10$, $y = -2$, so you think the answer is $(10,-2)$. When you check the answer by plugging these values into the actual system, you get

$2(10) + 6(-2)=8$ and $2(10) + 5.99999(-2) =7.99998$

This seems to be acceptable, but of course $(10,-2)$ is very far from the actual solution $(1,1)$. This indicates that the system is badly ill conditioned.

Here are some things to consider if you have an ill conditioned system:

- To identify if the matrix is ill conditioned, you can try 2 things. First, compute cond(A). This is relatively expensive and sometimes hard to interpret because the value may be in an intermediate range. Second, you can introduce deliberate "representation errors" by slightly perturbing one or more elements in A. Call the new matrix A^0, and solve $A^0x^0 = b$. If $x \approx x^0$, then there is probably no ill conditioning. The danger here is that you might be unlucky, and chose the wrong element to perturb. But if you try this several times with different elements and all the solutions are about the same, then you have confidence that the matrix is well conditioned.

EXAMPLE (Just Read)

If the system really is ill conditioned, there is no simple fix. Consider using Singular Value Decomposition (SVD Ill-Conditioned Matrices

Consider systems $x + y = 2$ THEN $x + 1.001y = 2$ $x + 1.001y = 2.001$
The system on the left has solution $x = 2$, $y = 0$ while the one on the right has solution $x = 1$, $y = 1$. The coefficient matrix is called ill-conditioned because a small change in the constant coefficients results in a large change in the solution. A condition number, defined in more advanced courses, is used to measure the degree of ill-conditioning of a matrix (≈ 4004 for the above).

In the presence of rounding errors, ill-conditioned systems are inherently difficult to handle. When solving systems where round-off errors occur, one must avoid ill-conditioned systems whenever possible; this means that the usual row reduction algorithm must be modified.

Consider the system: $.001x + y = 1$ AND $x + y = 2$

We see that the solution is $x = 1000/999 \approx 1$, $y = 998/999 \approx 1$ which does not change much if the coefficients are altered slightly (condition number ≈ 4).

The usual row reduction algorithm, however, gives an ill-conditioned system. Adding a multiple of the first to the second row gives the system on the left below, then dividing by −999 and rounding to 3 places on 998/999 = .99899 ≈ 1.00 gives the system on the right:

$$\begin{aligned} .001x + y &= 1 \\ -999y &= -998 \end{aligned} \qquad \begin{aligned} .001x + y &= 1 \\ y &= 1.00 \end{aligned}$$

The solution for the last system is $x = 0, y = 1$ which is wildly inaccurate (and the condition number is ≈ 2002).

This problem can be avoided using partial pivoting. Instead of pivoting on the first non-zero element, pivot on the largest pivot (in absolute value) among those available in the column.

In the example above, pivot on the x, which will require a permute first:

$$\begin{aligned} x + y &= 2 \\ .001x + y &= 1 \end{aligned} \qquad \begin{aligned} x + y &= 2 \\ .999y &= .998 \end{aligned} \qquad \begin{aligned} x + y &= 2 \\ y &= 1.00 \end{aligned}$$

where the third system is the one obtained after rounding. The solution is a fairly accurate $x = 1.00, y = 1.00$ (and the condition number is 4).

7.5 Error Bounds and Iterative Refinement

It seems intuitively reasonable that if $\tilde{\mathbf{x}}$ is an approximation to the solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ and the residual vector $\mathbf{r} = \mathbf{b} - A\tilde{\mathbf{x}}$ has the property that $\|\mathbf{r}\|$ is small, then $\|\mathbf{x} - \tilde{\mathbf{x}}\|$ would be small as well. This is often the case, but certain systems, which occur frequently in practice, fail to have this property.

mple 1 The linear system $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3.0001 \end{bmatrix}$$

has the unique solution $\mathbf{x} = (1, 1)^t$. Determine the residual vector for the poor approximation $\tilde{\mathbf{x}} = (3, -0.0001)^t$.

Solution We have

$$\mathbf{r} = \mathbf{b} - A\tilde{\mathbf{x}} = \begin{bmatrix} 3 \\ 3.0001 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -0.0001 \end{bmatrix} = \begin{bmatrix} 0.0002 \\ 0 \end{bmatrix},$$

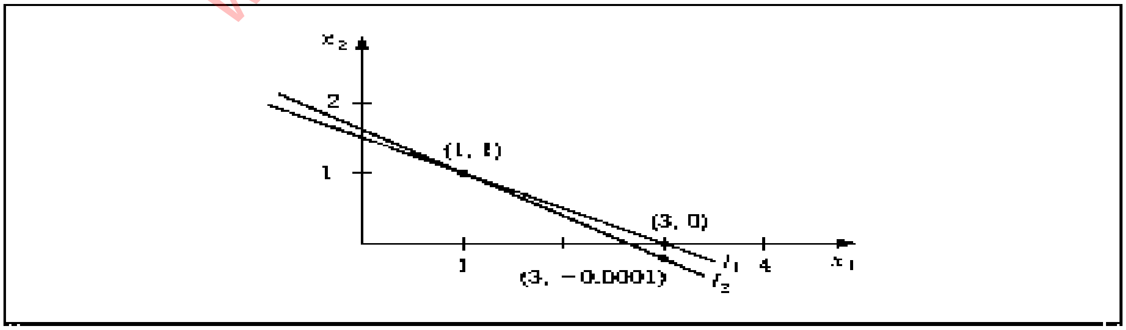
so $\|\mathbf{r}\|_\infty = 0.0002$. Although the norm of the residual vector is small, the approximation $\tilde{\mathbf{x}} = (3, -0.0001)^t$ is obviously quite poor; in fact, $\|\mathbf{x} - \tilde{\mathbf{x}}\|_\infty = 2$. ■

The difficulty in Example 1 is explained quite simply by noting that the solution to the system represents the intersection of the lines

$$l_1: \quad x_1 + 2x_2 = 3 \quad \text{and} \quad l_2: \quad 1.0001x_1 + 2x_2 = 3.0001.$$

The point $(3, -0.0001)$ lies on l_2, and the lines are nearly parallel. This implies that $(3, -0.0001)$ also lies close to l_1, even though it differs significantly from the solution of the system, given by the intersection point $(1, 1)$. (See Figure 7.7.)

ure 7.7



Example 1 was clearly constructed to show the difficulties that can—and, in fact, do—arise. Had the lines not been nearly coincident, we would expect a small residual vector to imply an accurate approximation.

In the general situation, we cannot rely on the geometry of the system to give an indication of when problems might occur. We can, however, obtain this information by considering the norms of the matrix A and its inverse.

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The approximation for the t-digit condition number $K(A)$ comes from consideration of the linear system

$$A\mathbf{y} = \mathbf{r}.$$

The solution to this system can be readily approximated because the multipliers for the Gaussian elimination method have already been calculated. So A can be factored in the form P^tLU as described in Section 5 of Chapter 6. In fact $\tilde{\mathbf{y}}$, the approximate solution of $A\mathbf{y} = \mathbf{r}$, satisfies

$$\tilde{\mathbf{y}} \approx A^{-1}\mathbf{r} = A^{-1}(\mathbf{b} - A\tilde{\mathbf{x}}) = A^{-1}\mathbf{b} - A^{-1}A\tilde{\mathbf{x}} = \mathbf{x} - \tilde{\mathbf{x}}; \tag{7.22}$$

and

$$\mathbf{x} \approx \tilde{\mathbf{x}} + \tilde{\mathbf{y}}.$$

So $\tilde{\mathbf{y}}$ is an estimate of the error produced when $\tilde{\mathbf{x}}$ approximates the solution $\mathbf{x}$ to the original system. Equations (7.21) and (7.22) imply that

$$\|\tilde{\mathbf{y}}\| \approx \|\mathbf{x} - \tilde{\mathbf{x}}\| = \|A^{-1}\mathbf{r}\| \le \|A^{-1}\| \cdot \|\mathbf{r}\| \approx \|A^{-1}\| \left(10^{-t}\|A\| \cdot \|\tilde{\mathbf{x}}\|\right) = 10^{-t}\|\tilde{\mathbf{x}}\| K(A).$$

This gives an approximation for the condition number involved with solving the system $A\mathbf{x} = \mathbf{b}$ using Gaussian elimination and the t-digit type of arithmetic just described:

$$K(A) \approx \frac{\|\tilde{\mathbf{y}}\|}{\|\tilde{\mathbf{x}}\|} 10^t. \tag{7.23}$$

Iterative Refinement

In Eq. (7.22), we used the estimate $\tilde{\mathbf{y}} \approx \mathbf{x} - \tilde{\mathbf{x}}$, where $\tilde{\mathbf{y}}$ is the approximate solution to the system $A\mathbf{y} = \mathbf{r}$. In general, $\tilde{\mathbf{x}} + \tilde{\mathbf{y}}$ is a more accurate approximation to the solution of the linear system $A\mathbf{x} = \mathbf{b}$ than the original approximation $\tilde{\mathbf{x}}$. The method using this assumption is called **iterative refinement**, or *iterative improvement*, and consists of performing iterations on the system whose right-hand side is the residual vector for successive approximations until satisfactory accuracy results.

If the process is applied using t-digit arithmetic and if $K_\infty(A) \approx 10^q$, then after k iterations of iterative refinement the solution has approximately the smaller of t and $k(t-q)$ correct digits. If the system is well-conditioned, one or two iterations will indicate that the solution is accurate. There is the possibility of significant improvement on ill-conditioned systems unless the matrix A is so ill-conditioned that $K_\infty(A) > 10^t$. In that situation, increased precision should be used for the calculations.

Iterative Refinement

To approximate the solution to the linear system $A\mathbf{x} = \mathbf{b}$:

INPUT the number of equations and unknowns n; the entries a_{ij}, $1 \leq i, j \leq n$ of the matrix A; the entries b_i, $1 \leq i \leq n$ of $\mathbf{b}$; the maximum number of iterations N; tolerance TOL; number of digits of precision t.

OUTPUT the approximation $\mathbf{xx} = (xx_i, \ldots, xx_n)^t$ or a message that the number of iterations was exceeded, and an approximation $COND$ to $K_\infty(A)$.

Step 0 Solve the system $A\mathbf{x} = \mathbf{b}$ for $x_1, \ldots, x_n$ by Gaussian elimination saving the multipliers m_{ji}, $j = i+1, i+2, \ldots, n$, $i = 1, 2, \ldots, n-1$ and noting row interchanges.

Step 1 Set $k = 1$.

Step 2 While ($k \leq N$) do Steps 3–9.

Step 3 For $i = 1, 2, \ldots, n$ (*Calculate* $\mathbf{r}$.)

$$\text{set } r_i = b_i - \sum_{j=1}^{n} a_{ij} x_j.$$

(*Perform the computations in double-precision arithmetic.*)

Step 4 Solve the linear system $A\mathbf{y} = \mathbf{r}$ by using Gaussian elimination in the same order as in Step 0.

Step 5 For $i = 1, \ldots, n$ set $xx_i = x_i + y_i$.

Step 6 If $k = 1$ then set $COND = \dfrac{\|\mathbf{y}\|_\infty}{\|\mathbf{xx}\|_\infty} 10^t$.

Step 7 If $\|\mathbf{x} - \mathbf{xx}\|_\infty < TOL$ then OUTPUT ($\mathbf{xx}$);
OUTPUT ($COND$);
(*The procedure was successful.*)
STOP.

Step 8 Set $k = k + 1$.

Step 9 For $i = 1, \ldots, n$ set $x_i = xx_i$.

Step 10 OUTPUT ('Maximum number of iterations exceeded');
OUTPUT ($COND$);
(*The procedure was unsuccessful.*)
STOP. ■

If t-digit arithmetic is used, a recommended stopping procedure in Step 7 is to iterate until $|y_i^{(k)}| \leq 10^{-t}$, for each $i = 1, 2, \ldots, n$.

JACOBI'S METHOD

Method is an **iterative method or simultaneous displacement method**.

We want to solve 'Ax = b' where "$\boldsymbol{A} \in \boldsymbol{R}^{n\times n}$" and 'n' is very large, 'A' is Sparse (with a large percent of zero entries) as well as 'A' is structured (i.e. the product 'Ax' can be computed efficiently). For this purpose, we can easily use Jacoby's.

ALGORITHM: We want to solve Ax=b writes it out

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots \dots \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots \dots \dots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad + \cdots \dots \dots \qquad = \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots \dots \dots + a_{nn}x_n = b_n \end{cases}$$

Rewrite it in another way

$$\begin{cases} x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots \dots \dots \dots - a_{1n}x_n) \\ x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots \dots \dots \dots - a_{2n}x_n) \\ \vdots \qquad \vdots \qquad \dots \dots \dots \qquad \vdots \\ x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots \dots \dots \dots - a_{n(n-1)}x_{n-1}) \end{cases}$$

Or in compact form $x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1, j\neq i}^{n} a_{ij}x_j\right) \quad i = 1, 2, 3 \dots \dots \dots n$

This gives the Jacoby's iteration.

Then Choose a start point (initial guess) i.e. $x^0 = (0, 0, 0)$

Apply $X^{k+1} = BX^k + C$ where $C_{ij} = \frac{b_i}{a_{ij}}$ and B can be defined as

$$B_{ij} = \begin{cases} -\frac{a_{ij}}{a_{ii}} & i \neq j \\ 0 & i = j \end{cases}$$

STOP CRITERIA

X^k close enough to X^{k-1} for example $\|X^k - X^{k-1}\| \leq \epsilon$ for certain vector norms.

Residual $r^k = Ax^k - b$ is small for example $\|r^k\| \leq \epsilon$

CONVERGENC CRITERIA: Sufficient condition for the convergence of Jacobi's is

$\|X\|_2 < 1 \quad or \quad |a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}| \qquad i = 1, 2, \dots \dots \dots \dots n$

Jacobi method also called method of simultaneous displacement why?

Because no element of $\boldsymbol{x_i^{k+1}}$ is in this iteration until every element is computed.

KEEP IN MIND

- Jacobi method is valid only when all "$a_{i's}$" are non-zeroes. (OR) the elements can rearrange for measuring the system according to condition. It is only possible if [A] is invertible i.e. inverse of 'A' exist.
- For fast convergence system should be diagonally dominant.
- Must make two vectors for the computation "$\boldsymbol{X^k}$" and "$\boldsymbol{X^{k+1}}$"
- System (method) is important for parallel computing.

QUESTION: Find the solution of the system of equation using Jacobi iterative method for the first five iterations.

$$\mathbf{83x + 11y - 4z = 95} \quad \dots\dots\dots\dots\dots\dots\dots\dots..(\boldsymbol{i})$$

$$\mathbf{3x + 52y + 13z = 104} \quad \dots\dots\dots\dots\dots\dots\dots\dots..(\boldsymbol{ii})$$

$$\mathbf{3x + 8y + 29z = 71} \quad \dots\dots..\dots\dots\dots\dots\dots\dots.(\boldsymbol{iii})$$

ANSWER

$$(i) \Longrightarrow \qquad x = \frac{95}{83} - \frac{11}{83}y + \frac{4}{83}z$$

$$(ii) \Longrightarrow \qquad y = \frac{104}{52} - \frac{7}{52}x - \frac{13}{52}z$$

$$(iii) \Longrightarrow \qquad z = \frac{71}{29} - \frac{8}{29}y - \frac{3}{29}x$$

Taking initial guess as (0, 0, 0) and using formula $X^{k+1} = BX^k + C$

Put k = 0 for first iteration

$$x^{(1)} = \frac{95}{83} - \frac{11}{83}(0) + \frac{4}{83}(0) = \frac{95}{83} = 1.1446$$

$$y^{(1)} = \frac{104}{52} - \frac{7}{52}(0) - \frac{13}{52}(0) = \frac{104}{52} = 2$$

$$z^{(1)} = \frac{71}{29} - \frac{8}{29}(0) - \frac{3}{29}(0) = \frac{71}{29} = 2.4483$$

$$\Longrightarrow \qquad \left(x^{(1)},\ y^{(1)},\ z^{(1)}\right) = (1.1446, \quad 2,\ 2.4483)$$

Put k = 1 for second iteration $x^{(2)} = \frac{95}{83} - \frac{11}{83}(2) + \frac{4}{83}(2.4483) = 0.9976$

$$y^{(2)} = \frac{104}{52} - \frac{7}{52}(1.4466) - \frac{13}{52}(2.4483) = 1.2339$$

$$z^{(2)} = \frac{71}{29} - \frac{8}{29}(2) - \frac{3}{29}(1.1446) = \frac{71}{29} = 1.7781$$

$$\Longrightarrow \qquad \left(x^{(2)},\ y^{(2)},\ z^{(2)}\right) = (0.9976,\quad 1.2339,\ 1.7781)$$

Put k = 2 for third iteration

$$x^{(3)} = \frac{95}{83} - \frac{11}{83}(1.2339) + \frac{4}{83}(1.7781) = 1.0668$$

$$y^{(3)} = \frac{104}{52} - \frac{7}{52}(0.9976) - \frac{13}{52}(1.7781) = 1.4212$$

$$z^{(3)} = \frac{71}{29} - \frac{8}{29}(1.2339) - \frac{3}{29}(0.9976) = 2.0046$$

$$\Longrightarrow \qquad \left(x^{(3)},\ y^{(3)},\ z^{(3)}\right) = (1.0668,\quad 1.4212,\ 2.0046)$$

Put k = 3 for fourth iteration

$$x^{(4)} = \frac{95}{83} - \frac{11}{83}(1.4212) + \frac{4}{83}(2.0046) = 1.0529$$

$$y^{(4)} = \frac{104}{52} - \frac{7}{52}(1.0668) - \frac{13}{52}(2.0046) = 1.3553$$

$$z^{(4)} = \frac{71}{29} - \frac{8}{29}(1.4212) - \frac{3}{29}(1.0668) = \frac{71}{29} = 1.9451$$

$$\Longrightarrow \qquad \left(x^{(4)},\ y^{(4)},\ z^{(4)}\right) = (1.0529,\ 1.3553,\ 1.9451)$$

Put k = 4 for fifth iteration

$$x^{(5)} = \frac{95}{83} - \frac{11}{83}(1.3551) + \frac{4}{83}(1.9451) = 1.0587$$

$$y^{(5)} = \frac{104}{52} - \frac{7}{52}(1.0529) - \frac{13}{52}(1.9451) = 1.3726$$

$$z^{(5)} = \frac{71}{29} - \frac{8}{29}(1.3553) - \frac{3}{29}(1.0529) = 1.9655$$

$$\Longrightarrow \qquad \left(x^{(5)},\ \ y^{(5)},\ z^{(5)}\right) = (1.0587,\ 1.3726,\ 1.9655)$$

GAUSS SEIDEL ITERATION METHOD

Gauss's Seidel method is an improvement of Jacobi's method. This is also known as method of successive displacement.

ALGORITHM:

In this method we can get the value of "x_1" from first equation and we get the value of "x_2" by using "x_1" in second equation and we get "x_3" by using "x_1" and "x_2" in third equation and so on.

ABOUT THE ALGORITHM

- Need only one vector for both "x^k" and "x^{k+1}" save memory space.
- Not good for parallel computing.
- Converge a bit faster than Jacobi's.

How Jacobi method is accelerated to get Gauss Seidel method for solving system of Linear Equations?

In Jacobi method the (r+1)th approximation to the system $\sum_{j=1, j\neq i}^{n} a_{ij}x_j = b_i$ is given by $x_i^{r+1} = \frac{b_i}{a_{ii}} - \sum_{j=1, j\neq i}^{n} \frac{a_{ij}}{a_{ii}} x_j^r \; ; r, j = 1, 2, 3, \ldots \ldots \ldots \ldots n$ from which we can observe that no element of x_i^{r+1} replaces x_i^r entirely for next cycle of computations. However, this is done in Gauss Seidel method. Hence called method of Successive displacement.

QUESTION:

Find the solutions of the following system of equations using Gauss Seidel method and perform the first five iterations.

$$x_1 - \frac{1}{4}x_2 - \frac{1}{4}x_3 = \frac{1}{2}$$

$$-\frac{1}{4}x_1 + x_2 - \frac{1}{4}x_4 = \frac{1}{2}$$

$$-\frac{1}{4}x_1 + x_3 - \frac{1}{4}x_4 = \frac{1}{4}$$

$$-\frac{1}{4}x_2 - \frac{1}{4}x_3 + x_4 = \frac{1}{4}$$

ANSWER

$$x_1 = 0.5 + 0.25x_2 + 0.25x_3$$

$$x_2 = 0.5 + 0.25x_1 + 0.25x_4$$

$$x_3 = 0.25 + 0.25x_1 + 0.25x_4$$

$$x_4 = 0.25 + 0.25x_2 + 0.25x_3$$

For first iteration using $(0, 0, 0, 0)$ we get

$$x_1^{(1)} = 0.5 + 0.25(0) + 0.25(0) = 0.5$$

$$x_2^{(1)} = 0.5 + 0.25(0.5) + 0.25(0) = 0.625$$

$$x_3^{(1)} = 0.25 + 0.25(0.5) + 0.25(0) = 0.375$$

$$x_4^{(1)} = 0.25 + 0.25(0.625) + 0.25(0.375) = 0.5$$

For second iteration using $(0.5, 0.625, 0.375, 0.5)$ we get

$$x_1^{(2)} = 0.5 + 0.25(0.625) + 0.25(0.375) = 0.75$$

$$x_2^{(2)} = 0.5 + 0.25(0.5) + 0.25(0.5) = 0.8125$$

$$x_3^{(2)} = 0.25 + 0.25(0.5) + 0.25(0.5) = 0.5625$$

$$x_4^{(2)} = 0.25 + 0.25(0.625) + 0.25(0.375) = 0.59375$$

For third iteration using $(0.75, 0.8125, 0.5625, 0.59375)$ we get

$$x_1^{(3)} = 0.5 + 0.25(0.8125) + 0.25(0.5625) = 0.84375$$

$$x_2^{(3)} = 0.5 + 0.25(0.75) + 0.25(0.59375) = 0.85938$$

$$x_3^{(3)} = 0.25 + 0.25(0.75) + 0.25(0.59375) = 0.60938$$

$$x_4^{(3)} = 0.25 + 0.25(0.8125) + 0.25(0.5625) = 0.61719$$

For fourth iteration using $(0.84375, 0.85938, 0.60938, 0.61719)$ we get

$$x_1^{(4)} = 0.5 + 0.25(0.85938) + 0.25(0.60938) = 0.86719$$

$$x_2^{(4)} = 0.5 + 0.25(0.84375) + 0.25(0.61719) = 0.87110$$

$$x_3^{(4)} = 0.25 + 0.25(0.84375) + 0.25(0.61719) = 0.62110$$

$$x_4^{(4)} = 0.25 + 0.25(0.85938) + 0.25(0.60938) = 0.62305$$

For fifth iteration using $(0.86719, 0.87110, 0.62110, 0.62305)$ we get

$$x_1^{(5)} = 0.5 + 0.25(0.87110) + 0.25(0.62110) = 0.87305$$

$$x_2^{(5)} = 0.5 + 0.25(0.86719) + 0.25(0.62305) = 0.87402$$

$$x_3^{(5)} = 0.25 + 0.25(0.86719) + 0.25(0.62305) = 0.62402$$

$$x_4^{(5)} = 0.25 + 0.25(0.87110) + 0.25(0.62110) = 0.62451$$

1) Solve using Gauss's Seidel method.

   i. $2\boldsymbol{x_1} - \boldsymbol{x_2} = 7$
   $-\boldsymbol{x_1} + \boldsymbol{x_2} - \boldsymbol{x_3} = 1$
   $-\boldsymbol{x_2} + 2\boldsymbol{x_3} = 1$ Perform first five iterations. **Ans:** $[6,5,3]^T = \begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix}$

   ii. $20x + y - 2z = 17$
   $3x + 20y - z = -18$
   $2x - 3y + 20z = 25$ Perform first three iterations.

7.4 SOR

SOR (Successive Over Relaxation) is a more general iterative method. It is based on Gauss-Seidal.

$$x_i^{k+1} = (1 - w)x_i^k + w \cdot \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij}x_j^k \right)$$

Note the second term is the Gauss-Seidal iteration multiplied with w.

w: relaxation parameter.

Usual value: $0 < w < 2$ (for convergence reason)

- $w = 1$: Gauss-Seidal
- $0 < w < 1$: under relaxation
- $1 < w < 2$: over relaxation

Example Try this on the same example with $w = 1.2$. General iteration is now:

$$\begin{cases} x_1^{k+1} &= -0.2x_1^k + 0.6x_2^k \\ x_2^{k+1} &= -0.2x_2^k + 0.6*(1 + x_1^{k+1} + x_3^k) \\ x_3^{k+1} &= -0.2x_3^k + 0.6*(2 + x_2^{k+1}) \end{cases}$$

With $x^0 = (0,\ 0.5,\ 1)^t$, we get

$$x^1 = (0.3, 1.28, 1.708)^t$$
$$x_2 = (0.708, 1.8290, 1.9442)^t$$

Recall the exact solution $x = (1, 2, 2)^t$.
Observation: faster convergence than both Jacobi and G-S.

EIGENVALUE , EIGNVECTOR: Suppose 'A' is a square matrix. The number '$\boldsymbol{\lambda}$' is called an Eignvalue of 'A' if there exist a non-zero vector 'x' such that

$$\boldsymbol{Ax = \lambda x \quad (or) \quad (A - \lambda I)x = 0}$$

And corresponding non-zero solution vector 'x' is called an Eigenvector.

Largest Eigenvalue is known as **Dominant Eigenvalue**.

CHARACTERISTIC POLYNOMIAL: The polynomial defined by "$\boldsymbol{P(\lambda) = \det(A - \lambda I)}$" is called characteristics polynomial.

SPECTRUM OF MATRIX: Set of all eignvalues of 'A' is called spectrum of 'A'.

SPECTRAL RADIUS: The Spectral radius P(A) of a matrix 'A' is defined by

$\boldsymbol{P(A) = max|\lambda|}$ Where '$\boldsymbol{\lambda}$' is an Eignvalue f 'A'.

SPECTRAL NORM: Let "$\boldsymbol{\lambda_i}$" be the largest Eigenvalue of $\boldsymbol{AA^*}$ or $\boldsymbol{A^*A}$ where $\boldsymbol{A^*}$ is the conjugate transpose of "$\boldsymbol{A}$" then the spectral norm of the matrix "$\boldsymbol{A}$" is defined as

$$\boldsymbol{\sigma(A) = \sqrt{\lambda_i}}$$

DETERMINANT OF A MATRIX:

The determinant of "$\boldsymbol{n \times n}$" matrix is the product of its Eigenvalues.

TRACE OF A MATRIX: The sum of diagonal elements of "$\boldsymbol{n} \times \boldsymbol{n}$" matrix is called the Trace of matrix "A"
This is also defined as the sum of Eigenvaluse of a matrix is Trace of it

QUESTION:

Write characteristic equation of $A = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 2 & -1 \\ 4 & 0 & -1 \end{bmatrix}$

SOLUTION:

Let $\lambda I - A = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 0 \\ -2 & 2 & -1 \\ 4 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda - 1 & 3 - & 0 \\ 2 & \lambda - 2 & 1 \\ -4 & 0 & \lambda + 1 \end{bmatrix}$

Now $\Delta_A(\lambda) = \ |\lambda I - A| = \begin{vmatrix} \lambda - 1 & 3 - & 0 \\ 2 & \lambda - 2 & 1 \\ -4 & 0 & \lambda + 1 \end{vmatrix} = \lambda^3 - \lambda^2 + 2\lambda + 28$

Then $\Delta_A(\lambda) = 0 = \lambda^3 - \lambda^2 + 2\lambda + 28$ is required characteristic equation.

PRACTICE: Write characteristic equation of $\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$

$|A - \lambda I|$ is known as the characteristic polynomial of the matrix A. Every root of eq (1) is called an eigenvalue of A.

Eigenvalues and eigenvectors are often called characteristic values and characteristic vectors respectively.

Example:-

Find the eigenvalues and corresponding eigenvectors of matrix

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$$

Sol:-

First, we find eigenvalues by $|A - \lambda I| = 0$

$$A - \lambda I = \begin{bmatrix} 4 & 1 & -2 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-\lambda & 1 & -2 \\ 2 & 3-\lambda & -1 \\ -2 & 1 & 5-\lambda \end{bmatrix}$$

Now

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 1 & -2 \\ 2 & 3-\lambda & -1 \\ -2 & 1 & 5-\lambda \end{vmatrix}$$

$$= (4-\lambda)\left[(3-\lambda)(5-\lambda)+1\right] - 1\left[2\right.$$

$(5-\lambda) - 2] - 2[-2 + 2(3-\lambda)]$

$|A - \lambda I| = (4-\lambda)[15 - 3\lambda - 5\lambda + \lambda^2 + 1] - [10 - 2\lambda - 2] - 2[2 + 6 - 2\lambda]$

$|A - \lambda I| = (4-\lambda)[\lambda^2 - 8\lambda + 16] - [8 - 2\lambda] - 2[8 - 2\lambda]$

$= (4-\lambda)[\lambda^2 - 2(4)\lambda + (4)^2] - 3[8-2\lambda]$

$= (4-\lambda)(4-\lambda)^2 - 3(8-2\lambda)$

$|A - \lambda I| = (4-\lambda)(4-\lambda)^2 - 6(4-\lambda)$

put $|A - \lambda I| = 0$

$\Rightarrow (4-\lambda)(4-\lambda)^2 - 6(4-\lambda) = 0$

$\Rightarrow (4-\lambda)[(4-\lambda)^2 - 6] = 0$

$\Rightarrow (4-\lambda)[16 + \lambda^2 - 8\lambda - 6] = 0$

$\Rightarrow (4-\lambda)[\lambda^2 - 8\lambda + 10] = 0$

$\Rightarrow (4-\lambda) = 0 \quad \lambda^2 - 8\lambda + 10 = 0$

$\Rightarrow 4 - \lambda = 0 \Rightarrow \lambda = \frac{8 \pm \sqrt{24}}{2}$

$\Rightarrow \lambda = 4, \quad \lambda = 1.5505, \lambda = 6.4495$

So, the eigenvalues are

$\lambda_1 = 1.5505 \quad \lambda_2 = 4, \quad \lambda_3 = 6.4495$

Now, find eigenvectors corresponding to eigenvalues 2 and 4, using $(A - \lambda I)V = 0$

For $\lambda_1 = 1.5505$

$$(A - \lambda I)V = 0 \Rightarrow \begin{bmatrix} 4-1.5505 & 1 & -2 \\ 2 & 3-1.5505 & -1 \\ -2 & 1 & 5-1.5505 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2.4495 & 1 & -2 \\ 2 & 1.4495 & -1 \\ -2 & 1 & 3.4495 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad m_{2,1} = \frac{2}{2.4495} = 0.8165, \quad m_{3,1} = \frac{-2}{0.4495} = -0.8165$$

Now $R_2 - m_{2,1}R_1$, $R_3 - m_{3,1}R_1$

$$\sim \begin{bmatrix} 2.4495 & 1 & -2 \\ 0 & 0.6330 & 0.6330 \\ 0 & 1.1865 & 1.1865 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - m_{3,2}R_2$

$$\sim \begin{bmatrix} 2.4495 & 1 & -2 \\ 0 & 0.6330 & 0.6330 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad m_{3,2} = \frac{1.1865}{0.6330} = 1.874$$

$\Rightarrow 2.4495x_1 + x_2 - 2x_3 = 0 \rightarrow (1)$

$0.6330(x_2 + x_3) = 0 \rightarrow (2)$

x_3 is trivial $\rightarrow (3)$

from (3) let

$x_3 = a$

put in (2)

$0.6330(x_2 + (a)) = 0$

$\Rightarrow x_2 = -a$

put in (1)

$2.4495x_1 - a - 2(a) = 0$

$\Rightarrow x_1 = 1.2247a$

The eigen vector is

$[1.2247 \quad 1 \quad -1]^t$

For $\lambda = 4$

$[A - \lambda I]V = 0$

$$\begin{bmatrix} 0 & 1 & -2 \\ 2 & -1 & -1 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

R_{12}

$$\sim \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -2 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - m_{3,1} R_1$ $\quad m_{3,1} = \frac{-2}{2} = -1$

$$\sim \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is trivial solution let $x_3 = a$ where a is arbitrary constant

Then, $x_3 = a \rightarrow (i)$

$2x_1 - x_2 - x_3 = 0 \rightarrow (ii)$

$x_2 - 2x_3 = 0 \rightarrow (iii)$

put $x_3 = a$ in (iii)

$x_2 - 2a = 0$

$\Rightarrow x_2 = 2a$

put in (ii)

$2x_1 - 2a - a = 0$

$\Rightarrow 2x_1 - 3a = 0$

$\Rightarrow x_1 = \frac{3a}{2}$

$\Rightarrow x_1 = 1.5a$

So, eigen vector is

$[1.5, 2, 1]^T$

For $\lambda = 6.4495$

$(A - \lambda I)V = 0$

$$\begin{bmatrix} -2.4495 & 1 & -2 \\ 2 & -3.4495 & -1 \\ -2 & 1 & -1.4495 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - m_{2,1}R_1 \quad , \quad R_3 - m_{3,1}R_1 \quad , \quad m_{2,1} = \frac{2}{-2.4495} = -0.8165$

$m_{3,1} = \frac{-2}{-2.4495} = 0.8165$

$$\sim \begin{bmatrix} -2.4495 & 1 & -2 \\ 0 & -2.6630 & -2.6330 \\ 0 & 0.1835 & 0.1835 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - m_{3,2}R_2 \quad , \quad m_{3,2} = \frac{0.1835}{-2.6630} = -0.0689$

$$\sim \begin{bmatrix} -2.4495 & 1 & -2 \\ 0 & -2.6330 & -2.6330 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is trivial sol. Let $x_3 = a$

and $-2.4495x_1 + x_2 - 2x_3 = 0 \rightarrow (i)$

$-2.6330x_2 - 2.6330x_3 = 0 \rightarrow (ii)$

put $x_3 = a$ in (ii)

$\Rightarrow -2.6330x_2 - 2.6330a = 0$

$\Rightarrow x_2 = -a$

put in (i)

$-2.4495x_1 - a - 2a = 0$

$\Rightarrow -2.4495x_1 - 3a = 0$

$\Rightarrow x_1 = -1.2247a$

The eigen vector is

$[-1.2247, -1, 1]^T$

Diagonalization of matrices:-

An $n \times n$ matrix A is said to be diagonalizable if it is similar to a diagonal matrix. That is, A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. The matrix P is said to diagonalize A.

Example:-

Find the diagonal matrix whose diagonal entries are the eigen values of A

$$D = P^{-1}AP$$

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$$

Sol:-

First, we find the eigen values and eigen vectors of the matrix A

For eigen value $|A - \lambda I| = 0$

$$A - \lambda I = \begin{bmatrix} 4 & 1 & -2 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 1 & -2 \\ 2 & 3-\lambda & -1 \\ -2 & 1 & 5-\lambda \end{vmatrix}$$

$$= (4-\lambda)[(3-\lambda)(5-\lambda)+1] - 1[2(5-\lambda)-2] - 2[2+2(3-\lambda)]$$

$$= (4-\lambda)[15-3\lambda-5\lambda+\lambda^2+1] - 1[10-2-2\lambda] - 2[2+6-2\lambda]$$

$$= (4-\lambda)[\lambda^2-8\lambda+16] - 1[8-2\lambda] - 2[8-2\lambda]$$

$$= (4-\lambda)[\lambda^2-8\lambda+16] - [8-2\lambda] - 2[8-2\lambda]$$

$$= (4-\lambda)[\lambda^2-8\lambda+16] - 3(8-2\lambda)$$

$$= (4-\lambda)(\lambda^2-8\lambda+16) - 6(4-\lambda)$$

$$= (4-\lambda)[\lambda^2-8\lambda+16-6]$$

$$= (4-\lambda)[\lambda^2-8\lambda+10]$$

For eigen values $|A-\lambda I| = 0$

So $(4-\lambda)(\lambda^2-8\lambda+10) = 0$

$\Rightarrow 4-\lambda = 0$, $\lambda^2 - 8\lambda + 10 = 0$

$\Rightarrow \lambda = 4$

$a = 1, \; b = -8, \; c = 10$

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \lambda = \frac{8 \pm \sqrt{24}}{2} \Rightarrow \lambda = 1.5505, \; \lambda = 6.4495$$

So, $\lambda_1 = 1.5505$, $\lambda_2 = 4$, $\lambda_3 = 6.4495$ are eigen values and their corresponding vectors are $[1.2249 \;\; -1 \;\; 1]^T$, $[1.5 \;\; 2 \;\; 1]^T$ and $[-1.2247 \;\; -1 \;\; 1]^T$

we get to normalize these vectors

$$\left[\frac{1.2249}{3.2249}, \frac{-1}{3.2249}, \frac{1}{3.2249}\right]^T = [0.3798, -0.3101, .3101]$$

$$\left[\frac{1.5}{6.5}, \frac{2}{6.5}, \frac{1}{6.5}\right]^T = [0.2308, 0.3078, 0.1538]^T$$

$$\left[\frac{-1.2247}{3.2247}, \frac{-1}{3.2247}, \frac{1}{3.2247}\right]^T = [-0.3798, -.3101 \;\; .3101]^T$$

Now

$$P = \begin{bmatrix} 0.3798 & 0.2308 & -0.3798 \\ -0.3101 & 0.3078 & -0.3101 \\ 0.3101 & 0.1538 & 0.3101 \end{bmatrix}$$

$$D = P^{-1}AP = P^TAP$$

So,

$$P^T = \begin{bmatrix} 0.3798 & -0.3101 & 0.3101 \\ 0.2308 & 0.3078 & 0.1538 \\ -0.3798 & -0.3101 & 0.3101 \end{bmatrix}$$

Now Solve P^TAP.

First, we take AP

$$AP = \begin{bmatrix} 4 & 1 & -2 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0.3798 & 0.2308 & -0.3798 \\ -0.3101 & 0.3078 & -0.3101 \\ 0.3101 & 0.1538 & 0.3101 \end{bmatrix}$$

$$AP = \begin{bmatrix} 0.5889 & 0.9234 & -2.4498 \\ -0.4808 & 1.2312 & -2 \\ 0.4808 & 0.6152 & 2 \end{bmatrix}$$

Now $P^{-1}AP = P^{T}AP$

$$P^{T}AP = \begin{bmatrix} 0.3798 & -0.3101 & 0.3101 \\ 0.2308 & 0.3078 & 0.1538 \\ -0.3798 & -0.3101 & 0.3101 \end{bmatrix} \begin{bmatrix} 0.5889 & 0.9234 & -2.4495 \\ -0.4808 & 1.2312 & -2 \\ 0.4808 & 0.6152 & 2 \end{bmatrix}$$

$$P^{T}AP = \begin{bmatrix} 0.5218 & 0.1597 & 0.3101 \\ 0.0619 & 0.6922 & -0.8733 \\ 0.0745 & -0.5417 & 2.1707 \end{bmatrix}$$

THE POWER METHOD

The power method is an iterative technique used to determine the dominant eigenvalue of a matrix. i.e the eigenvalue with the largest magnitude.

Method also called **RELEIGH POWER METHOD**

ALGORITHIM

i. Choose initial vector such that largest element is unity.
ii. This normalized vector $V^{(0)}$ ia premultiplied by 'nxn' matrix $[\boldsymbol{A}]$.
iii. The resultant vector is again normalized.
iv. Continues this process untill required accuracy is obtained.

At this point result looks like $\boldsymbol{U^{(k)} = [A]\, V^{(k-1)} = q_k\, V^{(k)}}$
Here '$\boldsymbol{q_k}$' is the desired largest Eigen value and '$\boldsymbol{v^{(k)}}$' is the corresponding EigenVector.

CONVERGENCE:

Power method Converges linearly , meaning that during convergence, the error decreases by a constant factor on each iteration step.

QUESTION: How to find smallest Eigen value using power method?

Answer: Consider $[A]X = \lambda X \Rightarrow [A^{-1}][A]X = \lambda[A^{-1}]X \Rightarrow X = \lambda[A^{-1}]X$

$$\Rightarrow [A^{-1}]X = \frac{1}{\lambda}X \quad Required$$

EXAMPLE:

Find the Eigen value of largest modulus and the associated eigenvector of the matrix by power method $[\boldsymbol{A}] = \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{2} \\ \mathbf{4} & \mathbf{3} & \mathbf{5} \\ \mathbf{3} & \mathbf{2} & \mathbf{9} \end{bmatrix}$

SOLUTION:

Let initial vector $V^{(0)}$ as $(0,0,1)^T$

You can take any other instead of $(0,0,0)$ which consist "0" and "1" like $(1,0,0)$ and $(1,1,1)$

(1). Using Formula $U^{(k)} = [A][V^{k-1}]$ for K=1

$$U^{(1)} = [A][V^{(0)}] = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} = 9\begin{bmatrix} 9/9 \\ 5/9 \\ 1 \end{bmatrix} = 9\begin{bmatrix} 0.222 \\ 0.556 \\ 1 \end{bmatrix} = q_1\, V^{(1)}$$

(2). Using Formula $U^{(k)} = [A][V^{k-1}]$ for K=2

$$U^{(2)} = [A]V^{(1)} = \begin{bmatrix} 2 & 3 & 3 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix}\begin{bmatrix} 0.222 \\ 0.556 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.112 \\ 7.556 \\ 10.778 \end{bmatrix} = 10.778\begin{bmatrix} 0.382 \\ 0.701 \\ 1 \end{bmatrix} = q_2 V^{(2)}$$

(3). Using Formula $U^{(k)} = [A][V^{k-1}]$ for K=3

$$U^{(3)} = [A]V^{(2)} = \begin{bmatrix} 2 & 3 & 3 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix}\begin{bmatrix} 0.382 \\ 0.701 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.867 \\ 8.631 \\ 11.548 \end{bmatrix} = 11.548\begin{bmatrix} 0.421 \\ 0.747 \\ 1 \end{bmatrix} = q_3 V^{(3)}$$

(4). Using Formula $U^{(k)} = [A][V^{k-1}]$ for K=4

$$U^{(4)} = [A]V^{(3)} = \begin{bmatrix} 2 & 3 & 3 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix}\begin{bmatrix} 0.421 \\ 0.747 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.083 \\ 8.925 \\ 11.757 \end{bmatrix} = 11.757\begin{bmatrix} 0.432 \\ 0.759 \\ 1 \end{bmatrix} = q_4 V^{(4)}$$

(5). Using Formula $U^{(k)} = [A][V^{k-1}]$ for K=5

$$U^{(5)} = [A]V^{(4)} = \begin{bmatrix} 2 & 3 & 3 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix}\begin{bmatrix} 0.432 \\ 0.759 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.141 \\ 9.005 \\ 11.814 \end{bmatrix} = 11.814\begin{bmatrix} 0.435 \\ 0.762 \\ 1 \end{bmatrix} = q_5 V^{(5)}$$

(6). Using Formula $U^{(k)} = [A][V^{k-1}]$ for K=6

$$U^{(6)} = [A]V^{(5)} = \begin{bmatrix} 2 & 3 & 3 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix}\begin{bmatrix} 0.435 \\ 0.762 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.156 \\ 9.026 \\ 11.829 \end{bmatrix} = 11.829\begin{bmatrix} 0.436 \\ 0.763 \\ 1 \end{bmatrix} = q_6 V^{(6)}$$

(7). Using Formula $U^{(k)} = [A][V^{k-1}]$ for K=7

$$U^{(7)} = [A]V^{(6)} = \begin{bmatrix} 2 & 3 & 3 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix}\begin{bmatrix} 0.436 \\ 0.763 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.161 \\ 9.033 \\ 11.834 \end{bmatrix} = 11.834\begin{bmatrix} 0.436 \\ 0.763 \\ 1 \end{bmatrix} = q_7 V^{(7)}$$

So largest Eigen value is $q = 11.834$ and corresponding Eigenvector is

$V=\begin{bmatrix} 0.436 \\ 0.763 \\ 1 \end{bmatrix}$ accurate to 3 decimals.

QUESTION: Find the smallest Eigen value of the matrix by power method $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ upto seven iterations.

SOLUTION: Put $A^{-1} = B \Longrightarrow \quad A^{-1} = \frac{adj\,A}{|A|} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

$a_{11} = (-1)^2(20+3) = +23$

$a_{12} = (-1)^3(20+6) = -26$

$a_{13} = (-1)^4(12-24) = -12$

$a_{21} = (-1)^3(-15-6) = 21$

$a_{22} = (-1)^4(5-12) = -7$

$a_{23} = (-1)^5(3+18) = -21$

$a_{31} = (-1)^4(-8+3) = -5$

$a_{32} = (-1)^5(-1-8) = 9$

$a_{33} = (-1)^6(4+12) = 16$

$$adjA = \begin{bmatrix} 23 & 21 & -5 \\ -26 & -7 & 9 \\ -12 & -21 & 16 \end{bmatrix}$$

$$and\ |A| = \begin{vmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{vmatrix} = 1(20+3) + 3(20+6) + 2(12-24) = 77$$

$$(1) \Longrightarrow B = A^{-1} = \begin{bmatrix} 23/77 & 21/77 & -5/77 \\ -26/77 & -7/77 & 9/77 \\ -12/77 & -21/77 & 16/77 \end{bmatrix}$$

Now Taking Initial vector as $V^{(0)} = (0,0,1)^T$

$$U^{(1)} = [B]V^{(0)} = \begin{bmatrix} \frac{23}{77} & \frac{21}{77} & -\frac{5}{77} \\ -\frac{26}{77} & -\frac{7}{77} & \frac{9}{77} \\ -\frac{12}{77} & -\frac{21}{77} & \frac{16}{77} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.06 \\ 0.12 \\ 0.21 \end{bmatrix} = 0.21 \begin{bmatrix} -0.29 \\ 0.57 \\ 1 \end{bmatrix} = q_1 V^{(1)}$$

$$U^{(2)} = [B]V^{(1)} = \begin{bmatrix} 0.30 & 0.27 & -0.06 \\ -0.34 & -0.09 & 0.12 \\ -0.16 & -0.27 & 0.21 \end{bmatrix} \begin{bmatrix} -0.29 \\ 0.57 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.01 \\ 0.17 \\ 0.10 \end{bmatrix} = 0.17 \begin{bmatrix} 0.06 \\ 1 \\ 0.59 \end{bmatrix} = q_2 V^2$$

$$U^{(3)} = [B]V^{(2)} = \begin{bmatrix} 0.30 & 0.27 & -0.06 \\ -0.34 & -0.09 & 0.12 \\ -0.16 & -0.27 & -0.21 \end{bmatrix} \begin{bmatrix} 0.06 \\ 1 \\ 0.59 \end{bmatrix} = \begin{bmatrix} 0.25 \\ -0.04 \\ -0.16 \end{bmatrix} = 0.25 \begin{bmatrix} 1 \\ 0.16 \\ 0.59 \end{bmatrix} = q_3 V^3$$

$$U^{(4)}= [B]V^{(3)} = \begin{bmatrix} 0.30 & 0.27 & -0.06 \\ -0.34 & -0.09 & 0.12 \\ -0.16 & -0.27 & 0.12 \end{bmatrix} \begin{bmatrix} 1 \\ 0.16 \\ 0.54 \end{bmatrix} = \begin{bmatrix} 0.30 \\ -0.28 \\ -0.07 \end{bmatrix} = 0.30 \begin{bmatrix} 1 \\ -0.93 \\ -0.23 \end{bmatrix} = q_4 V^{(4)}$$

$$U^{(5)} = [B]V^{(4)} = \begin{bmatrix} 0.30 & 0.27 & -0.06 \\ -0.34 & -0.09 & 0.12 \\ -0.16 & -0.27 & 0.12 \end{bmatrix} \begin{bmatrix} 1 \\ -0.93 \\ -0.23 \end{bmatrix} = \begin{bmatrix} 0.06 \\ -0.28 \\ 0.04 \end{bmatrix} = 0.06 \begin{bmatrix} 1 \\ -4.67 \\ 0.67 \end{bmatrix} = q_5 V^{(5)}$$

$$U^{(6)} = [B]V^{(5)} = \begin{bmatrix} 0.30 & 0.27 & -0.06 \\ -0.34 & -0.09 & 0.12 \\ -0.16 & -0.27 & 0.12 \end{bmatrix} \begin{bmatrix} 1 \\ -4.67 \\ 0.67 \end{bmatrix} = \begin{bmatrix} -1.00 \\ 0.16 \\ 1.24 \end{bmatrix} = 1.24 \begin{bmatrix} -0.81 \\ 0.13 \\ 1 \end{bmatrix} = q_6\, V^{(6)}$$

$$U^{(7)}=[B]V^{(6)} = \begin{bmatrix} 0.30 & 0.27 & -0.06 \\ -0.34 & -0.09 & 0.12 \\ -0.16 & -0.27 & 0.12 \end{bmatrix} \begin{bmatrix} -0.81 \\ 0.13 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.27 \\ 0.38 \\ 0.30 \end{bmatrix} = 0.38 \begin{bmatrix} -0.71 \\ 1 \\ 0.79 \end{bmatrix} = q_7\, V^{(7)}$$

So smallest Eigen value is $q = 0.38$ and corresponding Eigenvector is $V=\begin{bmatrix} -0.71 \\ 1 \\ 0.79 \end{bmatrix}$

EXERCISE: Find the Eigen value of largest modulus (Or dominant eigenvalue) and the associated eigenvector of the matrix by power method

i. $[A] = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & 1 \\ 6 & 3 & 5 \end{bmatrix}$ after six iterations.

ii. $[A] = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ after fourth iterations using initial vector as $V^{(0)} = [0,0,1]^T$

iii. $[A] = \begin{bmatrix} 8 & 1 & 2 \\ 0 & 10 & -1 \\ 6 & 2 & 15 \end{bmatrix}$ with unit vector as initial vector.

iv. $[A] = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ with unit vector as initial vector.

DIFFERENCE OPERATORS

DIFFERENCE EQUATION

Equation involving differences is called Difference Equation.

OR an equation that consist of an independent variable 't', a dependent variable 'y(t)' and one or more several differences of the dependent variable $\mathbf{y_t}$ as $\Delta \mathbf{y_t}, \Delta^2 \mathbf{y_t}, \dots \dots \dots \Delta^n \mathbf{y_t}$ is called difference equation.

The functional relationship of the difference equation is $\boldsymbol{F(t, y, \Delta y_t, \Delta^2 y_t, \dots \dots \dots \Delta^n y_t) = 0}$

Solution of differential equation will be sequence of $\mathbf{y_t}$ values for which the equation is true for some set of consecutive integer 't'.

Importance: difference equation plays an important role in problem where there is a quantity 'y' that depends on a continuous independent variable 't'

In polynomial dynamics, modeling the rate of change of the population or modeling the growth rate yields a differential equation for the population 'y' as the function of time 't'

i.e. $\boldsymbol{\frac{dy}{dt} = f(t, y)}$

In differential equation models, usually the population is assumed to vary continuously I time.

Difference equation model arise when the population is modeled only at certain discrete time.

DIFFERENCE OF A POLYNOMIAL

The "nth" difference of a polynomial of degree 'n' is constant, when the values of the independent variable are given at equal intervals.

EXAMPLES:

i. $\Delta^3 y_t + 3\Delta^2 y_t - 6\Delta y_t + y_t = 9t^2 + 6t$
ii. $\Delta^2 y_t + 3\Delta y_t - 7y_t = 0$

Order of differential equation is the difference between the largest and smallest argument 't' appearing in it.

For example; if $y_{t+2} + y_{t+1} - 7y_t = 0$ then $order = t + 2 - t = 2$

Degree of differential equation is the highest power of 'y'

For example; if $y_{t+2}{}^3 + y_{t+1}{}^2 - 7y_t = 0$ then $degree = 3$

FINITE DIFFERENCES: Let we have a following linear D. Equation

$$y''(t) + p(t)y' + q(t)y = r(t) \quad ; a \leq t \leq b$$

Subject to the boundary conditions $\quad y(a) = \propto \quad and \quad y(b) = \beta$

Then the finite difference method consists of replacing every derivative in above Equation by finite difference approximations such as the central divided difference approximations

$y'(t_i) \approx \frac{1}{2h}[y(t_i+1) - y(t_i-1)]$ and $y''(t_i) \approx \frac{1}{h^2}[y(t_i+1) - 2y(t) + y(t_i-1)]$

Shooting Method is a finite difference method.

FINITE DIFFERENCES OF DIFFERENT ORDERS: Supposing the argument equally spaced so that $\boldsymbol{x_{k+1} - x_k = h}$ the difference of the $'\boldsymbol{y_k}'$ values are denoted as

$\Delta \boldsymbol{y_k} = \boldsymbol{y_{k+1} - y_k}$ And are called First differences.

Second differences are as follows

$$\Delta^2 \boldsymbol{y_k} = \Delta(\Delta \boldsymbol{y_k}) = \Delta \boldsymbol{y_{k+1}} - \Delta \boldsymbol{y_k} = \boldsymbol{y_{k+2} - 2y_{k+1} + y_k}$$

In General: $\Delta^{\boldsymbol{n}} \boldsymbol{y_k} = \Delta^{\boldsymbol{n-1}} \boldsymbol{y_{k+1}} - \Delta^{\boldsymbol{n-1}} \boldsymbol{y_k}$ And are called n[th] differences

DIFFERENCE TABLE: The standard format for displaying finite differences is called difference table.

DIFFERENCE FORMULAS: Difference formulas for elementary functions somewhat parallel those of calculus. Example include the following

The differences of a constant function are zero. In symbol "$\Delta \boldsymbol{c} = \boldsymbol{0}$" where 'c' denotes a constant. For a constant time another function we have $\Delta(\boldsymbol{cu_k}) = \boldsymbol{c}\Delta \boldsymbol{u_k}$

The difference of a sum of two functions is the sum of their differences

$$\Delta(\boldsymbol{u_k + v_k}) = \Delta \boldsymbol{u_k} + \Delta \boldsymbol{v_k}$$

The 'linearity property' generalizes the two previous results.

$\Delta(\boldsymbol{c_1 u_k + c_2 v_k}) = \boldsymbol{c_1}\Delta \boldsymbol{u_k} + \boldsymbol{c_2}\Delta \boldsymbol{v_k}$ Where $\boldsymbol{c_1}$ and $\boldsymbol{c_2}$ are constants.

PROVE THAT $\Delta(\boldsymbol{cy_k}) = \boldsymbol{c}\Delta \boldsymbol{y_k}$

$$\Delta(\boldsymbol{cy_k}) = \boldsymbol{c}y_{k+1} - \boldsymbol{cy_k} = \boldsymbol{c(y_{k+1} - y_k)} = \boldsymbol{c}\Delta \boldsymbol{y_k}$$

<u>FOR A CONSTANT FUNCTION ALL DIFFERENCES ARE ZERO, PROVE!</u>

Let $\forall \boldsymbol{k}$; $\boldsymbol{c = y}$ then for all 'k'

$\Delta \boldsymbol{y_k} = \boldsymbol{y_{k+1} - y_k = c - c = 0}$ Where $\boldsymbol{y_k = c}$ is a constant function.

REMEMBER: The fundamental idea behind finite difference methods is the replace derivatives in the differential equation by discrete approximations, and evaluate on a grid to develop a system of equations.

COLLOCATION : Like the finite difference methods, the idea behind the collocation is to reduce the boundary value problem to a set of solvable algebraic equations.

However, instead of discretizing the differential equation by replacing derivative with finite differences, the solution is given a functional from whose parameters are fit by the method.

CRITERION OF APPROXIMATION: Some methods are as follows

i. collocation ii. Osculation iii. Least square

FORWARD DIFFERENCE OPERATOR 'Δ' (DELTA)

We define forward difference operator as $\Delta y_i = y_{i+1} - y_i$ $\quad i = 1, 2, \dots n-1$

Where y=f(x) $\quad (OR) \quad \Delta y_x = y_{x+h} - y_x$

<u>For first order:</u> Given function y=f(x) and a value of argument 'x' as x=a, a+h…… a+nh etc. Where 'h' is the step size (increment) first order Forward Difference Operator is

$\Delta f(a) = f(a+h) - f(a) \quad OR \quad \Delta y_i = y_{i+1} - y_i \quad \forall i = 1, 2, 3 \dots n-1$

<u>For Second Order</u>

Let $\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0)$

$= y_2 - 2y_1 + y_0$

For Third Order

$\Delta^3 y_0 = \Delta(\Delta^2 y_0) = \Delta(y_2 - 2y_1 + y_0) = \Delta y_2 - 2\Delta y_1 + \Delta y_0$

$= (y_3 - y_2) - 2(y_2 - y_1) + (y_1 - y_0) = y_3 - 3y_2 + 3y_1 - y_0$

$\Rightarrow \Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ **are called leading differences**

<u>In General</u>: $\Delta^n y_0 = y_n - {}^n_1C y_{n-1} + {}^n_2C y_{n-2} + \cdots + (-1)^n y_n$

<u>Remark:</u> ${}^n_rC = \frac{n!}{r!(n-r)!}$ *and* ${}^n_0C = {}^n_nC = 1$ *and* ${}^n_1C = {}^n_{n-1}C = n$

QUESTION: Express $\Delta^2 y_1$ and $\Delta^4 y_0$ in terms of the value of function y.

SOLUTION

$(I) \Rightarrow \quad \Delta^2 y_1 = \Delta y_2 - \Delta y_1 = (y_3 - y_1) - (y_2 - y_1) = y_3 - 2y_2 + y_1$

$(II) \Rightarrow \quad \Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 = \Delta^2 y_2 - \Delta^2 y_1 - (\Delta^2 y_1 - \Delta^2 y_0)$

$= \Delta y_3 - \Delta y_2 - (\Delta y_2 - \Delta y_1) - (\Delta y_2 - \Delta y_1) + (\Delta y_1 - \Delta y_0) = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$

QUESTION: Compute the missing values of y_n and Δy_n in the following table.

y_n	y_0	y_1	y_2	y_3=6	y_4	y_5	y_6
Δy_n	Δy_0	Δy_1	Δy_2=5	Δy_3	Δy_4	Δy_5	
$\Delta^2 y_n$	$\Delta^2 y_0$=1	$\Delta^2 y_1$=4	$\Delta^2 y_2$=13	$\Delta^2 y_3$=18	$\Delta^2 y_4$=24		

SOLUTION

$\Delta^2 y_0=1 \quad \Rightarrow \Delta y_1 - \Delta y_0 = 1 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots.(1)$

$\Delta^2 y_1=4 \quad \Rightarrow \Delta y_2 - \Delta y_1 = 4 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots.(2)$

$\Delta^2 y_2=13 \quad \Rightarrow \Delta y_3 - \Delta y_2 = 13 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots.(3)$

$\Delta^2 y_3=18 \quad \Rightarrow \Delta y_4 - \Delta y_3 = 18 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots.(4)$

$\Delta^2 y_4=24 \quad \Rightarrow \Delta y_5 - \Delta y_4 = 24 \quad \dots\dots\dots\dots\dots\dots\dots\dots.(5)$

(2) $\Rightarrow \quad \Delta y_2 - \Delta y_1=4$ and $\Delta y_2 = 5 \quad \Rightarrow \quad 5-\Delta y_1=4 \Rightarrow \Delta y_1 = 1$

(1) $\Rightarrow \quad \Delta y_1 - \Delta y_0 = 1 \Rightarrow 1 - \Delta y_0=1 \Rightarrow \Delta y_0=0$

(3) $\Rightarrow \quad \Delta y_3 - \Delta y_2 = 13$ And $\Delta y_2 = 5 \quad \Rightarrow \Delta y_3 - 5 = 13 \Rightarrow \Delta y_3=18$

(4) $\Rightarrow \Delta y_4 - \Delta y_3 = 18 \Rightarrow \Delta y_4 - 18 = 18 \Rightarrow \Delta y_4 = 36$

(5) $\Rightarrow \Delta y_5 - \Delta y_4 -= 18 \Rightarrow \Delta y_5 - 36 = 24 \Rightarrow \Delta y_5 = 60$

Now since we know that

$\Delta y_0 = y_1 - y_0 \quad \dots\dots\dots\dots\dots\dots\dots\dots.(6)$

$\Delta y_1 = y_2 - y_1 \quad \dots\dots\dots\dots\dots\dots\dots\dots.(7)$

$\Delta y_2 = y_3 - y_2 \quad \dots\dots\dots\dots\dots\dots\dots\dots.(8)$

$\Delta y_3 = y_4 - y_3 \quad \dots\dots\dots\dots\dots\dots\dots\dots.(9)$

$\Delta y_4 = y_5 - y_4 \quad \dots\dots\dots\dots\dots\dots\dots\dots.(10)$

$\Delta y_5 = y_6 - y_5 \quad \dots\dots\dots\dots\dots\dots\dots\dots.(11)$

Since By table $y_3 = 6$ and $\Delta y_2 = 5$

(8) $\Rightarrow 5 = 6 - y_2 \Rightarrow y_2 = 1$ and (7) $\Rightarrow \Delta y_1 = y_2 - y_1 \Rightarrow 1 = 1 - y_1 \Rightarrow y_1 = 0$

(6) $\Rightarrow \Delta\, y_0 = y_1 - y_0 \Rightarrow 0 = 0 - y_0 \Rightarrow y_0 = 0$

(9) $\Rightarrow \Delta y_3 = y_4 - y_3 \Rightarrow 18 = y_4 - 6 \Rightarrow y_4 = 24$

(10) $\Rightarrow \Delta y_4 = y_5 - y_4 \Rightarrow 36 = y_5 - 24 \Rightarrow y_5 = 60$

(11) $\Rightarrow \Delta y_5 = y_6 - y_5 \Rightarrow 60 = y_6 - 60 \quad \Rightarrow y_6 = 120$

QUESTION : Show that the value of 'y_n' can be expressed in terms of the leading value'$\boldsymbol{y_0}$ 'and the Binomial leading differences $\Delta \boldsymbol{y_0}, \Delta^2 \boldsymbol{y_0}$.....$\Delta^n \boldsymbol{y_0}$

SOLUTION

$$(1)\ \ldots\ldots\ldots\ldots\ldots\ldots \begin{cases} \Delta y_0 = y_1 - y_0 \quad OR \quad y_1 = y_0 + \Delta y_0 \\ \Delta y_1 = y_2 - y_1 \quad OR \quad y_2 = y_1 + \Delta y_1 \\ \Delta y_2 = y_3 - y_2 \quad OR \quad y_3 = y_2 + \Delta y_2 \\ \quad and\ so\ on \ldots\ldots\ldots\ldots\ldots \end{cases}$$

Similarly

$$(2)\ \ldots\ldots\ldots\ldots\ldots\ldots \begin{cases} \Delta^2 y_0 = \Delta(\Delta y_0) = \Delta y_1 - \Delta y_0 \quad OR \quad \Delta y_1 = \Delta y_0 + \Delta^2 y_0 \\ \Delta^2 y_1 = \Delta(\Delta y_1) = \Delta y_2 - \Delta y_1 \quad OR \quad \Delta y_2 = \Delta y_1 + \Delta^2 y_1 \\ \quad and\ so\ on \ldots\ldots\ldots\ldots\ldots \end{cases}$$

Similarly

$$(3)\ \ldots\ldots\ldots\ldots\ldots\ldots \begin{cases} \Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 \quad OR \quad \Delta^2 \Delta y_1 = \Delta^2 y_0 + \Delta^3 y_0 \\ \Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1 \quad OR \quad \Delta^2 y_2 = \Delta^2 y_0 + \Delta^3 y_0 \\ \quad and\ so\ on \ldots\ldots\ldots\ldots\ldots \end{cases}$$

Also from (2) and (3) we can write Δy_2 as

$$\Delta y_2 = (\Delta y_0 + \Delta^2 y_0) + (\Delta^2 y_0 + \Delta^3 y_0) = \Delta y_0 + 2\Delta^2 y_0 + \Delta^3 y_0 \ldots\ldots\ldots\ldots (4)$$

From (1) and (4) we can write y_3 as

$$y_3 = y_2 + \Delta y_2 = (y_1 + \Delta y_1) + (\Delta y_1 + \Delta^2 y_1)$$

$$= (y_0 + \Delta y_0) + 2(\Delta y_0 + \Delta^2 y_0) + (\Delta^2 y_0 + \Delta^3 y_0)$$

$$= y_0 + 3\Delta y_0 + 3\Delta^2 y_0 + \Delta^3 y_0 = (1+\Delta)^3 y_0$$

Similarly, we can symbolically write

$y_1 = (1+\Delta)y_0,\ y_2 = (1+\Delta)^2 y_0,\ y_3 = (1+\Delta)^3 y_0$ In general $y_n = (1+\Delta)^n y_0$

Hence $\quad y_n = y_0 + c_1^n \Delta y_0 + c_2^n \Delta^2 y_0 + ------ + c_n^n \Delta^n y_0 = \sum_{i=o}^{n} C_i^n \Delta^i y_0$

BACKWARD DIFFERENCE OPERATOR "∇" (NEBLA)

We Define Backward Difference Operator as

$\nabla \mathrm{y_n} = y_n - y_{n-1} \quad \forall\, n = 1,2,\ldots\ldots i$ (OR) $\nabla f(x) = f(x) - f(x-h)$

(OR) $\quad \nabla y_x = y_x - y_{x-h}$

QUESTION: Show that any value of 'y' can be expressed in terms of $'\boldsymbol{y_n}'$ and its backward differences.

SOLUTION: Since $y_{n-1} = y_n - \nabla y_n$ And $y_{n-2} = y_{n-1} - \nabla y_{n-1}$ ……………(1)

Also $\nabla y_{n-1} = \nabla y_n - \nabla^2 y_n$ ……………(2)

Thus $\nabla y_{n-1} = y_{n-1} - y_{n-2}$ (Rearranging Above)

(1) $\Rightarrow y_{n-2} = y_{n-1} - \nabla y_n + \Delta^2 y_n = y_n - \nabla y_n - \nabla y_n + \nabla^2 y_n$

$y_{n-2} = y_n - 2\nabla y_n + \Delta^2 y_n = (1 - 2\nabla + \nabla^2) y_n$

Similarly We Can Show That $y_{n-3} = y_n - 3\nabla y_n + 3\nabla^2 y_n - \nabla^3 y_n$

Symbolically above results can be written as $y_{n-1} = (1-\nabla)y_n$, $y_{n-2} = (1-\nabla)^2 y_n$ ... ...

In General $y_{n-r} = (1-\nabla)^r$

i.e. $y_{n-r} = y_n - {}^r_1C\, \nabla y_n + {}^r_2C\, \nabla^2 y_n - \dots \dots + (-1)^r \nabla^r y_n$

SHIFT OPERATOR "E": Shift Operator defined as for y=f(x)

$E^n y_i = y_{i+n} \quad \forall\, i = 1,2, \dots \dots,\; n = 1,2,3, \dots \dots$

OR $E^n f(x) = f(x + nh)$ **OR** $E^n y_x = y_{x+nh}$

"δ" CENTRAL DIFFERENT OPERATOR (DELTA IN LOWER CASE)

Central Different Operator for y=f(x) defined as

$\delta y_i = y_{i+\frac{1}{2}} - y_{i-\frac{1}{2}} \quad \forall\, i = 1,2, \dots . n$

(OR) $\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$ (OR) $\delta y_x = y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}}$

AVERAGE OPERATOR "μ"

For y=f(x) Differential Operator defined as

$\mu y_i = \frac{1}{2}\left[y_{i+\frac{1}{2}} + y_{i-\frac{1}{2}}\right] \quad \forall\, i = 1, 2, \dots \dots \dots n$

(OR) $\mu f(x) = \frac{1}{2}[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)]$ (OR) $\mu y_x = \frac{1}{2}[y_{x+\frac{h}{2}} + y_{x-\frac{h}{2}}]$

DIFFERENTIAL OPERATOR "D": For y=f(x) Differential Operator defined as

$$\boldsymbol{D^n f(x) = \frac{d^n}{dx^n}\, f(x) \quad \forall\, n}$$

SOME USEFUL RELATIONS

From the Definition of "Δ" and "E" we have

$\Delta y_x = y_{x+h} - y_x = Ey_x - y_x = (E-1)y_x \qquad \Longrightarrow \quad \Delta = \boldsymbol{E-1}$

Now by definitions of ∇ and E^{-1} we have

$\nabla y_x = y_x - y_{x-h} = y_x - E^{-1}y_x = (1-E^{-1})y_x \qquad \Longrightarrow \quad \boldsymbol{\nabla = 1 - E^{-1} = \frac{E-1}{E}}$

The definition of Operators 'δ'and 'E' gives

$\delta y_x = y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}} = E^{\frac{1}{2}}y_x - E^{\frac{-1}{2}}y_x = (E^{\frac{1}{2}} - E^{\frac{-1}{2}})y_x \quad \Longrightarrow \boldsymbol{\delta = E^{\frac{1}{2}} - E^{\frac{-1}{2}}}$

The definition of 'μ' and 'E' Yields

$\mu y_x = \frac{1}{2}\left[y_{x+\frac{h}{2}} + y_{x-\frac{h}{2}}\right] = \frac{1}{2}[E^{\frac{1}{2}} + E^{\frac{-1}{2}}]y_x \qquad \Longrightarrow \qquad \boldsymbol{\mu = \frac{1}{2}(E^{\frac{1}{2}} + E^{\frac{-1}{2}})}$

Now Relation between 'D' and 'E' is as follows

Since $Ey_x = y_{x+h} = f(x+h)$

Using Taylor series expansion, we have

$Ey_x = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \cdots \ldots\ldots\ldots = f(x) + hDf(x) + \frac{h^2}{2!}D^2 f(x) + \cdots \ldots.$

$Ey_x = [1 + \frac{hD}{1!} + \frac{h^2D^2}{2!} + \cdots \ldots\ldots\ldots\ldots..]\ f(x)$

$\boldsymbol{E\, y_x = e^{hD}\, y_x} \qquad \therefore\ e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \ldots\ldots\ldots$

Taking 'Log' on both sides we get $\boldsymbol{Log\ E = hD}$

Hence, all the operators are expressed in terms of 'E'

PROVE THAT $\boldsymbol{E\nabla = \Delta = \delta E^{\frac{1}{2}}}$

$E\nabla = E(1 - E^{-1}) \qquad \therefore\ \nabla = 1 - E^{-1} \ \Rightarrow E\nabla = E - EE^{-1} = E - 1 \ = \Delta$

$And \quad \delta E^{1/2} = (E^{\frac{1}{2}} - E^{-1/2})\ E^{1/2} \quad \because\ \delta = E^{1/2} - E^{-1/2} \ \Rightarrow \delta E^{1/2} = E - 1 = \Delta$

PROVE THAT $\boldsymbol{\delta = 2\sin h\left(\frac{hD}{2}\right)}$

$Since \quad \delta = \left(E^{1/2} - E^{-1/2}\right) \qquad \because\ log\ E = hD \Longrightarrow E = e^{hD}$

$= 2\left(\frac{E^{1/2} - E^{-1/2}}{2}\right) = 2\left(\frac{e^{hD/2} - e^{-hD/2}}{2}\right) = 2\,Sin\,h\left(\frac{hD}{2}\right)$

PROVE THAT $\boldsymbol{\mu = 2\cosh\left(\frac{hD}{2}\right)}$

Since $\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] = \frac{1}{2}\left[e^{\frac{hD}{2}} + e^{\frac{-hD}{2}}\right] = \cosh\left(\frac{hD}{2}\right)$

Show that $\boldsymbol{\delta, \mu, E, \Delta, \nabla}$ ***Commute***

$$\delta E f(x) = \delta f(x+h) = f(x+h+h) - f(x+h-h) = f(x+2h) - f(x)$$

$$E\delta f(x) = E[f(x+h) - f(x-h)] = E f(x+h) - Ef(x-h)$$

$$= f(x+h+h) - f(x-h+x) = f(x+2h) - f(x)$$

$$\Longrightarrow \delta E f(x) = E\delta f(x) \qquad \textit{Commute}$$

PROVE THAT $\boldsymbol{\Delta\nabla = \nabla\Delta = \Delta - \nabla}$

$$\Delta\nabla(y_x) = \Delta(y_x - y_{x-h}) = \Delta y_x - \Delta y_{x-h}$$

$$= (y_{x+h} - y_x) - (y_{x-h+h} - y_{x-h}) = y_{x+h} - y_x - y_x - y_{x-h} = y_{x+h} - y_x - (y_x + y_{x-h})$$

$$= \Delta y_x - \nabla y_x = \Delta - \nabla$$

And $\nabla\Delta(y_x) = \nabla(y_{x+h} - y_x) = \nabla y_{x+h} - \nabla y_x = (y_{x+h} - y_{x+h-h}) - (y_x - y_{x-h})$

$$= y_{x+h} - y_x - y_x + y_{x-h} = \Delta - \nabla \Longrightarrow \Delta y_x - \nabla y_x = \Delta - \nabla$$

PROVE THAT $\boldsymbol{\Delta\nabla = \nabla\Delta}$

$$\Delta\nabla(y_x) = \Delta(y_x - y_{x-h}) = \Delta y_x - \Delta y_{x-h}$$

$$= (y_{x+h} - y_x) - (y_{x-h+h} - y_{x-h}) = y_{x+h} - y_x - y_x - y_{x-h}$$

$$= y_{x+h} - 2y_x + y_{x-h}$$

And $\nabla\Delta(y_x) = \nabla(y_{x+h} - y_x) = \nabla y_{x+h} - \nabla y_x = (y_{x+h} - y_{x+h-h}) - (y_x - y_{x-h})$

$$= y_{x+h} - y_x - y_x + y_{x-h} = y_{x+h} - 2y_x + y_{x-h} \Longrightarrow \Delta\nabla(y_x) = \nabla\Delta(y_x) \;\; \textit{Commute}$$

PROVE THAT $\boldsymbol{\delta^2 = \Delta - \nabla}$

$$\delta^2(y_x) = \left(E^{1/2} - E^{-1/2}\right)^2 (y_x) = (E + E^{-1} - 2)(y_x) = E(y_x) + E^{-1}(y_x) - 2(y_x)$$

$$\delta^2(y_x) = y_{x+h} + y_{x-h} - 2(y_x) = (y_{x+h} - y_x) - (y_x - y_{x-h}) = \Delta - \nabla$$

PROVE THAT $\Delta + \boldsymbol{\nabla} = \frac{\Delta}{\boldsymbol{\nabla}} + \frac{\boldsymbol{\nabla}}{\Delta}$

$R.H.S = \frac{\Delta}{\nabla} + \frac{\nabla}{\Delta} = \frac{\Delta^2 - \nabla^2}{\nabla\Delta} = \frac{(\Delta - \nabla)(\Delta + \nabla)}{\nabla\Delta} = \frac{\nabla\Delta(\Delta + \nabla)}{\nabla\Delta} = (\Delta + \nabla) = L.H.S$

PROVE THAT $\boldsymbol{hD = log(1 + \Delta) = -log(1 - \nabla) = sin\, h^{-1}(\mu\delta)}$

$Since \quad hD = log\, E = log(1 + \Delta) \qquad \because E = 1 + \Delta$

$= -log\, E^{-1} = -log(1 - \nabla)$

$Also \qquad \mu\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right) = \frac{1}{2}(E - E^{-1})$

$= \frac{1}{2}(e^{hD} - e^{-hD}) \qquad \because E = e^{hD}\ , E^{-1} = e^{-hD}$

$\mu\delta = sin\, h(hD) \quad \Rightarrow hD = sin\, h^{-1}(\mu\delta)$

PROVE THAT $\boldsymbol{E = (1 - \nabla)^{-1}}$

Since $\nabla = 1 - E^{-1}$ therefore
$R.H.S. = [1 - (1 - E^{-1})]^{-1} = [1 - 1 + E^{-1}]^{-1} = [E^{-1}]^{-1} = E = L.H.S$

PROVE THAT $\boldsymbol{1 + \delta^2\mu^2 = (1 + \frac{\delta^2}{2})^2}$

$Since \quad \mu\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right) = \frac{1}{2}(E - E^{-1})$

$\mu^2\delta^2 = \frac{1}{4}(E - E^{-1})^2 \qquad \because Squqring\ both\ sides$

$\mu^2\delta^2 = \frac{1}{4}(E^2 + E^{-2} - 2)$

$1 + \mu^2\delta^2 = \frac{1}{4}(E^2 + E^{-2} - 2) + 1 \qquad \because Adding\ '1'\ on\ both\ sides$

$1 + \mu^2\delta^2 = \frac{E^2 + E^{-2} - 2 + 4}{4} = \frac{E^2 + E^{-2} + 2}{4} = \frac{(E + E^{-1})^2}{4}$

$1 + \mu^2\delta^2 = \left(\frac{E + E^{-1}}{2}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$

$Also \quad \delta = (E^{\frac{1}{2}} - E^{\frac{-1}{2}})$ then $\delta^2 = (E^{\frac{1}{2}} - E^{\frac{-1}{2}})^2 \quad \because Squaring\ Both\ sides.$

$\frac{\delta^2}{2} = \frac{1}{2}(E^{\frac{1}{2}} - E^{\frac{-1}{2}})^2 \quad \because deviding\ 2\ on\ Both\ sides$

$1+\frac{\delta^2}{2}=\frac{E+E^{-1}-2}{2}+1 \because Adding\ 1\ on\ Both\ sides$

$1+\frac{\delta^2}{2}=\frac{E+E^{-1}-2+2}{2}=\frac{1}{2}(E+E^{-1}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$

Combining (i) and (ii) we get the result.

PROVE THAT $\boldsymbol{E^{\frac{1}{2}}=\mu+\frac{\delta}{2}}$

since $\mu+\frac{\delta}{2}=\left(\frac{E^{\frac{1}{2}}+E^{\frac{-1}{2}}}{2}\right)+\frac{1}{2}\left(E^{\frac{1}{2}}-E^{\frac{-1}{2}}\right)=\frac{1}{2}[E^{\frac{1}{2}}+E^{\frac{-1}{2}}+E^{\frac{1}{2}}-E^{\frac{-1}{2}}=\frac{1}{2}\left(2E^{\frac{1}{2}}\right)=E^{\frac{1}{2}}$

PROVE THAT $\boldsymbol{\Delta=\frac{\delta^2}{2}+\delta.\sqrt{1+\frac{\delta^2}{4}}}$

since $\delta=E^{\frac{1}{2}}-E^{\frac{-1}{2}} \quad \Longrightarrow \delta^2=(E^{\frac{1}{2}}-E^{\frac{-1}{2}})^2 \because Squaring$

$\frac{\delta^2}{2}=\frac{1}{2}(E^{\frac{1}{2}}-E^{\frac{-1}{2}})^2 \because deviding\ by\ (2) \ldots\ldots\ldots\ldots\ldots..(i)$

Also $\frac{\delta^2}{4}=\frac{1}{4}(E^{\frac{1}{2}}-E^{\frac{-1}{2}})^2 \quad \Longrightarrow 1+\frac{\delta^2}{4}=1+\frac{1}{4}(E^{\frac{1}{2}}-E^{\frac{-1}{2}})^2 \quad \therefore adding\ one\ on\ side$

$\sqrt{1+\frac{\delta^2}{4}}=\sqrt{\left(\frac{4+E+E^{-1}-2}{4}\right)^2} \qquad \because taking\ squre$ root on both sides

$\sqrt{1+\frac{\delta^2}{4}}=\sqrt{\left(\frac{E^{1/2}+E^{-1/2}}{4}\right)^2} \quad \Longrightarrow \sqrt{1+\frac{\delta^2}{4}}=\frac{1}{2}\left(E^{\frac{1}{2}}+E^{\frac{1}{2}}\right)$

Now $\delta.\sqrt{1+\frac{\delta^2}{4}}=\delta\frac{\left(E^{\frac{1}{2}}+E^{\frac{-1}{2}}\right)}{2}=\frac{\left(E^{\frac{1}{2}}-E^{\frac{-1}{2}}\right)\left(E^{\frac{1}{2}}+E^{\frac{-1}{2}}\right)}{2}=\frac{E-E^{-1}}{2}\ldots\ldots\ldots\ldots\ldots.(\mathrm{ii})$

$(i)+(ii) \Rightarrow \frac{\delta^2}{2}+\delta.\sqrt{1+\frac{\delta^2}{4}}=\frac{(E^{\frac{1}{2}}-E^{\frac{-1}{2}})^2}{2}+\frac{(E-E^{-1})}{2}=\frac{E+E^{-1}-2+E-E^{-1}}{2}=\frac{2E-2}{2}=\frac{2(E-1)}{2}$

$=E-1=\Delta$ *forward difference operator*

PROVE THAT $\boldsymbol{\frac{\mu}{\sqrt{1+\frac{\delta^2}{4}}}=1}$

$\Longrightarrow \frac{\mu}{\sqrt{1+\frac{\delta^2}{4}}}=\mu\left[1+\frac{\delta^2}{4}\right]^{-\frac{1}{2}}=\mu\left[1+\frac{E^{\frac{1}{2}}-E^{-\frac{1}{2}}}{4}\right]^{-\frac{1}{2}}=\mu\left[\frac{4+\left(E^{\frac{1}{2}}\right)^2+\left(E^{-\frac{1}{2}}\right)^2-2}{4}\right]=\mu\left[\left(\frac{E^{\frac{1}{2}}+E^{-\frac{1}{2}}}{2}\right)^2\right]^{-\frac{1}{2}}$

$\Longrightarrow \mu.\mu^{-1}=1$

PROVE THAT $\boldsymbol{\mu\delta = \frac{\Delta E^{-1}}{2} + \frac{\Delta}{2}}$

$\mu\delta = \frac{1}{2}(E^{\frac{1}{2}} + E^{\frac{-1}{2}})(E^{\frac{1}{2}} - E^{\frac{-1}{2}}) = \frac{1}{2}(E - E^{-1})$

Now since $\Delta = E - 1$ *therefore* $E = 1 + \Delta$

$\mu\delta = \frac{1}{2}[1 + \Delta - E^{-1}] = \frac{\Delta}{2} + \frac{1}{2}(1 - E^{-1}) = \frac{\Delta}{2} + \frac{1}{2}\left(\frac{E-1}{E}\right) = \frac{\Delta}{2} + \frac{\Delta}{2E} = \frac{\Delta E^{-1}}{2} + \frac{\Delta}{2} \quad \because E - 1 = \Delta$

PROVE THAT $\boldsymbol{\mu\sigma = \frac{\Delta + \nabla}{2}}$

$\mu\sigma = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{\frac{1}{2}}\right)\left(E^{\frac{1}{2}} - E^{\frac{1}{2}}\right) = \frac{1}{2}(E - E^{-1}) \Rightarrow$ *Since* $\Delta = E - 1$ *and* $\nabla = 1 - E^{-1} = \frac{E-1}{E}$

therefore $\mu\sigma = \frac{1}{2}(1 + \Delta - 1 + \nabla) = \frac{\Delta + \nabla}{2}$

Show that operators "μ" and "E" commute

From the definition of "μ" and "E" $\Rightarrow \mu E y_0 = \mu y_1 = \frac{1}{2}(y_{\frac{3}{2}} + y_{\frac{1}{2}})$

While $E\mu y_0 = \mu y_1 = \frac{1}{2}E\left(y_{\frac{1}{2}} + y_{\frac{-1}{2}}\right) = \frac{1}{2}\left(y_{\frac{3}{2}} + y_{\frac{1}{2}}\right) \Longrightarrow \mu E = E\mu$

$\Longrightarrow$ *"μ" and "E" commute*

Question:-

Show that forward difference operator is linear operator.

Sol:-

(i)

$$\Delta(cf(x)) = \Delta(cf(x))$$
$$= [cf(x+h) - cf(x)]$$
$$= c[f(x+h) - f(x)]$$
$$= c\,\Delta f(x)$$

(ii)

$$\Delta[f(x)+g(x)] = [f(x+h)+g(x+h)] - [f(x)+g(x)]$$
$$\Delta[f(x)+g(x)] = f(x+h) - f(x) + g(x+h) - g(x)$$
$$= \Delta f(x) + \Delta g(x)$$

Therefore,
$$\Delta[cf(x)+g(x)] = \Delta[cf(x)] + \Delta g(x)$$
$$= c\Delta f(x) + \Delta g(x)$$

Hence, Δ is a linear operator.

Question:-

Show that Backward diff operator is linear operator.

Sol:-

(i) $\nabla(cf(x)) = [cf(x) - cf(x-h)]$
$$= c[f(x) - f(x-h)] = c\nabla f(x)$$

(ii)

$$\nabla[f(x)+g(x)] = f(x) - f(x-h) + g(x) - g(x-h)$$
$$= \nabla f(x) + \nabla g(x)$$

Therefore, $\nabla[cf(x)+g(x)] = \nabla[cf(x)] + \nabla g(x)$
$$= c\nabla f(x) + \nabla g(x)$$

Hence, ∇ is a linear operator.

Ex:-

If y_x is a polynomial of nth degree, then show that its nth order forward difference is constant and (n+1) order " " " is zero.

Sol:-

Let $y_x = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

$$\Delta y_x = y_{x+h} - y_x$$

$$\Delta y_x = a_0 + a_1(x+h) + a_2(x+h)^2 + \cdots + a_n(x+h)^n - a_0 - a_1 x - a_2 x^2 - \cdots - a_n x^n$$

$$\Delta y_x = a_1[(x+h) - x] + a_2[(x+h)^2 - x^2] + \cdots + a_n[(x+h)^n - x^n]$$

$$= a_1 h + a_2[x^2 + h^2 + 2hx - x^2] + \cdots + a_{n-1}\{x^{n-1} + x^{n-2}(n-1)h + \frac{(n-1)(n-2)}{2!}h^2 x^{n-3} + \cdots - x^{n-1}\} + a_n\{x^n + nhx^{n-1} + \frac{n(n-1)}{2!}h^2 x^{n-2} + \cdots - x^n\}$$

$$= a_1 h + a_2[h^2 + 2hx] + \cdots + a_{n-1}\{x^{n-2}(n-1)h + \frac{(n-1)(n-2)}{2!}h^2 x^{n-3} + \cdots\} + a_n\{nhx^{n-1} + \frac{n(n-1)}{2!}h^2 x^{n-2} + \cdots\}$$

$$\Delta y_x = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

$$\Delta^2 y_x = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-2} x^{n-2}$$

or

$$\Delta^2 y_x = \Delta . \Delta y_x$$
$$= \Delta[y_{x+h} - y_x]$$

Polynomial of degree $n-n = 0$, which is a constant.

$$\Delta^n y_x = a_n n! h^n$$

$$\Delta^{n+1} y_x = a_n n! h^n - a_n n! h^n$$
$$= 0$$

EXTRAPOLATION

The method of computing the values of 'y' for a given value of 'x' lying outside the table of values of 'x' is called Extrapolation.

REMEMBER THAT

RATE OF CONVERGENCE OF AN ITERATIVE METHOD: Suppose that the sequence (x_n) converges to "r" then the sequence (x_n) is said to converge to "r" with order of convergence "a" if there exist a positive constant "p" such that

$$\lim_{n\to\infty}\frac{|x_{n+1}-r|}{|x_n-r|^a} = \lim_{n\to\infty}\frac{\epsilon_{n+1}}{\epsilon_n^\alpha} = p\ (error\ constant\ )$$

AITKIN'S EXTRAPOLATION (no need) : Since we have rate of convergence of an iterative method $\lim_{n\to\infty}\frac{|x_{n+1}-r|}{|x_n-r|^a} = \lim_{n\to\infty}\frac{\epsilon_{n+1}}{\epsilon_n^\alpha} = p\ (error\ constant\ )$

Now suppose that $n > 1$ then it approximately yields;

$x_n - r \approx p(x_{n-1} - r)$ also $x_{n+1} - r \approx p(x_n - r)$

Dividing both we get $\frac{x_n-r}{x_{n+1}-r} \approx \frac{p(x_{n-1}-r)}{p(x_n-r)}$

$$\Rightarrow \frac{x_n-r}{x_{n+1}-r} \approx \frac{(x_{n-1}-r)}{(x_n-r)} \Rightarrow (x_n - r)^2 \approx (x_{n-1} - r)(x_{n+1} - r)$$

$$\Rightarrow {x_n}^2 + r^2 - 2x_n r \approx x_{n+1}x_{n-1} - r(x_{n+1} + x_{n-1}) + r^2$$

$$\Rightarrow {x_n}^2 - x_{n+1}x_{n-1} \approx -r(x_{n+1} + x_{n-1} - 2x_n)$$

$$\Rightarrow r \approx \frac{x_{n+1}x_{n-1}-{x_n}^2}{x_{n+1}+x_{n-1}-2x_n} = x_{n-1} - \frac{(x_n-x_{n-1})^2}{x_{n+1}+x_{n-1}-2x_n} \Rightarrow x_n = x_{n-1} - \frac{(x_n-x_{n-1})^2}{x_{n+1}+x_{n-1}-2x_n}$$

$$\boldsymbol{x_{n+1} = x_n - \frac{(x_{n+1}-x_n)^2}{x_{n+2}+x_n-2x_{n+1}} = x_{n+2} - \frac{(x_{n+2}-x_{n+1})^2}{x_{n+2}+x_n-2x_{n+1}} = x_n - \frac{(\Delta x_n)^2}{\Delta^2 x_n}}$$

NOTE: This method is used to accelerate convergence of sequence. i.e. sequence obtained from iterative method.

INTERPOLATION

For a given table of values $(\boldsymbol{x_k}, \boldsymbol{y_k})\ \forall\ \boldsymbol{K = 0, 1, 2, \ldots \ldots n}$. the process of estimating the values of "y=f(x)" for any intermediate values of "x = g(x)" is called "**interpolation**".

If g(x) is a Polynomial, Then the process is called "Polynomial" Interpolation.

ERROR OF APPROXIMATION

The deviation of g(x) from f(x) i.e. |f(x) – g (x)| is called Error of Approximation.

REMARK: A function is said to interpolate a set of data points if it passes through those points.

INVERSE INTERPOLATION

Suppose $f \epsilon\ C\ [a, b]$, $f'(x)\ \neq\ o$ on [a, b] and f has non- zero 'p' in [a, b]

Let "$x_0, x_1 \ldots \ldots \ldots . x_n$" be 'n+1' distinct numbers in [a, b] with $f\ (x_k)\ =\ y_k$ for each $k\ =\ 0,1,2\ \ldots . n$.

To approximate 'p' construct the interpolating polynomial of degree 'n' on the nodes "$y_0, y_1 \ldots \ldots \ldots y_n$" for "$f^{-1}$"

Since "$y_k = f\ (x_k)$" and f (p) =0, it follows that f^{-1} $(y_k) = X_k$ and p = f^{-1} (0).

"Using iterated interpolation to approximate $f^{-1}(0)$ is called iterated Inverse interpolation"

OR for given the set of values of x and y , the process of finding the values of x for certain values of y is called inverse interpolation.

LINEAR INTERPOLATION FORMULA

$$f(x) = p_1(x) = f_0 + p(f_1 - f_0) = f_0 + p\Delta f_0$$

Where $x = x_0 + ph \quad \Rightarrow p = \frac{x - x_0}{h} \qquad 0\ \leq P \leq 1$

QUADRATIC INTERPOLATION FORMULA

$$f(x) = p_2(x) = f_0 + p\Delta f_0 + \frac{P(P-1)}{2}\Delta^2 f_0$$

Where $x = x_0 + ph \quad \Rightarrow p = \frac{x - x_0}{h} \qquad 0\ \leq P \leq 2$

ERRORS IN POLYNOMIAL INTERPOLATION

Given a function f(x) and $a \leq x \leq b$, a set of distinct points $x_i\ i = 1, 2, \ldots \ldots . n$ and $x_i \epsilon$ [a, b]

Let $P_n(x)$ be a polynomial of degree ≤ n that interpolates f(x) at x_i

i.e. $P_n(x_i) = f(x_i)\quad ;\ i = 1, 2, 3, \ldots \ldots \ldots . . n$

Then Error define as "$\in (x)\ =\ f(x) - P_n(x)$"

REMARK

Sometime when a function is given as a data of some experiments in the form of tabular values corresponding to the values of independent variable 'X' then

1. Either we interpolate the data and obtain the function "f(x)" as a polynomial in 'x' and then differentiate according to the usual calculus formulas.
2. Or we use Numerical Differentiation which is easier to perform in case of Tabular form of the data.

DISADVANTAGES OF POLYNOMIAL INTERPOLATION

- n-time differentiable
- big error in certain intervals (especially near the ends)
- No convergence result
- Heavy to compute for large "n"

EXISTENCE AND UNIQUENESS THEOREM FOR POLYNOMIAL INTERPOLATION

Given $(x_i\,, y_i)_{i=0}^{n}$ with Xi's distinct there exists one and only one Polynomial $P_n(x)$ of degree $\leq n$ such that $P_n(x_i) = y_i \quad ; i = 1, 2, \dots \dots \dots . . n$

PROOF

Existence Ok from construction.

For Uniqueness:

Assume we have two polynomials P(x), q(x) of degree ≤ n both interpolate the data i.e.

$p(x_i) = y_i = q(x_i) \quad ; i = 1, 2, \dots \dots \dots . . n$

Now let $g(x) = P(x) - q(x)$ which will be a polynomial of degree ≤ n

Furthermore, we have $\quad g(x_i) = p(x_i) - q(x_i) = y_i - y_i = 0 \quad ; i = 0,1,2, \dots \dots \dots . . n$

So g(x) has 'n+1' Zeros. We must have g(x)≡ 0. Therefor p(x)≡ g(x).

REMEMBER:

Using Newton's Forward difference interpolation formula we find the n-degree polynomial $'P_n'$ which approximate the function $f(x)$ in such a way that $'P_n'$ and 'f' agrees at 'n+1' equally Spaced 'X' Values. So that

$P_n(x_0) = f_0, P_n(x_1) \cdots \dots \dots \dots \dots \dots , P_n(x_n) = f_n$

Where $\quad f_0 = f(x_0),\; f_1 = f(x_1) \dots \dots \dots \dots . \cdots f_n = f_{(x_n)}$ Are the values of ' f ' in table.

INTERPOLATION BY ITERATION:

Given the (n+1) points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots\dots\dots, (x_n, y_n)$ where the values of x need not necessarily be equally spaced, then to find the value of '$y = f(x)$' corresponding to any given value of 'x' we proceed iteratively as follows;

Obtain a first approximation to '$y = f(x)$' by considering the first two points only i.e.

$$\Delta_{01}(x) = f_0 + (x - x_0)f[x_0, x_1] = \frac{1}{x_1 - x_0}\begin{vmatrix} f_0 & x_0 & -x \\ f_1 & x_1 & -x \end{vmatrix}$$

$$\Delta_{02}(x) = f_1 + (x - x_1)f[x_1, x_2] = \frac{1}{x_2 - x_1}\begin{vmatrix} f_1 & x_1 & -x \\ f_2 & x_2 & -x \end{vmatrix}$$

$$\Delta_{03}(x) = f_2 + (x - x_2)f[x_2, x_3] = \frac{1}{x_3 - x_2}\begin{vmatrix} f_2 & x_2 & -x \\ f_3 & x_3 & -x \end{vmatrix}$$

And so on ………………

Then obtain a first approximation to '$y = f(x)$' by considering the first three points only i.e.

$\Delta_{012}(x) = \frac{1}{x_2 - x_1}\begin{vmatrix} \Delta_{01}(x) & x_1 & -x \\ \Delta_{02}(x) & x_2 & -x \end{vmatrix}$ Similarly $\Delta_{013}(x), \Delta_{014}(x)$ And so on ………………

And at the nth stage of approximation we obtain;

$$\Delta_{012\dots n}(x) = \frac{1}{x_n - x_{n-1}}\begin{vmatrix} \Delta_{012\dots(n-1)}(x) & x_{n-1} & -x \\ \Delta_{012\dots(n-2n)}(x) & x_n & -x \end{vmatrix}$$

Then using Table of Aitken's Scheme

X	Y= f(x)				
x_0	y_0				
		$\to \Delta_{01}(x)$			
x_1	y_1		$\to \Delta_{012}(x)$		
		$\to \Delta_{02}(x)$		$\to \Delta_{0123}(x)$	
x_2	y_2		$\to \Delta_{013}(x)$		$\to \Delta_{01234}(x)$
		$\to \Delta_{03}(x)$		$\to \Delta_{0124}(x)$	
x_3	y_3		$\to \Delta_{014}(x)$		
		$\to \Delta_{04}(x)$			
x_4	y_4				

QUESTION: Use Aitkin's iteration formula to find the value of $\boldsymbol{log_{10}(301)}$ as accurately as possible from the data

x	:	300	304	305	307
Y = f(x)	:	2.4771	2.4829	2.4843	2.4871

Solution:

X	Y= f(x)			
300	2.4771			
	$\longrightarrow$	2.47855		
304	2.4829	$\longrightarrow$	2.47858	
	$\longrightarrow$	2.47854	$\longrightarrow$	2.47860
305	2.4843	$\longrightarrow$	2.47857	
	$\longrightarrow$	2.47853		
307	2.4871			

Hence required solution is $log_{10}(301) = 2.47860$

QUESTION: Use Aitkin's iteration formula to find the value of $\boldsymbol{log(4.5)}$ as accurately as possible from the data

x	:	4	4.2	4.4	4.6
Y = f(x)	:	0.60206	0.62325	0.64345	0.66276

And Table of Neville's Scheme

X	Y= f(x)						
$\boldsymbol{x_0}$	$\boldsymbol{y_0}$						
	$\longrightarrow$	$\Delta_{01}(x)$					
$\boldsymbol{x_1}$	$\boldsymbol{y_1}$	$\longrightarrow$	$\Delta_{012}(x)$				
	$\longrightarrow$	$\Delta_{12}(x)$	$\longrightarrow$	$\Delta_{0123}(x)$			
$\boldsymbol{x_2}$	$\boldsymbol{y_2}$	$\longrightarrow$	$\Delta_{123}(x)$	$\longrightarrow$	$\Delta_{01234}(x)$		
	$\longrightarrow$	$\Delta_{23}(x)$	$\longrightarrow$	$\Delta_{1234}(x)$			
$\boldsymbol{x_3}$	$\boldsymbol{y_3}$	$\longrightarrow$	$\Delta_{234}(x)$				
	$\longrightarrow$	$\Delta_{34}(x)$					
$\boldsymbol{x_4}$	$\boldsymbol{y_4}$						

Inverse Interpolation

Given a set of values of x and y, the process of finding the value of x for a certain value of y is called *inverse interpolation*. When the values of x are at unequal intervals, the most obvious way of performing this process is by interchanging x and y in Lagrange's or Aitken's methods.

Example If $y_1 = 4, y_3 = 12, y_4 = 19$ and $y_x = 7$, find x. Compare with the actual value.

Solution

Aitken's scheme (see Table 1) is

y	x		
4	1		
		1.750	
12	3		1.857
		1.600	
19	4		

whereas Neville's scheme (see Table 2) gives

y	x		
4	1		
		1.750	
12	3		1.857
		2.286	
19	4		

In this examples both the schemes give the same result.

"δ" CENTRAL DIFFERENT OPERATOR (DELTA IN LOWER CASE)

Central Different Operator for y=f(x) defined as

$\delta y_i = y_{i+\frac{1}{2}} - y_{i-\frac{1}{2}} \qquad \forall\, i = 1,2,\ldots.n$

(OR) $\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$ (OR) $\delta y_x = y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}}$

TABLE

X	Y		δY		$\delta^2 Y$		$\delta^3 Y$
x_0	y_0						
		$\longrightarrow$	$\delta y_{\frac{1}{2}} = y_1 - y_0$				
x_1	y_1			$\longrightarrow$	$\delta^2 y_1$		
		$\longrightarrow$	$\delta y_{\frac{3}{2}}$			$\longrightarrow$	$\delta^3 y_{\frac{3}{2}}$
x_2	y_2			$\longrightarrow$	$\delta^2 y_2$		
		$\longrightarrow$	$\delta y_{\frac{5}{2}}$				
x_3	y_3						

CONSTRUCTION OF FORWARD DIFFERENCE TABLE (Also called Diagonal difference table)

X	Y		Δy		$\Delta^2 y$		$\Delta^3 y$		$\Delta^4 y$
x_0	y_0								
		$\longrightarrow$	$\Delta y_0 = y_1 - y_0$						
x_1	y_1			$\longrightarrow$	$\Delta^2 y_0$				
		$\longrightarrow$	$\Delta y_1 = y_2 - y_1$			$\longrightarrow$	$\Delta^3 y_0$		
x_2	y_2			$\longrightarrow$	$\Delta^2 y_1$			$\longrightarrow$	$\Delta^4 y_0$
		$\longrightarrow$	$\Delta y_2 = y_3 - y_2$			$\longrightarrow$	$\Delta^3 y_1$		
x_3	y_3			$\longrightarrow$	$\Delta^2 y_2$				
		$\longrightarrow$	$\Delta y_3 = y_4 - y_3$						
x_4	y_4								

QUESTION: Construct forward difference Table for the following value of 'X' and 'Y'

X	0.1	0.3	0.5	0.7	0.9	1.1	1.3
Y	0.003	0.067	0.148	0.248	0.370	0.518	0.697

SOLUTION

X	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0.1	0.003						
	→	0.064					
0.3	0.067	→	0.017				
	→	0.081	→	0.002			
0.5	0.148	→	0.019	→	0.001		
	→	0.100	→	0.003	→	0	
0.7	0.248	→	0.022	→	0.001	→	0
	→	0.122	→	0.004	→	0	
0.9	0.370	→	0.026	→	0.001		
	→	0.148	→	0.005			
1.1	0.518	→	0.031				
	→	0.179					
1.3	0.697						

Example **Construct the forward difference table for the data**

$$x: -2 \quad 0 \quad 2 \quad 4$$
$$y = f(x): \quad 4 \quad 9 \quad 17 \quad 22$$

The forward difference table is as follows:

x	y=f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$
-2	4			
		$\Delta y_0 = 5$		
0	9		$\Delta^2 y_0 = 3$	
		$\Delta y_1 = 8$		$\Delta^3 y_0 = -6$
2	17		$\Delta^2 y_1 = -3$	
		$\Delta y_2 = 5$		
4	22			

BACKWARD DIFFERENCE OPERATOR "∇" (NEBLA)

We Define Backward Difference Operator as

$\nabla y_n = y_n - y_{n-1} \quad \forall\, n = 1,2,\dots\dots i$ (OR) $\nabla f(x) = f(x) - f(x-h)$

(OR) $\nabla y_x = y_x - y_{x-h}$

BACKWARD DIFFERENCE TABLE

X	Y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		$\longrightarrow \nabla y_1 = y_1 - y_0$		
x_1	y_1		$\longrightarrow \nabla^2 y_2$	
		$\longrightarrow \nabla y_2 = y_2 - y_1$		$\longrightarrow \nabla^3 y_3$
x_2	y_2		$\longrightarrow \nabla^2 y_3$	
		$\longrightarrow \nabla y_3 = y_3 - y_2$		
x_3	y_3			

NEWTON FORWARD DIFFERENCE INTERPOLATION FORMULA

Newton's Forward Difference Interpolation formula is

$$f(x) = P_n(x)$$
$$= f(x_0) + P\Delta f(x_0) + \frac{P(P-1)}{2!}\Delta^2 f(x_0) + \cdots + \frac{P(P-1)\cdots(P-n+1)}{n!}\Delta^n f(x_0)$$

Where $x = x_0 + ph,\quad P = \frac{x-x_0}{h}\quad And\ 0 \le p \le n$

DERIVATION:

Let $y = f(x),\quad x_0 = f(x_0)\quad And\quad x_n = x_0 + nh\quad \Rightarrow x = x_0 + ph$

$$f(x) = f(x_0+ph) = E^p f(x_0) = (1+\Delta)^p f(x_0) \qquad \therefore E = 1+\Delta$$

$$= [1 + P\Delta + \frac{P(P-1)}{2!} + \cdots + \frac{P(P-1)\cdots(P-n+1)}{n!}]f(x_0)$$

$$f(x) = f(x_0) + P\Delta f(x_0) + \cdots + \frac{P(P-1)\cdots P-n+1)}{n!}f(x_0)$$

CONDITION FOR THIS METHOD

- Values of 'x' must have equal distance i.e. equally spaced.
- Value on which we find the function check either it is near to start or end.
- If near to start, then use forward method.
- If near to end, then use backward method.

QUESTION: Evaluate $f(15)$ given the following table of values

X	:	10	20	30	40	50
f(x)	:	46	66	81	93	101

SOLUTION: Here '15' nearest to starting point we use Newtown's Forward Difference Interpolation.

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	$\Delta^4 Y$
10	46				
		20			
20	66		-5		
		15		2	
30	81		-3		-3
		12		-1	
40	93		-4		
		8			
50	101				

$$f(x) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \frac{P(P-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$\therefore x = x_0 + ph \Rightarrow 15 = 10 + P(10) \Rightarrow P = 0.5$$

$$f(15) = 46 + (0.5)(20) + \frac{(0.5)(0.5-1)}{2!}(-5) + \frac{(0.5)(0.5-1)(0.5-2)}{3!}(2) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{4!}(-3)$$

$$\Rightarrow f(15) = 56.8672$$

Therefore, f (...)

Example 6.9 Find Newtons forward difference interpolating polynomial for the following data:

x	0.1	0.2	0.3	0.4	0.5
$y = f(x)$	1.40	1.56	1.76	2.00	2.28

Solution We shall first construct the forward difference table to the given data as indicated below:

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.1	1.40				
		0.16			
0.2	1.56		0.04		
		0.20		0.00	
0.3	1.76		0.04		0.00
		0.24		0.00	
0.4	2.00		0.04		
		0.28			
0.5	2.28				

Since, third and fourth leading differences are zero, we have Newton's forward difference interpolating formula as

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 \tag{1}$$

In this problem, $x_0 = 0.1$, $y_0 = 1.40$, $\Delta y_0 = 0.16$, $\Delta^2 y_0 = 0.04$, and

$$p = \frac{x - 0.1}{0.1} = 10x - 1$$

Substituting these values in Eq. (1), we obtain

$$y = f(x) = 1.40 + (10x - 1)(0.16) + \frac{(10x - 1)(10x - 2)}{2}(0.04)$$

That is, $y = 2x^2 + x + 1.28$. This is the required Newton's interpolating polynomial.

Example 6.10 Estimate the missing figure in the following table:

x	1	2	3	4	5
$y = f(x)$	2	5	7	–	32

Solution Since we are given four entries in the table, the function $y = f(x)$ can be represented by a polynomial of degree three. Using Theorem 6.1, we have

$$\Delta^3 f(x) = \text{Constant} \quad \text{and} \quad \Delta^4 f(x) = 0$$

for all x. In particular, $\Delta^4 f(x_0) = 0$. Equivalently, $(E - 1)^4 f(x_0) = 0$. Expanding, we have

$$(E^4 - 4E^3 + 6E^2 - 4E + 1) f(x_0) = 0$$

That is,

$$f(x_4) - 4f(x_3) + 6f(x_2) - 4f(x_1) + f(x_0) = 0$$

Using the values given in the table, we obtain

$$32 - 4f(x_3) + 6 \times 7 - 4 \times 5 + 2 = 0$$

which gives $f(x_3)$, the missing value equal to 14.

Example 6.11 Find a cubic polynomial in x which takes on the values −3, 3, 11, 27, 57 and 107, when $x = 0, 1, 2, 3, 4$ and 5 respectively.

Solution Here, the observations are given at equal intervals of unit width. To determine the required polynomial, we first construct the difference table as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	−3			
		6		
1	3		2	
		8		6
2	11		8	
		16		6
3	27		14	
		30		6
4	57		20	
		50		
5	107			

Since the fourth and higher order differences are zero, we have the required Newton's interpolation formula in the form

$$f(x_0 + ph) = f(x_0) + p\Delta f(x_0) + \frac{p(p-1)}{2}\Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{6}\Delta^3 f(x_0) \qquad (1)$$

Here,

$$p = \frac{x - x_0}{h} = \frac{x - 0}{1} = x, \quad \Delta f(x_0) = 6, \quad \Delta^2 f(x_0) = 2, \quad \Delta^3 f(x_0) = 6$$

Substituting these values into Eq. (1), we have

$$f(x) = -3 + 6x + \frac{x(x-1)}{2}(2) + \frac{x(x-1)(x-2)}{6}(6)$$

i.e. $f(x) = x^3 - 2x^2 + 7x - 3$ is the required cubic polynomial.

NEWTONS'S BACKWARD DIFFERENCE INTERPOLATION FORMULA

Newton's Backward Difference Interpolation formula is

$$y_x = f(x) \approx P_n\,(x)$$

$$= f(x_n) + P\nabla f(x_n) + \frac{P(P+1)}{2!}\nabla^2 f(x_n) + \cdots\cdots + \frac{P(P+1)(P+2)\cdots\cdots(P+n-1)}{n!}\nabla^n f(x_n)$$

$$Where\ x = x_n + ph,\ \ p = \frac{x-x_n}{h};\ \ -n \le P \le 0$$

DERIVATION: Let $y = f(x), x_n = f(x_n)$ and $x = x_n + Ph$ Then

$$f(x_n + Ph) = E^P f(x_n) = (E^{-1})^{-P} f(x_n) \quad = (1-\nabla)^{-P} f(x_n) \qquad \therefore E^{-1} = 1-\nabla$$

Using binomial expansion $f(x) = \left[1 + P\nabla + \frac{P(P+1)}{2!}\nabla^2 + \frac{P(P+1)(P+2)}{3!}\nabla^3 + \cdots\right] f(x_n)$

$$\boldsymbol{f(x) = f(x_n) + P\nabla f(x_n) + \frac{P(P+1)}{2!}\nabla^2 f(x_n) + \cdots\cdots}$$

This is required **Newton's Gregory** Backward Difference Interpolation formula.

QUESTION: For the following table of values estimate f(7.5)

X	1	2	3	4	5	6	7	8
f(x)	1	8	27	64	125	216	343	512

SOLUTION

Since '7.5' is nearest to End of table, So We use Newton's Backward Interpolation.

X	Y	∇Y	$\nabla^2 Y$	$\nabla^3 Y$	$\nabla^4 Y$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		0
		91		6	
6	216		36		0
		127		6	
7	243		42		
		169			

8 512

Since $P = \frac{x - x_n}{h} \Rightarrow P = \frac{7.5-8}{1} \Rightarrow P = -0.5$

Now $y = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n$

$$y = 512 + (-0.5)(169) + \frac{(-0.5)(-0.5+1)}{2!}(42) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}(6)$$

$$y = 512 - 84.5 - 5.26 - 0.375 = f(x) = 421.875$$

Example 6.13 The sales in a particular department store for the last five years is given in the following table:

Year	1974	1976	1978	1980	1982
Sales (in lakhs)	40	43	48	52	57

Estimate the sales for the year 1979.

Solution At the outset, we shall construct Newton's backward difference table for the given data as

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1974	40				
		3			
1976	43		2		
		5		−3	
1978	48		−1		5
		4		2	
1980	52		1		
		5			
1982	57				

In this example,

$$p = \frac{1979 - 1982}{2} = -1.5$$

and

$$\nabla y_n = 5, \quad \nabla^2 y_n = 1, \quad \nabla^3 y_n = 2, \quad \nabla^4 y_n = 5$$

Newton's interpolation formula gives

$$y_{1979} = 57 + (-1.5)5 + \frac{(-1.5)(-0.5)}{2}(1) + \frac{(-1.5)(-0.5)(0.5)}{6}(2) + \frac{(-1.5)(-0.5)(0.5)(1.5)}{24}(5)$$

$$= 57 - 7.5 + 0.375 + 0.125 + 0.1172$$

Therefore,

$$y_{1979} = 50.1172$$

LAGRANGE'S INTERPOLATION FORMULA

For points $x_0, x_1 \ldots\ldots\ldots\ldots x_n$ define the cardinal Function

$l_0, l_1 \ldots\ldots\ldots\ldots l_n \in P^n$ (polynomial of n-degree)

$$l_i(x_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \qquad \mathrm{i} = 0, 1, 2, \ldots\ldots \mathrm{n}$$

The Lagrange form of interpolation Polynomial is $p_n(x) = \sum_{i=0}^{n} l_i(x) y_i$

DERIVATION OF FORMULA

Let y=f (x) be a function which takes the values $y_0, y_1, y_2 \ldots\ldots\ldots y_n$ so we will obtain an n-degree polynomial $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots\ldots\ldots\ldots + a_n$

Now $(i)\ldots\ldots\ldots\ldots \begin{cases} y = f(x) = a_0(x-x_1)(x-x_2)\cdots\cdots(x-x_n) \\ +a_1(x-x_0)(x-x_2)\cdots\cdots(x-x_n) \\ +a_2(x-x_0)(x-x_1)(x-x_3)\cdots\cdots(x-x_n) \\ \vdots \\ \vdots \\ +a_n(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1}) \end{cases}$

Now we find the constants $a_0, a_1, \cdots\cdots a_n$

Put x= x_0 in (i)

$$(i) \Rightarrow \begin{cases} y = f(x) = a_0(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n) \\ +a_1(x_0-x_0)(x_0-x_2)\cdots\cdots(x_0-x_n) \\ +a_2(x_0-x_0)(x_0-x_1)(x_0-x_3)\cdots\cdots(x_0-x_n) \\ \vdots \\ \vdots \\ +a_n(x_0-x_0)(x_0-x_1)\cdots\cdots(x_0-x_{n-1}) \end{cases}$$

$\Rightarrow \quad y_o = a_o(x_o-x_1)(x_o-x_2)\cdots\cdots(x_o-x_n)$

$\Rightarrow \quad a_0 = y_1 \div [(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_1)]$

Now Put x=x_1 in

$y_1 = f(x_1) = a_1(x_1-x_o)(x_1-x_2)\cdots\cdots(x_1-x_n)$

$\Rightarrow \quad a_1 = y_1 \div [(x_n-x_0)(x_n-x_2)\cdots\cdots(x_n-x_{n-1})]$

Similarly $a_n = y_n \div [(x_n-x_0)(x_n-x_1)\cdots\cdots(x_n-x_{n-1})]$

Putting all the values in (i) we get

$y = f(x) = \ y_0 \frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)} \quad + \quad y_1 \frac{(x-x_0)(x-x_2)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)} \quad +\ldots\ldots\ldots\ldots\ldots$
$\ldots\ldots\ldots\ldots + y_n \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots\ldots(x_n-x_{n-1})}$

$\Rightarrow y = f(x) = l_0y_0 + l_1y_1 + l_2y_2 + \cdots\cdots + l_ny_n = \sum_{k=0}^{n} l_ky_k$

Where $l_k(x) = \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{k-1})(x-x_{k+1})\ldots\ldots(x-x_n)}{(x_k-x_0)(x_k-x_1)\ldots\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots\ldots(x_k-x_n)}$

ALTERNATIVELY DEFINE

$\pi(x) = (x-x_0)(x-x_1)\ldots\ldots(x-x_n)$

Then $\pi'(x) = (1-0)[(x-x_1)(x-x_2)\ldots\ldots(x-x_n)]$

$+(1-0)[(x-x_0)(x-x_2)(x-x_3)\ldots\ldots(x-x_n)]\ldots\ldots\ldots\ldots\ldots\ldots$

$+(1-0)[(x-x_0)(x-x_1)(x-x_2)\ldots..(x-x_{n-1})]$

$\pi'(x_k)=(x_k-x_0)(x_k-x_1)\ldots\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots\ldots(x_k-x_n)$

$l_k(x)=\frac{(x-x_k)}{(x-x_k)}\cdot\frac{(x-x_0)(x-x_1)\ldots\ldots..(x-x_{k-1})(x-x_{k+1})\ldots\ldots.(x-x_n)}{(x_k-x_0)(x_k-x_1)\ldots\ldots..(x_k-x_{k-1})(x-x_{k+1})\ldots\ldots.(x_k-x_n)}$

Then $l_k(x)=\frac{\pi(x)}{(x-x_k)\pi'(x)}$

CONVERGENCE CRITERIA

Assume a triangular array of interpolation nodes $x_i = x_i^{(n)}$ exactly '$n+1$' distinct nodes for "$n = 0,1,2\ldots\ldots..i$"

$x_0^{(0)}$

$x_0^{(1)}\quad x_1^{(1)}$

$x_0^{(2)}\quad x_1^{(2)}\quad x_2^{(2)}$

$x_0^{(n)}\quad x_1^{(n)}\quad x_2^{(n)}\ldots\ldots\ldots x_n^{(n)}$

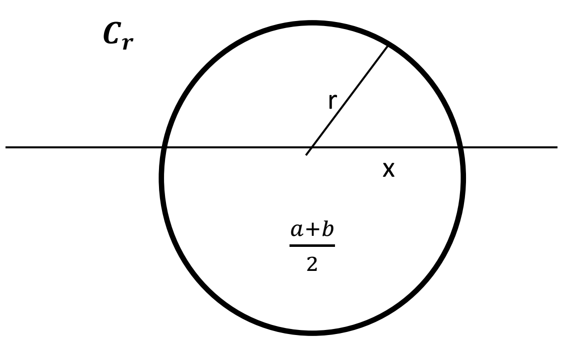
Further assume that all nodes $x_i^{(n)}$ are contained in finite interval $[a,b]$ then for each 'n' we define

$P_n(x)=P_n\left(f;\ \ x_0^{(n)},x_1^{(n)},,,,,,,,x_n^{(n)}\right),\quad x\epsilon[a,b]$

Then we say method "converges" if $P_n(x)\rightarrow f(x)$ as $n\rightarrow\infty$ uniformly for $x\in[a,b]$

(OR)

Lagrange's interpolation converges uniformly on [a, b] for on arbitrary triangular ret if nodes of 'f' is analytic in the circular disk '$\boldsymbol{C_r}$' centered at $\boldsymbol{\frac{a+b}{2}}$ and having radius 'r' sufficiently large. So that $\boldsymbol{r>\frac{3}{2}(b-a)}$ holds.



PROVE THAT $\int_a^b f(x)dx = \frac{b-a}{2}\,[f(a)+f(b)]$

PROOF: Using Lagrange's formula for $n = 1$ $\quad f(x) = \sum_{k=0}^{1} l_k(x) f(x_k) \quad k = 0,1$

$$f(x) = l_0(x)f(x_0) + l_1(x)f(x_1)$$

Integrating over $[a, b]$ when $x_o = a\,,\ x_1 = b$

$$\int_a^b f(x)\,dx = \int_a^b l_o(x)f(x_0)\,dx + \int_a^b l_1(x)f(x_1)dx$$

$$\int_a^b f(x)\,dx = f(x_0)\int_{x_0}^{x_1} l_0(x)dx + f(x_1)\int_{x_0}^{x_1} l_1(x)dx$$

Now $\quad l_0(x) = \frac{x-x_1}{x_0-x_1} \qquad l_1(x) = \frac{x-x_0}{x_1-x_0}$

$$\int_a^b f(x)\,dx = f(x_0)\int_{x_0}^{x_1} \frac{x-x_1}{x_0-x_1}dx + f(x_1)\int_{x_0}^{x_1} \frac{x-x_0}{x_1-x_0}dx$$

Let $\quad x = x_0 + ph \Rightarrow dx = hdp \quad as \quad x \to x_0 \quad then \quad p \to 0 \quad also \quad x \to x_1\ then \quad p \to 1$

$$\int_a^b f(x)\,dx = \int_0^1 \frac{x_0+ph-x_1}{a-b}hdp.f(x_0) + f(x_1)\int_0^1 \frac{x_0+ph-x_0}{b-a}hdp$$
$$\int_a^b f(x)\,dx = \int_0^1 \frac{a+ph-b}{a-b}hdp.f(x_0) + f(x_1)\int_0^1 \frac{x_0+ph-x_0}{b-a}hdp$$

$$f(x) = f(x_0)\int_0^1 \frac{a-b+ph}{-h}hdp + f(x_1)\int_0^1 \frac{ph.hdp}{h} \qquad \therefore x_0 = a_1 \quad x_1 = b$$

$$f(x) = f(x_0)\int_0^1 \frac{-h+ph}{-1}dp + f(x_1)\int_0^1 ph\,dp = -hf(x_0)\int_0^1 (p-1)dp + f(x_1)\,h\int_0^1 p\,dp$$

$$f(x) = -hf(x_0)\left[\left|\frac{p^2}{2}\right|_0^1 - |p|_0^1\right] + hf(x_1)\left|\frac{p^2}{2}\right|_0^1$$

$$f(x) = -hf(x_0)\left(\frac{1}{2}-1\right) + hf(x_1)\left(\frac{1}{2}\right) = -\frac{h}{2}f(x_0) + hf(x_0) + \frac{h}{2}f(x_1)$$

$$f(x) = \frac{h}{2}f(x_0) + \frac{h}{2}f(x_1) = \frac{h}{2}[f(x_0)+f(x_1)] = \frac{b-a}{2}[f(a)+f(b)]$$

Since $h = \frac{b-a}{n} = b - a \ \ for\ n = 1 \quad x_0 = a \quad and \ \ x_1 = b$ Hence the result .

PROS AND CONS OF LAGRANGE'S POLYNOMIAL

- Elegant formula $(+)$
- Slow to compute, each $\boldsymbol{l_i(x)}$ is different $(-)$

- Not flexible; if one change a point x_j , or add an additional point x_{n+1} one must re-compute all $\boldsymbol{l_{i's}}(-)$

INVERSE LAGRANGIAN INTERPOLATION : Interchanging 'x' and 'y' in Lagrange's interpolation formula we obtain the inverse given by $\boldsymbol{x \approx l_n(y) = \sum_{i=0}^{n} \frac{l_k(y)}{l_k(y_k)} x_k}$

QUESTION: Find langrage's Interpolation polynomial fitting the points $\boldsymbol{y(1) = -3}$, $\boldsymbol{y(3) = 0, y(4) = 30, y(6) = 132}$, Hence find $\boldsymbol{y(5)}$ =?

X:	$\boldsymbol{x_0}$=1	$\boldsymbol{x_1}$=3	$\boldsymbol{x_2}$=4	$\boldsymbol{x_3}$=6
Y:	-3	0	30	132

ANSWER: Since $y(x) = l_o y_o + l_1 y_1 + l_2 y_2 + l_3 y_3$

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_o + \frac{(x-x_o)(x-x_2)(x-x_3)}{(x_1-x_o)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_o)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_o)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

By putting values, we get

$$y(x) = \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)}(-3) + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)}(0) + \frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)}(30) + \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)}(132)$$

$$y(x) = \frac{1}{2}[-x^3 + 27x^2 - 92x + 60]$$

Put $x = 5$ to get $y(5) = \frac{1}{2}[-5^3 + 27(5^2) - 92(5) + 60] \implies$ Y (5) =75

Example 6.15 Given the following data, evaluate $f(3)$ using Lagrange's interpolating polynomial.

x	1	2	5
$f(x)$	1	4	10

Solution Using Lagrange's interpolation formula given by Eq. (6.37), we have

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Therefore,

$$f(3) = \frac{(3-2)(3-5)}{(1-2)(1-5)}(1) + \frac{(3-1)(3-5)}{(2-1)(2-5)}(4) + \frac{(3-1)(3-2)}{(5-1)(5-2)}(10) = 6.4999$$

DIVIDED DIFFRENCE: Assume that for a given value of $(x_1, y_1)(x_2, y_2) \ldots.(x_n, y_n)$

$y[x_0] = y(x_0) = y_0 \to y \; at \; x_0$

Then the first order divided Difference is defined as $y[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_o}$, $y[x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1} = a_1$

The 2nd Order Difference is $y[x_0, x_1, x_2] = \frac{y[x_1, x_2] - y[x_o, x_1]}{x_2, x_0} = a_2$

Similarly $y[x_0, x_1, x_2 \ldots \ldots x_n] = \frac{y[x_1, x_2, x_3 \ldots\ldots x_n] - y[x_o, x_1 \ldots\ldots x_{n-1}]}{x_n - x_o} = a_n$

DIVIDED DIFFERENCE IS SYMMETRIC

$y[x_o, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{-(y_o - y_1)}{-(x_0 - x_1)} = y[x_1, x_0]$

☺ Also Newton Divided Difference is Symmetric

NEWTON'S DIVIDED DIFFRENCE INTERPOLATION FORMULA

If $x_0, x_1 \ldots . x_n$ are arbitrarily Spaced (unequal spaced) Then the polynomial of degree 'n' through $(x_o, f_0)(x_1, f_1) \ldots \ldots.(x_n, f_n)$ where $f_j = f(x_j)$ is given by the newton's Devided difference Interpolation formula (Also known as Newton's General Interpolation formula) given by.

$$f(x) = (f_0) + (x - x_0)f[x_o, x_1] + (x - x_0)(x - x_1)f[x_o, x_1, x_2] + \cdots \ldots \ldots \ldots \ldots$$

$$\ldots + (x - x_0)(x - x_1) \ldots (x - x_{n-1})f[x_o, x_1 \ldots . x_n]$$

DERIVATION OF FORMULA

Let $y = f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots \ldots \ldots \ldots \ldots \ldots \ldots$
$\ldots . + a_n(x - x_0)(x - x_1)(x - x_2) \ldots (x - x_{n-1}) \quad \ldots \ldots \ldots \ldots (i)$

$Put \; x = x_0 \Rightarrow y_1 = f(x_0) = a_0 + a_1(x_0 - x_0) + 0 + 0 \ldots + 0 \;\Rightarrow a_0 = y_0 = y[x_o]$

Put $x = x_1 \quad \Longrightarrow y_1 = f(x_1) = a_0 + a_1(x_1 - x_0) + 0 + 0 \ldots + 0$

$\Longrightarrow a_1 = \frac{y_1 - a_0}{x_1 - x_0} = y[x_0, x_1] = \frac{y_1 - y_o}{x_1 - x_0} = \frac{y[x_1] - y[x_0]}{x_1 - x_0} \qquad \therefore a_0 = y_0$

Put $x = x_2 \qquad y_2 = f(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) + 0 + 0 \ldots + 0$

$\Longrightarrow y_2 - a_0 - a_1(x_2 - x_0) = a_2(x_2 - x_0)(x_2 - x_1)$

$y_2 - y_0 - \left(\frac{y_1 - y_0}{x_1 - x_0}\right)(x_2 - x_0) = a_2(x_2 - x_0)(x_2 - x_1) \qquad \therefore$ using above values

$\Rightarrow a_2 = \frac{y_2 - y_0 - (x_2 - x_0)y[x_0, x_1]}{(x_2 - x_0)(x_2 - x_1)} = \frac{(y_2 - y_1 + y_1 - y_0) - (x_2 - x_0)y[x_0, x_1]}{(x_2 - x_0)(x_2 - x_1)}$

$$a_2 = \frac{(y_2-y_1)+\frac{y_1-y_0}{x_1-x_0}(x_1-x_0)-(x_2-x_0)y[x_0,x_1]}{(x_2-x_0)(x_2-x_1)} = \frac{(y_2-y_1)+y[x_0\ ,\ x_1](x_1-x_0)-(x_2-x_0)y[x_0,x_1]}{(x_2-x_0)(x_2-x_1)}$$

$$a_2 = \frac{(y_2-y_1)+y[x_0\ ,\ x_1]\{x_1-x_0-x_2+x_0\}}{(x_2-x_0)(x_2-x_1)} = \frac{(y_2-y_1)+y[x_0\ ,\ x_1](x_1-x_2)}{(x_2-x_0)(x_2-x_1)}$$

$$a_2 = \frac{\frac{y_2-y_1}{x_2-x_1}+\frac{y[x_0\ ,\ x_1](x_1-x_2)}{x_2-x_1}}{(x_2-x_0)} = \frac{y[x_1,x_2]-\frac{y[x_0\ ,\ x_1]}{(x_1-x_2)}(x_1-x_2)}{(x_2-x_0)} = \frac{y[x_1-x_2]-y[x_0\ ,x_1]}{(x_2-x_0)} = y[x_0,x_1,x_2]$$

Similarly $a_3 = y[x_0,x_1,x_2,x_3] \dots\dots\dots\dots\dots\dots a_n = y[x_0,x_1,x_2 \dots a_n]$

$$(i) \Longrightarrow\ y = y[x_0] + (x-x_0)y[x_0,x_1]+..+(x-x_0)(x-x_1)..(x-x_{n-1})y[x_o,x_1..x_n]$$

TABLE

X	Y	1st Order	2nd Order	3nd Order
x_0	y_0			
	$\rightarrow$	$y[x_o,x_1]$		
x_1	y_1	$\rightarrow$	$y[x_o,x_1,x_2]$	
	$\rightarrow$	$y[x_1,x_2]$	$\rightarrow$	$y[x_o,x_1,x_2,x_3]$
x_2	y_2	$\rightarrow$	$y[x_1,x_2,x_3]$	
	$\rightarrow$	$y[x_2,x_3]$		
x_3	y_3	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
x_n	y_n	⋮	⋮	⋮

EXAMPLE:

X	Y	$Y[x_o,x_1]$	$Y[x_o,x_1,x_2]$	$Y[x_o,x_1,x_2,x_3]$
2	25			
		$\frac{40-25}{5-2}=5$		
5	40		$\frac{10-5}{7-2}=1$	
		10		$\frac{0-1}{10-2}=-\frac{1}{8}$
7	60		$\frac{10-10}{10-5}=0$	
		10		
10	90			

A RELATIONSHIP BETWEEN n^th DIVIDED DIFFERENCE AND THE n^th DARIVATIVE

Suppose "f" is n-time continuously differentiable and $x_0, x_1 \dots x_n$ are (n + 1) distinct numbers in [a, b] then there exist a number "§" in (a, b) such that $f[x_0, x_1 \dots x_n] = \frac{f^n(\S)}{n!}$

THEOREM : nth differences of a polynomial of degree 'n' are constant.

PROOF Let us consider a polynomial of degree 'n' in the form

$$y_x = a_o x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$$

Then $y_{x+h} = a_0(x+h)^n + a_1(x+h)^{n-1} + \cdots + a_{n-1}(x+n) + a_n$

We now examine the difference of polynomial $\Delta y_x = y_{x+h} - y_x$

$$\Delta y_x = a_0[(x+h)^n - x^n] + a_1[(x+h)^{n-1} - x^{n-1}] + \cdots \dots \dots + a_{n-1}[x+h-x]$$

Binomial expansion yields

$$\Delta y_x = a_0(x^n + {}^nC_1\, x^{n-1}\, h + {}^nC_2\, x^{n-2}\, h^2 + \cdots + h^n - x^n)$$

$$+a_1(x^{n-1} + {}^{n-1}C_1\, x^{n-2}\, h + {}^{n-2}C_2\, x^{n-3}\, h^2 + \cdots + h^{n-1} - x^n) + \cdots + a_{n-1}\, h$$

$$\Delta y_x = a_0 nhx^{n-1} + [a_0\, {}^n_2Ch^2 + a_1\, {}^{n-1}_{\ \ 1}Ch]x^{n-2} + \cdots \dots \dots \dots .. + a_{n-1}\, h$$

Therefore $\Delta y_x = a_0 nhx^{n-1} + b'x^{n-2} + c'x^{n-3} + \cdots + k'x + l'$

Where $\boldsymbol{b}', \boldsymbol{c}', \boldsymbol{k}', \boldsymbol{l}'$ are constants involving 'h' but not 'x'

Thus the first difference of a polynomial of degree 'n' is another polynomial of degree$(\boldsymbol{n-1})$

Similarly $\Delta^2 y_x = \Delta(\Delta y_x) = \Delta y_{x+h} - \Delta y_x$

$$= a_0\, nh[(x+h)^{n-1} - x^{n-1}] + b'[(x+h)^{n-2} - x^{n-2}] + \cdots \dots . + k'(x+h-x)$$

$$\Delta^2 y_x = a_0\, n(n-1)h^2 x^{n-2} + b''\, x^{n-2} + C''\, x^{n-4} + \cdots + q''$$

Therefore $\Delta^2 y_x$ is a polynomial of degree $(n-2)$ in 'x'

Similarly, we can find the higher order differences and every time we observe that the degree of polynomial is reduced by one.

After differencing n-time we get

$$\Delta^n y_x = a_0(n-1)(n-2) \dots (2)(1)h^n = a_0(n!)h^n = \ constant.$$

This constant is independent of 'x' since $\boldsymbol{\Delta^n y_x}$ is constant , $\boldsymbol{\Delta^{n+1}\, y_x = 0}$

Hence The $(\boldsymbol{n+1})\boldsymbol{th}$ and higher order differences of a polynomial of degree ‘n’ are zero.

NEWTION’S DIVIDED DIFERENCE FORMULA WITH ERROR TERM

$$y(x) = y_0 + (x - x_0)y[x, x_0] \quad \dots\dots\dots\dots..(i)$$

$$y[x, x_0] = y[x_o, x_1] + (x - x_1)y[x, x_o, x_1] \quad \dots\dots\dots\dots..(ii)$$

$$y[x, x_o, x_1] = y[x_o, x_1, x_2] + (x - x_2)y[x, x_o, x_1, x_2] \quad \dots\dots\dots\dots..(iii)$$

$$y[x, x_o, x_1, x_2] = y[x, x_o, x_1, x_2, x_3] + (x - x_3)y[x, x_o, x_1, x_2, x_3] \dots\dots\dots\dots..(iv)$$

$\vdots \qquad \qquad \vdots$

$\vdots \qquad \qquad \vdots$

$$y[x, x_o, x_1 \dots x_{n-1}] = y[x_0, x_1 \dots x_n] + (x, x_n)y[x, x_o, x_1 \dots x_n] \quad \dots\dots\dots\dots..(n)$$

Multiplying (ii) by $(x - x_0)$ (iii) $(x - x_0)\ (x - x_1) \dots\dots (n)$ By $(x - x_0)(x - x_1) \dots\dots (x_{n-1})$

And adding all Equation’s

$$y(x) = y_0 + (x - x_0)y[x_0, x_1] + (x - x_0)(x - x_1)y[x_0, x_1]$$

$+ \cdots \dots\dots\dots\dots + (x - x_0)(x - x_1) \dots (x - x_n)y[x_0, x_1, x_2 \dots x_n]$ Also last term will be $\in (x)$

LIMITATIONS OF NEWTON’S INTERPOLATION.

This formula used only when the values of independent variable ‘x’ are equally spaced. Also the differences of ‘y’ must ultimately become small. Its accuracy same as Lagrange’s Formula but has the advantage of being computationally economical in the sense that it involves less numbers of Arithmetic Operations.

ERROR TERM IN INTERPOLATION

As we know that

$$y(x) = y_0 + (x - x_0)y[x_0, x_1] + \cdots \dots\dots\dots\dots + (x - x_0)(x - x_1)..(x - x_{n-1})\ y[x_0, x_1 \dots x_n]$$

Approximated by polynomial $P_n(x)$ of degree ‘n’ the error term is

$$\in (x) = y(x) - P_n(x) \quad \dots\dots\dots\dots\dots\dots\dots..(i)$$

$$\in (x) = (x - x_0)(x - x_1) \dots\dots\dots\dots..(x - x_n)\ y[x, x_0, x_1 \dots x_n]$$

Let $\quad \in (x) = \pi\,(x)\ y[x, x_0, x_1 \dots x_n] = k\,\pi\,(x) \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots..(ii)$

And $F(x) = y(x) - P_n(x) - k\,\pi(x)$

$F(x)$ Vanish for $x_0, x_1 \dots x_n$ Choose arbitrarily $\bar{x}$ from them.

Consider an interval 'I' which span the points $\bar{x}, x_0, x_1 \dots x_n$. Total number of points $(n+2)$ Then $F(x)$ vanish $(n+2)$ time by *Roll's theorem*

$F'(x)$ Vanish $(n+1)$ time, $F''(x)$ vanish n-time. Hence $F^{n+1}(x)$ vanish 1-time choose arbitrarily $x = \S$

$$\Rightarrow F^{n+1}(\S) = y^{n+1}(\S) - P_n^{n+1}(\S) - k\frac{d^{n+1}}{dx^{n+1}}\pi(\S)$$

$$\Rightarrow 0 = y^{n+1}(\S) - 0 - k\pi^{n+1}(\S) \qquad \therefore y^{n+1}(\S) = 0 \;\; and \;\; P^{n+1}(\S) = 0$$

$$\Rightarrow y^{n+1}(\S) = k\pi^{n+1}(\S) \qquad \Rightarrow k = \frac{y^{n+1}(\S)}{\pi^{n+1}(\S)}$$

$$if \qquad \pi^{n+1}(x) = (n+1)! \qquad \Rightarrow k = \frac{y^{n+1}(\S)}{(n+1)!} \qquad \Rightarrow k = y[x_0, x_1 \dots x_n]$$

$$(ii) \Rightarrow \qquad \in(x) = \frac{y^{n+1}(\S)}{(n+1)!}\,\pi(x)$$

THEOREM:

NEWTON'S DIVIDED DIFFERENCE AND LAGRANGE'S INTERPOLATION FORMULA ARE IDENTICAL, PROVE!

PROOF: Consider y = f(x) is given at the sample points x_0, x_1, x_2

Since by Newton's divided difference interpolation for x_0, x_1, x_2 is given as

$$y = y_0 + (x - x_0)y[x_0, x_1] + (x - x_0)(x - x_1)y[x_0, x_1, x_2]$$

$$y = y_0 + (x - x_0)\left(\frac{y_1 - y_0}{x_1 - x_0}\right) + (x - x_0)(x - x_1)\frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0}$$

$$y = y_0 + (x - x_0)\left(\frac{y_1 - y_0}{x_1 - x_0}\right) + \left(\frac{(x - x_0)(x - x_1)}{x_2 - x_0}\right)\left(\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}\right)$$

$$y = y_0 + (x - x_0)\left(\frac{y_1 - y_0}{x_1 - x_0}\right) + \left(\frac{(x - x_0)(x - x_1)}{x_2 - x_0}\right)\left(\frac{y_2}{x_2 - x_1} + y_1\left\{\frac{-1 - 1}{(x_2 - x_1)(x_1 - x_0)}\right\} + \frac{y_0}{x_1 - x_0}\right)$$

$$y = y_0 + (x - x_0)\left(\frac{y_1 - y_0}{x_1 - x_0}\right) + \left(\frac{(x - x_0)(x - x_1)}{x_2 - x_0}\right)\left(\frac{y_2}{x_2 - x_1} + y_1\left\{\frac{-(x_1 - x_0) - (x_2 - x_1)}{(x_2 - x_1)(x_1 - x_0)}\right\} + \frac{y_0}{x_1 - x_0}\right)$$

$$y = y_0 + (x - x_0)\left(\frac{y_1 - y_0}{x_1 - x_0}\right) + \left(\frac{(x - x_0)(x - x_1)}{x_2 - x_0}\right)\left(\frac{y_2}{x_2 - x_1} + y_1\left\{\frac{x_0 - x_2}{(x_2 - x_1)(x_1 - x_0)}\right\} + \frac{y_0}{x_1 - x_0}\right)$$

$$y = y_0 + (x - x_0)\left(\frac{y_1 - y_0}{x_1 - x_0}\right) + \left(\frac{y_2(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} - \frac{y_1(x - x_0)(x - x_1)(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)(x_1 - x_0)} + \frac{y_0(x - x_0)(x - x_1)}{(x_2 - x_0)x_1 - x_0}\right)$$

$$y = y_0 + (x - x_0)\left(\frac{y_1 - y_0}{x_1 - x_0}\right) + \left(\frac{y_2(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} - \frac{y_1(x - x_0)(x - x_1)}{(x_2 - x_1)(x_1 - x_0)} + \frac{y_0(x - x_0)(x - x_1)}{(x_2 - x_0)x_1 - x_0}\right)$$

$$y = y_0\left[1 - \left(\frac{x - x_0}{x_1 - x_0}\right) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_1 - x_0)}\right] + y_1\left[\left(\frac{x - x_0}{x_1 - x_0}\right) - \frac{(x - x_0)(x - x_1)}{(x_2 - x_1)(x_1 - x_0)}\right] + y_2\left[\frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}\right]$$

$$y = y_0\left[\frac{(x_1 - x_0)(x_2 - x_0) - (x - x_0) + (x - x_0)(x - x_1)}{(x_2 - x_0)(x_1 - x_0)}\right] + y_1\left[\frac{(x - x_0)(x_2 - x_1) - (x - x_0)(x - x_1)}{(x_2 - x_1)(x_1 - x_0)}\right] + y_2\left[\frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}\right]$$

$$y = y_0\left[\frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}\right] + y_1\left[\frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}\right] + y_2\left[\frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}\right]$$

This is Lagrange’s form of interpolation polynomial.

Hence both Divided Difference and Lagrange’s are identical.

SPLINE

A function 'S' is called a spline of degree 'k' if it satisfied the following conditions.

(i) S is defined in the interval $[\boldsymbol{a}, \boldsymbol{b}]$

(ii) $\boldsymbol{S^r}$ is continuous on $[\boldsymbol{a}, \boldsymbol{b}]$; $\boldsymbol{0 \leq r \leq k-1}$

(iii) S is polynomial of degree $\boldsymbol{less\ than\ equals\ to\ 'k'}$ on each subinterval $[\boldsymbol{x_i}, \boldsymbol{x_{i+1}}]$; $\boldsymbol{i = 1, 2, \ldots, n-1}$

CUBIC SPLINE INTERPOLATION

A function $\boldsymbol{S(x)}$ denoted by $\boldsymbol{S_j(x)}$ over the interval $[\boldsymbol{x_j}, \boldsymbol{x_{j+1}}]$; $\boldsymbol{j = 0, 1, 2, \ldots, n-1}$

Is called a cubic spline interpolant if following conditions hold.

- $\boldsymbol{S_j(x_j) = f_j}$; $\boldsymbol{j = 0, 1, 2, \ldots \ldots \ldots . . n}$
- $\boldsymbol{S_{j+1}(x_{j+1}) = f_{j+1}}$; $\boldsymbol{j = 0, 1, 2, \ldots \ldots \ldots . . n-2}$
- $\boldsymbol{S_{j+1}{}'(x_{j+1}) = f'_{j+1}}$; $\boldsymbol{j = 0, 1, 2, \ldots \ldots \ldots . . n-2}$
- $\boldsymbol{S''_{j+1}(x_{j+1}) = f''_{j+1}}$; $\boldsymbol{j = 0, 1, 2, \ldots \ldots \ldots . . n-2}$

♠ A spline of degree "3" is cubic spline.

NATURAL SPLINE

A cubic spline satisfying these two additional conditions

$$\boldsymbol{S''_1(x_1) = 0 \quad and \quad S''_{n-1}(x_n) = 0}$$

HERMIT INTERPOLATION

In Hermit interpolation we use the expansion involving not only the function values but also its first derivative.

Hermit Interpolation formula is given as follows

$$P(x) = H_{2n+1}(x) = \sum_{j=0}^{n} H_{n,j}(x)\, f(x_j) + \sum_{j=0}^{n} \widehat{H}_{n,j}(x)\, f'(x_j)$$

Algorithm:

- Make table

k	x_k	$f(x_k)$	$f'(x_k)$
….	……	……	……
…..	……	……	……

- Find $L_{n,j}(x)$ j[th] Lagrange's differential polynomial of degree 'n'
- Find $H_{n,j}(x) = [1 - 2(x - x_j)L'_{n,j}(x_j)][L_{n,j}(x)]^2$ and $\widehat{H}_{n,j}(x) = (x - x_j)[L_{n,j}(x)]^2$
- Use formula $P(x) = H_{2n+1}(x) = \sum_{j=0}^{n} H_{n,j}(x)\, f(x_j) + \sum_{j=0}^{n} \widehat{H}_{n,j}(x)\, f'(x_j)$

EXAMPLE

Estimate the value of $\boldsymbol{f(1.5)}$ using hermit interpolation formula from the following data

k	x_k	$f(x_k)$	$f'(x_k)$
0	1.3	0.6200860	-0.5220232
1	1.6	0.4554022	-005698959
2	1.9	0.2810186	-0.5811571

Solution: we first compute Lagrange's Polynomials and their derivatives.

$L_{2,0}(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}$ and $L'_{2,0}(x) = \frac{100}{9}x - \frac{175}{9}$

$L_{2,1}(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = -\frac{100}{9}x^2 + \frac{320}{9}x - \frac{247}{9}$ and $L'_{2,1}(x) = -\frac{200}{9}x + \frac{320}{9}$

$L_{2,2}(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{50}{9}x^2 - \frac{145}{9}x + \frac{104}{9}$ and $L'_{2,2}(x) = \frac{100}{9}x^2 - \frac{145}{9}$

Now we will find the Polynomials $H_{n,j}(x)$ and $\widehat{H}_{n,j}(x)$ for n = 2 and j = 0,1,2

$$H_{2,0}(x) = [1 - 2(x - x_0)L'_{2,0}(x_0)][L_{2,0}(x)]^2 = [1 - 2(x - 1.3)(-1.5)]\left[\frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}\right]^2$$

$$H_{2,0}(x) = [10x - 12]\left[\frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}\right]^2$$

Similarly

$$H_{2,1}(x) = \left[-\frac{100}{9}x^2 + \frac{320}{9}x - \frac{247}{9}\right]^2 \quad \text{and} \quad H_{2,2}(x) = 10[2 - x]\left[\frac{50}{9}x^2 - \frac{145}{9}x + \frac{104}{9}\right]^2$$

Also $\widehat{H}_{2,0}(x) = (x - x_0)[L_{2,0}(x)]^2 = (x - 1.3)\left[\frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}\right]^2$

Similarly $\widehat{H}_{2,1}(x) = (x - x_1)[L_{2,1}(x)]^2 = (x - 1.6)\left[-\frac{100}{9}x^2 + \frac{320}{9}x - \frac{247}{9}\right]^2$

And $\widehat{H}_{2,2}(x) = (x - x_2)[L_{2,2}(x)]^2 = (x - 1.9)\left[\frac{50}{9}x^2 - \frac{145}{9}x + \frac{104}{9}\right]^2$

Finally using the formula

$$P(x) = H_{2n+1}(x) = \sum_{j=0}^{n} H_{n,j}(x)\, f(x_j) + \sum_{j=0}^{n} \widehat{H}_{n,j}(x)\, f'(x_j)$$

$$P(x) = H_{2(2)+1}(x) = H_5(x) = \sum_{j=0}^{2} H_{2,j}(x)\, f(x_j) + \sum_{j=0}^{2} \widehat{H}_{2,j}(x)\, f'(x_j)$$

$$P(x) = H_5(x) = H_{2,0}(x)f(x_0) + H_{2,1}(x)f(x_1) + H_{2,2}(x)f(x_2) + \widehat{H}_{2,0}(x)f'(x_0) + \widehat{H}_{2,1}(x)f'(x_1) + \widehat{H}_{2,2}(x)f'(x_2)$$

$$P(x) = H_5(1.5) = 0.5118277$$

EXAMPLE

Estimate the value of $\boldsymbol{y(1.05)}$ using hermit interpolation formula from the following data

X	Y = $f(x_k)$	$\boldsymbol{Y'}$ = $f'(x_k)$
1.00	1.00000	0.5000
1.10	1.04881	0.47673

Solution:

At first we compute $l_o(x) = \frac{x-x_1}{x_0-x_1} = \frac{1.05-1.10}{1.00-1.10} = 0.5$ and $l_0'(x) = \frac{1}{x_0-x_1} = \frac{1}{1.00-1.10} = -\frac{1}{0.10}$

And $l_1\ (x) = \frac{x-x_0}{x_1-x_0} = \frac{1.05-1.00}{1.10-1.00} = 0.5$ and $l_1'(x) = \frac{1}{x_1-x_0} = \frac{1}{1.10-1.00} = \frac{1}{0.1}$

Now putting the values in Hermit Formula

$$P(x) = \sum_{i=0}^{n}[1 - 2L_i'(x_i)(x - x_i)]\,[L_i(x_i)]^2 y_i + (x - x_i)[L_i(x_i)]^2 {y_i}'$$

We find

$y(1.05) =$

$$\left[1 - 2\left(-\frac{1}{0.1}\right)(0.05)\right]\left(\frac{1}{2}\right)^2(1) + (0.05)\left(\frac{1}{2}\right)^2(0.5) + \left[1 - 2\left(\frac{1}{0.1}\right)(-0.05)\right]\left(\frac{1}{2}\right)^2(1.04881) + (-0.05)\left(\frac{1}{2}\right)^2(0.47673)$$

$y(1.05) = 1.0247$ $\quad$ *required answer*

NUMARICAL DIFFERENTIATION

The problem of numerical differentiation is the determination of approximate values the derivatives of a function '***f***' at a given point.

FORWARD – BACKWARD DIFFERENCE FORMULAE

The formula given as follows is known as forward difference formula if $h > 0$ and backward difference formula if $h < 0$

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{h}{2} f^n(\xi)$$

Where $\frac{h}{2} f^n(\xi)$ is error bound.

METHOD:

- Write formula $f'(x_0) = \frac{f(x_0+h)-f(x_0)}{h}$
- Use values in given formula.
- To find error bound write absolute value of $\frac{h}{2} f^n(\xi)$ less than to its answer. i.e. $\left|\frac{h}{2} f^n(\xi)\right| < solution$
- Compare actual value of given function at given point if needed and approximated value.

EXAMPLE:

Use forward difference formula to approximate the derivative of $f(x) = lnx$ at $x_0 = 1.8$ using $h = 0.1$, $h = 0.05$ and $h = 0.01$ and determine bound for the approximation error.

SOLUTION:

Since we have $x_0 = 1.8, h = 0.1, f(x) = lnx$ and $f'(x_0) = \frac{f(x_0+h)-f(x_0)}{h}$

Then using given values; $f'(1.8) = \frac{f(1.8+0.1)-f(1.8)}{0.1} = \frac{ln(1.8+0.1)-ln(1.8)}{0.1} = \frac{ln(1.9)-ln(1.8)}{0.1}$

$f'(1.8) = \frac{0.64185389-0.58778667}{0.1} = 0.5406722$

Also for actual value $f'(1.8) = \frac{1}{1.8} = 0.55\overline{5}$

DIFFERENTIATION USING DIFFERENCE OPERATORS

We assume that the function $\boldsymbol{y = f(x)}$ is given for the equally spaced 'x' values $\boldsymbol{x_n = x_0 + nh}$ for $\boldsymbol{n = 0, 1, 2}, \ldots\ldots\ldots$ to find the darivatives of such a tabular function, we proceed as follows;

USING FORWARD DIFFERENCE OPERATOR $'\Delta'$

Since $hD = \log E = \log(1+\Delta) \qquad \therefore E = (1+\Delta)$

$\Rightarrow D = \frac{1}{h}[\log(1+\Delta)]$ Where D is differential operator.

$$\Rightarrow D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots\right] \quad \ldots\ldots\ldots\ldots\ldots(i) \quad using\ Maclaurin\ series$$

Therefore $D\,f(x_0) = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots\right] f(x_0) = f'(x_0)$

$$D\,f(x_0) = f'(x_0) = \frac{1}{h}\left[\Delta f(x_0) - \frac{\Delta^2}{2} f(x_0) + \frac{\Delta^3}{3} f(x_0) - \frac{\Delta^4}{4} f(x_0) + \cdots \ldots\ldots\ldots\ldots\right]$$

$$Dy_0 = y_0' = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2}{2} y_0 + \frac{\Delta^3}{3} y_0 - \frac{\Delta^4}{4} y_0 \ldots\ldots\ldots\right]$$

Similarly, for second derivative

$$(i) \Longrightarrow \qquad D^2 = \frac{1}{h^2}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \frac{\Delta^5}{5} \ldots\ldots\ldots\right]^2$$

$$D^2 = \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \cdots \ldots\ldots\right] \qquad \text{After solving}$$

$$D^2 y_0 = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots \ldots\ldots\right] = y_0''$$

USING BACKWARD DIFFERENCE OPERATOR "∇"

Since $hD = \log E = \log(E^{-1})^{-1} = -1 log\ E^{-1} = -1\log(1-\nabla)$

Since $log(1-\nabla) = -\nabla - \frac{\nabla^2}{2} - \frac{\nabla^3}{2} - \cdots \ldots\ldots\ldots\ldots\ldots$ therefore

$$\Rightarrow D = \frac{1}{h}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} - \frac{\nabla^4}{4} + \cdots\right] \quad \ldots\ldots\ldots\ldots\ldots(i)$$

Now $D\,f(x_n) = \frac{1}{h}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} - \frac{\nabla^4}{4} + \cdots\right] f(x_n) = f'(x_n)$

$$D\,f(x_n) = f'(x_n) = \frac{1}{h}\left[\nabla f(x_n) + \frac{\nabla^2}{2} f(x_n) + \frac{\nabla^3}{3} f(x_n) - \frac{\nabla^4}{4} f(x_n) + \cdots\right]$$

$$Dy_n = y_n' = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2}{2} y_n + \frac{\nabla^3}{3} y_n - \frac{\nabla^4}{4} y_n + \cdots\right]$$

Similarly, for second derivative squaring (i) we get

$$(i) \Rightarrow \qquad D^2 = \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \frac{5}{6}\nabla^5 + \cdots \dots\dots\dots\dots\right]$$

$$D^2 y_n = y_n'' = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots \dots\dots\dots\right]$$

TO COMPUTE DARIVATIVE OF A TABULAR FUNCTION AT POINT NOT FOUND IN THE TABLE

Since

$$y(x_n + ph) = f(x_n) + P\nabla f(x_n) + \frac{P(P+1)}{2!}\nabla^2 f(x_n) + \cdots\cdots + \frac{P(P+1)(P+2)\cdots\cdots(P+n-1)}{n!}\nabla^n f(x_n) \quad \dots\dots\dots\dots (i)$$

$$Where \quad x = x_n + ph \qquad \Rightarrow \quad p = \frac{x - x_n}{h}; \; -n \le P \le 0 \dots\dots\dots\dots\dots\dots\dots\dots (ii)$$

$$(i) \Rightarrow \qquad y = f(x) = f(x_n) + P\nabla f(x_n) + \frac{P(P+1)}{2!}\nabla^2 f(x_n) + \cdots \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots iii)$$

Differentiate with respect to 'x' and using (i) & (ii)

$$y' = \frac{dy}{dx} = \frac{dy}{dp}.\frac{dp}{dx} = \frac{d}{dp}\left[f(x_n) + P\nabla f(x_n) + \frac{P(P+1)}{2!}\nabla^2 f(x_n) + \cdots \dots\dots\dots\right]\frac{d}{dx}\left(\frac{x - x_n}{h}\right)$$

$$y' = \frac{d}{dp}\left[0 + \nabla f(x_n) + \frac{(2P+1)}{2}\nabla^2 f(x_n) + \cdots \dots\dots\dots\right]\left(\frac{1-0}{h}\right)$$

$$y' = \frac{1}{h}\left[\nabla f(x_n) + \frac{(2P+1)}{2}\nabla^2 f(x_n) + \left(\frac{3P^2+6P+2}{6}\right)\nabla^3 f(x_n) + \left(\frac{4P^3+18P^2+22P+6}{24}\right)\nabla^4 f(x_n) \dots\dots\dots\dots\right] \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (iv)$$

Differentiate y' with respect to 'x'

$$y'' = \frac{d^2y}{dx^2} = \frac{dy'}{dp}.\frac{dp}{dx} = \frac{1}{h}\left[\nabla^2 f(x_n) + (P+1)\nabla^3 f(x_n) + \left(\frac{6P^2+18P+11}{12}\right)\nabla^4 f(x_n) \dots\dots\dots\dots\right]$$

$$\dots\dots\dots\dots\dots\dots (v)$$

Equation (iv) & (v) are **Newton's backward interpolation formulae** which can be used to compute 1st and 2nd derivatives of a tabular function near the end of table similarly

Expression of Newton's forward interpolation formulae can be derived to compute the 1st, 2nd and higher order derivatives near the beginning of table of values.

DIFFERENTIATION USING CENTRAL DIFFERENCE OPERATOR (σ)

Since $\sigma = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$

Since $hD = log\ E\ and\ E = e$ therefore $\sigma = e^{\frac{hD}{2}} - e^{-\frac{hD}{2}}$

Also as $sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$ therefore $\sigma = 2\sin\left(\frac{hD}{2}\right)$

$\Rightarrow \frac{\sigma}{2} = sinh\left(\frac{hD}{2}\right) \quad \Rightarrow sinh^{-1}\left(\frac{\sigma}{2}\right) = \left(\frac{hD}{2}\right) \Rightarrow D = \frac{2}{h}\ sinh^{-1}\frac{\sigma}{2}$

Since by Maclaurin series

$$sinh^{-1}(x) = x - \frac{1}{2}\left(\frac{x^3}{3}\right) + \frac{1.3}{2.4}\frac{x^5}{5} - \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \cdots \ldots\ldots\ldots\ldots$$

$$\Rightarrow D = \frac{2}{h}\left[\frac{\sigma}{2} - \frac{1}{2}\left(\frac{\left(\frac{\sigma}{2}\right)^3}{3}\right) + \frac{1.3}{2.4}\frac{\left(\frac{\sigma}{2}\right)^5}{5} - \frac{1.3.5}{2.4.6}\frac{\left(\frac{\sigma}{2}\right)^7}{7} + \cdots\right]$$

$$D = \frac{1}{h}\left[\sigma - \frac{\sigma^3}{24} + \frac{3\sigma^5}{640}\ldots\ldots\ldots\ldots\ldots\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Similarly, for second derivatives squaring (i) and simplifying

$$D^2 = \frac{1}{h^2}\left[\sigma^2 - \frac{\sigma^4}{12} + \frac{\sigma^6}{90} - \cdots\ldots\ldots\ldots\ldots\right]$$

$$D^2y = y'' = \frac{1}{h^2}\left[\sigma^2 y - \frac{\sigma^4 y}{12} + \frac{\sigma^6 y}{90} - \cdots\ldots\ldots\ldots\ldots\right]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

For calculating first and second derivative at an inter tabular form (point) we use (i) and (ii) while 1st derivative can be computed by another convergent form for D_i which can derived as follows

Since $D = \frac{1}{h}\left[\sigma - \frac{\sigma^3}{24} + \frac{3\sigma^5}{640}\ldots\ldots\ldots\ldots\ldots\right]$

Multiplying R.H.S by $\frac{\mu}{\sqrt{1+\frac{\delta^2}{4}}} = 1$ which is unity and noting the binomial expansion

$$\left(1 + \frac{\delta^2}{4}\right)^{-1/2} = 1 - \frac{\sigma^2}{8} + \frac{3\sigma^4}{128} - \frac{15\sigma^6}{48\times 64}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

We get $D = \frac{\mu}{h}\left[1 - \frac{\sigma^2}{8} + \frac{3\sigma^4}{128}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right]\left[\sigma - \frac{\sigma^3}{24} + \frac{3\sigma^5}{640}\ldots\ldots\ldots\ldots\ldots\right]$

$$\Longrightarrow D = \frac{\mu}{h}\left[\sigma - \frac{\sigma^3}{68} + \frac{4\sigma^5}{120}\ldots\ldots\ldots\ldots\ldots\right]$$

Therefore $\Longrightarrow D' = Dy = \frac{\mu}{h}\left[\sigma y - \frac{\sigma^3}{68}y + \frac{4\sigma^5}{120}y\ldots\ldots\ldots\ldots\ldots\right]\ldots\ldots\ldots\ldots\ldots (iii)$

Equation (ii) and (iii) are called **STERLING FORMULAE** for computing the derivative of a tabular function. Equation (iii) can also be written as

$$D' = Dy = \frac{\mu}{h}\left[\sigma y - \frac{1^2}{3!}\sigma^3 y + \frac{1^2 2^2}{5!}\sigma^5 y - \frac{1^2 2^2 3^2}{7!}\sigma^7 y + \cdots \dots\dots\dots\dots..\right]$$

STERLING FORMULA

Sterling's formula is

$$y = y_0 + \frac{p}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}(\Delta^2 y_{-1}) + \frac{p(p^2-1^2)}{3!}\left[\frac{(\Delta^3 y_{-1} - \Delta^3 y_{-2})}{2}\right] + \frac{p^2(p^2-1^2)}{4!}(\Delta^4 y_{-2}) + \cdots \dots ....$$

Where $p = \frac{x - x_0}{h}$

DERIVATION:

Since by Gauss forward interpolation formula

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_{-1} + \frac{P(P-1)(P+1)}{3!}\Delta^3 y_{-1} \dots\dots\dots\dots(i)$$

Also by Gauss backward interpolation formula

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_{-1} + \frac{P(P-1)(P+1)}{3!}\Delta^3 y_{-2} \dots\dots\dots\dots(ii)$$

Taking mean of both values

$$y = y_0 + \frac{p}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}(\Delta^2 y_{-1}) + \frac{p(p^2-1^2)}{3!}\left[\frac{(\Delta^3 y_{-1} - \Delta^3 y_{-2})}{2}\right] + \frac{p^2(p^2-1^2)}{4!}(\Delta^4 y_{-2}) + \cdots \dots ....$$

TWO AND THREE POINT FORMULAE

Since $y_i' = \frac{\Delta}{h} y_i = \frac{y_{i+1} - y_i}{h} = \frac{y(x_i+h) - y(x_i)}{h}$ ……………………(i)

Similarly $y_i' = \frac{\nabla}{h} y_i = \frac{y_i - y_{i-1}}{h} = \frac{y(x_i) - y(x_i-h)}{h}$ ……………………(ii)

Adding (i) and (ii) we get

$$2y_i' = \frac{y(x_i+h) - y(x_i-h)}{h} \implies y_i' = \frac{1}{2h}[y(x_i+h) - y(x_i-h)] \text{ ……………….(iii)}$$

Subtracting (i) and (iii) we get two point formulae for the first derivative

Similarly, we know that

$$y_i'' = \frac{\Delta^2}{h^2} y_i = \frac{y_{i+2} - 2y_{i+1} + y_i}{h^2} = \frac{1}{h^2}[y(x_i+2h) - 2y(x_i+h) + y(x_i)] \text{ ……………….(iv)}$$

And $y''_i = \frac{\nabla^2}{h^2} y_i = \frac{y_i - 2y_{i-1} + y_{-2}}{h^2}$

$$y''_i = \frac{1}{h}[y(x_i) - 2y(x_i-h) + y(x_i-2h)] \text{ ………………………………(v)}$$

Similarly

$$y''_i = \frac{\sigma^2}{h^2} y_i = \frac{\sigma y_{i+\frac{1}{2}} - \sigma y_{i-\frac{1}{2}}}{h^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

$$y''_i = \frac{y(x_i-h) - 2y(x_i) + y(x_i-h)}{h^2} \text{ ……………………………….(vi)}$$

By subtracting (iv) and (vi) we get three point formulae for computing the 2nd derivative.

Example 7.1 Compute $f''(0)$ and $f'(0.2)$ from the following tabular data.

x	0.0	0.2	0.4	0.6	0.8	1.0
$f(x)$	1.00	1.16	3.56	13.96	41.96	101.00

Solution Since $x = 0$ and 0.2 appear at and near beginning of the table, it is appropriate to use formulae based on forward differences to find the derivatives. The difference table for the given data is depicted below:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0.0	1.00	0.16	2.24	5.76	3.84	0.00
0.2	1.16	2.40	8.00	9.60	3.84	
0.4	3.56	10.40	17.60	13.44		
0.6	13.96	28.00	31.04			
0.8	41.96	59.04				
1.0	101.00					

Using forward difference formula (7.5) for $D^2 f(x)$, i.e.

$$D^2 f(x) = \frac{1}{h^2}\left[\Delta^2 f(x) - \Delta^3 f(x) + \frac{11}{12}\Delta^4 f(x) - \frac{5}{6}\Delta^5 f(x)\right]$$

We obtain

$$f''(0) = \frac{1}{(0.2)^2}\left[2.24 - 5.76 + \frac{11}{12}(3.84) - \frac{5}{6}(0)\right] = 0.0$$

Also, using the formula (7.3) we have

$$Df(x) = \frac{1}{h}\left[\Delta f(x) - \frac{\Delta^2 f(x)}{2} + \frac{\Delta^3 f(x)}{3} - \frac{\Delta^4 f(x)}{4}\right]$$

Hence,

$$f'(0.2) = \frac{1}{0.2}\left(2.40 - \frac{8.00}{2} + \frac{9.60}{3} - \frac{3.84}{4}\right) = 3.2$$

Example 7.2 Find $y'(2.2)$ and $y''(2.2)$ from the table

x	1.4	1.6	1.8	2.0	2.2
$y(x)$	4.0552	4.9530	6.0496	7.3891	9.0250

Solution Since $x = 2.2$ occurs at the end of the table, it is appropriate to use backward difference formulae for derivatives. The backward difference table for the given data is shown as follows:

x	$y(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1.4	4.0552	0.8978	0.1988	0.0441	0.0094
1.6	4.9530	1.0966	0.2429	0.0535	
1.8	6.0496	1.3395	0.2964		
2.0	7.3891	1.6359			
2.2	9.0250				

Using backward difference formulae (7.8) and (7.9) for $y'(x)$ and $y''(x)$, we have

$$y'_n = \frac{1}{h}\left(\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4}\right)$$

Therefore,

$$y'(2.2) = \frac{1}{0.2}\left(1.6359 + \frac{0.2964}{2} + \frac{0.0535}{3} + \frac{0.0094}{4}\right) = 5(1.8043) = 9.0215$$

Also

$$y''_n = \frac{1}{h^2}\left(\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n\right)$$

Therefore,

$$y''(2.2) = \frac{1}{(0.2)^2}\left[0.2964 + 0.0535 + \frac{11}{12}(0.0094)\right] = 25\ (0.3585) = 8.9629$$

Example 7.3 From the following table of values, estimate $y'(2)$ and $y''(2)$ using appropriate central difference formula:

x	0	1	2	3	4
y	6.9897	7.4036	7.7815	8.1281	8.4510

Solution The central difference table for the given data is given below:

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
0	6.9897				
		0.4139			
1	7.4036		−0.0360		
		0.3779		0.0047	
2	7.7815		−0.0313		0.0029
		0.3466		0.0076	
3	8.1281		−0.0237		
		0.3229			
4	8.4510				

Now, using central difference formula (7.13), we shall compute the first derivative

$$y' = \frac{\mu}{h}\left(\delta y - \frac{1}{6}\delta^3 y + \frac{1}{30}\delta^5 y - \cdots\right)$$

In the present example

$$y'(2) = \frac{1}{1}\left(\frac{0.3779 + 0.3466}{2} - \frac{1}{6}\,\frac{0.0047 + 0.0076}{2}\right) = 0.3613$$

To compute the second derivative, we shall use formula (7.12). Thus,

$$y'' = \frac{1}{h^2}\left(\delta^2 y - \frac{1}{12}\delta^4 y + \frac{1}{90}\delta^6 y - \cdots\right)$$

In this example,

$$y''(2) = \frac{1}{1}\left(-0.0313 - \frac{0.0029}{12}\right) = -0.0315$$

Example 7.4 Find $y'(0.25)$ and $f''(0.25)$ from the following data using [illegible] based on divided differences:

x	0.15	0.21	0.23	0.27	0.32	[illegible]
$y=f(x)$	0.1761	0.3222	0.3617	0.4314	0.5051	[illegible]

Solution We first construct divided difference [illegible] follows:

x	y	1st divided difference	2nd divided difference	3rd divided difference	4th divided difference	[illegible]
$x_0 = 0.15$	0.1761					
		2.4350				
$x_1 = 0.21$	0.3222		−5.7500			
		1.9750		15.6250		
$x_2 = 0.23$	0.3617		−3.8750		−44.23	
		1.7425		8.1064		172.2
$x_3 = 0.27$	0.4314		−2.9833		−9.79	
		1.4740		6.7358		
$x_4 = 0.32$	0.5051		−2.1750			
		1.3000				
$x_5 = 0.35$	0.5441					

Using Newton's divided difference formula (7.21), we have

$$y(x) = p_5(x) = y[x_0] + (x - x_0)\, y[x_0, x_1] + (x - x_0)(x - x_1)\, y[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)(x - x_2)\, y[x_0, x_1, x_2, x_3]$$
$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)\, y[x_0, x_1, x_2, x_3, x_4]$$

Now, using values from the above table of divided differences, we obtain

$$y(x) = 0.1761 + (x - 0.15)\, 2.4350 + (x - 0.15)(x - 0.21)(-5.75)$$
$$+ (x - 0.15)(x - 0.21)(x - 0.23) 15.625$$
$$+ (x - 0.15)(x - 0.21)(x - 0.23)(x - 0.27)(-44.23)$$
$$+ (x - 0.15)(x - 0.21)(x - 0.23)(x - 0.27)(x - 0.32)\, 172.2 \quad (1)$$

Differentiating Eq. (1) with respect to x, we get

$$y'(x) = 2.4350 - (2x - 0.36)5.75 + 15.625(3x^2 - 1.18x + 0.1143)$$
$$- 44.23(4x^3 - 2.58x^2 + 0.5472x - 38.105 \times 10^{-3})$$
$$+ 172.2(5x^4 - 4.72x^3 + 1.6464x^2 - 0.2515x + 14.15 \times 10^{-3}) \quad (2)$$

Which immediately gives

$$y'(0.25) = 2.4350 - 0.805 + 0.10625 + 2.432 \times 10^{-3} - 7.5338 \times 10^{-3} = 1.7312$$

Now, differentiating Eq. (2) once again with respect to x, we obtain

$$y''(x) = 3444x^3 - 2969.112x^2 + 888.99696x - 91.700456$$

which gives at once

$$y''(0.25) = 53.8125 - 185.5695 + 222.24924 - 91.700456 = -1.208216$$

NUMERICAL INTEGRATION

The process of producing a numerical value for the defining integral $\int_a^b f(x)dx$ is called Numerical Integration. Integration is the process of measuring the Area under a function plotted on a graph. Numerical Integration is the study of how the numerical value of an integral can be found.

Also called Numerical Quadrature if $\int_a^b f(x)dx \approx \sum_{i=0}^{n} c_i f(x_i)$ which refers to finding a square whose area is the same as the area under the curve.

A GENERAL FORMULA FOR SOLVING NUMERICAL INTEGRATION

This formula is also called a general quadrature formula.

Suppose f(x) is given for equidistant value of 'x' say $a=x_0, x_0+h, x_0+2h \ldots x_0+nh = b$

Let the range of integration (a,b) is divided into 'n' equal parts each of width 'h' so that "b-a=nh".

By using fundamental theorem of numerical analysis It has been proved the general quadrature formula which is as follows

$$I = h\left[nf(x_0) + \frac{n^2}{2}\Delta f(x_0) + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 f(x_0)}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 f(x_0)}{3!} + \left(\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11}{3}n^3 - 3n^2\right)\frac{\Delta^4 f(x_0)}{4!} + \cdots \ldots\ldots\ldots\ldots\ldots.. + up\ to\ (n+1) terms\right]$$

Bu putting n into different values various formulae is used to solve numerical integration.

That are Trapezoidal Rule, Simpson's 1/3, Simpson's 3/8, Boole's, Weddle's etc.

IMPORTANCE: Numerical integration is useful when

- Function cannot be integrated analytically.
- Function is defined by a table of values.
- Function can be integrated analytically but resulting expression is so complicated.

COMPOSITE (MODIFIED) NUMERICAL INTEGRATION

Trapezoidal and Simpson's rules are limited to operating on a single interval. Of course, since definite integrals are additive over subinterval, we can evaluate an integral by dividing the interval up into several subintervals, applying the rule separately on each one and then totaling up. This strategy is called Composite Numerical Integration.

TRAPEZOIDAL RULE

Rule is based on approximating $f(x)$ by a piecewise linear polynomial that interpolates $f(x)$ at the nodes "$x_0, x_1, \dots \dots \dots x_n$"

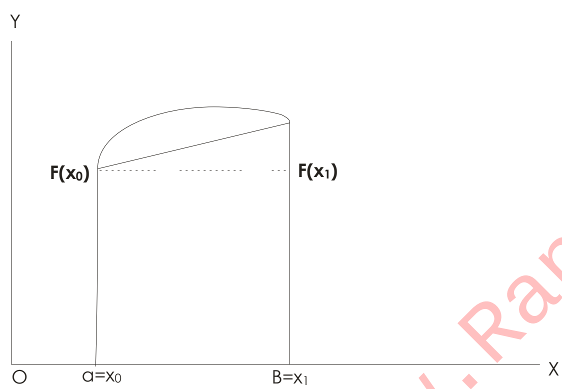
Trapezoidal Rule defined as follows

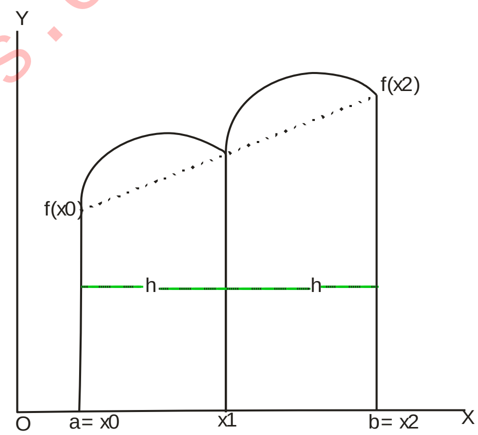
$\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}(y_0 + y_1) - \frac{h^3}{12}y''(a)$ And this is called Elementary Trapezoidal Rule.

Composite form of Trapezoidal Rule is $\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \cdots \dots + y_{n-1}) + y_n]$

DARIVATION (1st METHOD)

Consider a curve $y = f(x)$ bounded by $x_0 = a$ and $x_1 = b$ we have to find $\int_a^b f(x)dx$ i.e. Area under the curve $y = f(x)$ then for one Trapezium under the area i.e. n = 1





$$\int_a^b f(x)dx = Area\ of\ Trapezium = \frac{sum\ of\ parallel\ sides}{2} \times perpendicular$$

$$\int_a^b f(x)dx = \frac{f(x_0)+f(x_1)}{2} \times h = \frac{h}{2}[f(x_0) + f(x_1)]$$

For two trapeziums i. e. n = 2

$$\int_a^b f(x)dx = \frac{h}{2}[f(x_0) + f(x_1)] + \frac{h}{2}[f(x_1) + f(x_2)] = \frac{h}{2}[f(x_0) + 2f(x_1) + f(x_2)]$$

For n = 3 $\int_a^b f(x)dx = \frac{h}{2}[f(x_0) + f(x_1)] + \frac{h}{2}[f(x_1) + f(x_2)] + \frac{h}{2}[f(x_2) + f(x_3)]$

$$\int_a^b f(x)dx = \frac{h}{2}[f(x_0) + 2[f(x_1) + f(x_2)] + f(x_3)]$$

In general for n – trapezium the points will be "$x_0, x_1, \dots \dots \dots x_n$" and function will be "$y_0, y_1, \dots \dots \dots y_n$"

$\int_a^b f(x)dx = \frac{h}{2}[f(x_0) + 2[f(x_1) + f(x_2) + \cdots \ldots \ldots \ldots + f(x_{n-1})] + \ f(x_n)]$

$\int_a^b f(x)dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \cdots \ldots \ldots \ldots + y_{n-1}) + y_n]$

Trapezium rule is valid for n (number of trapezium) is even or odd.

The accuracy will be increase if number of trapezium will be increased OR step size will be decreased mean number of step size will be increased.

DARIVATION (2nd METHOD)

Define y = f(x) in an interval $[a, b] = [x_0, x_n]$ then

$\int_{x_0}^{x_0} f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots \ldots \ldots \ldots \ldots + \int_{x_{n-1}}^{x_n} f(x)dx$

$\int_{x_0}^{x_0} f(x)dx = \left[\frac{h}{2}(y_0 + y_1)\right] + \left[\frac{h}{2}(y_1 + y_2)\right] + \cdots \ldots \ldots \ldots + \left[\frac{h}{2}(y_{n-1} + y_n)\right] + \epsilon_n$

Where $\epsilon_n = -\frac{h^3}{12}[y''(a_1) + y''(a_2) + \cdots \ldots \ldots \ldots \ldots + y''(a_n)]$ is global error.

$\Rightarrow \epsilon_n = -\frac{h^3}{12}[ny''(a)]$

Therefore $\int_a^b f(x)dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \cdots \ldots \ldots \ldots + y_{n-1}) + y_n]$ Where $a = x_0 \text{ and } b = x_n$

REMEMBER: The maximum incurred in approximate value obtained by Trapezoidal Rule is nearly equal to $\frac{(b-a)^3 M}{12n^2}$ where $M = max|f''(x)|$ on $[a, b]$

EXAMPLE: Evaluate $I = \int_0^1 \frac{1}{1+x^2}dx$ using Trapezoidal Rule when $h = ¼$

SOLUTION

X	0	1/4	1/2	3/4	1
F(x)	1	0.9412	0.8000	0.6400	0.5000

Since by Trapezoidal Rule $\int_0^1 \frac{1}{1+x^2}dx = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)] = \mathbf{0.7828}$

SIMPSON'S $\left(\frac{1}{3}\right)$ RULE

Rule is based on approximating f(x) by a Quadratic Polynomial that interpolate f(x) at $x_{i-1}, x_i \text{ and } x_{i+1}$

Simpson's Rule is defined as for simple case $\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}[y_0 + 4y_1 + y_2] - \frac{h^5}{90} y^{iv}(\S)$

While in composite form it is defined as

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + \cdots \ldots \ldots + y_{2N-1}) + 2(y_2 + y_4 + \cdots \ldots \ldots + y_{2N-2}) + y_N]$$

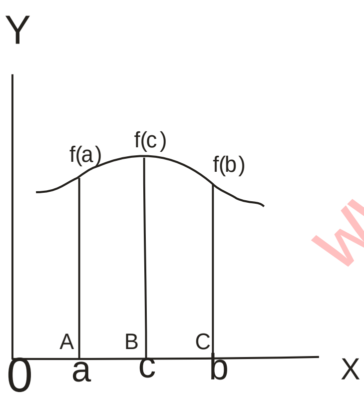
Global error for Simpson's Rule is defined as $\in = -\frac{x_{2N} - x_0}{180} h^4 y^{iv}(\S) = O(h^4)$

REMARK

In Simpson Rule number of trapezium must of Even and number of points must of Odd.

DERIVATION OF SIMPSON'S $\left(\frac{1}{3}\right)$ RULE (1st method)

Consider a curve bounded by x = a and x = b and let 'c' is the mid-point between a and b such that $a << b$ we have to find $\int_a^b f(x)dx$ i.e. Area under the curve.



Consider $X = C + Y \ldots\ldots\ldots\ldots (i) \Rightarrow \quad dx = dy$

Now $c = OB = OA + AB \Rightarrow c = a + h \Rightarrow a = c - h$

$b = OC = OB + BC \Rightarrow b = c + h$

$(i) \Rightarrow put\ x = a\ then\ a = c + y \Rightarrow c - h = c + y \Rightarrow -h = y$

$put\ x = b\ then\ c + h = c + y \Rightarrow h = y$

Now $\int_a^b f(x)dx = \int_{-h}^{+h} f(c+y)dy$ where y is small change

Using Taylor Series Formula $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots \ldots \ldots \ldots \ldots \ldots$

$$\int_{-h}^{+h} f(c+y)dy = \int_{-h}^{+h}\left[f(c) + yf'(c) + \frac{y^2}{2!}f''(c) + \cdots \ldots \ldots \ldots \ldots ..\right]dy$$

Neglecting higher derivatives

$$\int_{-h}^{+h} f(c+y)dy = \int_{-h}^{+h}\left[f(c) + yf'(c) + \frac{y^2}{2!}f''(c)\right]dy$$

$$\int_{-h}^{+h} f(c+y)dy = \left|yf(c) + \frac{y^2}{2}f'(c) + \frac{y^3}{2.3}f''(c)\right|_{-h}^{h} = 2h\left[f(c) + \frac{h^2}{6}f''(c)\right] \ldots \ldots \ldots \ldots \ldots \ldots .(i)$$

$$f(a) = f(c-h) = f(c) - hf'(c) + \frac{h^2}{2!}f''(c) + neglected$$

$$f(b) = f(c+h) = f(c) + hf'(c) + \frac{h^2}{2!}f''(c) + neglected$$

$$f(c-h) - f(c+h) = 2f(c) + 2\frac{h^2}{2!}f''(c)$$

$f(c-h) - f(c+h) - 2f(c) = h^2 f''(c)$ Put this value in (i)

$$\int_a^b f(x)dx = 2h[f(c) + \frac{1}{6}\{f(c-h) + f(c+h) - 2f(c)\}]$$

$$\int_a^b f(x)dx = \frac{2h}{6}[6f(c) + f(c-h) + f(c+h) - 2f(c)]$$

$$\int_a^b f(x)dx = \frac{h}{3}[4f(c) + f(c-h) + f(c+h)] = \frac{h}{3}[4f(c) + f(a) + f(b)]$$

$$\int_a^b f(x)dx = \frac{h}{3}[4f(x_1) + f(x_0) + f(x_2)] = \frac{h}{3}[4y_1 + y_0 + y_2]$$

For n = 4

$$\int_{x_0}^{x_4} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx = \frac{h}{3}[y_0 + 4y_1 + y_2] + \frac{h}{3}[y_2 + 4y_3 + y_4]$$

$$\int_{x_0}^{x_4} f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3) + 2y_2 + y_4]$$

In General

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 \ldots \ldots .. y_{2N-1}) + 2(y_2 + y_4 \ldots \ldots + y_{2N-2}) + y_{2N}]$$

DERIVATION OF SIMPSON'S $\left(\frac{1}{3}\right)$ RULE (2nd method)

$$\int_{x_0}^{x_{2N}} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots \ldots\ldots\ldots\ldots\ldots.\ldots\ldots + \int_{x_{2N-2}}^{x_{2N}} f(x)dx$$

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{h}{3}[y_0 + 4y_1 + y_2] + \frac{h}{3}[y_2 + 4y_3 + y_4] + \cdots \ldots\ldots\ldots\ldots\ldots.. + \frac{h}{3}[y_{2N-2} + 4y_{2N-1} + y_{2N}]$$

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 \ldots\ldots y_{2N-1}) + 2(y_2 + y_4 \ldots + y_{2N-2}) + y_{2N}]$$

This is required formula for Simpson's (1/3) Rule

EXAMPLE

Compute $I = \sqrt{\frac{2}{\pi}}\int_0^1 e^{-\frac{x^2}{2}}\mathbf{dx}$ using Simpson's (1/3) Rule when $h = 0.125$

SOLUTION

X	0	0.125	0.250	0.375	0.5	0.625	0.750	0.875	1
F(x)	0.798	0.792	0.773	0.744	0.704	0.656	0.602	0.544	0.484

Since by Simpson's Rule

$$\sqrt{\frac{2}{\pi}}\int_0^1 e^{-\frac{x^2}{2}}\mathbf{dx} = \frac{h}{3}[y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)] = 0.6827$$

EXERCISE:

i. Compute $I = \int_{1.0}^{1.8} \frac{e^x + e^{-x}}{2}\mathbf{dx}$ using Simpson's (1/3) Rule when $h = 0.2$

ii. Compute $I = \int_0^{\frac{\pi}{2}} \frac{dx}{\sin^2 x + \frac{1}{4}\cos^2 x}$ using Simpson's (1/3) Rule

iii. Compute $I = \int_1^2 \frac{dx}{x}$ using Simpson's (1/3) Rule and also obtain the error bound by taking $h = 0.25$

iv. Compute $I = \int_0^1 e^x dx$ using Simpson's (1/3) Rule by dividing the interval of integration into eight equal parts.

SIMPSON'S $(\frac{3}{8})$ RULE

Rule is based on fitting four points by a cubic.

Simpson's Rule is defined as for simple case

$$\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3] - \frac{3h^5}{80} y^{iv}(\S)$$

While in composite form ("n" must be divisible by 3) it is defined as

$$\int_{x_0}^{x_N} f(x)dx = \frac{3h}{8}[y_0 + 3(y_1 + y_2 + \cdots \ldots \ldots .. + y_{N-1}) + 2(y_3 + y_6 + \cdots \ldots \ldots .. + y_{N-3}) + y_N]$$

DERIVATION

$$\int_{x_0}^{x_N} f(x)dx = \int_{x_0}^{x_3} f(x)dx + \int_{x_3}^{x_6} f(x)dx + \cdots \ldots \ldots \ldots \ldots \ldots . \ldots \ldots + \int_{x_{N-3}}^{x_N} f(x)dx$$

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3] + \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6]$$

$$+ \cdots \ldots \ldots \ldots \ldots \ldots .. + \frac{3h}{8}[y_{N-3} + 3y_{N-2} + 3y_{N-1} + y_N]$$

$$\int_{x_0}^{x_N} f(x)dx = \frac{3h}{8}[y_0 + 3(y_1 + y_2 + \cdots \ldots \ldots .. + y_{N-1}) + 2(y_3 + y_6 + \cdots \ldots \ldots .. + y_{N-3}) + y_N]$$

This is required formula for Simpson's (3/8) Rule.

REMARK: Global error in Simpson's (1/3) and (3/8) rule are of the same order but if we consider the magnitude of error then Simpson (1/3) rule is superior to Simpson's (3/8) rule.

<u>IN NUMERICAL INTEGRATION, WHICH METHOD IS BETTER THAN OTHERS?</u>

Simpson 1/3 Rule is sufficiently accurate method. It based on fitting 3 points with a quadratic. It is best because it has low error than others. i.e. $\epsilon = -\frac{h^5}{90} y^{iv}(\S)$. The global error of Simpson 1/3 and 3/8 rule is same but if we consider the magnitude of error term we notice that Simpson 1/3 rule is superior than Simpson 3/8 rule.

TRAPEZOIDAL AND SIMPSON'S RULE ARE CONVERGENT

If we assume Truncation error, then in the case of Trapezoidal Rule

$I - A = -\frac{(b-a)h^2}{12} y^2(\S)$ Where "I" is the exact integral and "A" the approximation. If "$lmt\ h = 0$" then assuming "y^2" bounded

"$lmt\ (I - A) = 0$" (This the definition of convergence of Trapezoidal Rule)

For Simpson's Rule we have the similar result

$I - A = -\frac{(b-a)h^4}{180} y^4(\S)$

If "$lmt\ h = 0$" then assuming "y^4" bounded

"$lmt\ (I - A) = 0$" (This the definition of convergence of Simpson's Rule)

ERROR TERMS

Rectangular Rule	$\frac{h^2}{2!} y'(\S)$	$x_0 < \S < x_1$
Trapezoidal Rule	$-\frac{h^3}{12} y''(\S)$	$x_0 < \S < x_1$
Simpson's (1/3) Rule	$-\frac{h^5}{90} y^{iv}(\S)$	$x_0 < \S < x_1$
Simpson's (3/8) Rule	$-\frac{3h^5}{80} y^{iv}(\S)$	$x_0 < \S < x_1$

Example 7.6 Find the approximate value of

$$y = \int_0^{\pi} \sin x \, dx$$

using (i) trapezoidal rule, (ii) Simpson's 1/3 rule by dividing the range of integration into six equal parts. Calculate the percentage error from its true value in both the cases.

Solution We shall at first divide the range of integration $(0, \pi)$ into six equal parts so that each part is of width $\pi/6$ and write down the table of values:

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
$y = \sin x$	0.0	0.5	0.8660	1.0	0.8660	0.5	0.0

Applying trapezoidal rule, we have

$$\int_0^{\pi} \sin x \, dx = \frac{h}{2}[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

Here, h, the width of the interval is $\pi/6$. Therefore,

$$y = \int_0^{\pi} \sin x \, dx = \frac{\pi}{12}[0 + 0 + 2(3.732)] = \frac{3.1415}{6} \times 3.732 = 1.9540$$

Applying Simpson's 1/3 rule (7.41), we have

$$\int_0^{\pi} \sin x \, dx = \frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{\pi}{18}[0 + 0 + (4 \times 2) + (2)(1.732)] = \frac{3.1415}{18} \times 11.464 = 2.0008$$

But the actual value of the integral is

$$\int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi} = 2$$

Hence, in the case of trapezoidal rule

The percentage of error $= \frac{2 - 1.954}{2} \times 100 = 2.3$

While in the case of Simpson's rule the percentage error is

$$\frac{2 - 2.0008}{2} \times 100 = 0.04 \qquad \text{(sign ignored)}$$

Example 7.7 From the following data, estimate the value of

$$\int_1^5 \log x \, dx$$

using Simpson's 1/3 rule. Also, obtain the value of h, so that the value of the integral will be accurate up to five decimal places.

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$y = \log x$	0.0000	0.4055	0.6931	0.9163	1.0986	1.2528	1.3863	1.5041	1.6094

Solution We have from the data, $n = 0, 1, \ldots, 8$, and $h = 0.5$. Now using Simpson's 1/3 rule,

$$\int_1^5 \log x \, dx = \frac{h}{3}\left[y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$$

$$= \frac{0.5}{3}\left[(0 + 1.6094) + 4(4.0787) + 2(3.178)\right]$$

$$= \frac{0.5}{3}(1.6094 + 16.3148 + 6.356)$$

$$= 4.0467$$

The error in Simpson's rule is given by

$$E = \frac{x_{2N} - x_0}{180} h^4 y^{(iv)}(\xi) \quad \text{(ignoring the sign)}$$

Since

$$y = \log x, \qquad y' = \frac{1}{x}, \qquad y'' = -\frac{1}{x^2}, \qquad y''' = \frac{2}{x^3}, \qquad y^{(iv)} = -\frac{6}{x^4}$$

$$\max_{1 \le x \le 5} y^{(iv)}(x) = 6, \qquad \min_{1 \le x \le 5} y^{(iv)}(x) = 0.0096$$

Therefore, the error bounds are given by

$$\frac{(0.0096)(4)h^4}{180} < E < \frac{(6)(4)h^4}{180}$$

If the result is to be accurate up to five decimal places, then

$$\frac{24h^4}{180} < 10^{-5}$$

That is, $h^4 < 0.000075$ or $h < 0.09$. It may be noted that the actual value of integral is

$$\int_1^5 \log x \, dx = [x \log x - x]_1^5 = 5 \log 5 - 4$$

Example 7.8 Evaluate the integral

$$I = \int_0^1 \frac{dx}{1 + x^2}$$

using (i) trapezoidal rule, (ii) Simpson's 1/3 rule by taking $h = 1/4$. Hence, compute the approximate value of π.

Solution At first, we shall tabulate the function as

x	0	1/4	1/2	3/4	1
$y = \frac{1}{1+x^2}$	1	0.9412	0.8000	0.6400	0.5000

using trapezoidal rule, and taking $h = 1/4$

$$I = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)] = \frac{1}{8}[1.5 + 2(2.312)] = 0.7828 \quad (1)$$

using Simpson's 1/3 rule, and taking $h = 1/4$, we have

$$I = \frac{h}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2y_2] = \frac{1}{12}[1.5 + 4(1.512) + 1.6] = 0.7854 \quad (2)$$

But the closed form solution to the given integral is

$$\int_0^1 \frac{dx}{1+x^2} = [\tan^{-1} x]_0^1 = \frac{\pi}{4} \quad (3)$$

Equating (2) and (3), we get $\pi = 3.1416$.

Example 7.9 Compute the integral

$$I = \sqrt{\frac{2}{\pi}} \int_0^1 e^{-x^2/2} \, dx$$

using Simpson's 1/3 rule, taking $h = 0.125$.

Solution At the outset, we shall construct the table of the function as required.

x	0	0.125	0.250	0.375	0.5	0.625	0.750	0.875	1.0
$y = \sqrt{\frac{2}{\pi}} e^{-x^2/2}$	0.7979	0.7917	0.7733	0.7437	0.7041	0.6563	0.6023	0.5441	0.4839

Using Simpson's 1/3 rule, we have

$$I = \frac{h}{3}[y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{0.125}{3}[0.7979 + 0.4839 + 4(0.7917 + 0.7437 + 0.6563 + 0.5441)$$

$$+ 2(0.7733 + 0.7041 + 0.6023)]$$

$$= \frac{0.125}{3}(1.2818 + 10.9432 + 4.1594) = 0.6827$$

Hence, $I = 0.6827$.

Example 7.10 A missile is launched from a ground station. The acceleration during its first 80 seconds of flight, as recorded, is given in the following table:

t (s)	0	10	20	30	40	50	60	70	80
a (m/s^2)	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

compute the velocity of the missile when $t = 80$ s, using Simpson's 1/3 rule.

Solution Since acceleration is defined as the rate of change of velocity, we have

$$\frac{dv}{dt} = a \quad \text{or} \quad v = \int_0^{80} a\ dt$$

Using Simpson's 1/3-rule, we have

$$v = \frac{h}{3}[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{10}{3}[(30 + 50.67) + 4(31.63 + 35.47 + 40.33 + 46.69)$$

$$+ 2(33.34 + 37.75 + 43.25)]$$

$$= 3086.1 \text{ m/s}$$

Therefore, the required velocity is given by $v = 3.0861$ km/s.

Question:- Find the number M and step size h so that the error $E_T(f,h)$ for the composite trapezoidal Rule is less than 5×10^{-9} for the approximation

$$\int_2^7 \frac{dx}{x} \approx T(f,h).$$

Sol:-

$$|E_T(f,h)| = \left| -\frac{(b-a)}{12} f^{(2)}(c) h^2 \right| \rightarrow (1)$$

$$f(x) = \frac{1}{x}, \quad f^{(1)}(x) = -\frac{1}{x^2}, \quad f^{(2)}(x) = \frac{2}{x^3}$$

maximum value of $|f^{(2)}(x)|$ taken over $[2,7]$ occurs at the end points $x=2$ and thus we have the bound

$$|f^{(2)}(c)| \le |f^{(2)}(2)| = \frac{1}{4} \quad \text{for } 2 \le c \le 7$$

Now, put values in (1)

$$|E_T(f,h)| = \left| -\frac{(b-a) f^{(2)}(c) h^2}{12} \right|$$

$$\le \frac{(7-2)\frac{1}{4}h^2}{12} = \frac{5h^2}{48} \rightarrow (2)$$

$$\Rightarrow |E_T(f,h)| \le \frac{5h^2}{48} \rightarrow (2)$$

The step size h and number 'M' satisfy the relation $h = \frac{5}{M}$ ($h = \frac{b-a}{M}$) and this is used in (1) relation to get the relation $|E_T(f,h)| \le \frac{125}{48M^2}$

$$\le 5\times10^{-9} \rightarrow (3)$$

Now, re-write eq (3) so that its easier to solve for M:

$$\frac{25}{48} \times 10^{8} \leq M^{2} \quad (4)$$

Solving eq (4), we find that

$$M > 22821.77$$

Since, M must be an integer we choose

$$M = 22822$$

and the corresponding step size is

$$h = \frac{5}{22822} = 0.000219086$$

Error Analysis:

The significance of the next two results into understand that the error terms $E_T(f,h)$ and $E_s(f,h)$ for the composite Simpson's rule are of the order $|O(h^2)|$ and $O(h^4)$ respectively.

This shows that the error for Simpson's rule converges to the trapezoidal rule as the step size h decreases to zero. In cases, where the derivatives of $f(x)$ are known, the formulas

$$E_T(f,h) = -\frac{(b-a)f^{(2)}(c)h^2}{12} \quad \text{and}$$

$$E_s(f,h) = -\frac{(b-a)f^{(4)}(c)h^4}{180}$$

can be used to estimate the number of sub-intervals required to achieve a specified Accuracy.

Example:-

Find the number M and step size "h" so that the error $E_s(f,h)$ for the composite Simpson's rule is less than 5×10^{-9} for the approximation

$$\int_2^7 \frac{dx}{x} = S(f,h)$$

Sol:-

The integrant is $f(x) = \frac{1}{x}$ and

$$f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}, \quad f'''(x) = \frac{-6}{x^4}$$

$$f^{(4)}(x) = \frac{24}{x^3}$$

The maximum value of $|f^{(4)}(x)|$ taken over $[2, 7]$ occurs at the end point $x = 2$ and thus we have the bound

$$|f^{(4)}(c)| \leq |f^{(4)}(2)| = \frac{24}{32} = \frac{3}{4}$$

for $2 \leq c \leq 7$

This is used with formula obtain

$$|E_s(f,h)| = \left| \frac{-(b-a) f^{(4)}(c) h^4}{180} \right| \rightarrow (2)$$

$$\leq \frac{(7-2)\,^{3}\!/_{4}\, h^4}{180} = \frac{h^4}{48}$$

The stepsize 'h' and number M satisfy the relation $h = \frac{5}{2M}$ and this is used in (1) to get the relation

$$|E_s(f,h)| \leq \frac{625}{768144M^4} \leq 5\times10^{-9}$$

Now rewrite (2) so that is easier to solve for M:

$$\frac{125}{768}\times10^{9} \leq M^4 \rightarrow (3)$$

Solving (3), we find that

$$M \geq 112.95$$

Since M must be an integer we choose M = 113 and the corresponding stepsize is

$$h = \frac{5}{226} = 0.022123893.$$

WEDDLE'S

In this method "n" should be the multiple of 6. Rather function will not applicable. This method also called sixth order closed Newton's cotes (or) the first step of Romberg integration.

First and last terms have no coefficients and other move with 5, then 1, then 6.

Weddle's Rule is given by formula

$$\int_a^b f(x)dx = \frac{3h}{10}\begin{bmatrix} f(x_0)+5f(x_1)+f(x_2)+6f(x_3)+\cdots\ldots\ldots\ldots+ \\ +\cdots\ldots..+5f(x_{n-3})+f(x_{n-2})+6f(x_{n-1})+f(x_n)\end{bmatrix}$$

EXAMPLE: for $\int_{0.25}^{1.75}\frac{1}{1+x^2}\mathbf{dx}$ at n = 6

X	0.25	0.5	0.75	1	1.25	1.5	1.75
F(x)	0.9411	0.8	0.64	0.5	0.4	0.3	0.2

Now using formula $\int_{0.25}^{1.75}\frac{1}{1+x^2}\mathbf{dx} = \frac{3(0.25)}{10}[y_0+5y_1+y_2+6y_3+5y_4+y_5+y_6] = \mathbf{0.8310}$

BOOLE'S RULE

The method approximate $\int_{x_0}^{x_4} f(x)\mathbf{dx}$ for '5' equally spaced values. Rule is given by George Bool. Rule is given by following formula

$$\int_a^b f(x)\mathbf{dx} = \frac{2h}{45}[7y_0+32y_1+12y_2+32y_3+7y_4]$$

EXAMPLE: Evaluate $\int_{0.2}^{0.6}\frac{1}{1+x^2}\mathbf{dx}$ at n = 4 and h = 0.1

SOLUTION

X	0.2	0.3	0.4	0.5	0.6
F(x)	0.96	0.92	0.86	0.80	0.74

Now using formula $\int_{0.2}^{0.6}\frac{1}{1+x^2}\mathbf{dx} = \frac{2(0.1)}{45}[7y_0+32y_1+12y_2+32y_3+7y_4]$

$\int_{0.2}^{0.6}\frac{1}{1+x^2}\mathbf{dx} = \mathbf{0.3399}$ After putting the values.

RECTANGULAR RULE

Rule is also known as Mid-Point Rule. And is defined as follows for 'n + 1' points.

$$\int_a^b f(x)dx = h[f(x_0) + f(x_1) + \cdots \ldots \ldots \ldots \ldots . + f(x_n)]$$

In general $\int_a^b f(x)dx = h\sum_{i=0}^{n} f(x_i)$

REMEMBER

- As we increased 'n' or decreased 'h' the accuracy improved and the approximate solution becomes closer and closer to the exact value.
- If 'n' is given, then use it. If 'h' is given, then we can easily get 'n'.
- If 'n' is not given and only 'points' are discussed, then '1' less that points will be 'n'. For example, if '3' points are given then 'n' will be '2'.
- If only table is given, then by counting the points we can tell about 'n'.one point will be greater than 'n' in table.

EXAMPLE

Evaluate $\int_1^3 \frac{1}{x^2} dx$ for n = 4 using Rectangular Rule.

SOLUTION

Here a = 1, b = 3 then $h = \frac{b-a}{n} = 0.5$

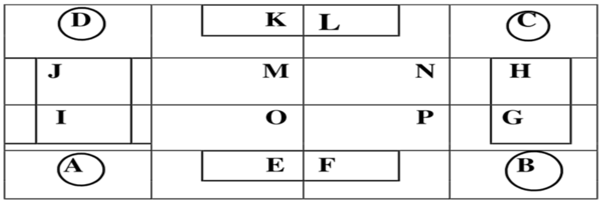
X	1	3/2	2	5/2	3
F(x)	1	4/9	1/4	4/25	1/5

Now using formula $\int_1^3 \frac{1}{x^2} dx = h[f(x_0) + f(x_1) + f(x_3)] = 0.925$

DOUBLE INTEGRATION

Double Integral
Trapezoidal Rule

Evaluate $\int_c^d \int_a^b f(x, y)\,dx\,dy$ where a, b, c, d are constants.



I= $\frac{hk}{4}$ **{[sum of values in]+2(sum of values in ▭ +4[sum of remaining values]}**

Simpson's Rule

$I = \frac{hk}{9}$ {(sum of the values of f at four corners)
+ 2(sum of the values of f at the odd positions on the boundary except the corners)
+ 4(sum of the values of f at the even positions on the boundary)
+ {4(sum of the values of f at the odd positions) +
8(sum of the values of f at the even positions)
on the odd row f of the matrix except boundary rows}+
{8(sum of the values of f at the odd positions) +
16(sum of the values of f at the even positions)
on the even row f of the matrix }}

Problems based on Double integrals

1. Evaluate $\int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\,dx\,dy$ using Trapezoidal and Simpson's rule. Verify your result by actual integration.

Solution:

Divide the range of x and y into 4 equal parts

$h = \frac{2.4 - 2}{4} = 0.1$

$k = \frac{1.4 - 1}{4} = 0.1$

Get the values of f(x, y) = $\frac{1}{xy}$ at nodal points

Y/X	2	2.1	2.2	2.3	2.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4545	0.4329	0.4132	0.3953	0.3788
1.2	0.4167	0.3968	0.3788	0.3623	0.3472
1.3	0.3846	0.3663	0.3497	0.3344	0.3205
1.4	0.3571	0.3401	0.3247	0.3106	0.2976

Now using previous formulae we get the required results

FOR TRAPEZOIDAL RULE: $I = 0.0614$ *FOR SIMPSON'S RULE*: $I = 0.0613$

Verify actual integration by yourself.

MUHAMMAD USMAN HAMID (0323-6032785)

QUESTION: Evaluate $\int_1^2 \int_1^2 \frac{dxdy}{x+y}$ by Trapezoidal rule for h = 0.25 = k

SOLUTION: $1 \le x \le 2 \implies x_0 = 1, x_1 = x_0 + h = 1.25, x_2 = 1.50, x_3 = 1.75, x_4 = 2$

And $1 \le y \le 2 \implies y_0 = 1, y_1 = y_0 + k = 1.25, y_2 = 1.50, y_3 = 1.75, y_4 = 2$

STEP – I: $f(x,y) = \frac{1}{x+y}$

Y/X	1	1.25	1.50	1.75	2
1	$\frac{1}{1+1}$ =0.5	0.4444	0.4	0.3636	0.3333
1.25	0.4444	0.4	0.3636	0.3333	0.3077
1.50	0.4	0.3636	0.3333	0.3077	0.2857
1.75	0.3636	0.3333	0.3077	0.2857	0.2667
2	0.3333	0.3077	0.2857	0.2667	0.25

STEP –I I:

$I_1 = \int_1^2 f(1,y)dy = \frac{k}{2}[f(1,y_0) + f(1,y_4) + 2[f(1,y_1) + f(1,y_2) + f(1,y_3)] = 0.4062$

$I_2 = \int_1^2 f(1.25,y)dy = \frac{k}{2}[f(1.25,y_0) + f(1.25,y_4) + 2[f(1.25,y_1) + f(1.25,y_2) + f(1.25,y_3)] = 0.3682$

$I_3 = \int_1^2 f(1.5,y)dy = \frac{k}{2}[f(1.5,y_0) + f(1.5,y_4) + 2[f(1.5,y_1) + f(1.5,y_2) + f(1.5,y_3)] = 0.3369$

$I_4 = \int_1^2 f(1.75,y)dy = \frac{k}{2}[f(1.75,y_0) + f(1.75,y_4) + 2[f(1.75,y_1) + f(1.75,y_2) + f(1.75,y_3)] = 0.3105$

$I_5 = \int_1^2 f(2,y)dy = \frac{k}{2}[f(2,y_0) + f(2,y_4) + 2[f(2,y_1) + f(2,y_2) + f(2,y_3)] = 0.2879$

STEP –III:

$I = \int_1^2 \int_1^2 \frac{dxdy}{x+y} = \frac{h}{2}[I_1 + I_5 + 2(I_2 + I_3 + I_4)] = 0..3407$

QUESTION: Evaluate $\int_0^{\pi/2}\int_0^{\pi/2}\sqrt{\sin(x+y)}\,dxdy$

SOLUTION: Take n = 4 (by own choice) then $h=\frac{b-a}{n}=\frac{\frac{\pi}{2}-0}{4}=\frac{\pi}{8}=k\ (also)$

$0\le x\le\frac{\pi}{2}\Rightarrow x_0=0, x_1=x_0+h=\frac{\pi}{8}, x_2=\frac{\pi}{4}, x_3=\frac{3\pi}{8}, x_4=\frac{\pi}{2}$

And $1\le y\le\frac{\pi}{2}\Rightarrow y_0=0, y_1=y_0+k=\frac{\pi}{8}, y_2=\frac{\pi}{4}, y_3=\frac{3\pi}{8}, y_4=\frac{\pi}{2}$

STEP – I: $f(x,y)=\sqrt{\sin(x+y)}$

Y/X	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
0	0	0.6186	0.8409	0.9612	1
$\frac{\pi}{8}$	0.6186	0.8409	0.9612	1	0.9612
$\frac{\pi}{4}$	0.8409	0.9612	1	0.9612	0.8409
$\frac{3\pi}{8}$	0.9612	1	0.9612	0.8409	0.6186
$\frac{\pi}{2}$	1	0.9612	0.8409	0.6186	0

STEP –I I:

$I_1=\int_0^{\frac{\pi}{2}}f(0,y)dy=\frac{k}{2}[f(0,y_0)+f(0,y_4)+2[f(0,y_1)+f(0,y_2)+f(0,y_3)]=1.1469$

$I_2=\int_0^{\frac{\pi}{2}}f\left(\frac{\pi}{8},y\right)dy=\frac{k}{2}[f\left(\frac{\pi}{8},y_0\right)+f\left(\frac{\pi}{8},y_4\right)+2\left[f\left(\frac{\pi}{8},y_1\right)+f\left(\frac{\pi}{8},y_2\right)+f\left(\frac{\pi}{8},y_3\right)\right]=1.4106$

$I_3=\int_0^{\frac{\pi}{2}}f\left(\frac{\pi}{4},y\right)dy=\frac{k}{2}[f\left(\frac{\pi}{4},y_0\right)+f\left(\frac{\pi}{4},y_4\right)+2\left[f\left(\frac{\pi}{4},y_1\right)+f\left(\frac{\pi}{4},y_2\right)+f\left(\frac{\pi}{4},y_3\right)\right]=1.4778$

$I_4=\int_0^{\frac{\pi}{2}}f\left(\frac{3\pi}{8},y\right)dy=\frac{k}{2}[f\left(\frac{3\pi}{8},y_0\right)+f\left(\frac{3\pi}{8},y_4\right)+2\left[f\left(\frac{3\pi}{8},y_1\right)+f\left(\frac{3\pi}{8},y_2\right)+f\left(\frac{3\pi}{8},y_3\right)\right]=1.4106$

$I_5=\int_0^{\frac{\pi}{2}}f\left(\frac{\pi}{2},y\right)dy=\frac{k}{2}[f\left(\frac{\pi}{2},y_0\right)+f\left(\frac{\pi}{2},y_4\right)+2\left[f\left(\frac{\pi}{2},y_1\right)+f\left(\frac{\pi}{2},y_2\right)+f\left(\frac{\pi}{2},y_3\right)\right]=1.1469$

STEP –III: $I=\int_0^{\pi/2}\int_0^{\pi/2}\sqrt{\sin(x+y)}\,dxdy=\frac{h}{2}[I_1+I_5+2(I_2+I_3+I_4)]=2.1386$

QUESTION: Evaluate $\iint_D \frac{dxdy}{x^2+y^2}$ where D is the square with comes at (1,1) , (2,1) ,(2,2) , (1,2)

SOLUTION: Take n = 4 (by own choice) then $h = \frac{b-a}{n} = \frac{\frac{\pi}{2}-0}{4} = \frac{\pi}{8} = k\ (also)$

$1 \le x \le 2$ Also $1 \le y \le 2$

STEP – I: $f(x,y) = \frac{1}{x^2+y^2}$

Y/X	1	1.25	1.50	1.75	2
1	0.5	0.3902	0.3077	0.2462	0.2
1.25	0.3902	0.3200	0.2623	0.2162	0.1798
1.50	0.3077	0.2623	0.2222	0.1882	0.1600
1.75	0.2462	0.2162	0.1882	0.1633	0.1416
2	0.2	0.1798	0.1600	0.1416	0.1250

STEP –I I:

$I_1 = \int_1^2 f(1,y)dy = \frac{k}{2}[f(1,y_0) + f(1,y_4) + 2[f(1,y_1) + f(1,y_2) + f(1,y_3)] = 0.3235$

$I_2 = \int_1^2 f(1.25,y)dy = \frac{k}{2}[f(1.25,y_0) + f(1.25,y_4) + 2[f(1.25,y_1) + f(1.25,y_2) + f(1.25,y_3)] = 0.2709$

$I_3 = \int_1^2 f(1.5,y)dy = \frac{k}{2}[f(1.5,y_0) + f(1.5,y_4) + 2[f(1.5,y_1) + f(1.5,y_2) + f(1.5,y_3)] = 0.2266$

$I_4 = \int_1^2 f(1.75,y)dy = \frac{k}{2}[f(1.75,y_0) + f(1.75,y_4) + 2[f(1.75,y_1) + f(1.75,y_2) + f(1.75,y_3)] = 0.1904$

$I_5 = \int_1^2 f(2,y)dy = \frac{k}{2}[f(2,y_0) + f(2,y_4) + 2[f(2,y_1) + f(2,y_2) + f(2,y_3)] = 0.1610$

STEP –III:

$I = \int_1^2 \int_1^2 \frac{dxdy}{x^2+y^2} = \frac{h}{2}[I_1 + I_5 + 2(I_2 + I_3 + I_4)] = 0.2325$

QUESTION: Evaluate $\int_0^1 \int_1^2 (x^2 + y^2)dxdy$ by using Simpson (1/3) rule

SOLUTION: Take n = 4 (by own choice) then $k = \frac{b-a}{n} = \frac{1-0}{4} = 0.25 = h\ (also)$

$\mathbf{1 \le x \le 2}$ Also $\mathbf{0 \le y \le 1}$

STEP – I: $\boldsymbol{f(x,y) = x^2 + y^2}$

Y/X	0	0.25	0.50	0.75	1
1	1	1.6250	1.25	1.5625	2
1.25	1.5625	1.6250	1.8125	2.1250	2.5625
1.50	2.25	2.3125	2.5	2.8125	3.25
1.75	3.0625	3.1250	3.3125	3.6250	4.0625
2	4	4.0625	4.2500	4.5625	5

STEP –I I:

$\boldsymbol{I_1 = \int_0^1 f(1,y)dy = \frac{k}{3}[f(1,y_0) + f(1,y_4) + 2f(1,y_2) + 4[f(1,y_1) + f(1,y_3)] = 1.3333}$

$\boldsymbol{I_2 = \int_0^1 f(1.25,y)dy = \frac{k}{3}[f(1.25,y_0) + f(1.25,y_4) + 2f(1.25,y_2) + 4[f(1.25,y_1) + f(1.25,y_3)] = 1.8958}$ $\boldsymbol{I_3 = \int_0^1 f(1.50,y)dy = \frac{k}{3}[f(1.50,y_0) + f(1.50,y_4) + 2f(1.50,y_2) + 4[f(1.50,y_1) + f(1.50,y_3)] = 2.5832}$ $\boldsymbol{I_4 = \int_0^1 f(1.75,y)dy = \frac{k}{3}[f(1.75,y_0) + f(1.75,y_4) + 2f(1.75,y_2) + 4[f(1.75,y_1) + f(1.75,y_3)] = 3.3958}$
$\boldsymbol{I_5 = \int_0^1 f(2,y)dy = \frac{k}{3}[f(2,y_0) + f(2,y_4) + 2f(2,y_2) + 4[f(2,y_1) + f(2,y_3)] = 4.3316}$

STEP –III:

$\boldsymbol{I = \int_0^1 \int_1^2 (x^2 + y^2)dxdy = \frac{h}{2}[I_1 + I_5 + 2I_3 + 4(I_2 + I_4)] = 2.6654}$

GUASSIAN QUADRATURE FORMULAE

DERIVATION OF TWO-POINT GAUSS QUADRATURE RULE

Method 1:

The two-point Gauss quadrature rule is an extension of the trapezoidal rule approximation where the arguments of the function are not predetermined as a and b, but as unknowns x_1 and x_2. So in the two-point Gauss quadrature rule, the integral is approximated as

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

There are four unknowns x_1, x_2, c_1 and c_2. These are found by assuming that the formula gives exact results for integrating a general third order polynomial,

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

Hence $\int_a^b f(x)dx = \int_a^b (a_0 + a_1 x + a_2 x^2 + a_3 x^3)dx$

$$\int_a^b f(x)dx = \left|(a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4})\right|_a^b$$

$$\int_a^b f(x)dx = \left[a_0(b-a) + a_1(\frac{b^2-a^2}{2}) + a_2(\frac{b^3-a^3}{3}) + a_3(\frac{b^4-a^4}{4})\right] \ldots\ldots\ldots\ldots (i)$$

The formula would then give

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) \approx c_1 f(x_1) + c_2 f(x_2)$$

$$= c_1(a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3) + c_2(a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

Equating Equations (i) and (ii) gives

$$\begin{bmatrix} a_0(b-a) + a_1\left(\frac{b^2-a^2}{2}\right) + a_2\left(\frac{b^3-a^3}{3}\right) + a_3\left(\frac{b^4-a^4}{4}\right) \\ = c_1(a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3) + c_2(a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3) \end{bmatrix}$$

This will give us

$$\int_a^b f(x)dx = a_0(c_1 + c_2) + a_1(c_1 x_1 + c_2 x_2) + a_2(c_1 x_1^2 + c_2 x_2^2) + a_3(c_1 x_1^3 + c_2 x_2^3)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$$

Since in Equation (iii), the constants a_0, a_1, a_2, and a_3 are arbitrary, the coefficients of a_0, a_1, a_2, and a_3 are equal. This gives us four equations as follows

$$(iv) \ldots\ldots\ldots\ldots\ldots\ldots \begin{cases} (b-a) = (c_1 + c_2) \\ \left(\frac{b^2-a^2}{2}\right) = (c_1 x_1 + c_2 x_2) \\ \left(\frac{b^3-a^3}{3}\right) = (c_1 x_1^2 + c_2 x_2^2) \\ \left(\frac{b^4-a^4}{4}\right) = (c_1 x_1^3 + c_2 x_2^3) \end{cases}$$

we can find that the above four simultaneous nonlinear equations have only one acceptable solution

$$c_1 = \frac{b-a}{2},\ c_2 = \frac{b-a}{2},\ \ x_1 = \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2},\ \ x_2 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

Hence

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) = \frac{b-a}{2} f\left[\left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right] + \frac{b-a}{2} f\left[\left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right]$$

Method 2 : We can derive the same formula by assuming that the expression gives exact values for the individual integrals of $\int_a^b 1dx$, $\int_a^b xdx$, $\int_a^b x^2dx$, and $\int_a^b x^3dx$. The reason the formula can also be derived using this method is that the linear combination of the above integrands is a general third order polynomial given by $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

These will give four equations as follows

$$\begin{cases} \int_a^b 1dx = (b-a) = (c_1 + c_2) \\ \int_a^b xdx = \left(\frac{b^2-a^2}{2}\right) = (c_1x_1 + c_2x_2) \\ \int_a^b x^2dx = \left(\frac{b^3-a^3}{3}\right) = (c_1x_1^2 + c_2x_2^2) \\ \int_a^b x^3dx .= \left(\frac{b^4-a^4}{4}\right) = (c_1x_1^3 + c_2x_2^3) \end{cases}$$

These four simultaneous nonlinear equations can be solved to give a single acceptable solution

$$c_1 = \frac{b-a}{2},\ c_2 = \frac{b-a}{2},\ \ x_1 = \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2},\ \ x_2 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

Hence $\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) = \frac{b-a}{2} f\left[\left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right] + \frac{b-a}{2} f\left[\left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right]$

Since two points are chosen, it is called the two-point Gauss quadrature rule. Higher point versions can also be developed.

HIGHER POINT GAUSS QUADRATURE FORMULAS

$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$ is called the three-point Gauss quadrature rule. The coefficients c_1, c_2 and c_3, and the function arguments x_1, x_2 and x_3 are calculated by assuming the formula gives exact expressions for integrating a fifth order polynomial

$$\int_a^b (a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5)dx$$

General n -point rules would approximate the integral

$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + \ldots\ldots.. + c_n f(x_n)$

A number of particular types of Gaussian formulae are given as follows.

GUASSIAN LEGENDER FORMULA: This formula takes the form $\int_a^b f(x)\mathrm{dx} = \sum_1^n A_i f(x_i)$

And Truncation error for formula is $E = \frac{1}{2n+1}[f(1) + f(-1) + I - \sum_1^n A_i x_i f'(x_i)$

Where "I" is the approximate integral obtained by n – point formula.

GUASS – LAGURRE FORMULA

This formula takes the form $\int_0^\infty e^{-x} f(x)\mathrm{dx} = \sum_1^n A_i f(x_i)$

GUASS – HERMITE FORMULA

This formula takes the form $\int_{-\infty}^\infty e^{-x^2} f(x)\mathrm{dx} = \sum_1^n A_i f(x_i)$

GUASS – CHEBYSHEV FORMULA

This formula takes the form $\int_{-1}^1 \frac{f(x)}{\sqrt{1-X^2}}\mathrm{dx} = \frac{\pi}{n}\sum_1^n f(x_i)$

Where "x_i" is zero n – Chebysheves polynomial

EXAMPLE: Find Gauss 2 and 3 point formula for $\int_0^1 e^{-x}dx$ and compare with exact value.

Solution: Firstly interval of integration are transformed to [0,1] to [-1,1]
For this , let $x = \frac{(b-a)t+a+b}{2} \Rightarrow t = \frac{2x-a-b}{b-a}$

$\Rightarrow |t|_{x=a=0} = \frac{0-0-1}{1-0} = -1 \; and \; |t|_{x=b=1} = \frac{2(1)-0-1}{1-0} = 1$

Since $x = \frac{(b-a)t+a+b}{2} \Rightarrow \; |x|_{a=0,b=1} = \frac{(1-0)t+0+1}{2} = \frac{t+1}{2} \Rightarrow x = \frac{t+1}{2} \Rightarrow dx = \frac{1}{2}dt$

Now $\int_0^1 e^{-x}dx = \int_{-1}^1 e^{-\left(\frac{t+1}{2}\right)}\frac{1}{2}dt = \frac{1}{2}\int_{-1}^1 e^{-\left(\frac{t+1}{2}\right)}dt$ …………(i)

Now Gauss 2 point formula is $\int_{-1}^1 f(t)dt = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$

$\Rightarrow \int_{-1}^1 e^{-\left(\frac{t+1}{2}\right)}dt = e^{-\left(\frac{1}{2\sqrt{3}}+\frac{1}{2}\right)} + e^{-\left(\frac{-1}{2\sqrt{3}}+\frac{1}{2}\right)} = e^{-(0.788675134)} + e^{-(0.211324865)}$

$\Rightarrow \int_{-1}^{1} e^{-\left(\frac{t+1}{2}\right)}dt = 0.454446477 + 0.809511042 = 1.263957519$

$(i) \Rightarrow \int_0^1 e^{-x}dx = \frac{1}{2}\int_{-1}^{1} e^{-\left(\frac{t+1}{2}\right)}dt = \frac{1}{2}(1.263957519\,) \Rightarrow \boldsymbol{\int_0^1 e^{-x}dx = 0.631978759}$

Also Gauss 3 point formula is $\int_{-1}^{1} f(t)dt = \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right)$

$\Rightarrow \int_{-1}^{1} e^{-\left(\frac{t+1}{2}\right)}dt = \frac{5}{9}e^{-\left(\sqrt{\frac{3}{5}}+\frac{1}{2}\right)} + \frac{8}{9}e^{-\left(\frac{1}{2}\right)} + \frac{5}{9}e^{-\left(-\sqrt{\frac{3}{5}}+\frac{1}{2}\right)}$

$\Rightarrow \int_{-1}^{1} e^{-\left(\frac{t+1}{2}\right)}dt = \frac{5}{9}e^{-(0.887298334)} + \frac{8}{9}e^{-(0.5)} + \frac{5}{9}e^{(0.112701665)}$

$\Rightarrow \int_{-1}^{1} e^{-\left(\frac{t+1}{2}\right)}dt = 0.228759281 + 0.539138364 + 0.496342865 = 1.26424054$

$(i) \Rightarrow \int_0^1 e^{-x}dx = \frac{1}{2}\int_{-1}^{1} e^{-\left(\frac{t+1}{2}\right)}dt = \frac{1}{2}(1.26424054) \Rightarrow \boldsymbol{\int_0^1 e^{-x}dx = 0.632120}$

Exact value: $\int_0^1 e^{-x}dx = \left|\frac{e^{-x}}{-1}\right|_0^1 = -\left[\frac{1}{e} - 1\right] = -[0.36787944 - 1] = \boldsymbol{0.632120558}$

$Error = exact - approximate = 0.632120558 - 0.632120 = 5.58 \times 10^{-07}$

EXAMPLE: Find Gauss quadrature formula for $\int_0^1 \frac{Sinx}{x}dx$

Solution: Firstly interval of integration are transformed to [0,1] to [-1,1]
For this , let $x = \frac{(b-a)t+a+b}{2}$

Since$x = \frac{(b-a)t+a+b}{2} \Rightarrow \ |x|_{a=0,b=1} = \frac{(1-0)t+0+1}{2} = \frac{t+1}{2} \Rightarrow x = \frac{t+1}{2} \Rightarrow dx = \frac{1}{2}dt$

Also $\Rightarrow |t|_{x=a=0} = \frac{0-0-1}{1-0} = -1 \ and \ |t|_{x=b=1} = \frac{2(1)-0-1}{1-0} = 1$

Now $\int_0^1 \frac{Sinx}{x}dx = \frac{1}{2}\int_{-1}^{1} \frac{Sin\left(\frac{t+1}{2}\right)}{\left(\frac{t+1}{2}\right)}dt \Rightarrow \int_0^1 \frac{Sinx}{x}dx = \int_{-1}^{1} \frac{Sin\left(\frac{t+1}{2}\right)}{t+1}dt$ …………..(i)

Now Gauss 2 point formula is $\int_{-1}^{1} f(t)dt = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$

$\Rightarrow \int_{-1}^{1} \frac{Sin\left(\frac{t+1}{2}\right)}{t+1}dt = \frac{Sin\left(\frac{\frac{1}{\sqrt{3}}+1}{2}\right)}{\frac{1}{\sqrt{3}}+1} + \frac{Sin\left(\frac{-\frac{1}{\sqrt{3}}+1}{2}\right)}{-\frac{1}{\sqrt{3}}+1}$

$\Rightarrow \int_{-1}^{1} \frac{Sin\left(\frac{t+1}{2}\right)}{t+1} dt = \frac{0.709420149}{1.577535.269} + \frac{0.2097554758}{0.4226497308} = 0.4497543526 + 0.4962867844$

$\Rightarrow \int_{-1}^{1} \frac{Sin\left(\frac{t+1}{2}\right)}{t+1} dt = 0.946041137$

NEWTON'S COTES FORMULA

A quadrature formula of the form $\int_a^b f(x)\mathrm{dx} \approx \sum_0^n C_i f(x_i)$ is called a Newton's Cotes Formula if the nodes "$x_0, x_1, \dots \dots \dots, x_n$" are equally spaced. Where $C_i = \int_a^b L_i(x)\mathrm{dx} = \int_a^b \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x-x_i}{x_i-x_j}\mathrm{dx}$

General Newton's Cotes Formula has the form

$$\int_a^b f(x)\mathrm{dx} = h\sum_0^n f(x_i) \int_0^n \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{t-j}{i-j}\mathrm{dt} + \frac{1}{(n+1)!}\int_a^b f^{n+1}(\S_x) \prod_{i=0}^{n}(x-x_i)\,\mathrm{dx}$$

REMARK: Trapezoidal and Simpson's Rule Are Close Newton Cotes formulae while Rectangular Rule is Open Newton Cotes formula.

LIMITATION OF NEWTON'S COTES: Newton's Cotes formulae (Simpson's, Rectangular Rule, and Trapezoidal Rule) are not suitable for Numerical integration over large intervals. Also Newton's Cotes formulas which are based on polynomial interpolation would be inaccurate over a large interval because of oscillatory nature of high degree polynomials. To solve this problem, we use composite Numerical integration.

FORMULA DARIVATION: We shall approximate the given tabulated function by a polynomial "$P_n(x)$" and then integrate this polynomial.

Suppose we are given the data (x_i, y_i) ; $i = 0, 1, 2, \dots \dots . n$ at equispaced points with spacing $h = x_{i+1} - x_i$ we can represent the polynomial by any standard interpolation polynomial.

Now by using Lagrange's formula $f(x) = \sum_0^n l_k(x) y_k \quad \dots \dots \dots \dots \dots \dots \dots . . (i)$

With associated error term $E(x) = \frac{\Pi(x)}{(n+1)!} y^{n+1}(\S) \quad \dots \dots \dots \dots \dots \dots \dots . (ii)$

And $l_k(x) = \frac{\Pi(x)}{(x-x_k)\Pi'(x)} \quad \dots \dots \dots \dots \dots \dots \dots \dots \dots (iii)$

Where $\Pi(x) = (x-x_0)(x-x_1) \dots \dots \dots \dots \dots \dots (x-x_n) \quad \dots \dots \dots \dots \dots \dots \dots (iv)$

Integrating (i) from $a = x_0 \; to \; b = x_n$ w. r. to 'x'

$$\int_a^b f(x)\,dx = \int_a^b \sum_0^n l_k(x) y_k\,dx = \int_a^b [l_0(x)y_0 + l_1(x)y_1 + \cdots \dots \dots + l_k(x)y_k]dx$$

$\int_a^b f(x)\,dx = \sum_0^n \int_a^b l_k(x) y_k\,dx = \sum_0^n \left(\int_a^b l_k(x)\right) y_k dx = \sum_0^n C_k y_k \;\; where\; C_k = \int_a^b l_k(x)\,dx$

And "C_k" are called Newton's Cotes ………………………….(v)

HOW TO FIND NEWTON'S COTES?

Let equispaced nodes are defined as $a = x_0 \; to \; b = x_n \; and \; h = \frac{b-a}{n}$ and $x_k = x_0 + kh$ change the variable $x = x_0 + ph$

Since $a = x_0 = x_0 + 0h, x_1 = x_0 + 1h, \ldots\ldots\ldots\ldots\ldots b = x_n = x_0 + nh$ And $x = x_0 + ph$

Using above values in (IV) we get

$$\Pi(x) = (x_0 + ph - x_0)(x_0 + ph - x_1)\ldots\ldots\ldots\ldots\ldots\ldots(x_0 + ph - x_n)$$

$$\Pi(x) = ph[x_0 + ph - (x_0 + h)][x_0 + ph - (x_0 + 2h)]\ldots\ldots\ldots\ldots\ldots[x_0 + ph - (x_0 + nh)]$$

$$\Pi(x) = ph(ph - h)(ph - 2h)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ph - nh)$$

$$\Pi(x) = h^{n+1}.p(p-1)(p-2)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(p-n) \quad \ldots\ldots\ldots\ldots\ldots..(vi)$$

So $\quad l_k(x) = \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{k-1})(x-x_{k+1})\ldots\ldots(x-x_n)}{(x_k-x_0)(x_k-x_1)\ldots\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots\ldots(x_k-x_n)}$

Now $x_k = x_0 + kh$ and $x_p = x_0 + ph \quad \Longrightarrow x_k - x_p = (k-p)h$

When $\quad p = 0 \Longrightarrow x_k - x_0 = (k-0)h = kh$

$$p = 1 \Longrightarrow x_k - x_1 = (k-1)h$$

$$\vdots \quad \vdots = \vdots$$

$$\vdots \quad \vdots = \vdots$$

$$p = k-1 \Longrightarrow x_k - x_{k-1} = h$$

$$p = k+1 \Longrightarrow x_k - x_{k+1} = -h$$

$$\vdots \quad \vdots = \vdots$$

$$p = n \Longrightarrow x_k - x_n = (k-n)h = -(n-k)h$$

If x = b, x_0 = a

$$\frac{x - x_0}{h} = p$$

$$\frac{x - x_0}{\frac{b-a}{n}} = p$$

n = p

Now putting in "$l_k(x)$" we get

$$l_k(x) = \frac{(x_0+ph-x_0)(x_0+ph-x_0-h)\ldots\ldots(x_0+ph-x_0-nh)}{(kh)(k-1)h(k-2)h\ldots\ldots h.(-h)(-2h)\ldots\ldots[-(n-k)h]}$$

$$l_k(x) = \frac{hp.h(p-1)h(p-2)\ldots\ldots h[p-(k-1)]h[p-(k+1)]\ldots\ldots h(p-n)}{(hk)h(k-1)h(k-2)\ldots\ldots h[k-(k-1)]h[k-(k+1)]\ldots\ldots h(k-n)}$$

$$l_k(x) = \frac{h^n.p(p-1)(p-2)\dots\dots[p-k+1][p-k-1]\dots\dots(p-n)}{h^n.[k(k-1)(k-2)\dots\dots 2.1].(-1)^{n-k}[1.2\dots\dots(n-k)]}$$

$$l_k(x) = \frac{p(p-1)(p-2)\dots\dots[p-k+1][p-k-1]\dots\dots(p-n)}{k!(-1)^{n-k}(n-k)!}$$

$$l_k(x) = \frac{p(p-1)(p-2)\dots\dots[p-k+1][p-k-1]\dots\dots(p-n)}{k!(-1)^{n-k}(n-k)!} \times \frac{(-1)^{n-k}}{(-1)^{n-k}}$$

$$l_k(x) = \frac{(-1)^{n-k}.p(p-1)(p-2)\dots\dots[p-k+1][p-k-1]\dots\dots(p-n)}{k!(-1)^{2(n-k)}(n-k)!}$$

$$l_k(x) = \frac{(-1)^{n-k}.p(p-1)(p-2)\dots\dots[p-k+1][p-k-1]\dots\dots(p-n)}{k!(n-k)!} \qquad \dots\dots\dots\dots\dots(vii)$$

Since $C_k = \int_a^b l_k(x)\,dx$ therefore after putting "$l_k(x)$" and "dx"

As "$x = x_0 + ph$" then $dx = hdp$ and if $x \to a$ $then$ $p \to 0$ $also$ $x \to b$ $then$ $p \to n$

$$C_k = \frac{(-1)^{n-k}}{k!(n-k)!}\int_0^n p(p-1)(p-2)\dots\dots(p-k+1)(p-k-1)\dots\dots(p-n).hdp$$

$$C_k = \frac{(-1)^{n-k}.h}{k!(n-k)!}\int_0^n p(p-1)(p-2)\dots\dots(p-k+1)(p-k-1)\dots\dots(p-n)\,dp$$

This is required formula for Newton Cotes.

ERROR TERM let $\in(x) = \frac{\Pi(x)}{(n+1)!}y^{n+1}(\S) \quad \dots\dots\dots\dots\dots\dots\dots(A)$

$$\Pi(x) = h^{n+1}.p(p-1)(p-2)\dots\dots(p-n) \qquad \dots\dots\dots\dots \quad (B)$$

Using (B) in (A) we get $\in(x) = \frac{h^{n+1}.p(p-1)\dots\dots\dots(p-n).y^{n+1}(\S)}{(n+1)!}$

Integrating both sides $\int_a^b \in(x)dx = \int_o^n \frac{h^{n+1}.p(n+1).p(p-1)\dots\dots\dots(p-n).y^{n+1}(\S)}{(n+1)!}\,hdp$

$$E(x) = \frac{h^{n+2}y^{n+1}(\S)}{(n+1)!}\int_0^n p(p-1)(p-2)\dots\dots\dots\dots\dots\dots\dots(p-n)dp$$

E(x) is called **integral error**.

ALTERNATIVE METHOD FOR DARIVATION OF TRAPEZOIDAL RULE AND ITS ERROR TERM

$f(x) = \sum_{k=0}^{n} l_k(x) y_k + \frac{\Pi(x)}{(n+1)!} y^{n+1}(\S)$

For trapezoidal rule put $n = 1$ $\quad f(x) = \sum_{k=0}^{1} l_k(x) y_k + \frac{\Pi(x)}{2!} \; y''(\S)$

$f(x) = l_o y_o + l_1 y_1 + \frac{(x-x_0)(x-x_1)}{2} \, y''(\S)$

Integrating both sides

$\int_{a=x_0}^{b=x_n=x_1} f(x)dx = y_o \int_{x_0}^{x_1} l_o(x)dx + y_1 \int_{x_0}^{x_1} l_1(x)dx + \frac{y''(\S)}{2} \int_{x_0}^{x_1} (x-x_0)(x-x_1)dx$

$\int_{x_0}^{x_1} f(x)dx = y_0 \int_{x_0}^{x_1} \frac{(x-x_1)}{(x_0-x_1)} dx + y_1 \int_{x_0}^{x_1} \frac{(x-x_0)}{(x_1-x_0)} dx + \frac{y''(\S)}{2} \int_{x_0}^{x_1} (x-x_0)(x-x_1)dx$

Now by changing variables

$x = x_0 + ph \quad then\; x \to x_0 \Rightarrow p \to 0 \;\; and \;\; x_1 = x_0 + 1h \; then \; x \to x_1 \Rightarrow p \to 1$

$\int_{x_0}^{x_1} f(x)dx = y_0 \int_0^1 \frac{(x_0+ph)-(x_0+1h).hdp}{x_0-(x_0+1h)} + y_1 \int_0^1 \frac{(x_0+ph)-x_0.hdp}{(x_0+ph)-x_0}$

$+ \frac{y''(\S)}{2} \int_0^1 (x_0 + ph - x_0)[(x_0 + ph) - (x_0 + 1h)].hdp$

$\int_{x_0}^{x_1} f(x)dx = y_0 \int_0^1 \frac{h(p-1)hdp}{-h} + y_1 \int_0^1 \frac{ph.hdp}{h} + \frac{y''(\S)}{2} \int_0^1 ph[h(p-1)]hdp$

$\int_{x_0}^{x_1} f(x)dx = y_0 h \left| \frac{(p-1)^2}{2} \right|_0^1 + y_1 h \left| \frac{p^2}{2} \right|_0^1 + \frac{y''(\S)}{2} h^3 \left| \frac{p^3}{3} - \frac{p^2}{2} \right|_0^1$

$\int_{x_0}^{x_1} f(x)dx = \frac{y_0 h}{2} + \frac{y_1 h}{2} - \frac{y''(\S)}{12} h^3$ As required.

SIMPSON'S RULE AND ERROR TERM

Since $\quad C_k = \frac{(-1)^{n-k}.h}{k!(n-k)!} \int_0^n p(p-1)(p-2) \dots \dots (p-k+1)(p-k-1) \dots \dots (p-n)\, dp$

$\dots \dots \dots \dots \dots \dots \dots \dots \dots .(i)$

And $\quad E_n(x) = \frac{h^{n+2} y^{n+1}(\S)}{(n+1)!} \int_0^n p(p-1)(p-2) \dots \dots \dots \dots \dots \dots . . \dots \dots (p-n)dp$

$\dots \dots \dots \dots \dots \dots \dots \dots \dots (ii)$

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Putting n = 2, k = 0 in (i) we get

$$C_0 = \frac{(-1)^{2-0}.h}{0!(2-0)!}\int_0^2 (p-1)(p-2)dp = \frac{h}{2}\int_0^2 (p-1)(p-2)dp = \frac{h}{2}\int_0^2 (p^2-3p+2)dp$$

$$C_0 = \frac{h}{2}\left|\frac{p^3}{3} - \frac{3p^2}{2} + 2p\right|_0^2 = \frac{h}{2}\left(\frac{2}{3}\right) = \frac{h}{3}$$

Now Putting n = 2, k = 1 in (i) we get

$$C_1 = \frac{(-1)^1.h}{1!(2-1)!}\int_0^2 p(p-2)dp = -h\int_0^2 (p^2-2p)dp = -h\left|\frac{p^3}{3} - p^2\right|_0^2 = \frac{4}{3}h$$

Now Putting n = 2, k =2 in (i) we get

$$C_2 = \frac{(-1)^{2-2}.h}{2!(2-2)!}\int_0^2 p(p-1)dp = \frac{h}{2}\left|\frac{p^3}{3} - \frac{p^2}{2}\right|_0^2 = \frac{h}{2}\left(\frac{8}{3} - 2\right) = \frac{h}{3}$$

ERROR TERM FOR SIMPSON'S RULE:

Now Putting n = 2 in (ii) we get

$$E_2(x) = \frac{h^{2+2}y^{2+1}(\S)}{(2+1)!}\int_0^2 p(p-1)(p-2)dp = \frac{h^4y^3(\S)}{3!}\int_0^2 p(p^2-3p+2)dp$$

$$E_2(x) = \frac{h^4y^3(\S)}{3!}\left|\frac{p^4}{4} - \frac{3p^3}{3} + \frac{2p^2}{2}\right|_0^2 = \frac{h^4y^3(\S)}{3!}\left(\frac{16}{4} - 8 + 4\right) = 0$$

Error term is zero so we find Global error term $E_2 = -\frac{h^3y^4(\S)}{90}$

Now for n = 3

$$\int_{x_0}^{x_3} f(x)dx = \sum_{k=0}^{3} C_k y_k = C_0y_0 + C_1y_1 + C_2y_2 + C_3y_3 + E_3(x) \quad \ldots\ldots\ldots..(i)$$

If $C_0 = \frac{3h}{8}, C_1 = \frac{9h}{8}, C_2 = \frac{9h}{8}, C_3 = \frac{3h}{8}, \quad E_3(x) = -\frac{3h^5y^4(\S)}{80}$ then (i) becomes

$$\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3] - \frac{3h^5y^4(\S)}{80}$$

DIFFERENTIAL EQUATIONS

DIFFERENCE EQUATION

Equation involving differences is called Difference Equation.

OR an equation that consist of an independent variable 't', a dependent variable 'y(t)' and one or more several differences of the dependent variable $\mathbf{y_t}$ as $\Delta\mathbf{y_t}, \Delta^2\mathbf{y_t}, \dots \dots .. \Delta^n\mathbf{y_t}$ is called difference equation.

The functional relationship of the difference equation is $\boldsymbol{F(t, y, \Delta y_t, \Delta^2 y_t, \dots \dots .. \Delta^n y_t) = 0}$

Solution of differential equation will be sequence of $\mathbf{y_t}$ values for which the equation is true for some set of consecutive integer 't'.

Importance: difference equation plays an important role in problem where there is a quantity 'y' that depends on a continuous independent variable 't'

In polynomial dynamics, modeling the rate of change of the population or modeling the growth rate yields a differential equation for the population 'y' as the function of time 't'

i.e. $\boldsymbol{\frac{dy}{dt} = f(t, y)}$

In differential equation models, usually the population is assumed to vary continuously I time.

Difference equation model arise when the population is modeled only at certain discrete time.

DIFFERENCE OF A POLYNOMIAL

The "nth" difference of a polynomial of degree 'n' is constant, when the values of the independent variable are given at equal intervals.

EXAMPLES:

i. $\Delta^3 y_t + 3\Delta^2 y_t - 6\Delta y_t + y_t = 9t^2 + 6t$
ii. $\Delta^2 y_t + 3\Delta y_t - 7y_t = 0$

ORDER AND DEGREE OF DE:

The highest derivative involved in the equation determines the order of Differential Equation and the power of highest derivative in Differential Eq. is called degree of D.E. for example $\boldsymbol{\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y = 0}$ has order "2" and degree "1"

Order of differential equation is the difference between the largest and smallest argument 't' appearing in it.

For example; if $y_{t+2} + y_{t+1} - 7y_t = 0$ then $order = t + 2 - t = 2$

Degree of differential equation is the highest power of 'y'

For example; if ${y_{t+2}}^3 + {y_{t+1}}^2 - 7y_t = 0$ then $degree = 3$

DIFFERENTIAL EQUATION: It is the relation which involves the dependent variable, independent variable and Differential co-efficient i.e.

$$f(t,y)=\frac{dy}{dt}=\frac{y-y_0}{t-t_0} \qquad \Rightarrow (t-t_0)\frac{dy}{dt}=y-y_0 \qquad \Rightarrow y=y_0+(t-t_0)\frac{dy}{dt}$$

ORDINARY DIFFERENTIAL EQUATION: If differential co-efficient of Differential Equation are total, then Differential Equation is called Ordinary Differential equation.

e. g. $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+5y=2x$

PARTIAL DIFFERENTIAL EQUATION: If differential co-efficient of Differential Equation are partial, then Differential Equation is called Ordinary Differential equation.

e. g. $\frac{\partial^2 y}{\partial x^2}+\frac{\partial^2 x}{\partial y^2}=0$

SOLUTION OF DIFFERENTIAL EQUATION: A solution to a difference equation is any function which satisfies it. It is the relation which satisfies the Differential Equation as consider $\frac{d^2y}{dx^2}+y=0$

Then $y=sinx, cosx, 3sinx, 20cosx$ Are all solution of above equation.

THE MOST GENERAL SOLUTION

It is the solution which contains as many arbitrary constants as the order of differential equation. e.g. $y''+y=0$ Is a 2nd order Differential Eq. with constant co-efficient and general solution is $y=c_1cosx+c_2sinx$

PARTICULAR SOLUTION: Solution which can be obtained from General Solution by giving different values to the arbitrary constants "c_1, c_2" in $y=c_1cosx+c_2sinx$ For example $y=4cosx+7sinx$

SINGULAR SOLUTION: Solution which cannot be obtained from General Solution by giving different values to the arbitrary constants.

HOMOGENOUS DIFFERENTIAL EQUATION: A differential equation for which "$u=0$" is a solution is called a Homogenous Differential Equation where 'u' is unknown function. In other words, a differential equation which always possesses the trivial solution "$u=0$" is called Homogenous Differential Equation.

NON-HOMOGENOUS DIFFERENTIAL EQUATION: A differential equation for which "$u\neq 0$" (i.e. Non-Trivial solution) is a solution is called a Nonhomogeneous Differential Equation where 'u' is unknown function.

INITIAL AND BOUNDARY CONDITIONS: To evaluate arbitrary constant in the General solution we need some conditions on the unknown function or solution corresponding to some values of the independent variables. Such conditions are called Boundary or Initial conditions.

If all the conditions are given at the same value of the independent variable, then they are called **Initial conditions.** If the conditions are given at the end points of the independent variable, then they are called **Boundary conditions.**

INITIAL VALUE PROBLEM: An initial value problem for a first order Ordinary Differential Equation is the equation together with an initial condition on a specific interval $a \le x \le b$

Such that $y' = f(x, y)$,, $y(a) = y_a$,, $and\ x \in [a, b]$

The equation is Autonomous if (y') is independent of 'x'

BOUNDARY VALUE PROBLEM: A problem in which we solve an Ordinary Differential Equation of order two subject to condition on $y(x)$ or $y'(x)$ at two different points is called a two point boundary value problem or simply a Boundary value problem.
OR A differential equation along with one or more boundary conditions defines a boundary value problem.

CONVEX SET: A set $D \subset R^2$ is said to be convex if whenever (t_1, y_1) and (t_2, y_2) belong to 'D' then $[(1-\lambda)t_1 + \lambda t_2, (1-\lambda)y_1 + \lambda y_2]$ also belong to 'D' for every "λ" in [0, 1]

LIPSCHITZ CONDITION: A function $f(t, y)$ is said to satisfy a Lipschitz condition in the variable 'y' on a set $D \subset R^2$ if a constant '$L > 0$' exists with

$|f(t, y_1) - f(t, y_2)| \le L|y_1 - y_2|$ Whenever (t, y_1) and (t, y_2) are in 'D' and 'L' is called Lipschitz constant for 'f'

WELL – POSED PROBLEM: The initial value problem $\frac{dy}{dx} = f(x, y)$; $a \le x \le b$; $y(a) = a$ is said to be a well - posed problem if **(i)** A unique solution y(x) to the problem exist.

There exist constants $\in_0 > 0$ and $k > 0$ such that for any " $\in$ " with $\in_0 > \in > 0$ whenever $\delta(x)$ is continuous with $[\delta(x)] < \in\ \forall\ x \in [a, b]$ and when $\delta_0 < \in$ the initial value problem $\frac{dz}{dx} = f(x, z) + \delta(x)$; $a \le x \le b$; $z(a) = a + \delta_0$ has a unique solution $z(x)$ that satisfies $|z(x) - y(x)| < k \in\ \ \forall\ \ x \in [a, b]$

The problem $\frac{dz}{dx}$ is called a **Perturbed** problem associated with $\frac{dy}{dx}$

SOME STANDARD TECHNIQUES FOR SOLVING ELEMENTARY DIFFERENTIAL EQUATIONS ANALYTICALLY

SECOND ORDER HOMOGENEOUS LINEAR DIFFERENCE EQUATION

The homogeneous difference equation of order 2 is

$\boldsymbol{y_{k+2} + a_1 y_{k+1} + a_2 y_k} = 0 \dots\dots\dots\dots(i)$

Suppose that $\boldsymbol{y_k} = Ab^k$ be the non – trivial solution of (i) then it must satisfies the eq (i) i.e.

$Ab^{k+2} + a_1 Ab^{k+1} + a_2 Ab^k = 0 \Rightarrow Ab^k(b^2 + a_1 b + a_2) = 0$

$\Rightarrow either\ Ab^k = 0\ \ or\ \ (b^2 + \boldsymbol{a_1} b + \boldsymbol{a_2}) = 0$

Since Ab^k is supposed to be non – trivial solution therefore $(\boldsymbol{b^2 + a_1 b + a_2}) = \mathbf{0} \dots\dots\dots(ii)$

This quadratic equation is called **auxiliary** equation or **characteristic** equation.

Let $b_1\ and\ b_2$ be the roots of equation (ii) then for these values there corresponds two solutions, $u_k\ and\ v_k$ which are given by $u_k = Ab_1{}^k\ and\ v_k = Ab_2{}^k$

Now **three cases** appear

CASE I : BOTH ROOTS $\boldsymbol{b_1\ and\ b_2}$ ARE REAL AND UNEQUAL(DISTINCT):

i.e. $discriminent = a_1{}^2 - 4a_2 > 0$ then general solution of (i) is $y_k = A_1 b_1{}^k + A_2 b_2{}^k$

or $y_k = A_1 b_1{}^k + A_2 b_2{}^k + A_3 b_3{}^k + \cdots \dots\dots.. + A_n b_n{}^k$

EXAMPLE : Solve the following homogeneous differential equation $y_{k+2} - 13y_{k+1} + 36y_k = 0$

SOLUTION : Given $y_{k+2} - 13y_{k+1} + 36y_k = 0 \dots\dots\dots(i)$

Let $\boldsymbol{y_k} = Ab^k$ be the non – trivial solution of (i) then it must satisfies the eq (i) i.e.

$Ab^{k+2} - 13Ab^{k+1} + 36Ab^k = 0 \Rightarrow Ab^k(b^2 - 13b + 36) = 0$

$\Rightarrow either\ Ab^k = 0\ \ or\ \ (b^2 - 13b + 36) = 0$

Since Ab^k is supposed to be non – trivial solution therefore $(b^2 - 13b + 36) = 0 \dots\dots\dots(ii)$

$\Rightarrow b^2 - 9b - 4b + 36 = 0 \Rightarrow b(b-9) - 4(b-9) = 0 \Rightarrow (b-4)(b-9) = 0 \Rightarrow b = 9,4$

So $y_k = A_1 4^k + A_2 9^k$

EXAMPLE : Solve the following homogeneous differential equation $y_{k+2} - 6y_{k+1} + 7y_k = 0$

SOLUTION : Given $y_{k+2} - 6y_{k+1} + 7y_k = 0 \ldots\ldots\ldots (i)$

Let $\boldsymbol{y_k} = Ab^k$ be the non – trivial solution of (i) then it must satisfies the eq (i) i.e.

$Ab^{k+2} - 6Ab^{k+1} + 7Ab^k = 0 \Rightarrow Ab^k(b^2 - 6b + 7) = 0$

$\Rightarrow either\ Ab^k = 0\ \ or\ \ (b^2 - 6b + 7) = 0$

Since Ab^k is supposed to be non – trivial solution therefore $(b^2 - 6b + 7) = 0 \ldots\ldots\ldots.(ii)$

Here $a = 0, b = -6b, c = 7$ then using quadratic formula we get $b = 3 \pm \sqrt{2}$

$\Rightarrow b_1 = 3 + \sqrt{2}\ and\ b_2 = 3 - \sqrt{2}$

then general solution of (i) is $y_k = A_1\left(3 + \sqrt{2}\right)^k + A_2\left(3 - \sqrt{2}\right)^k$

CASE II : BOTH ROOTS $\boldsymbol{b_1}$ *and* $\boldsymbol{b_2}$ ARE REAL AND EQUAL:

i.e. $discriminent = {a_1}^2 - 4a_2 = 0$

then general solution of (i) is $y_k = (A_1 + A_2k){b_1}^k$

or $y_k = (A_1 + A_2k + A_3k^2){b_1}^k + A_3{b_2}^k + \cdots\ldots\ldots.. + A_n{b_n}^k$ in general.

EXAMPLE : Solve the following homogeneous differential equation $9y_{k+2} - 12y_{k+1} + 4y_k = 0$

SOLUTION : Given $9y_{k+2} - 12y_{k+1} + 4y_k = 0 \ldots\ldots\ldots (i)$

Let $\boldsymbol{y_k} = Ab^k$ be the non – trivial solution of (i) then it must satisfies the eq (i) i.e.

$9Ab^{k+2} - 12Ab^{k+1} + 4Ab^k = 0 \Rightarrow Ab^k(9b^2 - 12b + 4) = 0$

$\Rightarrow either\ Ab^k = 0\ \ or\ \ (9b^2 - 12b + 4) = 0$

Since Ab^k is supposed to be non – trivial solution therefore $(9b^2 - 12b + 4) = 0 \ldots\ldots\ldots.(ii)$

$\Rightarrow (3b - 2)^2 = 0 \Rightarrow (3b - 2)(3b - 2) = 0 \Rightarrow b = \frac{2}{3}, \frac{2}{3}$

then general solution of (i) is $y_k = (A_1 + A_2k)\left(\frac{2}{3}\right)^k$

CASE III : BOTH ROOTS b_1 *and* b_2 ARE IMAGINARY:

i.e. $discriminent = {a_1}^2 - 4a_2 < 0$, the roots are complex (say) $m_1 \pm im_2$

let $b_1 = m_1 + im_2$ *and* $b_2 = m_1 - im_2$

then general solution of (i) is $y_k = (A_1 Cosk\theta + A_2 Sink\theta)R^k$

where $R = \sqrt{{m_1}^2 + {m_2}^2}$ and $\theta = Tan^{-1}\left(\frac{m_2}{m_1}\right)$

EXAMPLE : Solve the following homogeneous differential equation $y_{k+4} + 4y_k = 0$

SOLUTION : Given $y_{k+4} + 4y_k = 0 \ldots\ldots\ldots (i)$

Let $\boldsymbol{y_k} = Ab^k$ be the non – trivial solution of (i) then it must satisfies the eq (i) i.e.

$Ab^{k+4} + 4Ab^k = 0 \Rightarrow Ab^k(b^4 + 4) = 0 \Rightarrow either\ Ab^k = 0\ or\ (b^4 + 4) = 0$

Since Ab^k is supposed to be non – trivial solution therefore $(b^4 + 4) = 0 \ldots\ldots\ldots.(ii)$

$\Rightarrow (b^2)^2 + 2^2 + 2(b^2)(2) - 4b^2 = 0 \Rightarrow (b^2 + 2)^2 - (2b^2)^2 = 0$

$\Rightarrow (b^2 + 2 + 2b)(b^2 + 2 - 2b) = 0 \Rightarrow b^2 + 2 + 2b = 0\ and\ b^2 + 2 - 2b = 0$

$\Rightarrow b = -1 \pm i\ and\ b = 1 \pm i \Rightarrow b_1 = 1 \pm i\ \ and\ b_2 = -1 \pm i$

where $R = \sqrt{{m_1}^2 + {m_2}^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$ for both above

and $\theta = Tan^{-1}\left(\frac{m_2}{m_1}\right) = Tan^{-1}(1) = \frac{\pi}{4}$ also $\theta = Tan^{-1}\left(\frac{m_2}{m_1}\right) = Tan^{-1}(-1) = \frac{3\pi}{4}$

then general solution of (i) is $\boldsymbol{y_k = (A_1 Cosk\theta + A_2 Sink\theta)R^k = \left(A_1 Cosk\frac{\pi}{4} + A_2 Sink\frac{\pi}{4}\right)(\sqrt{2})^k + \left(B_1 Cos\frac{3k\pi}{4} + B_2 Sin\frac{3k\pi}{4}\right)(\sqrt{2})^k}$

EXAMPLE :

Solve the following homogeneous differential equation $y_{k+3} + y_{k+2} - y_{k+1} - y_k = 0$

SOLUTION : Given $y_{k+3} + y_{k+2} - y_{k+1} - y_k = 0 \ldots\ldots\ldots (i)$

Let $\boldsymbol{y_k} = Ab^k$ be the non – trivial solution of (i) then it must satisfies the eq (i) i.e.

$Ab^{k+3} + Ab^{k+2} - Ab^{k+1} - Ab^k = 0 \Rightarrow Ab^k(b^3 + b^2 - b - 1) = 0$

$\Rightarrow either\ Ab^k = 0\ or\ (b^3 + b^2 - b - 1) = 0$

Since Ab^k is supposed to be non – trivial solution therefore $(b^3 + b^2 - b - 1) = 0 \ldots\ldots\ldots(ii)$

-1	1	1 -1	-1 0	-1 1
	1	0	-1	0

$\Rightarrow b = -1 \; and \; \Rightarrow (b^2 - 1) = 0 \Rightarrow b = 1, -1$

then general solution of (i) is $y_k = A_1 1^k + (A_2 + A_3 k)(-1)^k$

EXAMPLE :

Solve the following homogeneous differential equation $2y_{k+2} - 5y_{k+1} + 2y_k = 0$ with $y_0 = 0, y_1 = 1$

SOLUTION : Given $2y_{k+2} - 5y_{k+1} + 2y_k = 0 \ldots\ldots\ldots(i)$

Let $\boldsymbol{y_k} = Ab^k$ be the non – trivial solution of (i) then it must satisfies the eq (i) i.e.

$2Ab^{k+2} - 5Ab^{k+1} + 2Ab^k = 0 \Rightarrow Ab^k(2b^2 - 5b + 2) = 0$

$\Rightarrow either \; Ab^k = 0 \; or \; (2b^2 - 5b + 2) = 0$

Since Ab^k is supposed to be non – trivial solution therefore $(2b^2 - 5b + 2) = 0 \ldots\ldots\ldots(ii)$

$\Rightarrow 2b^2 - 4b - b + 2 = 0 \Rightarrow 2b(b-2) - 1(b-2) = 0 \Rightarrow (2b-1)(b-2) = 0 \Rightarrow b = 2, \frac{1}{2}$

So $y_k = A_1 2^k + A_2 \left(\frac{1}{2}\right)^k$

Now using $y_0 = 0 \Rightarrow y_0 = A_1 2^0 + A_2 \left(\frac{1}{2}\right)^0 \Rightarrow A_1 + A_2 = 0$ ……….(iii) $\Rightarrow A_2 = -A_1$

$y_1 = 1 \Rightarrow y_1 = A_1 2^1 + A_2 \left(\frac{1}{2}\right)^1 \Rightarrow 2A_1 + \frac{1}{2}A_2 = 1$ ……….(iv)

$\Rightarrow 2A_1 - \frac{1}{2}A_1 = 1 \Rightarrow A_1 = \frac{3}{2}$ then $\Rightarrow A_2 = -A_1 \Rightarrow A_2 = -\frac{3}{2}$

then general solution of (i) is $y_k = \frac{3}{2}.2^k - \frac{3}{2}.\left(\frac{1}{2}\right)^k \Rightarrow y_k = \frac{1}{3}(2^{1+k} - 2^{1-k})$

EXAMPLE : Solve the following homogeneous differential equation $y_{k+2} + \frac{1}{4}y_k = 0$

SOLUTION : Given $y_{k+2} + \frac{1}{4}y_k = 0 \ldots\ldots\ldots(i)$

Let $\boldsymbol{y_k} = Ab^k$ be the non – trivial solution of (i) then it must satisfies the eq (i) i.e.

$Ab^{k+4} + \frac{1}{4}Ab^k = 0 \Rightarrow Ab^k\left(b^4 + \frac{1}{4}\right) = 0 \;\Rightarrow either\; Ab^k = 0 \;\; or \;\; \left(b^4 + \frac{1}{4}\right) = 0$

Since Ab^k is supposed to be non – trivial solution therefore $\left(b^4 + \frac{1}{4}\right) = 0 \ldots\ldots\ldots.(ii)$

$\Rightarrow b = \pm\frac{1}{2}i \;\Rightarrow b_1 = \frac{1}{2}i \quad and\; b_2 = -\frac{1}{2}i$

where $R = \sqrt{{m_1}^2 + {m_2}^2} = \sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \sqrt{2}$ also $R = \sqrt{{m_1}^2 + {m_2}^2} = \sqrt{1^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{2}$

and $\theta = Tan^{-1}\left(\frac{m_2}{m_1}\right) = Tan^{-1}\left(\frac{\frac{1}{2}}{0}\right) = Tan^{-1}(\infty) = \frac{\pi}{2}$

also $\theta = Tan^{-1}\left(\frac{m_2}{m_1}\right) = Tan^{-1}\left(\frac{-\frac{1}{2}}{0}\right) = Tan^{-1}(\infty) = \frac{\pi}{2}$

then general solution of (i) is $\boldsymbol{y_k = \left(A_1 Cosk\frac{\pi}{2} + A_2 Sink\frac{\pi}{2}\right)\left(\frac{1}{2}\right)^k}$

EXAMPLE :

Solve the following homogeneous differential equation
$y_{k+4} - 6y_{k+3} + 14y_{k+2} - 14y_{k+1} + 5y_k = 0$

SOLUTION : Given $y_{k+4} - 6y_{k+3} + 14y_{k+2} - 14y_{k+1} + 5y_k = 0$

Let $\boldsymbol{y_k} = Ab^k$ be the non – trivial solution of (i) then it must satisfies the eq (i) i.e. $\mathbf{Ab^{k+4} - 6Ab^{k+3} + 14Ab^{k+2} - 14Ab^{k+1} + 5Ab^k = 0 \Rightarrow Ab^k(b^4 - 6b^3 + 14b^2 - 14b + 5) = 0}$

$\Rightarrow either\; Ab^k = 0 \;\; or \; (b^4 - 6b^3 + 14b^2 - 14b + 5) = 0$

Since Ab^k is supposed to be non – trivial solution therefore

$(b^4 - 6b^3 + 14b^2 - 14b + 5) = 0 \ldots\ldots\ldots.(ii)$

-	1	-6	14	-14	5
1		1	-5	9	-5
	1	-5	9	-5	0
		1	-4	5	
1	1	-4	5	0	

$\Rightarrow b = 1{,}1 \; and \;\Rightarrow (b^2 - 4b + 5) = 0 \Rightarrow b = 2 \pm i$ using quadratic formula.

$\Rightarrow b_1 = 1\,, b_2 = 1\,, b_3 = 2 + i\,, b_4 = 2 - i$

where $R = \sqrt{{m_1}^2 + {m_2}^2} = \sqrt{2^2 + \;(1)^2} = \sqrt{5}$ also $R = \sqrt{{m_1}^2 + {m_2}^2} = \sqrt{2^2 + \;(-1)^2} = \sqrt{5}$

and $\theta = Tan^{-1}\left(\frac{m_2}{m_1}\right) = Tan^{-1}\left(\frac{1}{2}\right)$ also $\theta = Tan^{-1}\left(\frac{m_2}{m_1}\right) = Tan^{-1}\left(-\frac{1}{2}\right)$

EXAMPLE :

Solve the following homogeneous differential equation $y_{k+3} + 6y_{k+2} + 11y_{k+1} + 6y_k = 0$ with $y_0 = 0, y_1 = 1, y_2 = 0$

SOLUTION : Given $y_{k+3} + 6y_{k+2} + 11y_{k+1} + 6y_k \dots\dots\dots (i)$

Let $\boldsymbol{y_k} = Ab^k$ be the non – trivial solution of (i) then it must satisfies the eq (i) i.e.

$Ab^{k+3} + 6Ab^{k+2} + 11Ab^{k+1} + 6Ab^k = 0 \Rightarrow Ab^k(b^3 + 6b^2 + 11b + 6) = 0$

$\Rightarrow either\ Ab^k = 0\ \ or\ \ (b^3 + 6b^2 + 11b + 6) = 0$

Since Ab^k is supposed to be non – trivial solution therefore

$(b^3 + 6b^2 + 11b + 6) = 0 \dots\dots\dots. (ii)$

-1	1	6	11	6
		-1	-5	-6
	1	5	6	0

$\Rightarrow b = -1\ and\ \Rightarrow (b^2 + 5b + 6) = 0 \Rightarrow b = -2, -3 \Rightarrow b_1 = -1\ , b_2 = -2\ , b_3 = -3$

then general solution of (i) is $y_k = A_1(-1)^k + A_2(2)^k + A_3(-3)^k$

Now using $y_0 = 1 \Rightarrow y_0 = A_1(-1)^0 + A_2(2)^0 + A_3(-3)^0 \Rightarrow A_1 + A_2 + A_3 = 1$ ………..(iii)

$y_1 = 1 \Rightarrow y_1 = A_1(-1)^1 + A_2(2)^1 + A_3(-3)^1 \Rightarrow A_1 + 2A_2 + 2A_3 = -1$ ………..(iv)

$y_2 = 0 \Rightarrow y_2 = A_1(-1)^2 + A_2(2)^2 + A_3(-3)^2 \Rightarrow A_1 + 4A_2 + 9A_3 = 0$ ………..(v)

Eq (iii) – Eq (iv) $\Rightarrow A_2 + 2A_3 = -2$ ………..(vi) and Eq (iv) – Eq (v) $\Rightarrow A_2 + 3A_3 = \frac{1}{2}$ ………..(vii)

Eq (vi) – Eq (vii) $\Rightarrow A_3 = \frac{5}{3}$

Put in (vi) $\Rightarrow A_2 = -7$ Put in (iii) $\Rightarrow A_1 = \frac{11}{2}$

then general solution of (i) is $y_k = \frac{11}{2}(-1)^k - 7(-2)^k + \frac{5}{2}(-3)^k$

EXERCISE:

i. $y_{k+2} = y_{k+1} + y_k$; $y_0 = 0,\ y_1 = 1$
ii. $y_{k+2} - 2y_{k+1} + y_k = 0$

SECOND ORDER INHOMOGENEOUS LINEAR DIFFERENCE EQUATION

The non - homogeneous difference equation has the form as

$\boldsymbol{y_{n+k} + a_1 y_{n+(k-1)} + a_2 y_{n+(k-2)} + \cdots \ldots . + a_n y_k} = f(\boldsymbol{x}) \ldots \ldots \ldots \ldots . (i)$

If $\boldsymbol{y_k}$ is the solution of nth order homogeneous difference equation and $\boldsymbol{y_k^*}$ is the solution of non – homogeneous linear difference equation then $\boldsymbol{y_k = y_k + y_k^*}$ where $\boldsymbol{y_k}$ is also called complimentary function and $\boldsymbol{y_k^*}$ is particular integral.

TYPE I : When R.H.S of given non – homogeneous difference equation is constants. i.e. $f(\boldsymbol{k}) = \boldsymbol{constant}$ then to find $\boldsymbol{y_k^*}$ put $\boldsymbol{y_{k's} = A}$ and then evaluate the value of A.

EXAMPLE : Solve the following homogeneous differential equation $y_{k+2} - 3y_{k+1} - 4y_k = 2$

SOLUTION : Given $y_{k+2} - 3y_{k+1} - 4y_k = 2 \ldots \ldots \ldots (i)$

Firstly we find characteristic solution.

Let $\boldsymbol{y_k} = Ab^k$ be the non – trivial solution of (i) then it must satisfies the eq (i) i.e.

$Ab^{k+2} - 3Ab^{k+1} - 4Ab^k = 0 \Rightarrow Ab^k(b^2 - 3b - 4) = 0$
$\Rightarrow either\ Ab^k = 0\ or\ (b^2 - 3b - 4) = 0$

Since Ab^k is supposed to be non – trivial solution therefore $(b^2 - 3b - 4) = 0$

$\Rightarrow b^2 - 4b + b - 4 = 0 \Rightarrow b(b-4) + 1(b-4) = 0 \Rightarrow (b-4)(b+1) = 0 \Rightarrow b = -1,4$

So $y_k = A_1(-1)^k + A_2 4^k$
To find particular integral. Let $\boldsymbol{y_k = A}$ $\quad (i) \Rightarrow A - 3A - 4A = 2 \Rightarrow -6A = 2 \Rightarrow A = -\frac{1}{3}$
Now general solution is $\boldsymbol{y_k = y_k + y_k^*} = A_1(-1)^k + A_2 4^k - \frac{1}{3}$

EXAMPLE :

Solve the following homogeneous differential equation $y_{k+3} - 3y_{k+2} + 3y_{k+1} - y_k = 9$

SOLUTION : Given $y_{k+3} - 3y_{k+2} + 3y_{k+1} - y_k = 9 \ldots \ldots \ldots (i)$

Firstly we find characteristic solution.

Let $\boldsymbol{y_k} = Ab^k$ be the non – trivial solution of (i) then it must satisfies the eq (i) i.e.

$Ab^{k+3} - 3Ab^{k+2} + 3Ab^{k+1} - Ab^k = 0 \ldots \ldots \ldots (ii)$

$\Rightarrow Ab^k(b^3 - 3b^2 + 3b - 1) = 0 \Rightarrow either\ Ab^k = 0\ \ or\ \ (b^3 - 3b^2 + 3b - 1) = 0$

Since Ab^k is supposed to be non – trivial solution therefore

$(b^3 - 3b^2 + 3b - 1) = 0 \ldots\ldots\ldots(ii) \quad \Rightarrow (b-1)^3 = 0 \Rightarrow b = 1,1,1$

So $y_k = (A_1 + A_2k + A_3k^2)(1)^k$

TO FIND PARTICULAR INTEGRAL.: Since in characteristic solution $y_k = A_2k$ and $y_k = A_3k^2$ involves in homogeneous solution of given equation. So it does not particular integral for equation. So, we take higher power for 'k'' then put $y_k = Ak^3$ in (ii)

$(ii) \Rightarrow A(k+3)^3 - 3A(k+2)^3 + 3A(k+1)^3 - A(k)^3 = 9 \Rightarrow A = \frac{3}{2}$ (after solving)

Thus $\boldsymbol{y_k^*} = \frac{3}{2}k^3$

Now general solution is $\boldsymbol{y_k} = \boldsymbol{y_k} + \boldsymbol{y_k^*} = (A_1 + A_2k + A_3k^2)(1)^k + \frac{3}{2}k^3$

TYPE II : When R.H.S of given non – homogeneous difference equation has the form as $f(\boldsymbol{k}) = \boldsymbol{\beta} \propto^{\boldsymbol{k}}$ where $\propto$ *and* β are constants then to find $\boldsymbol{y_k^*}$ put $\boldsymbol{y_{k's}} = \boldsymbol{A} \propto^{\boldsymbol{k}}$ and then evaluate the value of A.

EXAMPLE : Solve the following homogeneous differential equation $y_{k+2} - 6y_{k+1} + 7y_k = \mathbf{3^k}$

SOLUTION : Given $y_{k+2} - 6y_{k+1} + 7y_k = \mathbf{3^k} \ldots\ldots\ldots(i)$

Firstly we find characteristic solution.

Let $\boldsymbol{y_k} = Ab^k$ be the non – trivial solution of (i) then it must satisfies the eq (i) i.e.

$Ab^{k+2} - 6Ab^{k+1} + 7Ab^k = 0 \ldots\ldots\ldots(ii)$

$\Rightarrow Ab^k(b^2 - 6b + 7) = 0 \Rightarrow either\ Ab^k = 0\ \ or\ \ (b^2 - 6b + 7) = 0$

Since Ab^k is supposed to be non – trivial solution therefore $(b^2 - 6b + 7) = 0$

$\Rightarrow b = 3 \pm \sqrt{2}$ using quadratic formula. So $y_k = A_1\left(3+\sqrt{2}\right)^k + A_2\left(3-\sqrt{2}\right)^k$

To find particular integral. Let $\boldsymbol{y_k} = \boldsymbol{A3^k}$

$(ii) \Rightarrow A3^{k+2} - 6A3^{k+1} + 7A3^k = 3^k \Rightarrow 3^k(9A - 18A + 7A) = 3^k \Rightarrow A = -\frac{1}{2}$

Now general solution is $\boldsymbol{y_k} = \boldsymbol{y_k} + \boldsymbol{y_k^*} = A_1\left(3+\sqrt{2}\right)^k + A_2\left(3-\sqrt{2}\right)^k - \frac{1}{2}3^k$

EXAMPLE :

Solve the following homogeneous differential equation $y_{k+3} - y_{k+2} + y_{k+1} - 7y_k = \mathbf{3^k}$

SOLUTION : Given $y_{k+3} - y_{k+2} + y_{k+1} - 7y_k = \mathbf{3^k}$ ... ... (i)

Firstly we find characteristic solution.

Let $\boldsymbol{y_k} = Ab^k$ be the non – trivial solution of (i) then it must satisfies the eq (i) i.e.

$Ab^{k+3} - Ab^{k+2} + Ab^{k+1} - Ab^k = 0$ ... ... (ii)

$\Rightarrow Ab^k(b^3 - b^2 + b - 1) = 0 \Rightarrow either\ Ab^k = 0\ \ or\ \ (b^3 - b^2 + b - 1) = 0$

Since Ab^k is supposed to be non – trivial solution therefore $(b^3 - b^2 + b - 1) = 0$

$\Rightarrow b^2(b-1) + 1(b-1) \Rightarrow (b^2 - 1)(b-1) \Rightarrow b = 1,\ \pm i$

where $R = \sqrt{{m_1}^2 + {m_2}^2} = \sqrt{0^2 + (1)^2} = \sqrt{1}$

and $\theta = Tan^{-1}\left(\frac{m_2}{m_1}\right) = Tan^{-1}\left(\frac{1}{0}\right) = Tan^{-1}(\infty) = \frac{\pi}{2}$

$y_k = A_1(1)^k + \left(A_2 Cosk\frac{\pi}{2} + A_3 Sink\frac{\pi}{2}\right)\sqrt{1}$

To find particular integral. Let $y_k = A_4 3^k$

$\Rightarrow A_4 3^{k+3} - A_4 3^{k+2} + A_4 3^{k+1} - A_4 3^k = 3^k \Rightarrow 3^k(27A_4 - 9A_4 + 3A_4 - A_4) = 3^k \Rightarrow A_4 = \frac{1}{20}$

Now general solution is $\boldsymbol{y_k} = \boldsymbol{y_k} + \boldsymbol{y_k^*} = A_1(1)^k + \left(A_2 Cosk\frac{\pi}{2} + A_3 Sink\frac{\pi}{2}\right)\sqrt{1} + \frac{1}{20}3^k$

EXAMPLE :

Solve the following homogeneous differential equation $y_{k+2} - 4y_{k+1} + 4y_k = 2.2^k$

SOLUTION : Given $y_{k+2} - 4y_{k+1} + 4y_k = 2.2^k$ ... ... (i)

Firstly we find characteristic solution.

Let $\boldsymbol{y_k} = Ab^k$ be the non – trivial solution of (i) then it must satisfies the eq (i) i.e.

$Ab^{k+2} - 4Ab^{k+1} + 4Ab^k = 0$ ... ... (ii)

$\Rightarrow Ab^k(b^2 - 4b + 4) = 0$

$\Rightarrow either\ Ab^k = 0\ \ or\ (b^2 - 4b + 4) = 0$

Since Ab^k is supposed to be non – trivial solution therefore $(b^2 - 4b + 4) = 0$

$\Rightarrow b(b-2) - 2(b-2) \Rightarrow (b-2)(b-2) \Rightarrow b = 2,\ 2$

$y_k = (A_1 + A_2 k)(2)^k$

To find particular integral. put $y_k = Ak^2 2^k$ in (i)

$(ii) \Rightarrow A(k+2)^2 2^{k+2} - 4A(k+1)^2 2^{k+1} + 4A(k)^2 2^k = 2.2^k \Rightarrow A = \frac{1}{4}$ (after solving)

Thus $\boldsymbol{y_k^*} = \frac{1}{4} k^2 2^k$

Now general solution is $\boldsymbol{y_k} = \boldsymbol{y_k} + \boldsymbol{y_k^*} = (A_1 + A_2 k)(2)^k + \frac{1}{4} k^2 2^k$

EXAMPLE :

Solve the following homogeneous differential equation $y_{k+2} + 6y_{k+1} + 25y_k = 2^k$

SOLUTION : Given $y_{k+2} + 6y_{k+1} + 25y_k = 2^k \dots\dots\dots (i)$

Firstly we find characteristic solution.

Let $\boldsymbol{y_k} = Ab^k$ be the non – trivial solution of (i) then it must satisfies the eq (i) i.e.

$Ab^{k+2} + 6Ab^{k+1} + 25Ab^k = 0 \dots\dots\dots (ii)$

$\Rightarrow Ab^k(b^2 + 6b + 25) = 0 \Rightarrow either\ Ab^k = 0\ or (b^2 + 6b + 25) = 0$

Since Ab^k is supposed to be non – trivial solution therefore $(b^2 + 6b + 25) = 0$

$\Rightarrow b = -3 \pm 4i$ using Quadratic formula.

where $R = \sqrt{{m_1}^2 + {m_2}^2} = \sqrt{(-3)^2 + (4)^2} = \sqrt{25} = 5$ and $\theta = Tan^{-1}\left(\frac{m_2}{m_1}\right) = Tan^{-1}\left(\frac{-4}{3}\right)$

$y_k = 5\left(A_1 Cosk\left(Tan^{-1}\left(\frac{-4}{3}\right)\right) + A_2 Sink\left(Tan^{-1}\left(\frac{-4}{3}\right)\right)\right)$

To find particular integral. put $y_k = A2^k$ in (i)

$(ii) \Rightarrow A2^{k+2} + 6A2^{k+1} + 25A2^k = 2^k \Rightarrow 4A + 12A + 25A = 0 \Rightarrow A = \frac{1}{41}$ (after solving)

Thus $\boldsymbol{y_k^*} = \frac{1}{41} 2^k$

Now general solution is

$$\boldsymbol{y_k} = \boldsymbol{y_k} + \boldsymbol{y_k^*} = 5\left(A_1 Cosk\left(Tan^{-1}\left(\frac{-4}{3}\right)\right) + A_2 Sink\left(Tan^{-1}\left(\frac{-4}{3}\right)\right)\right) + \frac{1}{41} 2^k$$

EXAMPLE :

Solve the following homogeneous differential equation $y_{k+2} - 6y_{k+1} + 8y_k = 4^k$

SOLUTION : Given $y_{k+2} - 6y_{k+1} + 8y_k = 4^k \dots \dots \dots (i)$

Firstly we find characteristic solution.

Let $\boldsymbol{y_k} = Ab^k$ be the non – trivial solution of (i) then it must satisfies the eq (i) i.e.

$Ab^{k+2} - 6Ab^{k+1} + 8Ab^k = 0 \dots \dots \dots (ii)$

$\Rightarrow Ab^k(b^2 - 6b + 8) = 0 \ \Rightarrow either\ Ab^k = 0\ \ or(b^2 - 6b + 8) = 0$

Since Ab^k is supposed to be non – trivial solution therefore $(b^2 - 6b + 8) = 0$

$\Rightarrow b = 2,4$ factorizing. So $y_k = A_1 2^k + A_2 4^k$

To find particular integral. put $y_k = Ak4^k$ in (i)

$(ii) \Rightarrow A(k+2)4^{k+2} - 6(k+1)A4^{k+1} + 8Ak4^k = 4^k$

$\Rightarrow 16Ak - 32 - 24Ak - 24 + 8Ak = 0 \Rightarrow A = \frac{1}{8}$ (after solving)

Thus $\boldsymbol{y_k^*} = \frac{1}{8}k4^k$

Now general solution is $\boldsymbol{y_k} = \boldsymbol{y_k} + \boldsymbol{y_k^*} = A_1 2^k + A_2 4^k + \frac{1}{8}k4^k$

TYPE III : When R.H.S of given non – homogeneous difference equation is a polynomial i.e $f(k) = 1 + k^2, 2 + 3k + 4k^2 + 9k^3$ etc. then to find $\boldsymbol{y_k^*}$ put $\boldsymbol{y_{k's}} = a_0 + a_1 k + a_2 k^2 + a_3 k^3$ ………. upto the highest power of 'k' defined in the difference equation and the evaluate the values of a_0, a_1, a_2, a_3……………..

EXAMPLE :

Solve the following homogeneous differential equation $y_{k+2} - 5y_{k+1} + 6y_k = k^2 - 1$

SOLUTION : Given $y_{k+2} - 5y_{k+1} + 6y_k = k^2 - 1 \dots \dots \dots (i)$

Firstly we find characteristic solution.

Let $\boldsymbol{y_k} = Ab^k$ be the non – trivial solution of (i) then it must satisfies the eq (i) i.e.

$Ab^{k+2} - 5Ab^{k+1} + 6Ab^k = 0 \dots \dots \dots (ii)$

$\Rightarrow Ab^k(b^2 - 5b + 6) = 0 \ \Rightarrow either\ Ab^k = 0\ \ or\ (b^2 - 5b + 6) = 0$

Since Ab^k is supposed to be non – trivial solution therefore $(b^2 - 5b + 6) = 0$

$\Rightarrow b^2 - 3b - 2b + 6 = 0 \Rightarrow b(b-3) - 2(b-3) = 0 \Rightarrow (b-3)(b-2) = 0 \Rightarrow b = 2,3$

So $y_k = A_1 2^k + A_2 3^k$

To find particular integral. Let $\boldsymbol{y_{k's}} = a_0 + a_1 k + a_2 k^2$ in (i)

$\Rightarrow [a_0 + a_1(k+2) + a_2(k+2)^2] - 5[a_0 + a_1(k+1) + a_2(k+1)^2] + 6[a_0 + a_1 k + a_2 k^2] = k^2 - 1$

$(i) \Rightarrow (2a_0 - 3a_1 - a_2) + (2a_1 - 6a_2)k + (2a_2)k^2 = k^2 - 1$

Comparing the coefficient of same powers of 'k' we get

$2a_0 - 3a_1 - a_2 = -1 \ldots\ldots\ldots (iii), 2a_1 - 6a_2 = 0 \ldots\ldots\ldots (iv), 2a_2 = 1 \ldots\ldots\ldots (v)$

$(v) \Rightarrow a_2 = \frac{1}{2}$ put in (iv) $(iv) \Rightarrow a_1 = \frac{3}{2}$ put in (iii) $(iii) \Rightarrow a_0 = 2$

Now general solution is $\boldsymbol{y_k} = \boldsymbol{y_k} + \boldsymbol{y_k^*} = A_1 2^k + A_2 3^k + \left(2 + \frac{3}{2}k + \frac{1}{2}k^2\right)$

EXERCISE : Solve the following homogeneous differential equation

- **i.** $y_{k+2} + 5y_{k+1} - 6y_k = k^2 + k + 1$
- **ii.** $y_{k+2} + y_{k+1} + y_k = k^2 + k + 1$
- **iii.** $4y_{k+2} + 4y_{k+1} + y_k = k + 1$

TYPE IV : When R.H.S of given non – homogeneous difference equation has the form as $f(k) = \propto^k g(k)$ where $g(k)$ is a polynomial in 'k' then to find $\boldsymbol{y_k^*}$ put $\boldsymbol{y_{k's}} = \propto^k (a_0 + a_1 k + a_2 k^2 + a_3 k^3)$ ……….. upto the highest power of 'k' defined in the difference equation and the evaluate the values of a_0, a_1, a_2, a_3……………..

EXAMPLE :

Solve the following homogeneous differential equation $y_{k+2} - 13y_{k+1} + 36y_k = 2^k(k^2+1)$

SOLUTION : Given $y_{k+2} - 13y_{k+1} + 36y_k = 2^k(k^2+1) \ldots\ldots\ldots (i)$

Firstly we find characteristic solution.

Let $\boldsymbol{y_k} = Ab^k$ be the non – trivial solution of (i) then it must satisfies the eq (i) i.e.

$Ab^{k+2} - 13Ab^{k+1} + 36Ab^k = 0 \ldots\ldots\ldots (ii)$

$\Rightarrow Ab^k(b^2 - 13b + 36) = 0 \Rightarrow either\ Ab^k = 0\ \ or\ (b^2 - 13b + 36) = 0$

Since Ab^k is supposed to be non – trivial solution therefore $(b^2 - 13b + 36) = 0$

$\Rightarrow b^2 - 9b - 4b + 36 = 0 \Rightarrow b(b-9) - 4(b-9) = 0 \Rightarrow (b-4)(b-9) = 0 \Rightarrow b = 4,9$

So $y_k = A_1 4^k + A_2 9^k$

To find particular integral. Let $\boldsymbol{y_{k's}} = 2^k(a_0 + a_1 k + a_2 k^2)$ in (i)

$(i) \Rightarrow 2^{k+2}[a_0 + a_1(k+2) + a_2(k+2)^2] - 13.2^{k+1}[a_0 + a_1(k+1) + a_2(k+1)^2] + 36.2^k[a_0 + a_1 k + a_2 k^2] = 2^k(k^2+1)$

$\Rightarrow 2^k\big[4[a_0 + a_1(k+2) + a_2(k+2)^2] - 13.2[a_0 + a_1(k+1) + a_2(k+1)^2] + 36[a_0 + a_1 k + a_2 k^2]\big] = 2^k(k^2+1)$

$\Rightarrow 2^k\big[4[a_0 + a_1(k+2) + a_2(k+2)^2] - 26[a_0 + a_1(k+1) + a_2(k+1)^2] + 36[a_0 + a_1 k + a_2 k^2]\big] = 2^k(k^2+1)$

$\Rightarrow \big[4[a_0 + a_1(k+2) + a_2(k+2)^2] - 26[a_0 + a_1(k+1) + a_2(k+1)^2] + 36[a_0 + a_1 k + a_2 k^2]\big] = (k^2+1)$

$(i) \Rightarrow (16a_0 - 18a_1 - 10a_2) + (66a_1 - 36a_2)k + (14a_2)k^2 = k^2 - 1$

Comparing the coefficient of same powers of ‘k’ we get

$16a_0 - 18a_1 - 10a_2 = -1 \dots\dots\dots (iii), 66a_1 - 36a_2 = 0 \dots\dots\dots (iv), 14a_2 = 1 \dots\dots\dots (v)$

$(v) \Rightarrow a_2 = \frac{1}{14}$ put in (iv) $(iv) \Rightarrow a_1 = \frac{18}{462}$ put in (iii) $(iii) \Rightarrow a_0 = \frac{78}{539}$

Now general solution is $\boldsymbol{y_k} = \boldsymbol{y_k} + \boldsymbol{y_k^*} = A_1 4^k + A_2 9^k + 2^k\left(\frac{78}{539} + \frac{18}{462}k + \frac{1}{14}k^2\right)$

EXERCISE : Solve the following homogeneous differential equation

- **i.** $y_{k+2} - 9y_{k+1} + 20y_k = 3^k(k^2 - 1)$
- **ii.** $y_{k+2} - 4y_{k+1} + 4y_k = 2^k(k^2 + k + 1)$
- **iii.** $y_{k+2} - 9y_{k+1} + 20y_k = 4^k(k^2 + 1)$
- **iv.** $y_{k+2} - 7y_{k+1} - 8y_k = 2^k(k^2 - k)$

TYPE V : When R.H.S of given non – homogeneous difference equation has the form as $f(k) = SinAk \ or \ CosBk$ where $A \ and \ B$ are constants then to find $\boldsymbol{y_k^*}$ put $\boldsymbol{y_{k's}} = c_1 SinAk + c_2 \ CosBk$ or $\boldsymbol{y_{k's}} = c_3 \ CosBk + c_4 SinBk$ then evaluate the values of c_1, c_2, c_3……………..

EXAMPLE :

Solve the following homogeneous differential equation $y_{k+2} - 7y_{k+1} + 12y_k = Sin3k$

SOLUTION : Given $y_{k+2} - 7y_{k+1} + 12y_k = Sin3k$ … … … (i)

Firstly we find characteristic solution.

Let $\boldsymbol{y_k} = Ab^k$ be the non – trivial solution of (i) then it must satisfies the eq (i) i.e.

$Ab^{k+2} - 7Ab^{k+1} + 12Ab^k = 0$ … … … (ii)

$\Rightarrow Ab^k(b^2 - 7b + 12) = 0 \Rightarrow either \ Ab^k = 0 \ or (b^2 - 7b + 12) = 0$

Since Ab^k is supposed to be non – trivial solution therefore $(b^2 - 7b + 12) = 0$

$\Rightarrow b^2 - 4b - 3b + 12 = 0 \Rightarrow b(b-4) - 3(b-4) = 0 \Rightarrow (b-4)(b-3) = 0 \Rightarrow b = 3,4$

So $y_k = c_1 3^k + c_2 4^k$

To find particular integral. Let $\boldsymbol{y_{k's}} = c_3 Sin3k + c_4 \ Cos3k$ in (i)

$(i) \Rightarrow [c_3 Sin3(k+2) + c_4 \ Cos3(k+2)] - 7[c_3 Sin3(k+1) + c_4 \ Cos3(k+1)] + 12[c_3 Sin3k + c_4 \ Cos3k] = Sin3k$

$\Rightarrow c_3[Sin3kCos6 + Cos3kSin6\,] + c_4[Cos3kCos6 - Sin3kSin6\,] - 7c_3[Sin3kCos3 + Cos3kSin3\,] - 7c_4[Cos3kCos3 - Sin3kSin3\,] + 12c_3 Sin3k + 12c_4 \ Cos3k = Sin3k$

$\Rightarrow c_3[Cos6 - 7Cos3 + 12\,]Sin3k + c_4[-Sin6 + 7Sin3\,]Sin3k + c_3[Sin6 - 7Sin3\,]Cos3k + c_4[Cos6 - 7Cos3 + 12\,]Cos3k = Sin3k$

$\Rightarrow [c_3(Cos6 - 7Cos3 + 12\,) - c_4(Sin6 - 7Sin3\,)]Sin3k + [c_3(Sin6 - 7Sin3\,) + c_4(Cos6 - 7Cos3 + 12\,)]Cos3k = Sin3k$

Comparing like terms

$\Rightarrow c_3(Cos6 - 7Cos3 + 12\,) - c_4(Sin6 - 7Sin3\,) = 1$ ……………….(iii)

$\Rightarrow c_3(Sin6 - 7Sin3\,) + c_4(Cos6 - 7Cos3 + 12\,) = 0$ ……………….(iv)

Let $(Cos6 - 7Cos3 + 12\,) = l_1$ and $(Sin6 - 7Sin3\,) = l_2$

Then (iii) and (iv) becomes

$\Rightarrow c_3 l_1 - c_4 l_2 = 1$ and $\Rightarrow c_3 l_2 + c_4 l_1 = 0$

$\Rightarrow \frac{c_3}{l_1} = \frac{c_4}{-l_2} = \frac{1}{{l_1}^2+{l_2}^2}$ then $\Rightarrow c_3 = \frac{l_1}{{l_1}^2+{l_2}^2}$ and $\Rightarrow c_4 = \frac{-l_2}{{l_1}^2+{l_2}^2}$

Then $\boldsymbol{y_k^*} = \frac{l_1}{{l_1}^2+{l_2}^2} Sin3k - \frac{l_2}{{l_1}^2+{l_2}^2}\ Cos3k$

Now general solution is $\boldsymbol{y_k} = \boldsymbol{y_k} + \boldsymbol{y_k^*} = c_1 3^k + c_2 4^k + \frac{l_1}{{l_1}^2+{l_2}^2} Sin3k - \frac{l_2}{{l_1}^2+{l_2}^2}\ Cos3k$

Where $(Cos6 - 7Cos3 + 12\ ) = l_1$ and $(Sin6 - 7Sin3\ ) = l_2$

EXERCISE : Solve the following homogeneous differential equation

i. $y_{k+2} - 4y_{k+1} + 4y_k = Sin4k$
ii. $y_{k+2} - 2y_{k+1} + y_k = Sin5k + Cos5k + 6$
iii. $y_{k+2} - 2y_{k+1} + y_k = Cos^2 k$

TYPE VI : When R.H.S of given non – homogeneous difference equation has the form as $f(k) = a^k SinAk\ or\ \ a^k CosBk$ where $A\ and\ B$ are constants then to find $\boldsymbol{y_k^*}$ put $\boldsymbol{y_{k's}} = a^k(c_1 SinAk + c_2\ CosBk)$ or $\boldsymbol{y_{k's}} = a^k(c_3\ CosBk\ + c_4 SinBk)$ then evaluate the values of c_1, c_2, c_3……………..

EXAMPLE :

Solve the following homogeneous differential equation $y_{k+2} + 13y_{k+1} + 3y_k = 3^k Cos4k$

SOLUTION : Given $y_{k+2} + 13y_{k+1} + 3y_k = 3^k Cos4k \ldots\ldots\ldots (i)$

Firstly we find characteristic solution.

Let $\boldsymbol{y_k} = Ab^k$ be the non – trivial solution of (i) then it must satisfies the eq (i) i.e.

$Ab^{k+2} + 13Ab^{k+1} + 3Ab^k = 0 \ldots\ldots\ldots (ii)$

$\Rightarrow Ab^k(b^2 + 13b + 3) = 0 \ \Rightarrow either\ Ab^k = 0\ \ or(b^2 + 13b + 3) = 0$

Since Ab^k is supposed to be non – trivial solution therefore $(b^2 + 13b + 3) = 0$

$\Rightarrow b = \frac{-13\pm\sqrt{157}}{2}$ using quadratic formula.

So $y_k = c_1\left(\frac{-13+\sqrt{157}}{2}\right)^k + c_2\left(\frac{-13-\sqrt{157}}{2}\right)^k$

To find particular integral. Let $\boldsymbol{y_{k's}} = 3^k(c_3\ Cos4k\ + c_4 Sin4k)$ in (i)

$(i) \Rightarrow 3^{k+2}[c_3 Cos4(k+2) + c_4 Sin4(k+2)] + 13.3^{k+1}[c_3 Cos4(k+1) + c_4\ Sin4(k+1)] +$
$+3.3^k[c_3 Cos4k + c_4\ Sin4k] = 3^k Cos4k$

$\Rightarrow 3^k\big[3^2[c_3 Cos4(k+2) + c_4 Sin4(k+2)] + 13.3[c_3 Cos4(k+1) + c_4\ Sin4(k+1)] +$
$3[c_3 Cos4k + c_4\ Sin4k]\big] = 3^k Cos4k$

$\Rightarrow 9[c_3 Cos4(k+2) + c_4 Sin4(k+2)] + 39[c_3 Cos4(k+1) + c_4\ Sin4(k+1)] +$
$3[c_3 Cos4k + c_4\ Sin4k] = Cos4k$

$\Rightarrow 9c_3[Cos4kCos8 - Sin4kSin8\ ] + 9c_4[Sin4kCos8 + Cos4kSin8\ ] + 39c_3[Cos4kCos4 -$
$Sin4kSin4\ ] + 39c_4[Sin4kCos4 + Cos4kSin4\ ] + 3c_3 Cos4k + 3c_4\ Sin4k = Cos4k$

$\Rightarrow c_3[9Cos8 + 39Cos4 + 3\ ]Cos4k + c_4[9Cos8 + 39Cos4 + 3\ ]Sin4k - c_3[9Sin8 +$
$39Sin4\ ]Sin4k + c_4[9Sin8 + 39Sin4\ ]Cos4k = Sin3k$

$\Rightarrow [c_3(9Cos8 + 39Cos4 + 3) + c_4(9Sin8 + 39Sin4\ )]Cos4k + [-c_3(9Sin8 + 39Sin4\ ) +$
$c_4(9Cos8 + 39Cos4 + 3\ )]Sin4k = Sin3k$

Comparing like terms

$\Rightarrow c_3(9Cos8 + 39Cos4 + 3) + c_4(9Sin8 + 39Sin4\ ) = 1$ ……………….(iii)

$\Rightarrow -c_3(9Sin8 + 39Sin4\ ) + c_4(9Cos8 + 39Cos4 + 3\ ) = 0$ ……………….(iv)

Let $(9Cos8 + 39Cos4 + 3) = l_1$ and $(9Sin8 + 39Sin4\ ) = l_2$

Then (iii) and (iv) becomes $\Rightarrow c_3 l_1 + c_4 l_2 = 1$ and $\Rightarrow -c_3 l_2 + c_4 l_1 = 0$

$\Rightarrow \frac{c_3}{l_1} = \frac{c_4}{l_2} = \frac{1}{{l_1}^2 + {l_2}^2}$ then $\Rightarrow c_3 = \frac{l_1}{{l_1}^2 + {l_2}^2}$ and $\Rightarrow c_4 = \frac{l_2}{{l_1}^2 + {l_2}^2}$

Then $\boldsymbol{y_k^*} = 3^k\left(\frac{l_1}{{l_1}^2 + {l_2}^2}\ Cos4k + \frac{l_2}{{l_1}^2 + {l_2}^2}\ Sin4k\right)$

Now general solution is

$$\boldsymbol{y_k} = \boldsymbol{y_k} + \boldsymbol{y_k^*} = c_1\left(\frac{-13+\sqrt{157}}{2}\right)^k + c_2\left(\frac{-13-\sqrt{157}}{2}\right)^k + 3^k\left(\frac{l_1}{{l_1}^2 + {l_2}^2}\ Cos4k + \frac{l_2}{{l_1}^2 + {l_2}^2}\ Sin4k\right)$$

Where $(9Cos8 + 39Cos4 + 3) = l_1$ and $(9Sin8 + 39Sin4\ ) = l_2$

EXERCISE : Solve the following homogeneous differential equation

i. $y_{k+2} - 4y_{k+1} + 4y_k = (2)^k Sink\quad with\ y(o) = o = y(1)$
ii. $y_{k+2} - 7y_{k+1} + 12y_k = 12k + 8\quad with\ y(o) = o = y(1)$
iii. $y_{k+2} + y_{k+1} + y_k = (2)^k SinkCos3k$
iv. $y_{k+4} - 2y_{k+3} + 2y_{k+2} - 2y_{k+1} + y_k = n^2$

Examples from Vedamurthy's Book

Example # 1:-

Form difference eq corresponding to family of curves $y = ax^2 + bx - 3$

Sol:-

Since $y_x = ax^2 + bx - 3 \rightarrow (1)$

where a and b are arbitrary constts.

$\therefore \; y_{x+1} = a(x+1)^2 + b(x+1) - 3$

$y_{x+1} = a(x^2 + 2x + 1) + bx + b - 3$

$= ax^2 + 2ax + a + bx + b - 3$

$y_{x+1} = ax^2 + (2a + b)x + (a + b - 3) \rightarrow (2)$

and $y_{x+2} = a(x+2)^2 + b(x+2) - 3$

$= a(x^2 + 4x + 4) + bx + 2b - 3$

$y_{x+2} = ax^2 + (4a + b)x + (4a + 2b - 3) \rightarrow (3)$

Now eq(2) − eq(1)

$y_{x+1} - y_x = (2a + b)x - bx + a + b - 3 + 3$

$\Delta y_x = 2ax + a + b$

$\Delta y_x = (2x + 1)a + b \rightarrow (4)$

Now $\Delta^2 y_x = y_{x+2} - 2y_{x+1} + y_x$

$$-x^2 y_{x+2} + 2x^2 y_{x+1} - x^2 y_x - \frac{x}{2} y_{x+2} + x y_{x+1} - \frac{x}{2} y_x - 3 = 0$$

$$\Rightarrow \left(\frac{x^2}{2} - \frac{x}{2} - x^2\right) y_{x+2} + (-x^2 + x + 2x^2 + x) y_{x+1} + \left(\frac{x^2}{2} - x - 1 - x^2 - \frac{x}{2}\right) y_x - 3 = 0$$

$$\Rightarrow (x^2 - x - 2x^2) y_{x+2} + (-2x^2 + 2x + 4x^2 + 2x) y_{x+1} + (x^2 - 2x - 2 - 2x^2 - x) y_x - 6 = 0$$

$$\Rightarrow (-x^2 - x) y_{x+2} + (2x^2 + 4x) y_{x+1} + (-x^2 - 3x - 2) y_x - 6 = 0$$

$$\Rightarrow (x^2 + x) y_{x+2} - 2(x^2 + 2x) y_{x+1} + (x^2 + 3x + 2) y_x + 6 = 0$$

is required difference eq.

Example # 2: Form the difference eq given that $y_n = A3^n + B5^n$

Sol:

Given $y_n = A3^n + B5^n \rightarrow (1)$

where A and B are arbitrary constants.

So, $y_{n+1} = A3^{n+1} + B5^{n+1} \rightarrow (2)$

$$y_{n+2} = A3^{n+2} + B5^{n+2} \rightarrow (3)$$

$$\Delta^2 y_x = ax^2 + 4ax + bx + 4a + 2b - 3 - 2ax^2 - 4ax - 2bx + 2a - 2b + 6 + ax^2 + bx - 3$$

$$\Delta^2 y_x = 2a$$

$$\Rightarrow \quad a = \frac{1}{2}\Delta^2 y_x \rightarrow (5)$$

put in eq (4)

$$\Delta y_x = (2x+1)\frac{1}{2}\Delta^2 y_x + b$$

$$b = \Delta y_x - \frac{1}{2}(2x+1)\Delta^2 y_x \rightarrow (6)$$

put these in (1)

$$y_x = \left(\frac{1}{2}\Delta^2 y_x\right)x^2 + \left(\Delta y_x - \frac{1}{2}(2x+1)\Delta^2 y_x\right)x - 3$$

$$y_x = \left(\frac{1}{2}\Delta^2 y_x\right)x^2 + x(y_{x+1} - y_x) - \frac{x(2x+1)\Delta^2 y_x}{2} - 3$$

$$y_x = \left(\frac{1}{2}\Delta^2 y_x\right)x^2 + xy_{x+1} - xy_x - \frac{x(2x+1)}{2}\Delta^2 y_x - 3 = 0$$

$$\Rightarrow y_x = \frac{1}{2}\{y_{x+2} - 2y_{x+1} + y_x\}x^2 + xy_{x+1} - xy_x \left(-\frac{2x^2}{2} - \frac{x}{2}\right)\{y_{x+2} - 2y_{x+1} + y_x\} - 3 = 0$$

$$\Rightarrow x^2\left(\frac{1}{2}y_{x+2} - \frac{2}{2}y_{x+1} + \frac{1}{2}y_x\right) + xy_{x+1} - xy_x - y_x$$

Eliminate A and B from (1), (2) and (3).
For this eq (1), (2) and (3) becomes

$$y_n = A3^n + B5^n$$

$$y_{n+1} = 3A3^n + 5B5^n$$

$$y_{n+2} = 9A3^n + 25B5^n$$

So

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 3 & 5 \\ y_{n+2} & 9 & 25 \end{vmatrix} = 0$$

$$\Rightarrow y_n(75-45) - 1(25y_{n+1} - 5y_{n+2}) + 1(9y_{n+1} - 3y_{n+2}) = 0$$

$$\Rightarrow 30y_n - 25y_{n+1} + 5y_{n+2} + 9y_{n+1} - 3y_{n+2} = 0$$

$$\Rightarrow 2y_{n+1} - 16y_{n+1} + 30y_n = 0$$

$$\Rightarrow y_{n+1} - 8y_{n+1} + 15y_n = 0$$

is required difference equation.

Example # 3 :-

Solve $y_{x+1} - 2y_x \cos\alpha + y_{x-1} = 0$

Sol:- Equation can be written as

$$(E^2 - 2E\cos\alpha + 1)\, y_{x-1} = 0$$

its Auxiliary eq is

$$E^2 - 2E\cos\alpha + 1 = 0.$$

$$E = \frac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4}}{2}$$

$$= \frac{2\cos\alpha \pm \sqrt{4(\cos^2\alpha - 1)}}{2}$$

$$= \frac{2\cos\alpha \pm \sqrt{4(-\sin^2\alpha)}}{2}$$

$$= \frac{2\cos\alpha \pm 2i\sin\alpha}{2}$$

$$E = \cos\alpha \pm i\sin\alpha$$

$$R = \sqrt{m_1^2 + m_2^2}$$

$$R = \sqrt{\cos^2\alpha + \sin^2\alpha} = 1$$

$$\theta = \tan^{-1}\left(\frac{m_2}{m_1}\right) = \tan^{-1}\left(\frac{\sin\alpha}{\cos\alpha}\right) = \alpha$$

So, solution is

$$y_{x-1} = [C_1 \cos\alpha(x-1) + C_2 \sin\alpha(x-1)](1)$$

or $$y_x = C_1 \cos\alpha x + C_2 \sin\alpha x$$

Example #4:- Solve $y_{n+4} - 4y_{n+3} + 8y_{n+2} - 8y_{n+1} + 4y_n = 0 \rightarrow (1)$

Sol:- Eq (1) can be written as

$$(E^4 - 4E^3 + 8E^2 - 8E + 4)\,y_n = 0$$

$$\Rightarrow E^4 - 4E^3 + 8E^2 - 8E + 4 = 0$$

$$\Rightarrow E^4 - 2E^3 - 2E^3 + 4E^2 + 2E^2 + 2E^2 - 4E - 4E + 4 = 0$$

$$\Rightarrow E^4 - 2E^3 + 2E^2 - 2E^3 + 4E^2 - 4E + 2E^2 - 4E + 4 = 0$$

$$\Rightarrow E^2(E^2 - 2E + 2) - 2E(E^2 - 2E + 2) + 2(E^2 - 2E + 2) = 0$$

$$\Rightarrow (E^2 - 2E + 2)(E^2 - 2E + 2) = 0$$

$$\Rightarrow E^2 - 2E + 2 = 0, \quad E^2 - 2E + 2 = 0$$

$$E = \frac{2 \pm \sqrt{4-4(1)(2)}}{2}$$

$$E = \frac{2 \pm \sqrt{-4}}{2}$$

$$E = \frac{2 \pm 2i}{2}$$

$$E = 1 \pm i \;, \quad E = 1 \pm i$$

$$R = \sqrt{m_1^2 + m_2^2}$$

$$= \sqrt{1+1} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{m_2}{m_1}\right)$$

$$\theta = \tan^{-1}(1) = \frac{\pi}{4}$$

So, solution is

$$y_n = \left\{ (c_1 + c_2 n) \cos \frac{n\pi}{4} + (c_3 + c_4 n) \sin \frac{n\pi}{4} \right\} (\sqrt{2})^n$$

Simultaneous Linear Difference Equations:-

Example:- Solve the following system of difference equations

$$3y_{n+1} - 2y_n + z_n = n^2 \rightarrow (1)$$

$$4z_{n+1} - 3z_n + y_n = 3^n \rightarrow (2)$$

Sol:- The given system of linear difference equations can be written in operator form as

$$(3E-2)y_n + z_n = n^2 \rightarrow (3)$$

$$(4E-3)z_n + y_n = 3^n \rightarrow (4)$$

Operating $(3E-2)$ to eq(4) and then subtracting from eq(3), gives

$$(3E-2)y_n + z_n = n^2$$

$$\pm(3E-2)y_n \pm (3E-2)(4E-3)z_n = \pm 3^n(3E-2)$$

$$(1-(3E-2)(4E-3))z_n = n^2 - (3E-2)3^n$$

$$z_n[1-(12E^2-9E-8E+6)] = n^2 - 3\cdot 3^{n+1} + 2\cdot 3^n$$

$$\Rightarrow z_n[1-12E^2+17E-6] = n^2 + 3^n(2-9)$$

$$\Rightarrow z_n - 12z_{n+2} + 17z_{n+1} - 6z_n = +n^2 \pm 3^n\cdot 7$$

$$\Rightarrow 12z_{n+2} - 17z_{n+1} + 5z_n = 7\cdot 3^n - n^2 \rightarrow (5)$$

Consider the homogeneous difference equation corresponding to difference eq(5), as

$$12Z_{n+2} - 17Z_{n+1} + 5Z_n = 0 \rightarrow (6)$$

Let $Z_n = Ab^n$ be non-trivial solution of eq(6). So, it satisfies eq(6) i.e

$$12Ab^{n+2} - 17Ab^{n+1} + 5Ab^n = 0$$

$$Ab^n(12b^2 - 17b + 5) = 0$$

$$\Rightarrow 12b^2 - 17b + 5 = 0$$

$$\Rightarrow 12b^2 - 12b - 5b + 5 = 0$$

$$\Rightarrow 12b(b-1) - 5(b-1) = 0$$

$$(b-1)(12b-5) = 0$$

$$\Rightarrow b-1=0, \quad 12b-5=0$$

$$\Rightarrow b=1, \quad b=\frac{5}{12}$$

So, $C.F = Z_n = C_1(1)^n + C_2\left(\frac{5}{12}\right)^n$

To find P.I of eq(5). Substitute

$$Z_n = C_3 \cdot 3^n + n(C_4 + C_5 n + C_6 n^2) \text{ in eq(5)}$$

gives

$$12[C_3 3^{n+2} + (n+2)(C_4 + C_5(n+2) + C_6(n+2)^2] -$$

$$17[C_3 3^{n+1} + (n+1)(C_4 + C_5(n+1) + C_6(n+1)^2] +$$

$$5[C_3 3^n + n(C_4 + C_5 n + C_6 n^2)] = 7 \cdot 3^n - n^2$$

$12\left(c_3 3^{n+2} + (n+2)\left(c_4 + nc_5 + 2c_5 + c_6(n^2+4n+4)\right)\right)$
$-17\left(c_3 3^{n+1} + (n+1)\left(c_4 + nc_5 + c_5 + c_6(n^2+2n+1)\right)\right)$
$+5\left(c_3 3^n + n\left(c_4 + c_5 n + c_6 n^2\right)\right) = 7\cdot 3^n - n^2$

$12\left(c_3 3^{n+2} + (n+2)\left(c_4 + nc_5 + 2c_5 + n^2c_6 + 4nc_6 + 4c_6\right)\right)$
$-17\left(c_3 3^{n+1} + (n+1)\left(c_4 + nc_5 + c_5 + c_6 n^2 + 2c_6 n + c_6\right)\right)$
$+5\left(c_3 3^n + n\left(c_4 + c_5 n + c_6 n^2\right)\right) = 7\cdot 3^n - n^2$

equating coefficients of n^3

$12(c_6) - 17(c_6) + 5(c_6) = 0$

equating coefficients of n^2

$12(c_5 + 4c_6 + 2c_6) - 17(c_5 + 2c_6 + c_6) + 5(c_5) = -1$

equating coefficients of n

$12(c_4 + 2c_5 + 4c_6 + 2c_5 + 8c_6) - 17(c_4 + c_5 + 2c_6 + c_5 + 2c_6)$
$+ 5(c_4) = 0$

equating constant terms

$12(2c_4 + 4c_5 + 8c_6) - 17(c_4 + c_5 + 2c_6) = 0$

Equating coefficients of 3^n

$12(3^2 c_3) - 17(c_3(3)) + 5c_3 = 7$

The general solution of eq(5) is

$$Z_n = C_1(1)^n + C_2\left(\frac{5}{12}\right)^n + \frac{7}{62}3^n + n\left(-\frac{1777}{2058} + \frac{31}{98}n - \frac{1}{21}n^2\right) \rightarrow (7)$$

using eq(7) in eq(2) leads to

$$y_n = 3^n - 4Z_{n+1} + 3Z_n$$

$$= 3^n - 4\left[C_1(1)^{n+1} + C_2\left(\frac{5}{12}\right)^{n+1} + \frac{7}{62}3^{n+1} + (n+1)\left(-\frac{1777}{2058} + \frac{31}{98}(n+1) - \frac{1}{21}(n+1)^2\right)\right] + 3\left[C_1(1)^n + C_2\left(\frac{5}{12}\right)^n + \frac{7}{62}3^n + (n)\left(-\frac{1777}{2058} + \frac{31}{98}n - \frac{1}{21}n^2\right)\right]$$

$$y_n = 3^n - 4C_1(1)^{n+1} - 4C_2\left(\frac{5}{12}\right)^{n+1} - \frac{28}{62}3^{n+1} - 4(n+1)\left[-\frac{1777}{2058} + \frac{31}{98}(n+1) - \frac{1}{21}(n+1)^2\right] + 3C_1(1)^n + 3C_2\left(\frac{5}{12}\right)^n + \frac{7}{62}3^n + n\left[-\frac{1777}{2058} + \frac{31}{98}n - \frac{1}{21}n^2\right] \rightarrow (8)$$

eq(7) and (8) are General solution for given system.

(ii) $4Z_{n+1} - 13Z_n + 2y_n = \sin n$

$3y_{n+1} - 2y_n + Z_n = n^2$

Sol:-

(i) $Z_{n+1} - Z_n + y_n = 2^n \rightarrow (1)$

$3y_{n+1} + 2y_n + Z_n = 7 \rightarrow (2)$

The given system of linear difference equations can be written in operator form as

$(E-1)Z_n + y_n = 2^n \rightarrow (3)$

$(3E+2)y_n + Z_n = 7 \rightarrow (4)$

Operating eq (3) by operator $(3E+2)$ and subtract from (4)

$(3E+2)y_n + Z_n = 7$

$\pm(3E+2)y_n \pm (3E+2)(E-1)Z_n = \pm(3E+2)2^n$

$Z_n\{1-(3E+2)(E-1)\} = 7-(3E+2)2^n$

$Z_n\{1-(3E^2-3E+2E-2)\} = 7-3\cdot 2^{n+1}+2^{n+1}$

$Z_n\{1-3E^2+E+2\} = 7-(3+1)2^{n+1}$

$Z_n\{-3E^2+E+3\} = 7-2\cdot 2^{n+1}$

$-Z_n\{3E^2-E-3\} = -(2^{n+2}-7)$

$$Z_n[3E^2 - E - 3] = 2^{n+2} - 7$$

$$\Rightarrow 3Z_{n+2} - Z_{n+1} - 3Z_n = 2^{n+2} - 7 \rightarrow (5)$$

Consider the homogeneous difference eq corresponding to eq(5) is

$$3Z_{n+2} - Z_{n+1} - 3Z_n = 0 \rightarrow (6)$$

Let $Z_n = Ab^n$ be non-trivial sol of eq(6). So, it satisfies eq(6) i.e.

$$3Ab^{n+2} - Ab^{n+1} - 3Ab^n = 0$$

$$Ab^n(3b^2 - b - 3) = 0$$

$$\Rightarrow 3b^2 - b - 3 = 0$$

$$b = \frac{1 \pm \sqrt{1 - 4(3)(-3)}}{2(6)}$$

$$b = \frac{1 \pm \sqrt{1+36}}{6}$$

$$b = \frac{1 \pm \sqrt{37}}{2}$$

$$C.F = C_1\left(\frac{1+\sqrt{37}}{6}\right)^n + C_2\left(\frac{1-\sqrt{37}}{6}\right)^n$$

To find P.I substitute

$Z_n = C_3 2^{n+2} - nC_4$ in (5)

$$3(C_3 2^{n+4} - (n+2)C_4) - (C_3 2^{n+3} - (n+1)C_4) -$$

$$3(C_3 2^{n+2} - nC_4) = 2^{n+2} - 7$$

$$12C_3 2^{n+2} - 3nC_4 - 6C_4 - 2C_3 2^{n+2} + nC_4 + C_4$$
$$-3C_3 2^{n+2} - 3nC_4 = 2^{n+2} - 7$$

$$\Rightarrow (12C_3 - 2C_3 - 3C_3) 2^{n+2} + (-3C_4 + C_4 - 3C_4)n$$
$$+ (-6C_4 + C_4) = 2^{n+2} - 7$$

comparing like terms

$$12C_3 - 2C_3 - 3C_3 = 1$$
$$\Rightarrow (12 - 5) C_3 = 1$$
$$\Rightarrow 7C_3 = 1$$
$$\Rightarrow C_3 = \frac{1}{7}$$

$$-6C_4 + C_4 = -7$$
$$-5C_4 = -7$$
$$\Rightarrow C_4 = \frac{7}{5}$$

P.I is $(\frac{1}{7}) 2^{n+2} - \frac{7}{5} n$

General solution of eq (5) is

$$Z_n = C_1 \left(\frac{1+\sqrt{37}}{2}\right)^n + C_2 \left(\frac{1-\sqrt{37}}{2}\right)^n + \frac{2^{n+2}}{7} - \frac{7}{5} n \rightarrow (6)$$

using eq (3) in (1) yields.

$$c_1\left(\frac{1+\sqrt{37}}{2}\right)^{n+1} + c_2\left(\frac{1-\sqrt{37}}{2}\right)^{n+1} + \frac{2^{n+3}}{7} - \frac{7}{5}(n+1)$$

$$- c_1\left(\frac{1+\sqrt{37}}{2}\right)^{n} - c_2\left(\frac{1-\sqrt{37}}{2}\right)^{n} - \frac{2^{n+2}}{7} - \frac{7}{5}n$$

$$+ y_n = 2^n$$

$$\Rightarrow y_n = 2^n + c_1\left(\frac{1+\sqrt{37}}{2}\right)^{n} + c_2\left(\frac{1-\sqrt{37}}{2}\right)^{n} + \frac{2^{n+2}}{7}$$

$$+ \frac{7}{5}n - c_1\left(\frac{1+\sqrt{37}}{2}\right)^{n+1} - c_2\left(\frac{1-\sqrt{37}}{2}\right)^{n+1}$$

$$- \frac{2^{n+3}}{7} + \frac{7}{5}(n+1) \rightarrow (7)$$

Eq(6) and (7) are general solutions of given system.

(ii)

$$4z_{n+1} - 13z_n + 2y_n = \sin n \rightarrow (1)$$

$$3y_{n+1} - 2y_n + z_n = n^2 \rightarrow (2)$$

The given system of linear difference eqs can be operated as

$$(4E-13)z_n + 2y_n = \sin n \rightarrow (3)$$

$$(3E-2)y_n + z_n = n^2 \rightarrow (4)$$

Operating eq(4) by $(4E-13)$ and multiplying ... subract from (3)

$$(4E-13)z_n + 2y_n = \sin n$$

$$\pm(4E-13)z_n \pm y_n(3E-2)(4E-13) = \pm n^2 \cdot (4E-13)$$

$$y_n[2-(3E-2)(4E-13)] = \sin n - n^2(4E-13)$$

$$y_n[2-(12E^2-39E-8E+26)] = \sin n - 4n^{2+1} + 13n^2$$

$$\Rightarrow y_n[2-12E^2+39E+8E-26] = \sin n - 4n^3 + 13n^2$$

$$\Rightarrow y_n[-12E^2+47E-24] = \sin n + 13n^2 - 4n^3$$

$$\Rightarrow [12y_{n+2} - 47y_{n+1} + 24y_n] = 4n^3 - 13n^2 - \sin n \rightarrow (5)$$

Consider the homogeneous difference eq corresponding to eq (5) is

$$12y_{n+2} - 47y_{n+1} + 24y_n = 0 \rightarrow (6)$$

Let $y_n = Ab^n$ be non-trivial sol of eq(6). So, it satisfies eq(6) i-e

$$12Ab^{n+2} - 47Ab^{n+1} + 24Ab^n = 0$$

$$Ab^n[12b^2 - 47b + 24] = 0$$

$$\Rightarrow 12b^2 - 47b + 24 = 0.$$

$$b = \frac{47 \pm \sqrt{2209 - 4(12)(24)}}{2(12)}$$

$$b = \frac{47 \pm \sqrt{2209 - 1152}}{24}$$

$$b = \frac{47 \pm \sqrt{1057}}{24}$$

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$$C.F = c_1\left(\frac{47+\sqrt{1057}}{24}\right)^n + c_2\left(\frac{47-\sqrt{1057}}{24}\right)^n$$

To find P.I. Substitute $y_n = n(C_3 + C_4 n + C_5 n^2 + C_6 n^3) - C_7 \sin n - C_8 \cos n$ in (5)

$12[(n+2)(C_3 + C_4(n+2) + C_5(n+2)^2 + C_6(n+2)^3) - C_7 \sin(n+2) - C_8\cos(n+2)] - 47[(n+1)(C_3 + C_4(n+1) + C_5(n+1)^2 + C_6(n+1)^3) - C_7\sin(n+1) - C_8\cos(n+1)] + 24[n(C_3 + C_4 n + C_5 n^2 + C_6 n^3) - C_7 \sin n - C_8 \cos n] = 4n^3 - 13n^2 - \sin n$

$\Rightarrow [(12n+24)(C_3 + nC_4 + 2C_4 + n^2C_5 + 4nC_5 + 4C_5 + n^3C_6 + 12nC_6 + 6n^2C_6 + 8C_6) - C_7 12(\sin n\cos 2 + \cos n \sin 2) - C_8(\cos n\cos 2 - \sin n \sin 2)12] + [(-47n - 47)(C_3 + C_4 n + C_4 + n^2C_5 + 2nC_5 + C_5 + n^3C_6 + 3n^2C_6 + 3nC_6 + C_6) + 47C_7(\sin n\cos 1 + \cos n \sin 1) + 47C_8(\cos n \cos 1 - \sin n \sin 1) + 24nC_3 + 24n^2C_4 + 24n^3C_5 + 24n^4C_6 - 24C_7\sin n - 24C_8\cos n = 4n^3 - 13n^2 - \sin n$

$\Rightarrow 12nC_3 + 12n^2C_4 + 24nC_4 + 12n^3C_5 + 48n^2C_5 + 48nC_5 + 12n^4C_6 + 144n^2C_6 + 72n^2C_6 + 96nC_6 + 24C_3$

STIFF DIGFFERENTIAL EQUATIONS: Those equations whose exact solutions has a term of the form e^{-ct} where c is a large positive constant.

REGION OF ABSOLUTE STABILITY: Region R of absolute stability for a one step method is $R = \{h\lambda \in \mathbb{C}: |Q(h\lambda)| < 1\}$ and for a multistep method it is

$$R = \{h\lambda \in \mathbb{C}: |\beta_k| < 1, for\ all\ zeros\ \beta_k\ of Q(z, h\lambda)\}$$

A-STABLE METHOD: A numerical method is said to be A-Stable method if its region R of absolute stability contains the entire left half plane.

***<u>HOW TO REMOVE STIFFNESS OF IVP</u>*?:** by increasing magnitude of derivative but not of solution. IN this situation error can grow so large that it dominates the calculations IVP.

METHODS FOR NUMERICAL SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

SINGLE STEP METHODS: A series for 'y' in terms of power of 'x' form which the value of 'y' at a particular value of 'x' can be obtained by direct substitution e.g. Taylor's, Picard's, Euler's, Modified Euler's Method.

MULTI – STEP METHODS: In multi-step methods, the solution at any point 'x' is obtained using the solution at a number of previous points.

(Predictor- corrector method, Adam's Moulton Method, Adam's Bash forth Method)

REMARK: There are some ODE that cannot be solved using the standard methods. In such situations we apply numerical methods. These methods yield the solutions in one of two forms.

(i) A series for 'y' in terms of powers of 'x' from which the value of 'y' can be obtained by direct substitution. e.g. Taylor's and Picard's method

(ii) A set of tabulated values of 'x' and 'y'. e.g. and Euler's, Runge Kutta

ADVANTAGE/DISADVANTAGE OF MULTI - STEP METHODS

They are not self-starting. To overcome this problem, the single step method with some order of accuracy is used to determine the starting values.

Using these methods one step method clears after the first few steps.

LIMITATION (DISADVANTAGE) OF SINGLE STEP METHODS.

For one step method it is typical, for several functions evaluation to be needed.

IMPLICIT METHODS: Method that does not directly give a formula to the new approximation. A need to get it, need an implicit formula for new approximation in term of known data. These methods also known as close methods. It is possible to get stable 3rd order implicit method.

EXPLICIT METHODS; Methods that not directly give a formula to new approximation and need an explicit formula for new approximation "y_{i+1}" in terms of known data. These are also called open methods.

Most Authorities proclaim that it is not necessary to go to a higher order method. Explain.

Because the increased accuracy is offset by additional computational effort.

If more accuracy is required, then either a smaller step size. OR an adaptive method should be used.

CONSISTENT METHOD: A multi-step method is consistent if it has order at least one "1"

TAYLOR'S SERIES EXPANSION

Given (x) , smooth function. Expand it at point $x = c$ then

$$f(x) = f(c) + (x-c)f'(c) + \frac{(x-c)^2}{2!}f''(c) + \cdots \ldots \ldots \ldots.$$

$$\Rightarrow f(c) = \sum_{k=0}^{\infty} \frac{(x-c)^k}{k!} f^k$$ This is called Taylor's series of 'f' at 'c'

If $x_0 - c = h$ *and* $f(x) = y$ *then* $\Rightarrow c = x_0 + h$

$$y(x_0 + h) = y(x_0) + hy'(x_0) + \frac{h^2}{2!}\ y''(x_0) + \cdots \ldots \ldots \ldots \ldots \ldots \ldots$$

MECLAURIN SERIES FROM TAYLOR'S: If we put $c = 0$ in Taylor's series then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^2}{3!}\ f'''(0) + \cdots \ldots \ldots = \sum_{k=0}^{\infty}\ \frac{x^k}{k!}\ f^k(0)$$

ADVANTAGE OF TAYLOR'S SERIES

(1) One step, Explicit.
(2) Can be high order.
(3) Easy to show that global error is the same as local truncation error.
(4) Applicable to keep the error small.

DISADVANTAGE: Need to explicit form of the derivatives of function. That is why not practical.

ERROR IN TAYLOR'S SERIES

Assume $f^k(x)$ $(0 \le k \le n)$ are continuous functions. Call

$$f_n(x) = \sum_{k=0}^{n} \frac{(x-c)^k}{k!}\ f^k(c)$$ Then first $(n+1)$ term is Taylor series

Then the error is

$$E_{n+1} = f(x) - f_n(x) \sum_{k=n+1}^{\infty} \frac{(x-c)^k}{k!}\ f^k(c) = \frac{(x-c)^{n+1}}{(n+1)!}\ f^{n+1}\ (\S)$$

Where '§' is some point between 'x' and 'c' .

CONVERGENCE

A Taylor's series converges rapidly if 'x' is nears 'c' and slowly (or not at all) if 'x' is for away form 'c'.

EXAMPLE: Obtain numerically the solution of $\boldsymbol{y' = t^2 + y^2}$; $\boldsymbol{y(1) = 0}$ using Taylor Series method to find 'y' at 1.3

SOLUTION

$y' = t^2 + y^2 \ldots\ldots\ldots.(i)$
$y'' = 2t + 2yy' \ldots\ldots\ldots.(ii)$ $\quad y''' = 2[1 + y'^2 + yy''] \ldots\ldots\ldots.(iii)$

$y''' = 2[yy''' + 3y'y''] \ldots\ldots\ldots.(iv)$ ....and so on.

where $y_0 = 0$ and $t_0 = 0$, $h = t - t_0 = 0.3$

therefore $(i) \Rightarrow y_0' = 1, (ii) \Rightarrow y_0'' = 2$, $(iii) \Rightarrow y_0''' = 4$, $(iv) \Rightarrow y_0^{iv} = 12$,..........................

Now by using formula $y(t_0 + h) = y(t_0) + hy'(t_0) + \frac{h^2}{2!} y''(t_0) + \cdots \ldots\ldots\ldots\ldots\ldots$
we get $y(1 + 0.3) = y(1.3) = 0.4132$ as required.

QUESTION: Explain Taylor Series method for solving an initial value problem described by $\boldsymbol{\frac{dy}{dx} = f(x,y)}$; $\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\boldsymbol{i})$ $\boldsymbol{with}$ $\boldsymbol{y(x_0) = y_0}$

SOLUTION: Here we assume that f(x,y) is sufficiently differentiable with respect to 'x' and 'y' If y(x) is exact solution of (i) we can expand by Taylor Series about the point $x = x_0$ and obtain
$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \cdots \ldots\ldots\ldots\ldots\ldots\ldots$
Since the solution is not known, the derivatives in the above expansion are known explicitly. However 'f' is assume to be sufficiently differentiable and therefore the derivatives can be obtained directly from the given differentiable equation itself. Noting that 'f' is an implicit function of 'y' . we have $y' = f(x,y)$

$$\Rightarrow y'' = \frac{d}{dx}(y') = \frac{d}{dx} f(x,y) = \frac{df}{dx} = \frac{\partial f}{\partial x}.\frac{dx}{dx} + \frac{\partial f}{\partial y}.\frac{dy}{dx} = f_x + f_y.f$$

$$\Rightarrow y''' = \frac{d}{dx}(y'') = \frac{d}{dx}(f_x) + \frac{d}{dx}(f_y.f) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

Now $\frac{d}{dx}(f_x) = \frac{\partial f_x}{\partial x}.\frac{dx}{dx} + \frac{\partial f_x}{\partial y}.\frac{dy}{dx} = f_{xx} + f_{xy}.f \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (a)$

$$\frac{d}{dx}(f_y.f) = f_y.\frac{df}{dx} + f.\frac{df_y}{dx} = f_y.f_x + ff_y^2 + ff_{yx} + f^2 f_{yy} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (b)$$

Using (a) and (b) in (ii) we get $\Rightarrow y''' = f_{xx} + f_{xy}.f \; + f_y.f_x + ff_y^2 + ff_{yx} + f^2 f_{yy}$

$$\Rightarrow y''' = f_{xx} + 2ff_{xy} \quad + f_y.\left[f_x + ff_y\right] + f^2 f_{yy} \qquad \therefore f_{xy} = f_{yx}$$

Continuing in this manner we can express any derivative of 'y' in term of $f(x,y)$ and its partial derivatives.

EULER'S METHOD

To find the solution of the given Differential Equation in the form of a recurrence relation $\boldsymbol{y_{m+1} = y_m + hf\left(t_{m,}y_m\right)}$ Is called Euler Method

FORMULA DERIVATION

Consider the differential Equation of the first order

$$\boldsymbol{\frac{dy}{dx} = f(t,y) \qquad and \quad y(t_0) = y_0}$$

Let $(\boldsymbol{t_0, y_0})$ and$(\boldsymbol{t_1, y_1})$ be two points of approximation curve. Then

$$\boldsymbol{y_1 - y_0 = m(x_1 - x_0)} \qquad \ldots\ldots\ldots..(\boldsymbol{i}) \quad (\boldsymbol{point\ Slope\ form})$$

Given That $\quad \boldsymbol{\frac{dy}{dt} = f(t,y) \Longrightarrow \frac{dy}{dt}\Big|_{(t_0,y_0)} = f\left(t_{0,}y_0\right) \Longrightarrow m = f\left(t_{0,}y_0\right)}$

$$(\boldsymbol{i}) \ \Longrightarrow\ \boldsymbol{y_1 - y_0 = f(t_0, y_0)(x_1 - x_0) \Longrightarrow y_1 = y_0 + (x_1 - x_0)f(t_0, y_0)}$$

Similarly

$$\boldsymbol{y_2 = y_1 + (x_2 - x_1)f(t_1, y_1)}$$

$$\boldsymbol{y_3 = y_2 + (x_3 - x_2)f(t_2, y_2)}$$

$$\vdots \qquad \vdots \qquad\qquad \vdots$$

$$\boldsymbol{y_{m+1} = y_m + (x_{m+1} - x_m)f(t_m, y_m)}$$

$$\Longrightarrow \boldsymbol{y_{m+1} = y_m + hf(t_m, y_m) \qquad is\ called\ Euler\ Method.}$$

BASE OF EULER'S METHOD

In this method we use the property that in a small interval, a curve is nearly a Straight Line. Thus at $(\boldsymbol{t_0, y_0})$ We approximate the Curve by a tangent at that point.

OBJECT (PURPOSE) OF METHOD

The object of Euler's Method is to obtain approximations to the well posed initial value problem $\frac{dy}{dt} = f(t, y) \quad ; a \leq t \leq b \; ; y(a) = a$

GEOMETRICAL INTERPRETATION

Geometrically, this method has a very simple meaning. The desired function curve is approximated by a polygon train. Where the direction of each part is determined by the value of the function $f(t, y)$ at its starting point

Also $y_{m+1} = y_m + hf(t_m, y_m)$ Shows that the next approximation y_{m+1} is obtained at the point where the tangent to the graph of $y(t)$ $at\ t = t_i$ interest with the vertical line $t = t_{m+1}$

LIMITATION OF EULER METHOD

There is too much inertia in Euler Method. One should not follow the same initial slope over the whole interval of length "h".

EULER METHOD IN VECTOR NOTATION

Consider the system $\frac{dY}{dt} = F(Y)$ where $Y = (x, y), \frac{dY}{dt} = (\frac{dx}{dt}, \frac{dy}{dt})$ and $F(Y) = (f(x, y), g(x, y))$ if we are given the initial condition $Y_0 = (x_0, y_0)$ then Euler method approximate a solution (x, y) by $(x_{k+1}, y_{k+1}) = (x_k, y_k) + \Delta t F(x_k, y_k)$

ADVANTAGE/DISADVANTAGE OF EULER METHOD

The advantage of Euler's method is that it requires only one slope evaluation and is simple to apply, especially for discretely sampled (experimental) data points. The disadvantage is that errors accumulate during successive iterations and the results are not very accurate.

EXAMPLE: Obtain numerically the solution of $y' = t^2 + y^2$; $y(0) = 0.5$ using simple Euler method to find 'y' at 0.1

SOLUTION: $y' = t^2 + y^2 = f(t, y)$ where $y_0 = 0.5\ and\ t_0 = 0$

Then $n = \frac{t - t_0}{h} = \frac{0.1 - 0}{0.1} = 1\ (number\ of\ steps)$ $\quad \therefore h = t - t_0$

Now by using formula $y_{m+1} = y_m + hf(t_m, y_m)$ we get

$y(0.1) = y_1 = y_0 + hf(t_0, y_0) = 0.525$ as required.

For illustration, an example follows.

$h = \frac{0.1}{5} = 0.02$

Example 8.2 Given

$$\frac{dy}{dt} = \frac{y - t}{y + t}$$

with the initial condition $y = 1$ at $t = 0$. Find y approximately at $t = 0.1$, in five steps, using Euler's method.

Solution Since the number of steps involved are five, we shall march in steps of $0.1/5 = 0.02$. Therefore, taking step size $h = 0.02$, we shall compute the value of y at $t = 0.02, 0.04, 0.06, 0.08$ and 0.1. Thus,

$$y_1 = y_0 + hf(t_0, y_0), \qquad \text{where } y_0 = 1,\ t_0 = 0$$

Therefore,

$$y_1 = 1 + 0.02\,\frac{1-0}{1+0} = 1.02$$

Similarly

$$y_2 = y_1 + hf(t_1, y_1) = 1.02 + 0.02\,\frac{1.02 - 0.02}{1.02 + 0.02} = 1.0392$$

$$y_3 = y_2 + hf(t_2, y_2) = 1.0392 + 0.02\,\frac{1.0392 - 0.04}{1.0392 + 0.04} = 1.0577$$

$$y_4 = y_3 + hf(t_3, y_3) = 1.0577 + 0.02\,\frac{1.0577 - 0.06}{1.0577 + 0.06} = 1.0756$$

$$y_5 = y_4 + hf(t_4, y_4) = 1.0756 + 0.02\,\frac{1.0756 - 0.08}{1.0756 + 0.08} = 1.0988$$

Hence, the value of y corresponding to $t = 0.1$ is 1.0988.

MODIFIED EULER METHOD

Modified Euler's Method is given by the iteration formula

$$y_{m+1} = y_m + \frac{h}{2}\left[f(t_m, y_m) + f\left(t_{m+1}, y_{m+1}^{(1)}\right)\right]$$

Method also known as Improved Euler method sometime known as Runge Kutta method of order 2

CONVERGENCE FOR EULER METHOD: Assume that f (t, y) has a Lipschitz constant L, for the variable 'y, and that the solution y_i of the initial value problem $y' = f(t, y), t \in [a, b], y(a) = y_a$ at t_i is Approximated by $w_i = y(t_i)$ using Euler Method

Let 'M' be an upper bound for $|y^n(t)|$ on $[a, b]$ then $[w_i - y_i] \le \frac{Mh}{2l}\left(e^{L(t_i - a)} - 1\right)$

DARIVATION OF MODIFIED EULER METHOD

Consider the differential Equation of 1st order $\frac{dy}{dt} = f(t,y)$ and $y(t_0) = y_0$

Then by Euler's Method

$y_1 = y_0 + hf(t_0, y_0) \qquad \because h = t_{i+1} - t_i$

$y_1 = y_0 + \frac{h}{2}[f(t_0, y_0) + f\left(t_1, y_1^{(1)}\right)]$

$y_{2=}y_1 + \frac{h}{2}\left[f\left(t_{1,}y_1\right) + f\left(t_{2,}y_2^{(1)}\right)\right]$

$\vdots \qquad \vdots \qquad\qquad \vdots$

$y_{m+1} = y_m + \frac{h}{2}\left[f\left(t_{m,}y_m\right) + f\left(t_{m+1,}y_{m+1}^{(1)}\right)\right]$

EXAMPLE: Obtain numerically the solution of $\boldsymbol{y' = log(t+y)}$; $\boldsymbol{y(0) = 1}$ using modified Euler method to find 'y' at 0.2

SOLUTION: Take h = 0.1 (own choice) and $t_0 = 0\,, t_1 = t_0 + h = 0.1\,, t_2 = 0.2$

Now using Euler's method $y_1^{(1)} = y_0 + hf(t_0, y_0) = 1$

Then by using Euler's modified method

$y_1 = y_0 + \frac{h}{2}\left[f(t_0, y_0) + f\left(t_1, y_1^{(1)}\right)\right] = 1.002069$

Again using Euler's method $y_2^{(1)} = y_1 + hf(t_1, y_1) = 1.006289$

Then by using Euler's modified method

$y_2 = y_1 + \frac{h}{2}\left[f(t_1, y_1) + f\left(t_2, y_2^{(1)}\right)\right] = 1.008175 \Longrightarrow y_2 = y(0.2) \approx 1.0082$

Example 8.3 Using modified Euler's method, obtain the solution of the differential equation

$$\frac{dy}{dt} = t + \sqrt{y} = f(t, y)$$

with the initial condition $y_0 = 1$ at $t_0 = 0$ for the range $0 \le t \le 0.6$ in steps of 0.2.

Solution At first, we use Euler's method to get

$$y_1^{(1)} = y_0 + hf(t_0, y_0) = 1 + 0.2\,(0 + 1) = 1.2$$

then, we use modified Euler's method to find

$$y(0.2) = y_1 = y_0 + h\,\frac{f(t_0, y_0) + f(t_1, y_1^{(1)})}{2}$$

$$= 1.0 + 0.2\,\frac{1 + \left(0.2 + \sqrt{1.2}\right)}{2} = 1.2295$$

Similarly proceeding, we have from Euler's method

$$y_2^{(1)} = y_1 + hf(t_1, y_1) = 1.2295 + 0.2\,(0.2 + \sqrt{1.2295}) = 1.4913$$

Using modified Euler's method, we get

$$y_2 = y_1 + h\,\frac{f(t_1, y_1) + f(t_2, y_2^{(1)})}{2}$$

$$= 1.2295 + 0.2\,\frac{\left(0.2 + \sqrt{1.2295}\right) + \left(0.4 + \sqrt{1.4913}\right)}{2} = 1.5225$$

Finally

$$y_3^{(1)} = y_2 + hf\,(t_2, y_2) = 1.5225 + 0.2\,\left(0.4 + \sqrt{1.5225}\right) = 1.8493$$

Now, modified Euler's method gives

$$y(0.6) = y_3 = y_2 + h\,\frac{f(t_2, y_2) + f(t_3, y_3^{(1)})}{2}$$

$$= 1.5225 + 0.1[(0.4 + \sqrt{1.5225}) + (0.6 + \sqrt{1.8493})] = 1.8819$$

Hence, the solution to the given problem is given by

t	0.2	0.4	0.6
y	1.2295	1.5225	1.8819

RUNGE KUTTA METHODS

Basic idea of Runge Kutta Methods can be explained by using Modified Euler's Method by Equation $\boldsymbol{y_{m+1} = y_n + h}$ $(\boldsymbol{average\ of\ slopes})$

Here we find the slope not only at '$\boldsymbol{t_n}$' but also at several other interior points and take the weighted average of these slopes and add to '$\boldsymbol{y_n}$ ' to get '$\boldsymbol{y_{n+1}}$'.

ALSO RK-Approach is to aim for the desirable features for the Taylor Series method but with the replacement of the requirement for the evaluation of the higher order derivatives with the requirement to evaluate $\boldsymbol{f(x,y)}$ at some points with in the steps $'\boldsymbol{x_i}'$ to '$\boldsymbol{x_{i+1}}$'

IMPORTANCE: Quite Accurate, Stable and easy to program but requires four slopes evaluation at four different points of (x,y): these slope evaluations are not possible for discretely sampled data points, because we have is what is given to us and we do not get to choose at will where to evaluate slopes. These methods do not demand prior computation of higher derivatives of $\boldsymbol{y(t)}$ as in Taylor Series Method. Easy for automatic Error control. Global and local errors have same order in it.

DIFFERENCE B/W TAYLOR SERIES AND RK-METHOD (ADVANTAGE OF RK OVER TAYLOR SERIES)

Taylor Series needs to explicit form of derivative of $\boldsymbol{f\ (t,y)}$ but in RK-method this is not in demand. RK-method very extensively used.

SECOND ORDER RUNGE KUTTA METHOD

WORKING RULE: For a given initial value problem of first order $\boldsymbol{y' = f(x,y)}$, $\boldsymbol{y(x_0) = y_0}$

Suppose "$x_0, x_1, x_2 \ldots \ldots \ldots$" be equally spaced 'x' values with interval 'h'

i.e. $x_1 = x_0 + h$, $x_2 = x_1 + h$, ……………..

Also denote $y_0 = y(x_0)$, $y_1 = y(x_1)$, $y_2 = y(x_2)$ ………………..

Then for "$n = 0, 1, 2 \ldots \ldots \ldots ..$" until termination do:

$x_{n+1} = x_n + h$,, $k_n = hf(x_n, y_n)$,, $l_n = hf(x_{n+1}, y_n + k_n)$

Then $y_{n+1} = y_n + \frac{1}{2}(k_n + l_n)$ Is the formula for second order RK-method.

REMARK: Modified Euler Method is a special case of second order RK-Method.

IN ANOTHER WAY:

If $\boldsymbol{k_1 = hf(x_k, y_k)}$, $\boldsymbol{k_2 = hf(x_{k+1}, y_k + k_1)}$

Then Equation for second order method is $\boldsymbol{y_{k+1} = y_k + \frac{1}{2}(k_1 + k_2)}$

This is called Heun's Method(third order RK method)

ANOTHER FORMULA FOR SECOND ORDER RK-METHOD

$\boldsymbol{y_{n+1} = y_n + \frac{1}{3}(2k_1 + k_2)}$ Where $\boldsymbol{k_1 = hf(t_n, y_n)}$, $k_2 = \boldsymbol{hf\left(t_n + \frac{3}{2}h, \ y_n + \frac{3}{2}k_1\right)}$

LOCAL TRUNCATION ERROR IN RK-METHOD.

LTE in RK-method is the error that arises in each step simply because of the truncated Taylor series. This error is inevitable. Error of Runge Kutta method of order two involves an error of O($\boldsymbol{h^3}$).

In General RK-method of order 'm' takes the form $\boldsymbol{x_{k+1} = x_k + w_1k_1 + w_2k_2 + \cdots + w_mk_m}$

Where $\boldsymbol{k_1 = h.f(t_k, x_k)}$,$\boldsymbol{k_2 = hf(t_k + a_2h, x + b_2k_1)}$

$\boldsymbol{k_3 = h\,f(t_k + a_3h, x + b_3k_1 + c_3k_2)}$ ……………………….. $\boldsymbol{k_m = h.f\left(t_k + a_mh, x + \sum_{i=1}^{m-1} \emptyset_i\, k_i\right)}$

MULTI STEP METHODS OVER RK-METHOD (PREFRENCE):

Determination of $\boldsymbol{y_{i+1}}$ require only on evaluation of $\boldsymbol{f\,(t, y)}$ per step. Whereas RK-method for $\boldsymbol{n \geq 3}$ require four or more function evaluations. For this reason, multi-step methods can be twice as fast as RK-method of comparable Accuracy.

EXAMPLE:

use second order RK method to solve $\boldsymbol{\frac{dy}{dx} = \frac{y+x}{y-x} = f(x, y)}$; $\boldsymbol{y(0) = 1}$ at x =0.4 and h=0.2

SOLUTION: $\boldsymbol{\frac{dy}{dx} = \frac{y+x}{y-x} = f(x, y)}$ ……………….$\boldsymbol{(i)}$

If 'h' is not given then use by own choice for 4 – step take h=0.1 and for 1 – step take h=0.4

Given that h=0.2 , $x_0 = 0, x_1 = x_0 + h = 0.2, x_2 = 0.4$

Now using formula of order two

$y_{n+1} = y_n + \frac{1}{3}(2k_1 + k_2)$ Where $k_1 = hf(x_n, y_n)$, $k_2 = hf\left(x_n + \frac{3}{2}h,\ y_n + \frac{3}{2}k_1\right)$

$k_1 = hf(x_0, y_0) = 0.2,\quad k_2 = hf\left(x_n + \frac{3}{2}h, y_n + \frac{3}{2}k_1\right) = 0.32$

$\therefore n = \frac{x - x_0}{h}$

n = 2 steps

For n = 0; $k_1 = hf(x_0, y_0) = 0.2,\quad k_2 = hf\left(x_0 + \frac{3}{2}h, y_0 + \frac{3}{2}k_1\right) = 0.32$

$(i) \Rightarrow y_1 = y_0 + \frac{1}{3}(2k_1 + k_2) = 1.24 \Rightarrow y(0.2) = 1.24$

For n = 1; $k_1 = hf(x_1, y_1) = 0.2769,\quad k_2 = hf\left(x_1 + \frac{3}{2}h, y_1 + \frac{3}{2}k_1\right) = 0.3731$

$(i) \Rightarrow y_2 = y_1 + \frac{1}{3}(2k_1 + k_2) = 1.54897 \Rightarrow y(0.4) = 1.54897$

Kutta methods of ...

Example 8.4 Use the following second order Runge–Kutta method described by

$$y_{n+1} = y_n + \frac{1}{3}(2k_1 + k_2)$$

where

$$k_1 = hf(x_n, y_n) \qquad \text{and} \qquad k_2 = hf\left(x_n + \frac{3}{2}h,\, y_n + \frac{3}{2}k_1\right)$$

and find the numerical solution of the initial value problem described as

$$\frac{dy}{dx} = \frac{y + x}{y - x}, \qquad y(0) = 1$$

at $x = 0.4$ and taking $h = 0.2$.

Solution In the present problem

$$f(x, y) = \frac{y + x}{y - x}, \qquad h = 0.2, \qquad x_0 = 0, \quad y_0 = 1$$

We calculate

$$k_1 = hf(x_0, y_0) = 0.2\,\frac{1 + 0}{1 - 0} = 0.2$$

$$k_2 = hf[x_0 + 0.3,\, y_0 + (1.5)(0.2)] = hf(0.3, 1.3) = 0.2\,\frac{1.3 + 0.3}{1.3 - 0.3} = 0.32$$

Now, using the given R–K method, we get

$$y(0.2) = y_1 = 1 + \frac{1}{3}(0.4 + 0.32) = 1.24$$

Now, taking $x_1 = 0.2, y_1 = 1.24$, we calculate

$$k_1 = hf(x_1, y_1) = 0.2\,\frac{1.24 + 0.2}{1.24 - 0.2} = 0.2769$$

$$k_2 = hf\left(x_1 + \frac{3}{2}h, y_1 + \frac{3}{2}k_1\right) = hf(0.5, 1.6554) = 0.2\,\frac{1.6554 + 0.5}{1.6554 - 0.5} = 0.3731$$

Again using the given R–K method, we obtain

$$y(0.4) = y_2 = 1.24 + \frac{1}{3}[2(0.2769) + 0.3731] = 1.54897$$

Example 8.5 Solve the following differential equation

$$\frac{dy}{dt} = t + y$$

with the initial condition $y(0) = 1$, using fourth-order Runge–Kutta method from

$$t = 0 \text{ to } t = 0.4 \qquad \text{taking } h = 0.1$$

Solution The fourth-order Runge–Kutta method is described as

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{1}$$

where

$$k_1 = hf(t_n, y_n)$$

$$k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(t_n + h, y_n + k_3)$$

In this problem, $f(t, y) = t + y$, $h = 0.1$, $t_0 = 0$, $y_0 = 1$. As a first step, we calculate

$$k_1 = hf(t_0, y_0) = 0.1\ (1) = 0.1$$

$$k_2 = hf(t_0 + 0.05, y_0 + 0.05) = hf(0.05, 1.05) = 0.1\ [0.05 + 1.05] = 0.11$$

$$k_3 = hf(t_0 + 0.05, y_0 + 0.055) = 0.1\ (0.05 + 1.055) = 0.1105$$

$$k_4 = 0.1\ (0.1 + 1.1105) = 0.12105$$

Now, we compute from, Eq. (1) that

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(0.1 + 0.22 + 0.2210 + 0.12105)$$

$$= 1.11034$$

Therefore, $y(0.1) = y_1 = 1.11034$. In the second step, we have to find $y_2 = y(0.2)$. Knowing that $t_1 = 0.1$, $y_1 = 1.11034$, we compute

$$k_1 = hf(t_1, y_1) = 0.1\,(0.1 + 1.11034) = 0.121034$$

$$k_2 = hf\left(t_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1[0.15 + (1.11034 + 0.060517)] = 0.13208$$

$$k_3 = hf\left(t_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1[0.15 + (1.11034 + 0.06604)] = 0.132638$$

$$k_4 = hf(t_1 + h, y_1 + k_3) = 0.1[0.2 + (1.11034 + 0.132638)] = 0.1442978$$

and then from Eq. (1), we see that

$$y_2 = 1.11034 + \frac{1}{6}[0.121034 + 2(0.13208) + 2(0.132638) + 0.1442978] = 1.2428$$

Similarly by calculating,

$$k_1 = hf(t_2, y_2) = 0.1\,[0.2 + 1.2428] = 0.14428$$

$$k_2 = hf\left(t_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = 0.1[0.25 + (1.2428 + 0.07214)] = 0.156494$$

$$k_3 = hf\left(t_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = 0.1[0.25 + (1.2428 + 0.078247)] = 0.1571047$$

$$k_4 = hf(t_2 + h, y_2 + k_3) = 0.1[0.3 + (1.2428 + 0.1571047)] = 0.16999047$$

Using Eq. (1), we compute

$$y(0.3) = y_3 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.399711$$

Finally, we calculate

$$k_1 = hf(t_3, y_3) = 0.1[0.3 + 1.3997] = 0.16997$$

$$k_2 = hf\left(t_3 + \frac{h}{2}, y_3 + \frac{k_1}{2}\right) = 0.1[0.35 + (1.3997 + 0.084985)] = 0.1834685$$

$$k_3 = hf\left(t_3 + \frac{h}{2}, y_3 + \frac{k_2}{2}\right) = 0.1[0.35 + (1.3997 + 0.091734)] = 0.1841434$$

$$k_4 = hf(t_3 + h, y_3 + k_3) = 0.1[0.4 + (1.3997 + 0.1841434)] = 0.19838434$$

Using them in Eq. (1), we get

$$y(0.4) = y_4 = y_3 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.58363$$

which is the required result.

CLASSICAL RUNGE KUTTA METHOD (RK – METHOD OF ORDER FOUR)

ALORITHM: Given the initial value problem of first order $y' = f(x,y)$, $y(x_0) = y_0$

Suppose "$x_0, x_1, x_2 \dots \dots \dots$" be equally spaced 'x' values with interval 'h'

i.e. $x_1 = x_0 + h$, $x_2 = x_1 + h$, …………..
Also denote $y_0 = y(x_0),\ \ y_1 = y(x_1),\ \ y_2 = y(x_2) \dots \dots \dots \dots \dots$

Then for "$n = 0, 1, 2 \dots \dots \dots ..$" until termination do:

$x_{n+1} = x_n + h$,, $k_1 = hf(x_n, y_n)$ $k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$
$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$ $k_4 = hf(x_n + h, y_n + k_3)$

Then $y_{n+1} = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + y_n$

Is the formula for Runge Kutta method of order four and its error is "$O(h^5)$"

ADVANTAGE OF METHOD

- Accurate method.
- Easy to compute for the use of computer.
- It lakes in estimating the error.
- Easy to program and is efficient.

COMPUTATIONAL COMPARISON:

The main computational effort in applying the Runge Kutta method is the evaluation of '***f***'. In RK – 2 the cost is two function evaluation per step. In RK – 4 require four evaluations per step.

EXAMPLE:

use 4th order RK method to solve $\frac{dy}{dx} = t + y; y(0) = 1$ from t =0 to 0.4 taking h = 0.4

SOLUTION: $\frac{dy}{dx} = t + y \quad \dots \dots \dots \dots \dots \dots .(i)$
$h = 0.1, t_0 = 0, t_1 = t_0 + h = 0.1$, $t_2 = 0.2$, $t_3 = 0.3, t_4 = 0.4$

Now using formulas for the RK method of 4th order
$y_{n+1} = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + y_n \quad \dots \dots \dots \dots \dots \dots (ii)$

Where $k_1 = hf(t_n, y_n)$, $k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$, $k_3 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$ $k_4 = hf(t_n + h, y_n + k_3)$

STEP I : for n=0;

$k_1 = hf(t_0, y_0) = 0.1$, $k_2 = hf\left(t_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.11$

$k_3 = hf\left(t_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1105$ $k_4 = hf(t_0 + h, y_0 + k_3) = 0.12105$

$(ii) \Rightarrow y_1 = y(0.1) = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + y_0 = 1.11034$

STEP II : for n=1;

$k_1 = hf(t_1, y_1) = 0.121034$, $k_2 = hf\left(t_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.13208$

$k_3 = hf\left(t_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.132638$ $k_4 = hf(t_1 + h, y_1 + k_3) = 0.1442978$

$(ii) \Rightarrow y_1 = y(0.2) = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + y_1 = 1.2428$

STEP III : for n=2;

$k_1 = hf(t_2, y_2) = 0.14428$, $k_2 = hf\left(t_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = 0.156494$

$k_3 = hf\left(t_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = 0.1571047$ $k_4 = hf(t_2 + h, y_2 + k_3) = 0.16999047$

$(ii) \Rightarrow y_1 = y(0.3) = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + y_2 = 1.399711$

this is required answer

Example 8.6 Solve the following van der Pol's equation

$$y'' - (0.1)(1 - y^2)y' + y = 0$$

using fourth order Runge-Kutta method for $x = 0.2$, with the initial values $y(0) = 1, y'(0) = 0$.

Solution Let

$$\frac{dy}{dx} = p = f_1(x, y, p)$$

Then

$$\frac{dp}{dx} = (0.1)(1 - y^2)p - y = f_2(x, y, p)$$

Thus, the given van der Pol's equation reduced to two first-order equations.

In the present problem, we are given that $x_0 = 0$, $y_0 = 1$, $p_0 = y_0' = 0$. Taking $h = 0.2$, we compute

$$k_1 = hf_1(x_0, y_0, p_0) = 0.2(0.0) = 0.0$$

$$l_1 = hf_2(x_0, y_0, p_0) = 0.2(0.0 - 1) = -0.2$$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, p_0 + \frac{l_1}{2}\right) = hf_1(0.1, 1.0, -0.1) = -0.02$$

$$l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, p_0 + \frac{l_1}{2}\right) = hf_2(0.1, 1.0, -0.1) = 0.2(0 - 1) = -0.2$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, p_0 + \frac{l_2}{2}\right) = hf_1(0.1, 0.99, -0.1) = 0.2(-0.1) = -0.02$$

$$l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, p_0 + \frac{l_2}{2}\right) = hf_2(0.1, 0.99, -0.1)$$

$$= 0.2[(0.1)(0.0199)(-0.1) - 0.99] = -0.1980$$

$$k_4 = hf_1(x_0 + h, y_0 + k_3, p_0 + l_3), = hf_1(0.2, 0.98, -0.1980) = -0.0396$$

$$l_4 = hf_2(x_0 + h, y_0 + k_3, p_0 + l_3) = hf_2(0.2, 0.98, -0.1980)$$

$$= 0.2[(0.1)(1 - 0.9604)(-0.1980) - 0.98] = -0.19616.$$

Now, $y(0.2) = y_1$ is given by

$$y(0.2) = y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1 + \frac{1}{6}[0.0 + 2(-0.02) + 2(-0.02) + (-0.0396)]$$

$$= 1 - 0.019935 = 0.9801$$

and

$$y'(0.2) = p_1 = p_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$= 0 + \frac{1}{6}[-0.2 + 2(-0.2) + 2(-0.1980) + (-0.19616)]$$

$$= -0.19869 \ (= -0.1987)$$

Therefore, the required solution is

$$y(0.2) = 0.9801, \qquad y'(0.2) = -0.1987$$

PREDICTOR - CORRECTOR METHODS

A predictor corrector method refers to the use of the predictor equation with one subsequent application of the corrector equation and the values so obtained are the final solution at the grid point.

PREDICTOR FORMULA: The explicit (open) formula used to predict approximation "y_{i+1}^{n}"is called a predictor formula.

CORRECTOR FORMULA: The implicit (closed) formula used to determine "y_{i+1}^{n}"is called Corrector Formula. This used to improve "y_{i+1}"

IN GENERAL: Explicit and Implicit formula are used as pair of formulas. The explicit formula is called 'predictor' and implicit formula is called 'corrector'

<u>Implicit methods are often used as 'corrector' and Explicit methods are used as 'predictor' in predictor-corrector method. why?</u>

Because the corresponding Local Truncation Error formula is smaller for implicit method on the other hand the implicit methods has the inherent difficulty that extra processing is necessary to evaluate implicit part.

REMARK

- Truncation Error of predictor is $E_p = \frac{14}{45}h^5 y_{k-1}^{(5)}$ OR $\frac{28}{90}h\Delta^4 y_0'$
- Local Truncation Error of Adam's Predictor is $\frac{251}{720}h^5 y^{(5)}$
- Truncation Error of Corrector is $\frac{1}{90}h\Delta^4 y_0'$

<u>Why Should one bother using the predictor corrector method When the Single step method are of the comparable accuracy to the predictor corrector methods are of the same order?</u>

A practical answer to that relies in the actual number of functional evaluations. For example, RK - Method of order four, each step requires four evaluations where the Adams Moulton method of the same order requires only as few as two evaluations. For this reason, predictor corrector formulas are in General considerably more accurate and faster than single step methods.

REMEMBER: In predictor corrector method if values of "$y_0, y_1, y_2 \dots\dots\dots$" against the values of "$x_0, x_1, x_2 \dots\dots\dots$" are given the we use symbol predictor corrector method and in this method we use given values of"$y_0, y_1, y_2 \dots\dots\dots$"

If "$y_0, y_1, y_2 \dots\dots\dots$" Are not given against the values of "$x_0, x_1, x_2 \dots\dots\dots$" then we first find values of "$y_0, y_1, y_2 \dots\dots\dots$" by using RK - method
OR By using formula $\forall\ j = 1, 2, 3 \dots\dots\dots n$

$$y_j = y_0 + (jh)y_0' + \frac{(jh)^2}{2!} y_0'' + \frac{(jh)^3}{3!} y_0''' + \dots\dots\dots$$

8.5 PREDICTOR–CORRECTOR METHODS

The methods presented in Sections 8.2 to 8.4 are in general known as single-step methods, where we have seen that the computation of y at t_{n+1}, that is y_{n+1} requires the knowledge of y_n only. In *predictor–corrector methods* which we discuss below, also known as *multi-step methods*, we require to know the solution y at t_n, t_{n-1}, t_{n-2}, etc., to compute the value of y at t_{n+1}. Thus, a *predictor formula* is used to predict the value of y at t_{n+1} and then a *corrector formula* is used to improve the value of y_{n+1}. For example, consider a differential equation

$$\frac{dy}{dt} = f(t, y)$$

with the initial condition $y(t_n) = y_n$. Using simple Euler's and modified Euler's method, we can write down a simple predictor–corrector pair (P–C) as

$$\left.\begin{aligned} \text{P}: y_{n+1}^{(0)} &= y_n + hf(t_n, y_n) \\ \text{C}: y_{n+1}^{(1)} &= y_n + \frac{h}{2}\left[f(t_n, y_n) + f\left(t_{n+1}, y_{n+1}^{(0)}\right)\right] \end{aligned}\right\} \tag{8.32}$$

Here, $y_{n+1}^{(1)}$ is the first corrected value of y_{n+1}. The corrector formula may be used iteratively as defined below:

$$y_{n+1}^{(r)} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f\left(t_{n+1}, y_{n+1}^{(r-1)}\right)\right], \qquad r = 1, 2, \dots \tag{8.33}$$

The iteration is terminated when two successive iterates agree to the desired accuracy. In this pair, to extrapolate the value of y_{n+1}, we have approximated the solution curve in the interval (t_n, t_{n+1}) by a straight line passing through (t_n, y_n) and (t_{n+1}, y_{n+1}). The accuracy of the predictor formula can be improved by considering a quadratic curve through the equispaced points (t_{n-1}, y_{n-1}), (t_n, y_n), (t_{n+1}, y_{n+1}). Suppose, we fit a quadratic curve of the form

$$y = a + b\,(t - t_{n-1}) + c\,(t - t_n)\,(t - t_{n-1}) \tag{8.34}$$

where a, b, c are constants to be determined. Since the curve passes through (t_{n-1}, y_{n-1}) and (t_n, y_n) and satisfies

$$\left(\frac{dy}{dt}\right)_{(t_n, y_n)} = f(t_n, y_n)$$

we obtain

$$y_{n-1} = a, \qquad y_n = a + bh = y_{n-1} + bh$$

Therefore,

$$b = \frac{y_n - y_{n-1}}{h}$$

and

$$\left(\frac{dy}{dt}\right)_{(t_n, y_n)} = f(t_n, y_n) = \{b + c[(t - t_{n-1}) + (t - t_n)]\}_{(t_n, y_n)}$$

Which gives $$f(t_n, y_n) = b + c(t_n - t_{n-1}) = b + ch$$

or

$$c = \frac{f(t_n, y_n)}{h} - \frac{(y_n - y_{n-1})}{h^2}$$

Substituting these values of a, b and c into the quadratic equation (8.34), we get

$$y_{n+1} = y_{n-1} + 2(y_n - y_{n-1}) + 2[hf(t_n, y_n) - (y_n - y_{n-1})]$$

That is,

$$y_{n+1} = y_{n-1} + 2hf(t_n, y_n) \qquad (8.35)$$

Thus, instead of considering the predictor-corrector pair (8.32), we may consider the predictor-corrector pair given by

$$\left.\begin{aligned} P: y_{n+1} &= y_{n-1} + 2hf(t_n, y_n) \\ C: y_{n+1} &= y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})] \end{aligned}\right\} \qquad (8.36)$$

The essential difference between them is, the predictor of (8.36) is more accurate. However, the predictor of (8.36) cannot be used to predict y_{n+1} for a given initial value problem. The reason being, its use require the knowledge of past two points. In such a situation, a Runge–Kutta method is generally used to start the predictor method.

BASE (MAIN IDEA) OF PREDICTOR CORRECTOR METHOS

In predictor corrector methods a predictor formula is used to predict the value of 'y' at t_{n+1} and then a corrector formula is used to improve the value of y_{n+1}

Following are predictor – corrector methods

1. Milne's Method
2. Adam – Moulton method

MILNE'S METHOD

It's a multi-step method. In General, Milne's Predictor – Corrector pair can be written as

P: $y_{n+1} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n) \quad n \geq 3$

C: $y_{n+1} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1}) \quad n \geq 3$

REMARK: Magnitude of truncation error in Milne's corrector formula is $\frac{1}{90}h\,\Delta^4 y'_0$ and truncation error in Milne's predictor formula is $\frac{28}{90}h\,\Delta^4 y'_0$

stable, convergent, efficient, accurate, compeer friendly.

ALGORITHM

- First predict the value of y_{n+1} by above predictor formula.
  Where derivatives are computed using the given differential equation itself.
- Using the predicted value "y_{n+1}" we calculate the derivative y'_{n+1} from the given differential Equation.
- Then use the corrector formula given above for corrected value of y_{n+1}. Repeat this process.

8.5.1 Milne's Method

It is also a *multi-step method* where we assume that the solution to the given initial value problem is known at the past four equispaced points t_0, t_1, t_2 and t_3. To derive Milne's predictor-corrector pair, we proceed as follows:

Let us consider the typical differential equation

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0$$

On integration between the limits t_0 and t_4, we get

$$\int_{t_0}^{t_4} \frac{dy}{dt}\,dt = \int_{t_0}^{t_4} f(t, y)\,dt \tag{8.37}$$

That is,

$$y_4 - y_0 = \int_{t_0}^{t_4} f(t, y)\,dt \tag{8.38}$$

To carry out integration, we employ a quadrature formula such as *Newton's forward difference formula* (6.33), so that

$$f(t, y) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{6}\Delta^3 f_0 + \cdots \tag{8.39}$$

where

$$s = \frac{t - t_0}{h}, \qquad t = t_0 + sh$$

Substituting Eq. (8.39) into Eq. (8.38), we obtain

$$y_4 = y_0 + \int_{t_0}^{t_4}\left[f_0 + s\Delta f_0 + \frac{s(s-1)}{2}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{6}\Delta^3 f_0 + \frac{s(s-1)(s-2)(s-3)}{24}\Delta^4 f_0 + \cdots\right]dt$$

Now, by changing the variable of integration (from t to s), the limits of integration also changes (from 0 to 4), and thus the above expression becomes

$$y_4 = y_0 + h\int_0^4\left[f_0 + s\Delta f_0 + \frac{s(s-1)}{2}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{6}\Delta^3 f_0 + \frac{s(s-1)(s-2)(s-3)}{24}\Delta^4 f_0 + \cdots\right]ds \quad (8.40)$$

which simplifies to

$$y_4 = y_0 + h\left[4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \frac{8}{3}\Delta^3 f_0 + \frac{28}{90}\Delta^4 f_0\right] \quad (8.41)$$

Substituting the differences, such as $\Delta f_0 = f_1 - f_0$, $\Delta^2 f_0 = f_2 - 2f_1 + f_0$, etc., Eq. (8.41) can be further simplified to

$$y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) + \frac{28}{90}h\Delta^4 f_0 \quad (8.42)$$

Alternatively, it can also be written as

$$y_4 = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') + \frac{28}{90}h\Delta^4 y_0' \quad (8.43)$$

This is known as *Milne's predictor formula*.

Similarly, integrating Eq. (8.37) over the interval t_0 to t_2 or $s = 0$ to 2 and repeating the above steps, we get

$$y_2 = y_0 + \frac{h}{3}(y_0' + 4y_1' + y_2') - \frac{1}{90}h\Delta^4 y_0' \quad (8.44)$$

which is known as *Milne's corrector formula*.

In general, Milne's predictor–corrector pair can be written as

$$\left.\begin{aligned} P: y_{n+1} &= y_{n-3} + \frac{4h}{3}(2y_{n-2}' - y_{n-1}' + 2y_n') \\ C: y_{n+1} &= y_{n-1} + \frac{h}{3}(y_{n-1}' + 4y_n' + y_{n+1}') \end{aligned}\right\} \quad (8.45)$$

EXAMPLE: use Milne's method to solve $\frac{dy}{dx} = \mathbf{1} + \boldsymbol{y}^{\mathbf{2}}$; $\boldsymbol{y}(\mathbf{0}) = \mathbf{0}$ and compute y(0.8)

SOLUTION: $h = 0.2, x_0 = 0, x_1 = x_0 + h = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$ $also$ $y_0 = 0$

Now by using Euler's method $\Rightarrow y_{m+1} = y_m + hf(t_m, y_m)$

$for\ m = 0;$ $\Rightarrow y_1 = y_0 + hf(t_0, y_0) = 0.2 = y(0.2)$

$for\ m = 1;$ $\Rightarrow y_2 = y_1 + hf(t_1, y_1) = 0.48 = y(0.4)$

$for\ m = 2;$ $\Rightarrow y_3 = y_2 + hf(t_2, y_2) = 0.73 = y(0.6)$

Now $y_n' = 1 + y_n^2$ For n=1 $\Rightarrow y_1' = 1 + y_1^2 = 1.04$

For n=2 $\Rightarrow y_2' = 1 + y_2^2 = 1.16$ For n=3 $\Rightarrow y_3' = 1 + y_3^2 = 1.36$

Now using Milne's Predictor formula

$$P: y_{n+1} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n) \quad n \geq 3$$

$$y_4 = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3) = 0.98 \implies y'_4 = 1 + y_4^2 = 1.9604$$

Now using corrector formula

$$C: \quad y_{n+1} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1}) \quad n \geq 3$$

$$y_4 = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4) = 1.05 = y(0.8)$$

examples.

Example 8.7 Find $y(2.0)$ if $y(t)$ is the solution of

$$\frac{dy}{dt} = \frac{1}{2}(t + y)$$

assuming $y(0) = 2$, $y(0.5) = 2.636$, $y(1.0) = 3.595$ and $y(1.5) = 4.968$ using Milne predictor–corrector method.

Solution Taking $t_0 = 0.0$, $t_1 = 0.5$, $t_2 = 1.0$, $t_3 = 1.5$, where we are given y_1, y_2 and y_3, we have to compute y_4, the solution of the given differential equation corresponding to $t = 2.0$. The Milne's P–C pair is given as

$$P: y_{n+1} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n)$$

$$C: y_{n+1} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1})$$

From the given differential equation, we have $y' = (t + y)/2$. Therefore,

$$y'_1 = \frac{t_1 + y_1}{2} = \frac{0.5 + 2.636}{2} = 1.5680$$

$$y'_2 = \frac{t_2 + y_2}{2} = \frac{1.0 + 3.595}{2} = 2.2975$$

$$y'_3 = \frac{t_3 + y_3}{2} = \frac{1.5 + 4.968}{2} = 3.2340$$

Now, using predictor formula, we compute

$$y_4 = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3)$$

$$= 2 + \frac{4(0.5)}{3}[2(1.5680) - 2.2975 + 2(3.2340)]$$

$$= 6.8710.$$

Using this predicted value, we shall compute the improved value of y_4 from corrector formula

$$y_4 = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4)$$

in the following steps. Now using the available predicted value y_4 and the initial values, we compute

$$y'_4 = \frac{t_4 + y_4}{2} = \frac{2 + 6.8710}{2} = 4.4355$$

$$y'_3 = \frac{t_3 + y_3}{2} = \frac{1.5 + 4.968}{2} = 3.2340$$

and

$$y_2' = 2.2975$$

Thus, the first corrected value of y_4 is given by

$$y_4^{(1)} = 3.595 + \frac{0.5}{3}[2.2975 + 4(3.234) + 4.4355] = 6.8731667$$

Suppose, we apply the corrector formula again, then we have

$$y_4^{(2)} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + \left(y_4^{(1)}\right)'\right]$$

$$= 3.595 + \frac{0.5}{3}\left[2.2975 + 4(3.234) + \frac{2 + 6.8731667}{2}\right]$$

$$= 6.8733467$$

Finally, the value of y at $t = 2.0$ is given by $y(2.0) = y_4 = 6.8734$.

Example 8.8 Tabulate the solution of

$$\frac{dy}{dt} = t + y, \qquad y(0) = 1$$

in the interval $0 \le t \le 0.4$, with $h = 0.1$, using Milne's predictor–corrector method.

Solution Milne's P-C method demands the solution at the first four points t_0, t_1, t_2 and t_3. As it is not a self-starting method, we shall use Runge–Kutta method of fourth order (why?) to get the required solution and then switch over to Milne's P-C method. Thus, taking $t_0 = 0$, $t_1 = 0.1$, $t_2 = 0.2$, $t_3 = 0.3$ we get the corresponding y values using Runge–Kutta method of fourth order; that is, $y_0 = 1$, $y_1 = 1.1103$, $y_2 = 1.2428$ and $y_3 = 1.3997$ (as obtained in Example 8.5). Now, we compute

$$y_1' = t_1 + y_1 = 0.1 + 1.1103 = 1.2103$$

$$y_2' = t_2 + y_2 = 0.2 + 1.2428 = 1.4428$$

$$y_3' = t_3 + y_3 = 0.3 + 1.3997 = 1.6997$$

Using Milne's predictor formula

$$\text{P} : y_4 = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$$

$$= 1 + \frac{4(0.1)}{3}[2(1.2103) - 1.4428 + 2(1.6997)]$$

$$= 1.58363$$

Before using corrector formula, we compute

$$y_4' = t_4 + y_4 \text{ (predicted value)} = 0.4 + 1.5836 = 1.9836$$

ADAM'S MOULTON METHOD

The predictor – corrector formulas for Adam's Moulton method are given as follows

$$P:\ y_{n+1} = y_n + \frac{h}{24}[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}]$$

$$C:\ y_{n+1} = y_n + \frac{h}{24}[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}]$$

REMARK

- The predictor Truncation Error is $\left(\frac{251}{720}\right)h\nabla^4 y'_n$ and Corrector Truncation Error is $\left(-\frac{19}{720}\right)h\nabla^4 y'_{n+1}$
- Truncation Error in Adam's predictor is approximately 13-time more than that in the corrector. OF course with Opposite Sign.
- In the predictor Corrector methods if $'y_0'$ is given another $(y_1, y_2 \ldots .)$ or not Then Using Euler or RK or any other method we can find these. For example, by Euler method

$$y_{m+1} = y_m + hf(t_m, y_m) \quad \text{For m=0} \quad \Rightarrow y_1 = y_0 + hf(t_0, y_0)$$

EXAMPLE: use Adam's Moulton method to solve $\frac{dy}{dt} = y - t^2$; $y(0) = 1$ at t = 1.0 taking h=0.2 and compare it with analytic solution.

SOLUTION: in order to use Adam's Moulton method we require the solution of the given Differential equation at the past four equispaced points for which we have RK-4th order method which is self-starting.

Using RK method we get $y_1 = 1.21859, y_2 = 1.46813, y_3 = 1.73779$ where $h = 0.2, y_0 = 1, t_0 = 0, t_1 = t_0 + h = 0.2$, $t_2 = 0.4$, $t_3 = 0.6$

Also in easy way we can find y_1, y_2, y_3 by using Euler's method with some error as follows Now by using Euler's method $\Longrightarrow y_{m+1} = y_m + hf(t_m, y_m)$

$for\ m = 0;\quad \Longrightarrow y_1 = y_0 + hf(t_0, y_0) = 1.20000$

$for\ m = 1;\quad \Longrightarrow y_2 = y_1 + hf(t_1, y_1) = 1.4320$

$for\ m = 2;\quad \Longrightarrow y_3 = y_2 + hf(t_2, y_2) = 1.6864$

now as $y'_n = y_n - t_n^2$

$\Rightarrow y'_0 = y_0 - t_0^2 = 1 \Rightarrow y'_1 = y_1 - t_1^2 = 1.16 \Rightarrow y'_2 = y_2 - t_2^2 = 1.27 \Rightarrow y'_3 = y_3 - t_3^2 = 1.32$

Now using Adam's Predictor formula

$$P:\ y_{n+1} = y_n + \frac{h}{24}[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}]$$

$$y_4 = y_3 + \frac{h}{24}[55y'_3 - 59y'_2 + 37y'_1 - 9y'_0] = 1.95 = y(0.8) \Rightarrow y_4^{(p)'} = y_4 - t_4^2 = 1.3105$$

Now using Adam's Corrector formula
C: $y_{n+1} = y_n + \frac{h}{24}[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}]$
$\Rightarrow$ $y_4 = y_3 + \frac{h}{24}[9y'_4 + 19y'_3 - 5y'_2 + y'_1] = 1.9530 \Rightarrow y_4^{(c)'} = y_4{}^{(c)} - t_4^2 = 1.3103$

Proceeding in similar way we can get $y_5{}^{(p)} = 2.2039$, $y_5^{(p)'} = 1.2039, y_5{}^{(c)} = 2.2034 = y(1.0)$

Now the analytic solution can be seen in the following steps $\frac{dy}{dt} - y = -t^2$ Then using integrating factor $e^{-t} \Rightarrow \frac{d}{dt}(ye^{-t}) = -te^{-t} \Rightarrow ye^{-t} = -\int te^{-t}$
$\Rightarrow y = t^2 + 2t + 2 + ce^t$
now using initial conditions $y(0) = 1$ we get c = -1 therefor analytic solution is

$\Rightarrow y = t^2 + 2t + 2 - e^t$ $\Rightarrow y(1.0) = 2.2817$

EXERCISES

Explain Taylor's series method of solving an initial value problem described by

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Using Taylor's series method, find the solution of the differential equation

$xy' = x - y$ given that $y(2) = 2$, at $x = 2.1$.

8.3 Using Taylor's series method, find the solution of

$$\frac{dy}{dt} = t^2 + y^2$$

with the initial values $t_0 = 1$, $y_0 = 0$, at $t = 1.3$.

8.4 Using Taylor's series method, solve

$$y' = y \sin x + \cos x$$

subject to $x = 0$, $y = 0$ for some x.

8.5 Using modified Euler's method, obtain the solution of

$$\frac{dy}{dt} = 1 - y, \qquad y(0) = 0$$

for the range $0 \le t \le 0.2$, by taking $h = 0.1$.

8.6 Solve the initial value problem

$$yy' = x, \qquad y(0) = 1.5$$

using simple Euler's method, taking $h = 0.1$ and hence find y (0.2).

8.7 Obtain numerically the solution of

$$y' = x^2 + y^2, \qquad y(0) = 0.5$$

using Euler's method to find y at $x = 0.1$ and 0.2

8.8 Use Runge–Kutta method of fourth order to solve numerically the initial value problem

$$10\frac{dy}{dx} = x^2 + y^2, \qquad y(0) = 1$$

and find y in the interval $0 \le x \le 0.4$, taking $h = 0.1$.

8.9 Solve $y'' = x(y')^2 - y^2$, using fourth order Runge–Kutta method for $x = 0.2$ correct to four decimal places with the initial conditions $y(0) = 1$, $y'(0) = 0$.

8.10 Using R–K method of fourth order, solve $y'' = xy' + y^2$, given that $y(0) = 1$, $y'(0) = 2$. Take $h = 0.2$ and find y and y' at $x = 0.2$.

8.11 Use fourth order Runge–Kutta method to solve numerically the following initial value problem

$$\frac{dy}{dt} = y^2 - 100 \exp[-100(t-1)^2], \qquad y(0.8) = 4.9491$$

and find y in the interval $0.8 \le t \le 0.9$ taking $h = 0.01$.

8.12 Find $y(0.8)$ using Milne's P-C method, if $y(x)$ is the solution of the differential equation

$$\frac{dy}{dx} = -xy^2, \qquad y(0) = 2$$

assuming $y(0.2) = 1.92308$, $y(0.4) = 1.72414$, $y(0.6) = 1.47059$.

8.13 Explain the principle of predictor–corrector methods. Derive Milne's predictor–corrector formulae to solve an initial value problem

$$\frac{dy}{dx} = f(x, y), \qquad y(0) = y_0$$

8.14 Using Adam's predictor–corrector method, find y at $t = 4.4$ from the differential equation

$$5t\frac{dy}{dt} + y^2 = 2,$$

given that

t	4.0	4.1	4.2	4.3
y	1.0	1.0049	1.0097	1.0143

8.15 It is well known in the theory of beams that the radius of curvature is given by

$$\frac{\text{EI}\, y''}{(1 + y'^2)^{3/2}} = M(x)$$

where $M(x)$ is the bending moment. For a cantilever beam, it is known that $y(0) = y'(0) = 0$. Express the above equation into two first order simultaneous differential equations.

8.16 The resonant spring system with a periodic forcing function is given by

$$\frac{d^2y}{dt^2} + 64y = 16 \cos 8t, \qquad y(0) = y'(0) = 0$$

Determine the displacement at $t = 0.1, 0.2, \ldots, 0.5$ using Adam's–Moulton method after getting the required starting values by Runge–Kutta fourth order method.

8.17 Solve the initial value problem

$$\frac{dy}{dx} = 3x^2 + y, \qquad y(0) = 4$$

for the range $0.1 \le x \le 0.5$, using Euler's method by taking $h = 0.1$.

8.18 Using Euler's method, obtain the solution to the initial value problem

$$y' = x + y + xy, \qquad y(0) = 1$$

at $x = 0.1$, by taking $h = 0.025$.

8.19 Solve the initial value problem

$$\frac{dy}{dx} = \log(x + y), \qquad y(0) = 1$$

using modified Euler method and find $y(0.2)$.

8.20 Using fourth order Runge–Kutta method find the solution of the initial value problem

$$y' = 1/(x + y), \qquad y(0) = 1$$

in the range $0.5 \le x \le 2.0$, by taking $h = 0.5$.

8.21 Using fourth order Runge–Kutta method, find the solution of

$$x(dy + dx) = y(dx - dy), \qquad y(0) = 1$$

at $x = 0.1$ and 0.2, by taking $h = 0.1$.

8.22 Find the solution of

$$y' = y(x + y), \qquad y(0) = 1$$

using Milne's P-C method at $x = 0.4$ given that $y(0.1) = 1.11689$, $y(0.2) = 1.27739$ and $y(0.3) = 1.50412$.

8.23 Using Adam's–Moulton P-C method, find the solution of

$$x^2 y' + xy = 1, \qquad y(1) = 1.0$$

at $x = 1.4$, given that $y(1.1) = 0.996$, $y(1.2) = 0.986$, $y(1.3) = 0.972$

8.24 Find the solution of the initial value problem

$$y' = y^2 \sin t \qquad y(0) = 1$$

using Adam's–Moulton P-C method, in the interval $(0.2, 0.5)$, given that $y(0.05) = 1.00125$, $y(0.1) = 1.00502$, $y(0.15) = 1.01136$.

8.25 Solve the following system of differential equations

$$\frac{dx}{dt} = x + 2y, \qquad x(0) = 6$$

$$\frac{dy}{dt} = 3x + 2y, \qquad y(0) = 4$$

over the interval $(0.02, 0.06)$ using Runge–Kutta method, with step size $h = 0.02$.

THE END

GOOD LUCK

REFERENE BOOKS

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- Best Numerical Methods by Sri.Nanda Kumar M.
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