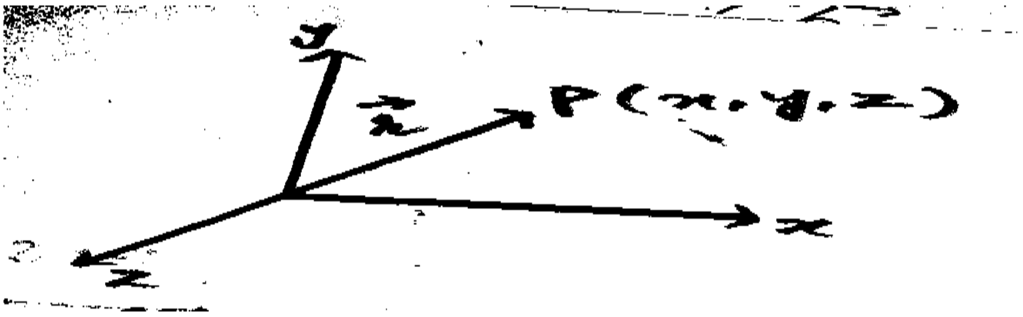
REVIEW OF VECTORS AND SCALARS

VECTOR: A quantity with magnitude, unit and direction is called vectors. e.g. force, velocity, acceleration etc.

GRAPHICALLY: vector represented graphically by directed line segment. If P(x,y,z) be a point in R^3(space) then the vector $\overrightarrow{OP}$ from the origin to the point P is called the Position vector of the point P.



Usually we denote vector $\overrightarrow{OP}$ by $\vec{r}$ and (x,y,z) is called the coordinate representation. Thus we can write $\vec{r} = (x, y, z)$ and the magnitude of $\vec{r}$ is denoted by $|\vec{r}|$ or simply '**r**' and from geometry we know that $|\vec{r}| = \boldsymbol{r} = \sqrt{x^2 + y^2 + z^2}$

COMPONENTS OF VECTOR: In certain coordinate system, a three dimensional vector having initial point $P(x_1, y_1, z_1)$ and terminal point $Q(x_2, y_2, z_2)$ is represented by $\vec{r} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$ where $x_2 - x_1, y_2 - y_1, z_2 - z_1$ are called components of $\vec{r}$

VECTOR ADDITION:
if $\vec{r}_1 = (x_1, y_1, z_1)$ and $\vec{r}_2 = (x_2, y_2, z_2)$ then $\vec{r}_1 + \vec{r}_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

SCALAR MULTIPLICATION:
If $\vec{r} = (x, y, z)$ be a vector and λ is any scalar then $\lambda\vec{r} = (\lambda x, \lambda y, \lambda z)$
also $(\lambda + u)\vec{r} = \lambda\vec{r} + u\vec{r}$ and $(\lambda u)\vec{r} = \lambda(u\vec{r}) = u(\lambda\vec{r})$

SCALAR: A quantity which needs only magnitude and unit and no need to direction is called scalar. e.g. temperature, time, volume, length etc.

SCALAR PRODUCT (DOT PRODUCT):
If $\vec{a}_1$ and $\vec{a}_2$ be two vectors then their scalar product can be denoted by $\vec{a}_1 . \vec{a}_2$ And defined as $\vec{a}_1 . \vec{a}_2 = \boldsymbol{a_1}\, \boldsymbol{a_2}\, Cos\theta$ Where θ is the angle between $\vec{a}_1$ and $\vec{a}_2$ and $Cos\theta = \frac{\vec{a}_1 . \vec{a}_2}{\boldsymbol{a_1}\, \boldsymbol{a_2}}$

UNIT VECTORS: If $\hat{\imath}, \hat{\jmath}, \hat{k}$ denote the unit vectors along $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ respectively then

- $\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1$ and $\hat{\imath}.\hat{\jmath} = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\imath} = 0$
- The scalar product is distributive over addition i.e. $\vec{a}.(\vec{b} + \vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$
- If $\vec{a} = (a_1, a_2, a_3) = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = (b_1, b_2, b_3) = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ then $\vec{a}.\vec{b} = (a_1b_1 + a_2b_2 + a_3b_3)$
- $\vec{a}.\vec{a} = \vec{a}^2 = |\vec{a}|^2$ i.e. square of vector equals to square of its modulus.
- The necessary and sufficient condition that two vectors be perpendicular is that their scalar product vanishes.
- $\vec{r} = \vec{a} + t\vec{b}$ is called vector equation of straight line. With 't' a +ve or –ve number.
- The vector of constant length is perpendicular to its derivative. i.e. $\vec{n}.\vec{n}' = 0$
- $\vec{r}.\vec{r}' = |\vec{r}|\vec{r}'$

VECTOR PRODUCT (CROSS PRODUCT):

If $\vec{a}_1$ and $\vec{a}_2$ be two vectors then their vector product can be denoted by $\vec{a}_1 \times \vec{a}_2$ and defined as $\vec{a}_1 \times \vec{a}_2 = \boldsymbol{a_1}\, \boldsymbol{a_2}\, Sin\theta \hat{n}$ where θ is the angle between $\vec{a}_1$ and $\vec{a}_2$ and $\hat{n}$ is the unit vector in the direction of movement of a Right handed screw when turned from $\vec{a}_1$ to $\vec{a}_2$ Also $Sin\theta = \frac{\vec{a}_1 \times \vec{a}_2}{\boldsymbol{a_1}\, \boldsymbol{a_2}}$

- $\vec{a}_1 \times \vec{a}_2 = -\vec{a}_2 \times \vec{a}_1$
- Vector product is distributive over addition.
- $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$
- $\hat{\imath} \times \hat{\jmath} = \hat{k}, \quad \hat{\jmath} \times \hat{k} = \hat{\imath}, \quad \hat{k} \times \hat{\imath} = \hat{\jmath}$
- $\hat{\jmath} \times \hat{\imath} = -\hat{k}, \quad \hat{k} \times \hat{\jmath} = -\hat{\imath}, \quad \hat{\imath} \times \hat{k} = -\hat{\jmath}$
- If $\vec{a}_1$ *and* $\vec{a}_2$ are parallel then $\vec{a}_1 \times \vec{a}_2 = 0$
- If $\vec{a}_1 = (a_1, a_2, a_3) = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{a}_2 = (b_1, b_2, b_3) = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ then
$$\vec{a}_1 \times \vec{a}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{\imath} + (a_3b_1 - a_1b_3)\hat{\jmath} + (a_1b_2 - a_2b_1)\hat{k}$$
- The necessary and sufficient condition that two vectors be parallel is that their cross product vanishes.

SCALAR TRIPLE PRODUCT: $\vec{a}.(\vec{b} \times \vec{c})$ is called Scalar Triple Product or Box Product. Its value is numerically equal to the volume of the parallelepiped whose edges are determined by the vectors $\vec{a}, \vec{b}$ *and* $\vec{c}$

- $\vec{a}.(\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
- $\vec{a}.(\vec{b} \times \vec{c}) = \vec{b}.(\vec{c} \times \vec{a}) = \vec{c}.(\vec{a} \times \vec{b})$
- $|\vec{a} \times \vec{b}|$ is the area of parallelogram with sides $\vec{a}$ *and* $\vec{b}$
- $\vec{a}.(\vec{b} \times \vec{c})$ is denoted by $[\vec{a}\vec{b}\vec{c}]$ where $[\vec{a}\vec{b}\vec{c}] = -[\vec{a}\vec{c}\vec{b}]$
- The necessary and sufficient condition that three vectors be coplanar is that their scalar triple product vanishes.
- Right handed screw, positive sense for rotation or anti – clockwise, all same.

VECTOR PRODUCT OF MORE THAN TWO VECTORS

- $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$
- $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$
- $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{b}.\vec{c})(\vec{a}.\vec{d})$

  Proof: let $\vec{a} \times \vec{b} = \vec{I}$ then
  $\vec{I}.(\vec{c} \times \vec{d}) = [\vec{I}\vec{c}\vec{d}] = [\vec{c}\vec{d}\vec{I}] = \vec{c}.(\vec{d} \times \vec{I}) = \vec{c}.[\vec{d} \times (\vec{a} \times \vec{b})]$
  $= \vec{c}.[(\vec{d}.\vec{b})\vec{a} - (\vec{d}.\vec{a})\vec{b}] = [(\vec{d}.\vec{b})(\vec{c}.\vec{a}) - (\vec{d}.\vec{a})(\vec{c}.\vec{b})] = (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{b}.\vec{c})(\vec{a}.\vec{d})$
- $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\vec{c}\vec{d}]\vec{b} - [\vec{b}\vec{c}\vec{d}]\vec{a}$

  Proof: let $\vec{c} \times \vec{d} = \vec{I}$ then
  $(\vec{a} \times \vec{b}) \times \vec{I} = (\vec{a}.\vec{I})\vec{b} - (\vec{b}.\vec{I})\vec{a} = [\vec{a}.(\vec{c} \times \vec{d})]\vec{b} - [\vec{b}.(\vec{c} \times \vec{d})]\vec{a} = [\vec{a}\vec{c}\vec{d}]\vec{b} - [\vec{b}\vec{c}\vec{d}]\vec{a}$
- $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\vec{b}\vec{d}]\vec{c} - [\vec{a}\vec{b}\vec{c}]\vec{d}$

  Second Proof: let $\vec{a} \times \vec{b} = \vec{I}$ then
  $\vec{I} \times (\vec{c} \times \vec{d}) = (\vec{I}.\vec{d})\vec{c} - (\vec{I}.\vec{c})\vec{d} = [(\vec{a} \times \vec{b}).\vec{d}]\vec{c} - [(\vec{a} \times \vec{b}).\vec{c}]\vec{d} = [\vec{a}\vec{b}\vec{d}]\vec{c} - [\vec{a}\vec{b}\vec{c}]\vec{d}$

SURFACE: A surface $S\ of\ R^3$ is the locus of the point whose coordinates are functions of two independent parameters 'u' and 'v'

Thus $x = f_1(u,v), y = f_2(u,v), z = f_3(u,v)$ are parametric equations of surfaces.

OR: A surface in $R^3$is a set of all points whose coordinates satisfy a single equation $f(x,y,z) = 0$

OR: A surface may be regarded as the locus of a point whose position vector $\vec{r}$ is a function of two independent parameters u and v.

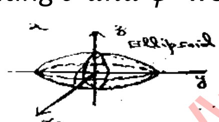
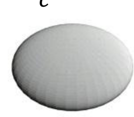
We not that any relation between the parameters $(f\ (u,v) = 0)$ represents a curve on the surface, because $\vec{r}$ than becomes a function of only one independent parameter.

In particular, the curve on the surface, along which one of the parameters remains constant are called **parametric curves**.

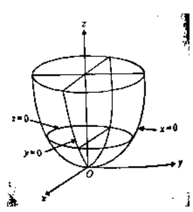
Position of any point on the surface is uniquely determined by the values of 'u' and 'v'. So that the parameters 'u' and 'v' constitute a system of coordinates which are called **curvilinear coordinates**.

EXAMPLES:

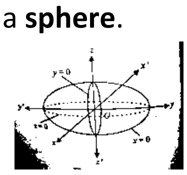
1) The parametric equation of the sphere with Centre at origin and radius 'a' are
$x = aSin\theta Cos\varphi, y = aSin\theta Sin\varphi, z = aCos\theta$
$\Longrightarrow\ x^2 + y^2 + z^2 = (aSin\theta Cos\varphi)^2 + (aSin\theta Sin\varphi)^2 + (aCos\theta)^2$
$\Longrightarrow\ x^2 + y^2 + z^2 = (aSin\theta)^2[(Cos\varphi)^2 + (Sin\varphi)^2] + (aCos\theta)^2$
$\Longrightarrow\ x^2 + y^2 + z^2 = (a)^2[(Sin\theta)^2 + (Cos\theta)^2]$
$\Longrightarrow\ x^2 + y^2 + z^2 = a^2$ is the implicit form of the sphere
2) The parametric equation of the ellipsoid are
$x = aCos\theta Cos\varphi, y = bCos\theta Sin\varphi, z = cSin\theta$
eliminating $\theta\ and\ \varphi$ we have $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ which is the equation of the **ellipsoid**.

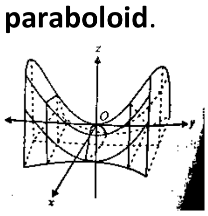
3) A surface in R^3defined by the equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz$ is called an **elliptical paraboloid**.



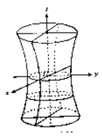
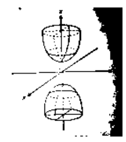

4) A surface in R^3 defined by the equation of the form
$x^2 + y^2 + z^2 + ux + vy + wz + d = 0 \Longrightarrow (x-h)^2 + (y-k)^2 + (z-I)^2 = r^2$ is called a **sphere**.



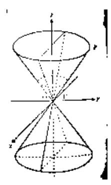
5) A surface in R^3 defined by the equation of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = cz$ is called a **hyperbolic paraboloid**.




6) A surface in R^3 defined by the equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1\ or\ \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$ is called a **hyperboloid**. Hyperboloid defined by first equation is called Hyperboloid of one sheet and Hyperboloid defined by second equation is called Hyperboloid of two sheet

7) A surface in R^3 defined by the equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=cz\,;c>0$ is called a **elliptical cone**.

8) The parametric equation of the cone are
$x=\mu Sin\varphi Cos\Psi, y=\mu Sin\varphi Sin\Psi, z=\mu Cos\varphi$
$\Longrightarrow\ x^2+y^2=(\mu Sin\varphi Cos\Psi)^2+(\mu Sin\varphi Sin\Psi)^2$
$\Longrightarrow\ x^2+y^2=(\mu Sin\varphi)^2[(\varphi Cos\Psi)^2+(Sin\Psi)^2]=(\mu Sin\varphi)^2=\frac{\mu^2 Sin^2\varphi}{Cos^2\varphi}.Cos^2\varphi$
$\Longrightarrow\ x^2+y^2=\mu^2Tan^2\varphi Cos^2\varphi \Longrightarrow\ x^2+y^2=z^2Tan^2\varphi$

9) A surface $z\ =\ 0$ represents $xy-\ plane$ **OR** the set of all points (x,y,z) satisfying a linear equation $ax+by+cz=0$ where a,b,c are not all zero determines a linear surface called a **<u>Plane</u>**.

10) The set of all points (x,y,z) satisfying a linear equation $ax^2+by^2+cz^2+fyz+gxz+hxy+ux+vy+wz+k=0$ where a,b,c,f,g are not all zero determines a quadratic surface

11) If $ax^2+by^2+cz^2+fyz+gxz+hxy+ux+vy+wz+k=0$ factorize into two linear functions then equation defines a pair of planes and then surface is called **<u>Degenerate Surface</u>**.

12) $x\ =aCos\theta Cos\varphi,\quad y\ =aCos\theta Sin\varphi,\quad z\ =\ aSin\theta$ Represents a sphere with centre at origin and radius "a". The surface of the sphere is the function of $\theta\ and\ \varphi$ the parametric curves are $\theta\ =\ constant$, which are small circles parallel to $xy-\ plane$ and $\varphi\ =\ constant$,which are great circles revolving about the $z-axis$

SYMMETRY: A surface is symmetric with respect to yz - plane if its equation is unchanged when 'x' is replace by '-x' . It is symmetric with respect to z – axis if its equation is unchanged when 'x' is replace by '-x' and y is replaced with '-y' .A surface is symmetric with respect to origin if its equation is unchanged when 'x' is replace by '-x' , 'y' is replaced with '-y' and 'z' is replaced with '-z'. e.g. the equation $x^2+y^2+z^2=a^2$ is symmetric about each coordinate plane , about each axis and about origin.

TRACE AND INTERCEPTS: The section of a surface intersected by a coordinate plane is called the **<u>trace of the surface</u>** in that plane. If a coordinate axis intersects a surface, such a point of intersection is called as **<u>intercept</u>**.

Consider a Hyperboloid type surface $x^2+y^2-z^2-2x=0$
then its trace in yz – plane is $y^2-z^2=0$ which are two straight lines as $y=\pm z$
its trace in xz – plane is $x^2-z^2-2x=0$ which is hyperbola.
its trace in xy – plane is $x^2+y^2-2x=0$ which is circle.

The x – intercepts are 0 and 2, y – intercepts is 0 and the z – intercept is 0.

CURVES WITH TORSION

CURVE: A curve is a locus of a point whose position vector $\vec{r}$ relative to the fixed origin may be represented as a function of single variable parameter (say 't') then its Cartesian coordinates are also the function of same parameter.
The equations of the curve in parametric forms are $x = x(t)\ , y = y(t)\ \ , z = z(t)$
where $x(t)\ , y(t)\ \ , z(t)$ are any functions of parameter 't'

LEVEL CURVE: A level curve is a function $U(x, y)$ define by the locus of a point (x, y) in domain D such that $U(x, y) = constant$ where C is a constant.

PARAMETERIZED CURVE "$\vec{r}(t)$": A parameterized curve in R^n is a mapping $\vec{\boldsymbol{r}}: (\boldsymbol{\alpha}, \boldsymbol{\beta}) \to \boldsymbol{R^n}$ for some α, β with $-\infty \le \alpha < \beta \le \infty$
the symbol (α, β) denote the open interval $(\alpha, \beta) = \{t\epsilon R: \alpha < t < \beta\}$
where $\vec{r}(t) = \{\vec{r}_1(t), \vec{r}_2(t), \dots \dots .. \vec{r}_n(t)\}$
PARAMETERIZATION OF LEVEL CURVE: A parameterized curve which is contained in a level curve 'C' is called Parameterization of level curve.

Example: How we can define a parameterization $\vec{r}(t)$ of parabola $y = x^2$?
Answer: If $\vec{r}(t) = \{\vec{r}_1(t), \vec{r}_2(t)\}$ then components $\vec{r}_1(t), \vec{r}_2(t)$ must satisfy $\vec{r}_2(t) = [\vec{r}_1(t)]^2$ now if $\vec{r}_1(t) = t$ then $\vec{r}_2(t) = t^2$ and so parameterization will b $\vec{r}(t) = \{t, t^2\}$
$\Rightarrow \vec{r}(t): (-\infty, \infty) \to R^2$ where $t\epsilon(-\infty, \infty)$
Example: Is $\vec{r}(t) = \{t^2, t^4\}$ a parameterization of the parabola $y = x^2$?
Answer: Yes $\vec{r}(t) = \{t^2, t^4\}$ a parameterization of the parabola $y = x^2$ because $x = t^2$ then $y = [t^2]^2 = t^4$
Example: How we can define a parameterization $\vec{r}(t)$ of parabola $x^2 + y^2 = 1$?
Answer: If $\vec{r}(t) = \{\vec{r}_1(t), \vec{r}_2(t)\}$ then components $\vec{r}_1(t), \vec{r}_2(t)$ must satisfy $\vec{r}_1^{\,2} + \vec{r}_2^{\,2} = 1$
now if $\vec{r}_1(t) = t$ then $t^2 + [\vec{r}_2(t)]^2 = 1 \Rightarrow [\vec{r}_2(t)]^2 = 1 - t^2 \Rightarrow \vec{r}_2(t) = \sqrt{1 - t^2}$
and so parameterization will be $\vec{r}(t) = \{t, \sqrt{1 - t^2}\} \Rightarrow \vec{r}(t): (-\infty, \infty) \to R^2$ where $t\epsilon(-\infty, \infty)$
REMARK: We can use $\vec{r}_1(t) = Cost, , \vec{r}_2(t) = Sint$ must satisfy $Cos^2t + Sin^2t = 1$ then $\vec{r}(t) = \{Cost, Sint\}$
Example: Show that the parameterized $\vec{r}(t) = \{Cos^3t, Sin^3t\}$; $t\epsilon R$ represent an equation of asteroid $x^{2/3} + y^{2/3} = 1$
Answer: Put $x = Cos^3t, y = Sin^3t$ then $(Cos^3t)^{2/3} + (Sin^3t)^{2/3} = Cos^2t + Sin^2t = 1$
Example: Find parameterization $\vec{r}(t)$ of the curve $y^2 - x^2 = 1$?
Answer: If $\vec{r}(t) = \{\vec{r}_1(t), \vec{r}_2(t)\}$ then components $\vec{r}_1(t), , \vec{r}_2(t)$ must satisfy $\vec{r}_1^{\,2} - \vec{r}_2^{\,2} = 1$ now if $\vec{r}_1(t) = t$ then $t^2 - [\vec{r}_2(t)]^2 = 1 \Rightarrow [\vec{r}_2(t)]^2 = 1 - t^2 \Rightarrow \vec{r}_2(t) = \sqrt{1 - t^2}$
and so parameterization will be $\vec{r}(t) = \{t, \sqrt{1 - t^2}\} \Rightarrow \vec{r}(t): (-\infty, \infty) \to R^2$ where $t\epsilon(-\infty, \infty)$
Example: Find parameterization $\vec{r}(t)$ of the curve $\frac{x^2}{4} + \frac{y^2}{9} = 1$?
Answer: If $\vec{r}(t) = \{\vec{r}_1(t), \vec{r}_2(t)\}$ then components $\vec{r}_1(t), , \vec{r}_2(t)$ must satisfy $\frac{\vec{r}_1^{\,2}}{4} + \frac{\vec{r}_2^{\,2}}{9} = 1$ now if $\vec{r}_1(t) = 2t$ then $\frac{(2t)^2}{4} + \frac{\vec{r}_2^{\,2}}{9} = 1 \Rightarrow \frac{\vec{r}_2^{\,2}}{9} = 1 - t^2 \Rightarrow \vec{r}_2^{\,2} = 9(1 - t^2) \Rightarrow \vec{r}_2(t) = 3\sqrt{1 - t^2}$
and so parameterization will be $\vec{r}(t) = \{2t, 3\sqrt{1 - t^2}\}$
Example: Consider the ellipse $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$ verify that $\vec{r}(t) = \{pCost, qSint\}$ is the parameterization of ellipse.
Answer: Put $x = pCost\ and\ y = qSint$ then $\Rightarrow \frac{x^2}{p^2} + \frac{y^2}{q^2} = \frac{(pCost)^2}{p^2} + \frac{(qSint)^2}{q^2} = 1$
Hence $\vec{r}(t) = \{pCost, qSint\}$ is the parameterization of ellipse.

ALLOWABLE CHANGE OF PARAMETER OF LEVEL CURVE: A real valued function $t = t(\theta)$ on an interval I_θ is an allowable change of parameter if

i. $t(\theta)$ is of class C in I_θ (1^{st} time differentiable)
ii. $\frac{dt}{d\theta} \neq 0 \quad \forall \theta \; in \; I_\theta$ i.e. $t = t(\theta)$ is one – to – one mapping of I_θ onto an interval $I_t = t(I_\theta)$ and the inverse of $\theta = \theta(t)$ is an allowable change of parameter on I_t

Example: Show that $t = \frac{\theta^2}{\theta^2+1}$ is an allowable change of parameter on $0 < \theta < \infty$ and takes the interval $0 < \theta < \infty$ onto $0 < t < 1$

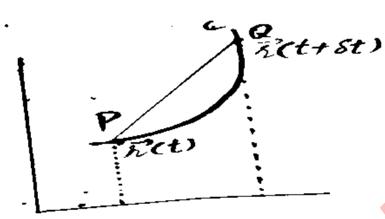
Answer: If $t = \frac{\theta^2}{\theta^2+1} \Rightarrow \frac{dt}{d\theta} = \frac{2\theta}{(\theta^2+1)^2}$ is continuous and $\frac{dt}{d\theta} \neq 0$ on $0 < \theta < \infty$ hence it is an allowable change of parameter on $0 < \theta < \infty$ and Since $\frac{\theta^2}{\theta^2+1} = 0 \; for \; \theta = 0$ and $\lim_{\theta\to 0} \frac{\theta^2}{\theta^2+1} = 1$ it shows that interval $0 < \theta < \infty$ onto $0 < t < 1$

DERIVATIVES OF "$\vec{r}(t)$": Since $\vec{r}: (\alpha, \beta) \to R^n$ then taking derivatives w.r.to 't' we get

$$\vec{r}'(t) = \frac{d\vec{r}}{dt} = \left(\frac{d\vec{r}_1}{dt}, \frac{d\vec{r}_2}{dt}, \frac{d\vec{r}_3}{dt} \ldots \ldots \ldots, \frac{d\vec{r}_m}{dt}\right)$$
$$\vec{r}''(t) = \frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2\vec{r}_1}{dt^2}, \frac{d^2\vec{r}_2}{dt^2}, \frac{d^2\vec{r}_3}{dt^2} \ldots \ldots \ldots, \frac{d^2\vec{r}_m}{dt^2}\right)$$
$$\vdots = \vdots = \vdots \qquad \vdots$$
$$\vec{r}^n(t) = \frac{d^n\vec{r}}{dt^n} = \left(\frac{d^n\vec{r}_1}{dt^n}, \frac{d^n\vec{r}_2}{dt^n}, \frac{d^n\vec{r}_3}{dt^n} \ldots \ldots \ldots, \frac{d^n\vec{r}_m}{dt^n}\right)$$

TANGENT VECTOR OF "$\vec{r}(t)$": if $\vec{r}$ is a parameterized curve then its first derivative $\vec{r}'(t) = \frac{d\vec{r}}{dt}$ is called the tangent vector of $\vec{r}$ at point $\vec{r}(t)$.

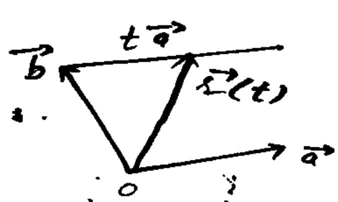
GEMETRICAL INTERPRETATION OF "$\vec{r}'(t)$" OR GEMETRICAL INTERPRETATION OF TANGENT VECTOR OF "$\vec{r}(t)$":



consider two points $\vec{r}(t)$ and $\vec{r}(t+\delta t)$ on the curve 'C' then joining these two points by a chord '$\widehat{PQ}$' then $\frac{\vec{r}(t+\delta t)-\vec{r}(t)}{\delta t} \parallel \widehat{PQ} \Rightarrow \vec{r}'(t) = \frac{d\vec{r}}{dt} = \lim_{\delta t\to 0} \frac{\vec{r}(t+\delta t)-\vec{r}(t)}{\delta t}$

PREPOSITION: If the tangent vector of a parameterized curve is constant, then image of the curve is a straight line.

Answer: Given that $\vec{r}'(t) = tangent\ vector = constant = \vec{a}(say) \Rightarrow \frac{d\vec{r}}{dt} = \vec{a} \Rightarrow d\vec{r} = \vec{a}dt$
$\Rightarrow \int d\vec{r} = \vec{a} \int dt \;\; \Rightarrow \vec{r}(t) = \vec{a}t + \vec{b} \Rightarrow$ image of the curve is a straight line.
Now here two cases we can discuss.



Case I: if $\vec{a} \neq 0$ then this is the parametric equation of straight line parallel to $\vec{a}$ and passes through $\vec{b}$.
Case II: if $\vec{a} = 0$ then image $\vec{r}$ of is a single point (namely $\vec{b}$)

Example: Find the Cartesian equation of the curve $\vec{r}(t) = \{Cos^2t, Sin^2t\}$ and also find the tangent vector.

Answer: Put $x = Cos^2t, y = Sin^2t$ then $x + y = Cos^2t + Sin^2t = 1 \Longrightarrow x + y = 1$
also $\vec{r}'(t) = (-2CostSint,\ 2SintCost)$

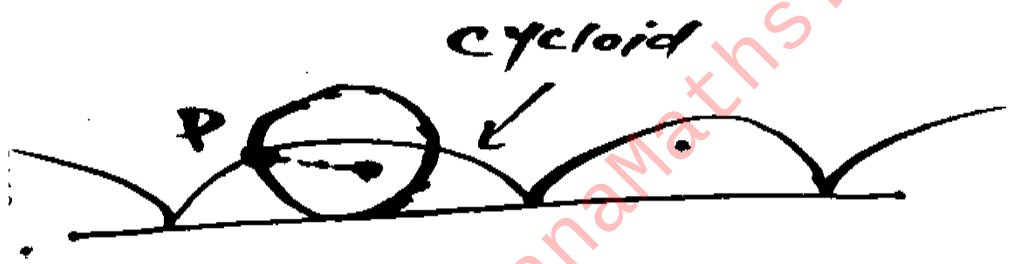
Example: Find the Cartesian equation of the curve $\vec{r}(t) = \{e^t, t^2\}$ and also find the tangent vector.

Answer: Put $x = e^t \ldots.(i)$ $\quad, y = t^2 \ldots.(ii)$ then $(ii) \Longrightarrow t = \sqrt{y}$ and $(i) \Longrightarrow x = e^{\sqrt{y}}$
also $\vec{r}'(t) = (e^t,\ 2t)$

Example: Calculate the tangent vector of asteroid $x^{2/3} + y^{2/3} = 1$ at each point. And at which point tangent vector will be zero?

Answer: when $x = Cos^3t, y = Sin^3t$ then $x^{2/3} + y^{2/3} = (Cos^3t)^{2/3} + (Sin^3t)^{2/3} = 1$ and we gent parameterization $\vec{r}(t) = \{Cos^3t, Sin^3t\}$
$\Longrightarrow \vec{r}'(t) = (-3Cos^2tSint,\ 3Sin^2tCost)$ a required tangent vector
now $\vec{r}'(t) = 0\ if\ 0 \quad \Longrightarrow -3Cos^2tSint = 0 \Longrightarrow\ Sint = 0 \Longrightarrow\ t = Sin^{-1}(0) \Longrightarrow\ t = 0$
$and\ \ also\ \ 3Sin^2tCost = 0 \Longrightarrow Cos = 0 \Longrightarrow\ t = Cos^{-1}(0) \Longrightarrow\ t = 90^0$

CYCLOID: A cycloid is the plane curve traced out by the point on the circumference of the circle as it rolls without slipping along a straight line.

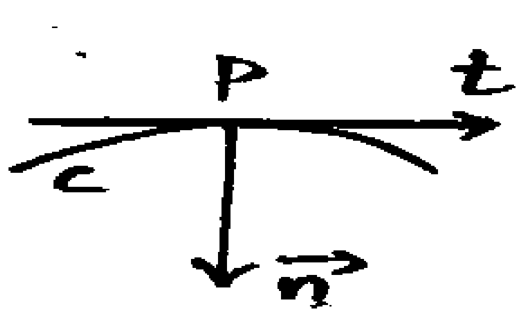


QUESTION: Show that if the straight line is the x – axis and circle has radius $a > 0$ then the cycloid can be parameterized by $\vec{r}(t) = [a(t - Sint), a(1 - Cost)]$

Answer: let $x = a(t - Sint),\quad y = a(1 - Cost)$
Now if straight line is x – axis then $y = 0 \Longrightarrow a(1 - Cost) = 0 \Longrightarrow a \neq 0, (1 - Cost) = 0$
$\Longrightarrow Cost = 1 \Longrightarrow t = Cos^{-1}(1) \Longrightarrow t = 2n\pi \qquad \forall n = 0,1,2, \ldots \ldots$

Then $x = a(t - Sint) \Longrightarrow x = a(2n\pi - Sin2n\pi) \Longrightarrow \boldsymbol{x = 2n\pi a}$ with $a > 0$

NORMAL LINE: the normal line to a curve at 'P' is the straight line passing through 'P' and perpendicular to the tangent line at 'P'.



QUESTION: Find the tangent and normal line passing through $P(x,y)$ to the curve $\vec{r}(t) = [2Cost - Cos2t, 2Sint - Sin2t]$ at the point corresponding $t = \frac{\pi}{4}$

Answer: Given $\vec{r}(t) = [2Cost - Cos2t, 2Sint - Sin2t]$

$\Rightarrow \vec{r}\left(\frac{\pi}{4}\right) = \left[2Cos\left(\frac{\pi}{4}\right) - Cos2\left(\frac{\pi}{4}\right), 2Sin\left(\frac{\pi}{4}\right) - Sin2\left(\frac{\pi}{4}\right) \Rightarrow \vec{r}\left(\frac{\pi}{4}\right) = [\sqrt{2}\ ,\ \sqrt{2}-1]\right]$

let $x = 2Cost - Cos2t,\quad y = 2Sint - Sin2t \quad \Rightarrow dx = -2Sint + 2Sin2t,\quad dy = 2Cost - 2Cos2t$

$\Rightarrow \frac{dy}{dx} = \frac{2Cost-2Cos2t}{-2Sint+2Sin2t} \Rightarrow \left(\frac{dy}{dx}\right)_{\frac{\pi}{4}} = m_1 = \frac{2Cos\left(\frac{\pi}{4}\right)-2Cos2\left(\frac{\pi}{4}\right)}{-2Sin\left(\frac{\pi}{4}\right)+2Sin2\left(\frac{\pi}{4}\right)} \Rightarrow m_1 = \frac{\sqrt{2}}{-\sqrt{2}+2} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}-\sqrt{2}} = \frac{1}{\sqrt{2}-1}$

And $m_2 = -\frac{1}{m_1} \Rightarrow m_2 = -(\sqrt{2}-1) \Rightarrow m_2 = -\sqrt{2}+1 = (1-\sqrt{2})$

Now equation of tangent at $[\sqrt{2}\ ,\ \sqrt{2}-1]$ is $Y - y_1 = m(X - x_1) \Rightarrow Y - y_1 = m_1(X - x_1)$

$\Rightarrow Y - (\sqrt{2}-1) = \frac{1}{\sqrt{2}-1}(X - \sqrt{2}) \Rightarrow (\sqrt{2}-1)Y - (\sqrt{2}-1)^2 = (X - \sqrt{2})$

$\Rightarrow X + (\sqrt{2}-1)Y + 3(\sqrt{2}-1) = 0$ is required tangent line. (after solving)

Now equation of Normal at $[\sqrt{2}\ ,\ \sqrt{2}-1]$ is $Y - y_1 = -\frac{1}{m}(X - x_1) \Rightarrow Y - y_1 = m_2(X - x_1)$

$\Rightarrow Y - (\sqrt{2}-1) = (1-\sqrt{2})(X-\sqrt{2}) \Rightarrow Y + (1-\sqrt{2}) = X(1-\sqrt{2}) - \sqrt{2}(1-\sqrt{2})$

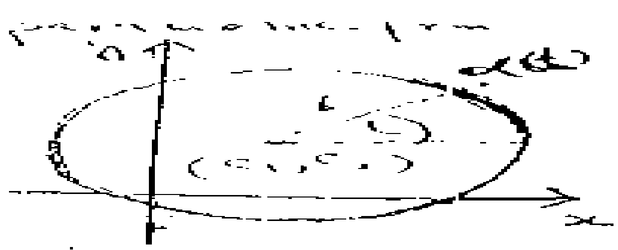
$\Rightarrow (1-\sqrt{2})X - Y + 1 = 0$ is required Normal line. (after solving)

Example: Find the representation of the intersection of the cylinder $x_3^2 = x_1, {x_2}^2 = 1 - x_1$ that does not involve radius.

Answer: given $x_3^2 = x_1$ and ${x_2}^2 = 1 - x_1 \Rightarrow x_1 = 1 - {x_2}^2$ then $\Rightarrow x_3^2 = 1 - {x_2}^2$ or $\Rightarrow x_3^2 + {x_2}^2 = 1$ now if we put $\boldsymbol{x_2 = Cos\theta}$ **and** $\boldsymbol{x_3 = Sin\theta}$ then $\boldsymbol{x_1} = 1 - Cos^2\theta = \boldsymbol{Sin^2\theta}$ then equation of the resulting curve is $\propto(\theta) = (Sin^2\theta, Cos\theta\ , Sin\theta)$

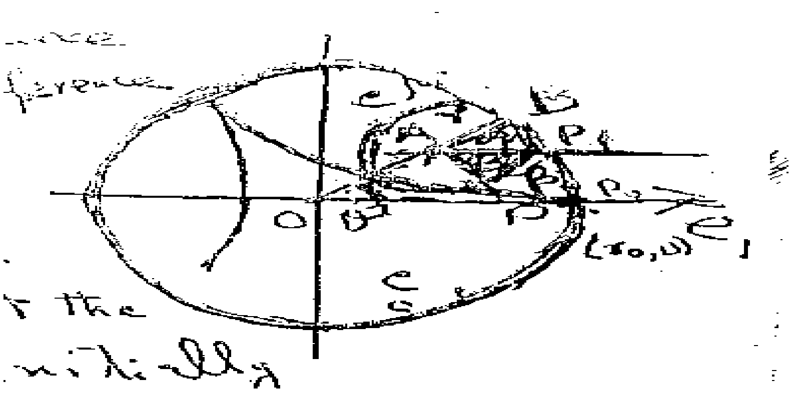
Example: Write the equation of circle centered at (c_1, c_2) having radius 'r' in parametric form.

Answer: $\propto(t) = (c_1, c_2) + (rCost\ , rSint) = (x = c_1 + rCost\ , y = c_2 + rSint)$



Example: The hypocycloid is the plane curve generated by the point 'P' on the circumference of the circle 'C' as 'C' rolls without sliding on the interior of the fixed circle C_0 as shown in the figure. If 'C' has radius 'r' and C_0 as at origin with radius r_0 and 'P' is initially located at $P_0(r_0, 0)$ then find a representation of hypocycloid.

Answer: let 'A' denote the center of 'C' and $d\theta$ is the angle that 'OA' makes with e_1 then



$\overrightarrow{OA} = |OA|Cos\theta e_1 + |OA|Sin\theta e_2 = (r_0 - r)Cos\theta e_1 + (r_0 - r)Sin\theta e_2$

If 'B' is the angle that $\overrightarrow{AP}$ makes with e_1 then

$\widehat{PP_1B} = \widehat{PP_1} + \widehat{P_1B} \Rightarrow r_0\theta = r\beta + r\theta \Rightarrow r\beta = (r_0 - r)\theta \Rightarrow \beta = \left(\frac{r_0 - r}{r}\right)\theta$

Since β is in clockwise direction therefore we will have $\beta = -\left(\frac{r_0-r}{r}\right)\theta$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = [(r_0 - r)Cos\theta e_1 + (r_0 - r)Sin\theta e_2] + \left[rCos\left(-\left(\frac{r_0-r}{r}\right)\theta\right)e_1 + rSin\left(-\left(\frac{r_0-r}{r}\right)\theta\right)e_2\right]$

$\Rightarrow \overrightarrow{OP} = \left[(r_0 - r)Cos\theta + rCos\left(\left(\frac{r_0-r}{r}\right)\theta\right), (r_0 - r)Sin\theta - rSin\left(\left(\frac{r_0-r}{r}\right)\theta\right)\right]$

ARC LENGTH OF PARAMETERIZED CURVE: the arc length of a curve $\vec{r}$ starting from a point $\vec{r}(t_0)$ is the function $s(t)$ and given by $s(t) = \int_{t_0}^{t}\|\vec{r}'(u)\|du$

QUESTION: Find the arc length of a logarithmic spiral $\vec{r}(t) = [e^{kt}Cost, e^{kt}Sint]$

Answer: Given $\vec{r}(t) = [e^{kt}Cost, e^{kt}Sint]$

$\Rightarrow \vec{r}'(t) = \frac{d\vec{r}}{dt} = [-e^{kt}Sint + ke^{kt}Cost, \quad e^{kt}Cost + ke^{kt}Sint]$

$\Rightarrow \|\vec{r}'(t)\| = \sqrt{e^{2kt}(kCost - Sint)^2 + e^{2kt}(kSint + Cost)^2} \Rightarrow \|\vec{r}'(t)\| = e^{kt}\sqrt{k^2+1}$

Then for arc length

$s(t) = \int_{t_0}^{t}\|\vec{r}'(u)\|du \Rightarrow s(t) = \int_0^t e^{ku}\sqrt{k^2+1}du \Rightarrow s(t) = \sqrt{k^2+1}\int_0^t e^{ku}du$

$\Rightarrow s(t) = \sqrt{k^2+1}\left|\frac{e^{ku}}{k}\right|_0^t \Rightarrow \boldsymbol{s(t) = \frac{\sqrt{k^2+1}}{k}(e^{kt} - 1)}$

QUESTION: Find the arc length as the function of θ along the epicycloid $\vec{r}(\theta) = \left[(r_0 + r)Cos\theta - rCos\left(\frac{r_0+r}{r}\theta\right), (r_0 + r)Sin\theta - rSin\left(\frac{r_0+r}{r}\theta\right), 0\right]$ and calculate the arc length when $r_0 = 4, r = 2, \theta = \pi$

Answer: Given $\vec{r}(\theta) = \left[(r_0 + r)Cos\theta - rCos\left(\frac{r_0+r}{r}\theta\right), (r_0 + r)Sin\theta - rSin\left(\frac{r_0+r}{r}\theta\right), 0\right]$

$$\Rightarrow \vec{r}'(\theta) = \left[-(r_0 + r)Sin\theta + \frac{r_0+r}{r}rSin\left(\frac{r_0+r}{r}\theta\right)\ , (r_0 + r)Cos\theta - \frac{r_0+r}{r}rCos\left(\frac{r_0+r}{r}\theta\right)\right]$$

$$\Rightarrow \|\vec{r}'(\theta)\| = \sqrt{\left(-(r_0 + r)Sin\theta + \frac{r_0+r}{r}rSin\left(\frac{r_0+r}{r}\theta\right)\right)^2 + \left((r_0 + r)Cos\theta - \frac{r_0+r}{r}rCos\left(\frac{r_0+r}{r}\theta\right)\right)^2}$$

$$\Rightarrow \|\vec{r}'(\theta)\| = (r_0 + r)\sqrt{\left(-Sin\theta + Sin\left(\frac{r_0+r}{r}\theta\right)\right)^2 + \left(Cos\theta - Cos\left(\frac{r_0+r}{r}\theta\right)\right)^2}$$

$$\Rightarrow \|\vec{r}'(\theta)\| = (r_0 + r)\sqrt{2 - 2Cos\left(\frac{r_0}{r}\theta\right)} = 2(r_0 + r)Sin\left(\frac{r_0}{2r}\theta\right)$$

Then for arc length $s(\theta) = \int_{t_0}^{t}\|\vec{r}'(\theta)\|d\theta \Rightarrow s(\theta) = 2(r_0 + r)\int_0^\theta Sin\left(\frac{r_0}{2r}\theta\right)d\theta$

$$\Rightarrow s(\theta) = \left|4\frac{r_0+r}{r_0}rCos\left(\frac{r_0}{2r}\theta\right)\right|_0^\theta = 4\frac{r_0+r}{r_0}r\left[Cos\left(\frac{r_0}{2r}\theta\right) - 1\right]$$

Now when $r_0 = 4, r = 2, \theta = \pi \Rightarrow s(\theta) = -24units \Rightarrow s(\theta) = 24units$

PRACTICE: find the arc length of followings;

i. Circular helix $\vec{r}(t) = [rCost, rSint,\ ht]$; $r > 0, h > 0\ for\ 0 \le t \le 10$
ii. $\vec{r}(t) = [2Cosh3t, -2Sinh3t\ ,6t\ ] = 2[Cosh3t, -Sinh3t\ ,3t\ ]$; $0 \le t \le 5$
iii. $\vec{r}(t) = [2\lambda, \lambda^2\ , ln\lambda\ ]$; $\lambda > 0$ between the points $P(2,1,0)$ and $Q(4,4,ln2)$

UNIT SPEED CURVE: if $\vec{\boldsymbol{r}}: (\boldsymbol{\alpha}, \boldsymbol{\beta}) \to \boldsymbol{R^n}$ is a parameterized curve, its speed at point $\vec{r}(t)$ is $\|\vec{r}'(t)\|$ and $\vec{r}$ is said to be a unit speed curve if $\vec{r}'(t)$ is a unit vector for all $t\epsilon(a, b)$
i.e. $\|\vec{r}'(t)\| = 1$

QUESTION: Show that $\vec{r}(t) = \left[\frac{1}{3}(1 + t)^{\frac{3}{2}},\ \frac{1}{3}(1 - t)^{\frac{3}{2}},\ \frac{t}{\sqrt{2}}\right]$ is unit speed curve.

Answer: Given $\vec{r}(t) = \left[\frac{1}{3}(1 + t)^{\frac{3}{2}},\ \frac{1}{3}(1 - t)^{\frac{3}{2}},\ \frac{t}{\sqrt{2}}\right]$

$$\Rightarrow \vec{r}'(t) = \frac{d\vec{r}}{dt} = \left[\frac{1}{2}(1 + t)^{\frac{1}{2}},\ -\frac{1}{2}(1 - t)^{\frac{1}{2}},\ \frac{1}{\sqrt{2}}\right]$$

$$\Rightarrow \|\vec{r}'(t)\| = \sqrt{\frac{1}{4}(1 + t) + \frac{1}{4}(1 - t) + \frac{1}{2}} = \sqrt{\frac{1}{4}(1 + t + 1 - t) + \frac{1}{2}} \Rightarrow \|\vec{r}'(t)\| = 1$$

$\Rightarrow \vec{r}'(t)$ is unit vector $\Rightarrow \vec{r}(t)$ is unit Speed curve.

PRACTICE: Show that $\vec{r}(t) = \frac{1}{2}\left[s + \sqrt{s^2 + 1},\ \frac{1}{s+\sqrt{s^2+1}},\ \sqrt{2}ln\left(s + \sqrt{s^2 + 1}\right)\right]$ is unit speed curve

REPARAMETERIZATION CURVE: a parameterized curve $\tilde{\boldsymbol{r}}: (\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) \to \boldsymbol{R^n}$ is a reparameterization of a parameterized curve $\vec{\boldsymbol{r}}: (\boldsymbol{\alpha}, \boldsymbol{\beta}) \to \boldsymbol{R^n}$ if there is a smooth bijective mapping $\boldsymbol{\varphi}: (\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) \to (\boldsymbol{\alpha}, \boldsymbol{\beta})$ such that the inverse mapping $\boldsymbol{\varphi^{-1}}: (\boldsymbol{\alpha}, \boldsymbol{\beta}) \to (\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}})$ is also smooth and $\tilde{\boldsymbol{r}}(\tilde{\boldsymbol{t}}) = \vec{\boldsymbol{r}}[\boldsymbol{\varphi}(\tilde{\boldsymbol{t}})] \quad \forall\ \tilde{\boldsymbol{t}}\boldsymbol{\epsilon}(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}})$

Since φ has a smooth inverse therefore $\vec{r}$ is a reparameterization of $\tilde{r}$ i.e.
$\varphi(\tilde{t}) = t \Rightarrow \tilde{t} = \varphi^{-1}(t)$

And $\quad \tilde{r}(\tilde{t}) = \vec{r}[\varphi(\tilde{t})] \Rightarrow \tilde{r}(\varphi^{-1}(t)) = \vec{r}[\varphi(\varphi^{-1}(t))] \Rightarrow \tilde{r}(\varphi^{-1}(t)) = \vec{r}(t)$

QUESTION (BS;2018): Find the unit speed reparameterization of logarithmic spiral

$\vec{r}(t) = [e^t Cost, e^t Sint]$ or $\vec{r}(t) = [e^{kt} Cost, e^{kt} Sint]$

Answer: Given $\vec{r}(t) = [e^t Cost, e^t Sint]$

$\Longrightarrow \vec{r}'(t) = \dfrac{d\vec{r}}{dt} = [-e^t Sint + e^t Cost, \quad e^t Cost + e^t Sint]$

$\Longrightarrow \|\vec{r}'(t)\| = \sqrt{e^{2t}(Cost - Sint)^2 + e^{2t}(Sint + Cost)^2} \Longrightarrow \|\vec{r}'(t)\| = e^t\sqrt{1+1} = \sqrt{2}e^t$

Then for arc length $s(t) = \int_{t_0}^{t} \|\vec{r}'(u)\| du \Longrightarrow s(t) = \int_0^t \sqrt{2}e^u du \Longrightarrow s(t) = \sqrt{2}\int_0^t e^u du$

$\Longrightarrow s(t) = \sqrt{2}|e^u|_0^t \Longrightarrow s(t) = \sqrt{2}(e^t - 1) \Longrightarrow \frac{s(t)}{\sqrt{2}} = (e^t - 1) \Longrightarrow e^t = \frac{s(t)}{\sqrt{2}} + 1$

$\Longrightarrow t = ln\left(\frac{s(t)}{\sqrt{2}} + 1\right)$

Now unit speed reparameterization of curve is

$\tilde{r}(s) = \left[\left(\frac{s(t)}{\sqrt{2}} + 1\right) Cos\left(ln\left(\frac{s(t)}{\sqrt{2}} + 1\right)\right) \;,\; \left(\frac{s(t)}{\sqrt{2}} + 1\right) Cos\left(ln\left(\frac{s(t)}{\sqrt{2}} + 1\right)\right)\right]$

$\Longrightarrow \|\tilde{r}'(s)\| = 1$ and it is unit speed reparameterization curve.

PRACTICE: Find unit speed reparameterization of the followings;

- **i.** $\vec{r}(t) = [e^t Cost \,, e^t Sint, e^t\,]$
- **ii.** $\vec{r}(t) = [Cosht \,, Sinht \,, t\,]$
- **iii.** $\vec{r}(t) = [rCost, rSint, \; ht] \quad ; r > 0, h > 0 \,, 0 \le t < \infty$
- **iv.** $\vec{r}(t) = [e^{kt} Cost, e^{kt} Sint]$
- **v.** Find reparameterization of the regular curve
  $x = e^t[a(Cost)e_1 + a(Sint)e_2 + be_3] \,; -\infty < t < \infty$

NATURAL REPRESENTATION: when a curve is presented by its arc length then the representation is called natural, Since the speed of such curve is 1 it is therefore called unit speed curve.

QUESTION: Obtained the equation of circular helix $\vec{r}(t) = [rCost, rSint, \; ht] \quad ; r > 0, h > 0 \,, 0 \le t < \infty$ referred to arc length as a parameter and show that length of one complete turn of the helix is $2\pi c$ where $c = \sqrt{a^2 + b^2}$ and is called pitch.

Answer: Given $\vec{r}(t) = [rCost, rSint, \; ht] \quad ; r > 0, h > 0 \,, 0 \le t < \infty \; \ldots..(i)$

$\Longrightarrow \vec{r}'(t) = \frac{d\vec{r}}{dt} = [-rSint, rCost, \; h] \;\; \Longrightarrow \|\vec{r}'(t)\| = \sqrt{a^2 + b^2} = c$

Then for arc length $s(t) = \int_0^t \|\vec{r}'(t)\| dt \Longrightarrow s(t) = \int_0^t c dt = ct \Longrightarrow t = \frac{s}{c}$

$(i) \Longrightarrow \vec{r}(t) = \left[rCos\left(\frac{s}{c}\right), rSin\left(\frac{s}{c}\right), \; h\left(\frac{s}{c}\right)\right]$ this is required parameterization.

For one completer turn 't' varies from $0 \; to \; 2\pi \;\; \Longrightarrow s(t) = ct = 2\pi c$ where $c = \sqrt{a^2 + b^2}$

REGULAR POINT AND SINGULAR POINT: a point $\vec{r}(t)$ of a parameterized curve $\vec{r}$ is called regular point if $\vec{r}'(t) \neq \vec{0}$ otherwise $\vec{r}(t)$ is called singular point of $\vec{r}$.

Remember: as $s(t) = \int_{t_0}^{t} \|\vec{r}'(u)\| du \Longrightarrow \frac{ds}{dt} = \|\vec{r}'(u)\|$

REMARK: A curve is regular if all of its points are regular.

Example: Show that the representation $\vec{r}(t) = t\, e_1 + (t^2+2)\, e_2 + (t^3+t)\, e_3$ is regular for all 't' and sketch the projection on the $\vec{r}_1\,\vec{r}_3$ and $\vec{r}_3\,\vec{r}_1$ planes.

Answer: If $\vec{r}(t) = t\, e_1 + (t^2+2)\, e_2 + (t^3+t)\, e_3$

$\Longrightarrow \vec{r}'(t) = e_1 + 2t\, e_2 + (3t^2+1)\, e_3 \Longrightarrow \|\vec{r}'(t)\| = \sqrt{1+(2t)^2+(3t^2+1)^2} \neq 0\ \forall t$
$\Longrightarrow$ the representation is regular. And the projection onto the $\vec{r}_1\,\vec{r}_3$ plane is
$\vec{r}_1 = t,\ \vec{r}_2 = 0,\ \vec{r}_3 = t^3+t$ or $\vec{r}_3 = \vec{r}_1^{\ 3} + \vec{r}_1,\ \vec{r}_2 = 0$ also the projection onto the $\vec{r}_3\,\vec{r}_1$ plane is $\vec{r}_1 = t,\ \vec{r}_2 = t^2+2,\ \vec{r}_3 = 0$ or $\vec{r}_2 = \vec{r}_1^{\ 2} + 2,\ \vec{r}_3 = 0$

PRACTICE: Determine whether the following curves are regular?

i. Circular helix $\vec{r}(t) = [rCost, rSint,\ ht]\quad ; r > 0, h > 0$

ii. $\vec{r}(t) = [t^2, 0, 0]$

iii. $\vec{r}(t) = [t^3+t, 0, 0]$

iv. $\vec{r}(t) = [Cos\theta, 1 - Cos\theta - Sin\theta, -Sin\theta]$

v. $\vec{r}(t) = [2Sin^2\theta, 2Sin^2\theta Tan\theta, 0]$

vi. $\vec{r}(t) = [Cos\theta, Cos^2\theta, Sin\theta]$

vii. $(\boldsymbol{BS}; \mathbf{2018})\ \sigma(u,v) = [u+v, u-v, u^2+v^2]$

QUESTION: Find the singular point of the hypocycloid and sketch

$x_1 = (r_0 - r)Cos\theta + rCos\left(\left(\frac{r_0-r}{r}\right)\theta\right), x_2 = (r_0 - r)Sin\theta - rSin\left(\left(\frac{r_0-r}{r}\right)\theta\right)$ with $r_0 = 5, r = 2$

Answer: If $x_1 = (r_0 - r)Cos\theta + rCos\left(\left(\frac{r_0-r}{r}\right)\theta\right), x_2 = (r_0 - r)Sin\theta - rSin\left(\left(\frac{r_0-r}{r}\right)\theta\right)$

$\Rightarrow \boldsymbol{x_1(r_0 = 5, r = 2) = \left(3Cos\theta + 2Cos\tfrac{3}{2}\theta\right)}$ ***and*** $\boldsymbol{x_2(r_0 = 5, r = 2) = \left(3Sin\theta - 2Sin\tfrac{3}{2}\theta\right)}$

$\Longrightarrow x_1'(\theta) = -3\left(Sin\theta + Sin\frac{3}{2}\theta\right)$ *and* $x_2'(\theta) = 3\left(Cos\theta - Cos\frac{3}{2}\theta\right)$

$\Longrightarrow |r'(\theta)| = \sqrt{(x_1')^2 + (x_2')^2} = 6Sin\frac{5}{4}\theta$

And for singular points $|r'(\theta)| = 0 \Longrightarrow 6Sin\frac{5}{4}\theta = 0 \Longrightarrow \theta = \left(\frac{4}{5}\right)n\pi\ ; n = 0 \pm 1, \pm 2 \ldots \ldots$

PREPOSITION: Any reparameterization of a regular curve is regular.

PROOF: Suppose $\tilde{r}: (\tilde{\alpha}, \tilde{\beta}) \to R^n$ is a reparameterization of a regular curve $\vec{r}: (\alpha, \beta) \to R^n$ then there exist smooth bijective mapping $\varphi: (\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta)$ whose inverse $\varphi^{-1}: (\alpha, \beta) \to (\tilde{\alpha}, \tilde{\beta})$ is also smooth and $\tilde{r}(\tilde{t}) = \vec{r}[\varphi(\tilde{t})]$

Now let $t = \varphi(\tilde{t})$ and $\Psi = \varphi^{-1}$ then $\tilde{t} = \varphi^{-1}(t) \Longrightarrow \tilde{t} = \Psi(t)\ \ldots\ldots\ldots (i)$

Consider $\varphi\left(\varphi^{-1}(t)\right) = t \Longrightarrow \varphi(\Psi(t)) = t \ldots\ldots\ldots (ii)$

Now $\varphi(\tilde{t}) = t \Longrightarrow \frac{d\varphi(\tilde{t})}{dt} = 1 \Longrightarrow \frac{d\varphi(\tilde{t})}{d\tilde{t}}.\frac{d\tilde{t}}{dt} = 1 \Longrightarrow \frac{d\varphi(\tilde{t})}{d\tilde{t}}.\frac{d\Psi}{dt} = 1 \Longrightarrow \frac{d\varphi(\tilde{t})}{d\tilde{t}} \neq 0,, \frac{d\Psi}{dt} \neq 0$
now we have to show that $\frac{d\tilde{r}}{d\tilde{t}} \neq 0\quad \forall\, \tilde{t}\epsilon\left(\tilde{\alpha}, \tilde{\beta}\right)$

For this consider $\tilde{r}(\tilde{t}) = \vec{r}[\varphi(\tilde{t})] \Longrightarrow \frac{d\tilde{r}}{d\tilde{t}} = \frac{d\vec{r}}{d\tilde{t}} \Longrightarrow \frac{d\tilde{r}}{d\tilde{t}} = \frac{d\vec{r}}{dt}.\frac{dt}{d\tilde{t}} \Longrightarrow \frac{d\tilde{r}}{d\tilde{t}} = \frac{d\vec{r}}{dt}.\frac{d\varphi}{d\tilde{t}}$
now since $\frac{d\vec{r}}{dt} \neq 0$ *and* $\frac{d\varphi}{d\tilde{t}} \neq 0 \Longrightarrow \frac{d\tilde{r}}{d\tilde{t}} \neq 0\quad \forall\, \tilde{t}\epsilon\left(\tilde{\alpha}, \tilde{\beta}\right)$ hence proved.

PREPOSITION: if $\vec{r}(t)$ is a regular curve, its arc length is 's' starting at any point $\vec{r}$, is a smooth function of 't' .

PROOF: Given that $\vec{r}(t)$ is regular and $s(t) = \int_{t_0}^{t} \|\vec{r}'(u)\| du$ is its arc length

$\Rightarrow \frac{ds}{dt} = \|\vec{r}'(t)\| > 0 \qquad \therefore \vec{r}(t) \text{ is regular}$

Assume that $\vec{r}(t) = \big(u(t), v(t)\big)$ is plane curve.

$\Rightarrow \vec{r}'(t) = \frac{d\vec{r}}{dt} = \big(u'(t), v'(t)\big) \qquad \therefore u \;\&v \text{ are smooth functions}$

$\Rightarrow \frac{ds}{dt} = \|\vec{r}'(t)\| = \sqrt{(u')^2 + (v')^2} > 0$

Since $u \;\&v$ *are smooth functions* then $u'(t), v'(t)$ and $(u')^2 + (v')^2$ are also smooth functions of t

$\Rightarrow \frac{ds}{dt} = \|\vec{r}'(t)\| = \sqrt{(u')^2 + (v')^2}$ is also a smooth function.

Now $\vec{r}''(t) = \frac{d^2 s}{dt^2} = \frac{u'.u''}{\sqrt{(u')^2+(v')^2}} + \frac{v'.v''}{\sqrt{(u')^2+(v')^2}}$ which exists, similarly for derivatives, we have smooth functions of 't' This the completer proof.

PREPOSITION: a parameterized curve has a unit speed reparameterization if and only if it is regular.

PROOF: Suppose $\tilde{r}: (\tilde{\alpha}, \tilde{\beta}) \rightarrow R^n$ be a unit speed reparameterization of a regular curve $\vec{r}: (\alpha, \beta) \rightarrow R^n$ i.e. $\|\tilde{r}'(t)\| = 1$ then there exist smooth bijective mapping $\varphi: (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ which has a smooth inverse $\varphi^{-1}: (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$ is also smooth such that
$\tilde{r}(\tilde{t}) = \vec{r}[\varphi(\tilde{t})] \ldots\ldots\ldots (i)$

Now let $\varphi(\tilde{t}) = t \Rightarrow \frac{d\varphi(\tilde{t})}{dt} = 1 \Rightarrow \frac{d\varphi}{d\tilde{t}} . \frac{d\tilde{t}}{dt} = 1 \ldots\ldots\ldots (ii)$

$(i) \Rightarrow \frac{d\tilde{r}(\tilde{t})}{d\tilde{t}} = \frac{d\vec{r}}{d\tilde{t}} \Rightarrow \frac{d\tilde{r}(\tilde{t})}{d\tilde{t}} = \frac{d\vec{r}}{dt} . \frac{dt}{d\tilde{t}} \Rightarrow \frac{d\tilde{r}(\tilde{t})}{d\tilde{t}} = \frac{d\vec{r}}{dt} . \frac{d\varphi}{d\tilde{t}} \qquad \therefore \varphi(\tilde{t}) = t \Rightarrow \frac{d\varphi(\tilde{t})}{d\tilde{t}} = \frac{dt}{d\tilde{t}}$

$\Rightarrow \left\|\frac{d\tilde{r}}{d\tilde{t}}\right\| = \left\|\frac{d\vec{r}}{dt} . \frac{d\varphi}{d\tilde{t}}\right\| \Rightarrow \|\tilde{r}'(t)\| = \left\|\frac{d\vec{r}}{dt}\right\| \left\|\frac{d\varphi}{d\tilde{t}}\right\| \Rightarrow 1 = \left\|\frac{d\vec{r}}{dt}\right\| \left\|\frac{d\varphi}{d\tilde{t}}\right\|$

$\Rightarrow \left\|\frac{d\vec{r}}{dt}\right\| \left\|\frac{dt}{d\tilde{t}}\right\| = 1 \neq 0 \Rightarrow \left\|\frac{d\vec{r}}{dt}\right\| \neq 0 \Rightarrow \frac{d\vec{r}}{dt} \neq 0 \quad ; \forall t \Rightarrow \vec{r}'(t) \neq 0 \quad ; \forall t \Rightarrow$ $\boldsymbol{\vec{r}(t)}$ **is regular**.

CONVERSLY: Suppose that $\vec{r}(t)$ is regular. $\Rightarrow \frac{d\vec{r}}{dt} = \vec{r}'(t) \neq 0 \quad ; \forall t \Rightarrow \frac{ds}{dt} = \|\vec{r}'(t)\| > 0$

$\Rightarrow s: (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$ has a local smooth inverse $s^{-1}: (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$

If $\tilde{r}$ is the corresponding reparameterization then $\tilde{r}\big(s(t)\big) = \vec{r}(t)$

$\Rightarrow \frac{d\tilde{r}}{dt} = \frac{d\vec{r}}{dt} \Rightarrow \frac{d\tilde{r}}{ds} . \frac{ds}{dt} = \frac{d\vec{r}}{dt} \Rightarrow \left\|\frac{d\tilde{r}}{ds} . \frac{ds}{dt}\right\| = \left\|\frac{d\vec{r}}{dt}\right\| \Rightarrow \left\|\frac{d\tilde{r}}{ds}\right\| \left\|\frac{ds}{dt}\right\| = \|\vec{r}'(t)\|$

$\Rightarrow \|\tilde{r}(t)\| \|\vec{r}'(t)\| = \|\vec{r}'(t)\| \Rightarrow \|\tilde{r}(t)\| = 1 \Rightarrow$ $\boldsymbol{\tilde{r}(t)}$ **is a unit speed curve**.

QUESTION: Show that given is regular also find unit speed reparameterization of curve where curve is $\vec{r}(t) = [Cos^2 t, Sin^2 t] \quad ; t\epsilon R$

Answer: Given $\vec{r}(t) = [Cos^2 t, Sin^2 t]$

$\Rightarrow \vec{r}'(t) = \frac{d\vec{r}}{dt} = [-2CostSint, \quad 2SintCost] = [-Sin2t, \quad Sin2t] = Sin2t[-1, \quad 1]$

Now as $t\epsilon R$ so it can be any multiple of $\frac{\pi}{2}$ and on $t = \frac{\pi}{2}$ we have $\vec{r}'(t) = 0$ so given curve is not regular. And we have no need to find unit speed reparameterization of curve.

QESTION: Show that given is regular also find unit speed reparameterization of curve where curve is $\vec{r}(t) = [Cos^2t, Sin^2t]$; $for\ 0 < t < \frac{\pi}{2}$

Answer: Given $\vec{r}(t) = [Cos^2t, Sin^2t]$

$\Longrightarrow \vec{r}'(t) = \frac{d\vec{r}}{dt} = [-2CostSint, \quad 2SintCost] = [-Sin2t, \quad Sin2t] = Sin2t[-1, \quad 1]$

Now as $0 < t < \frac{\pi}{2}$ we have $\vec{r}'(t) \neq 0$ so given curve is **regular**. And we will find unit speed reparameterization of curve. $\Longrightarrow \vec{r}'(t) = \frac{d\vec{r}}{dt} = [-Sin2t, \quad Sin2t]$

$\Longrightarrow \|\vec{r}'(t)\| = \sqrt{Sin^2 2t + Sin^2 2t} = \sqrt{2Sin^2 2t} = \sqrt{2}Sin2t \Longrightarrow \|\vec{r}'(t)\| = \sqrt{2}Sin2t$

Then for arc length $s(t) = \int_{t_0}^{t} \|\vec{r}'(u)\| du \Longrightarrow s(t) = \int_0^t \sqrt{2}Sin2udu \Longrightarrow s(t) = \sqrt{2}\int_0^t Sin2udu$

$\Longrightarrow s(t) = \sqrt{2}\left|\frac{-Cos2u}{2}\right|_0^t \Longrightarrow s(t) = \sqrt{2}\left(\frac{-Cos2t}{2} + \frac{Cos20}{2}\right) = \frac{\sqrt{2}}{2}(1 - Cos2t)$

$\Longrightarrow s(t) = \frac{1}{\sqrt{2}}(1 - Cos2t) \Longrightarrow \sqrt{2}s(t) = 2Sin^2t \Longrightarrow \frac{\sqrt{2}}{2}s(t) = Sin^2t \Longrightarrow Sin^2t = \frac{1}{\sqrt{2}}s(t)$

now since $Cos^2t = 1 - Sin^2t = 1 - \frac{1}{\sqrt{2}}s$

Hence Reparameterization curve is $\tilde{r}(s) = \left[1 - \frac{1}{\sqrt{2}}s \, , \, \frac{1}{\sqrt{2}}s\right]$

QUESTION: Show that given is regular also find unit speed reparameterization of curve where curve is $\vec{r}(t) = [t, Cosht]$; $for\ t\epsilon R$

Answer: Given $\vec{r}(t) = [t, Cosht]$ $\Longrightarrow \vec{r}'(t) = \frac{d\vec{r}}{dt} = [1, \quad Sinht]$

Now as $t\epsilon R$ we have $\dot{\gamma}(t) \neq 0$ so given curve is **regular**. And we will find unit speed reparameterization of curve. $\Longrightarrow \vec{r}'(t) = \frac{d\vec{r}}{dt} = [1, \quad Sinht]$

$\Longrightarrow \|\vec{r}'(t)\| = \sqrt{1 + Sinh^2t} = \sqrt{Cosh^2t} = Cosht \Longrightarrow \|\vec{r}'(t)\| = Cosht$

Then for arc length $s(t) = \int_{t_0}^{t} \|\vec{r}'(u)\| du \Longrightarrow s(t) = \int_0^t Coshudu$

$\Longrightarrow s(t) = |Sinhu|_0^t \Longrightarrow s(t) = Sinht \Longrightarrow s(t) = \frac{e^t - e^{-t}}{2} \Longrightarrow e^{2t} - 1 = 2se^t$

$\Longrightarrow (e^t)^2 - 2se^t - 1 = 0 \Longrightarrow e^t = \frac{2s \pm \sqrt{4s^2 - 4}}{2} = s \pm \sqrt{s^2 - 1} \Longrightarrow t = ln\left(s \pm \sqrt{s^2 - 1}\right)$

$\Longrightarrow Cosh^2u = 1 + Sinh^2u \Longrightarrow Coshu = \sqrt{1 + s^2}$

Hence Reparameterization curve is $\tilde{r}(s) = \left[ln\left(s \pm \sqrt{s^2 - 1}\right), \, \sqrt{1 + s^2}\right]$

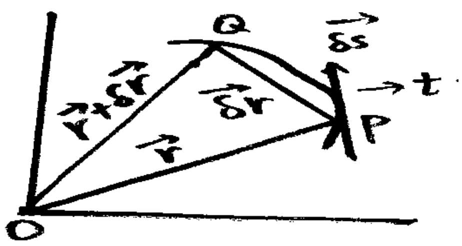
VELOCITY VECTOR: The velocity vector of regular curve $\vec{r} = \vec{r}(t)$ at $t = t_0$ is the derivative $\vec{r}'(t)$ evaluated at $t = t_0$ and the velocity vector field is the vector $\vec{r}'(t)$. The speed of $\vec{r}(t)$ at $t = t_0$ is the length of the velocity vector i.e. $|\vec{r}'(t)|_{t=t_0}$

PLANE CURVES: the set of all points lies in same plane then the curves are called Plane curves.

SPACE CURVE OR TWISTED CURVE OR SKEW CURVE OR TORTOUS CURVE:
when all the points of a curve do not lie in the same plane then it is said to be a space curve otherwise a plane curve.

Example: $\vec{r}(t) = aCost\hat{\imath} + bSint\hat{\jmath} + 0\hat{k}$ comparing this equation with general equation of the curve $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ then we get $x = aCost$, $y = bSint$, $z = 0$ then $\frac{x^2}{a^2} = Cos^2t$ ………(i) $\frac{y^2}{b^2} = Sin^2t$ ………(ii) then $\frac{x^2}{a^2} + \frac{y^2}{b^2} = Cos^2t + Sin^2t = 1$ which is the equation of ellipse.

EQUATION OF TANGENT OF A PLANE TO THE CURVE



Let ‘p’ and ‘θ’ be two points on the given curve ‘C’ whose position vectors are $\vec{r}$ and $\vec{r}+\delta\vec{r}$ corresponding to the values of $'s'$ and "$s+\delta s$" of parameters. Then we note that $\vec{r}' = \frac{dr}{ds} = \lim_{\delta s \to \delta r} \frac{\delta r}{\delta s} = \vec{t}$ is the unit vector parallel to the tangent to the curve at P. this unit vector is denoted by $\vec{t}$ and is called **unit tangent**. The equation of tangent to the curve at any point P is $\vec{R}-\vec{r} = u\vec{t} \implies \vec{R} = \vec{r}+u\vec{t} \implies \vec{R} = \vec{r}+u\vec{r}'$ where $\vec{R}$ is the position vector of any point on the tangent ‘t’ and is called the current point and ‘u’ is any scalar.

Question: Find the equation of the tangent to the curve whose coordinates are $x = aCos\theta\ ,\ y = aSin\theta\ ,\ z = 0$.

Solution: Since $\vec{r}'(t) = x'(t)\hat{\imath} + y'(t)\hat{\jmath} + z'(t)\hat{k}$ and $x' = -aSin\theta\theta'$, $y' = aCos\theta\theta'$, $z' = 0$ so using these in equation of tangent

$\vec{R}-\vec{r} = u\vec{t} \implies \vec{R} = \vec{r}+u\vec{t} \implies \vec{R} = \vec{r}+u\vec{r}' \implies \vec{R}-\vec{r} = u\vec{r'} \implies \frac{\vec{R}-\vec{r}}{\vec{r}'} = u$ ....(i)

now $\vec{R} = X\hat{\imath} + Y\hat{\jmath} + Z\hat{k}$ and $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

then equation (i) becomes $\frac{X-x}{x'} = \frac{Y-y}{y'} = \frac{Z-z}{z'} = u$

$\implies \frac{X-aCos\theta}{-aSin\theta\theta'} = \frac{Y-aSin\theta}{aCos\theta\theta'} = \frac{Z-0}{0} = u \qquad \implies \frac{Z-0}{0} = u = \infty$

and $\frac{X-aCos\theta}{-Sin\theta} = \frac{Y-aSin\theta}{Cos\theta} \implies (X-aCos\theta)Cos\theta = (Y-aSin\theta)(-Sin\theta)$

$\implies XCos\theta + YSin\theta = aCos^2\theta + aSin^2\theta \qquad \implies XCos\theta + YSin\theta = a$

Question: Find the equation of the tangent vector and tangent line to the curve $x = te_1 + t^2e_2 + t^3e_3$ at $t = 1 \implies x(t=1) = e_1+e_2+e_3$

Solution: Since $x'(t) = e_1 + 2te_2 + 3t^2e_3 \implies x'(t=1) = e_1 + 2e_2 + 3e_3$

$\implies |x'(t=1)| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$ and $T = \frac{x'(t=1)}{|x'(t=1)|} = \frac{1}{\sqrt{14}}(e_1+2e_2+3e_3)$

Now $X - x = uT \implies X = x + uT$ where $-\infty < u < \infty$

$\implies X = (e_1+e_2+e_3) + u\left[\frac{1}{\sqrt{14}}(e_1+2e_2+3e_3)\right]$

$\implies X = \left(1+\frac{k}{\sqrt{14}}\right)e_1 + \left(1+\frac{2k}{\sqrt{14}}\right)e_2 + \left(1+\frac{3k}{\sqrt{14}}\right)e_3 = (1+t)e_1 + (1+2t)e_2 + (1+3t)e_3$

$\therefore \frac{k}{\sqrt{14}} = t\ (say)$ Then in Cartesian coordinate we get parametric form

$\implies X = (x_1, x_2, x_3) = (1+t), \quad (1+2t), \quad (1+3t)$

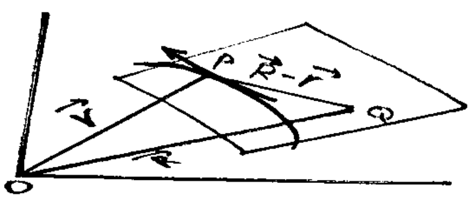
$\implies x_1 = (1+t), \quad x_2 = (1+2t), \quad x_3 = (1+3t)$

$\implies x_1 - 1 = t, \quad \frac{x_2-1}{2} = t, \quad \frac{x_3-1}{3} = t$

$\implies x_1 - 1 = \frac{x_2-1}{2} = \frac{x_3-1}{3}$ is the tangent line to the curve at t = 1 i.e. at $P(1,1,1)$

NORMAL PLANE: the plane normal to the tangent to a curve at the point 'P' of contact is called the normal plane at that point P.

EQUATION OF NORMAL PLANE:



let $\vec{R}$ be the position vector of the current point 'Q' on the normal plane and $\vec{r}$ be the position vector of the current point 'P' thus $\vec{R}-\vec{r}$ is the position vector of any line in the plane as shown in figure. Thus according to the definition of the normal plane $\vec{R}-\vec{r}$ and $\vec{t}$ are perpendicular to each other. i.e. $(\vec{\boldsymbol{R}}-\vec{\boldsymbol{r}}).\vec{\boldsymbol{t}}=\mathbf{0}$ Which is the equation of normal plane. Thus every line through P in this plane is normal to the curve.

Question: Let $\propto(s)$ be a unit speed curve such that every normal plane to $\propto(s)$ goes through a given fixed point $x_0 \epsilon R^3$ then the image of $\propto(s)$ lies on sphere.

Solution: Since the equation of normal plane is $(\vec{R}-\vec{r}).\vec{t}=0 \Rightarrow (x_0-\propto(s)).\vec{t}=0$
$(\propto(s)-x_0).\vec{t}=0$ and $\Rightarrow (\propto(s)-x_0).\vec{r}'=0 \Rightarrow (\propto(s)-x_0).\propto'(s)=0$

Where $\left(\propto_1(s)-x_{0_1}, \propto_2(s)-x_{0_2}, \propto_3(s)-x_{0_3}\right).\left(\propto_1'(s), \propto_2'(s), \propto_3'(s)\right)=0$

$$\Rightarrow \left[\left(\propto_1(s)-x_{0_1}\right).\propto_1'(s)+\left(\propto_2(s)-x_{0_2}\right).\propto_2'(s)+\left(\propto_3(s)-x_{0_3}\right).\propto_3'(s)\right]=0$$

Integrating w.r.to 's' $\Rightarrow \frac{\left(\propto_1(s)-x_{0_1}\right)^2}{2}+\frac{\left(\propto_2(s)-x_{0_2}\right)^2}{2}+\frac{\left(\propto_3(s)-x_{0_3}\right)^2}{2}=costant$

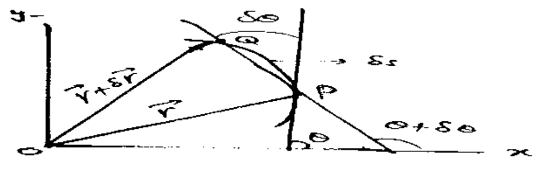
$$\Rightarrow \left(\propto_1(s)-x_{0_1}\right)^2+\left(\propto_2(s)-x_{0_2}\right)^2+\left(\propto_3(s)-x_{0_3}\right)^2=2\times costant=costant=a^2(say)$$

This is the required equation of curve lying on sphere.

CURVATURE: (it tells how much does a curve, curve)
curvature of the curve at any point is defined as **'the arc rate of rotation of the tangent"** it is the ratio of change in turning to the distance travelled. It is sometime called first curvature or circular curvature. It is denoted by $\boldsymbol{K(keppa)}$**.** It measures the extent to which a curve is not contained in a straight line.

OR if $\vec{\boldsymbol{r}}$ is a unit speed curve with parameter 's' then its curvature denoted $K(s)$ at point $\vec{r}(s)$ is defined as $\boldsymbol{K(s)}=\|\vec{\boldsymbol{r}}''(\boldsymbol{s})\|$ Also remember: $\kappa=\left|\frac{dT}{ds}\right|=\left|\frac{dT/dt}{ds/dt}\right|=\left|\frac{\vec{T}'}{\vec{r}'}\right|$

DERIVATION OF EXPRESSION FOR CURVATURE:



let 'C' be a curve and 'CX' be a fixed direction. Let position vectors of 'P' and 'Q' are $\vec{r}$ and $\vec{r}+\delta\vec{r}$ respectively. Let tangent at 'P' makes angle 'θ' with x-axis or OX and tangent at 'Q' makes angle '$\theta+\delta\theta$' with OX. Where the angle '$\delta\theta$' is the angle between tangent at P and Q and clearly $\widehat{PQ}=\delta s$ then $\frac{\delta\theta}{\delta s}$ is average curvature of arc $\widehat{PQ}$ when $\delta s \to 0$ then its limiting value is the curvature at the point P. i.e. $\boldsymbol{K}=\lim_{\delta s\to 0}\frac{\delta\theta}{\delta s}=\frac{d\theta}{ds}=\boldsymbol{\theta}'$

REMARK:

- The circle passing through three points on the curve coincident at P is called **circle of curvature** at P. it is the circle that best describes how C behave near P; it shares the same tangent, normal and curvature at P.
- $\boldsymbol{K}(\boldsymbol{keppa})$ is considered as the positive quantity.
- the reciprocal of curvature is called **Radius of Curvature** and is denoted by $'\rho'$ so $\rho = \frac{1}{K}$ and $K = \frac{1}{\rho}$
- if $K = 0$ or $\|\vec{r}''(s)\| = 0$ everywhere then $\vec{r}$ is a part of straight line.
- curvature of straight line should be zero.
- Curvature should be independent of reparameterization. It depends only on the shape of curve
- Large circle should have small curvature and vice versa.

QUESTION: if $K = 0$ or $\|\vec{r}''(s)\| = 0$ everywhere then $\vec{r}$ is a part of straight line.

ANSWER: if $K = 0 \Longrightarrow \|\vec{r}''(s)\| = 0 \Longrightarrow \vec{r}''(s) = 0 \Longrightarrow \vec{r}'(s) = c \Longrightarrow \vec{r}(s) = cs + b$
$\Longrightarrow$ curve is a straight line.

QUESTION: Curvature of straight line should be zero.

ANSWER: let $\vec{r}(s)$ curve is a straight line
$\Longrightarrow \vec{r}(s) = as + b \Longrightarrow \vec{r}'(s) = a \Longrightarrow \vec{r}''(s) = 0 \Longrightarrow \|\vec{r}''(s)\| = K = 0$.

QUESTION (BS;2016) :

A regular curve of class ≥ 2 is a straight line iff its curvature is identically zero.

ANSWER: let $\vec{r}(s)$ curve is a straight line
$\Longrightarrow \vec{r}(s) = as + b \Longrightarrow \vec{r}'(s) = a \Longrightarrow \vec{r}''(s) = 0 \Longrightarrow \|\vec{r}''(s)\| = K = 0$.
Conversely: if $K = 0 \Longrightarrow \|\vec{r}''(s)\| = 0 \Longrightarrow \vec{r}''(s) = 0 \Longrightarrow \vec{r}'(s) = c \Longrightarrow \vec{r}(s) = cs + b$
$\Longrightarrow$ curve is a straight line.

- Possible Question: A curve of class ≥ 2 is straight line if all tangent lines have a common intersection.

QUESTION: Prove that $\rho = \frac{1}{K}$ or $K = \frac{1}{\rho}$.

PROOF: Consider a circle in R^2 with centre at $(0,0)$ and radius is ρ and this will be a unit speed parameterization. i.e.

$$\vec{r}(s) = \left[\rho Cos\left(\frac{s}{\rho}\right), \rho Sin\left(\frac{s}{\rho}\right)\right] \quad \because \vec{r}(s) = [rCos\theta, rSin\theta] \ and \ s = r\theta \Longrightarrow s = \rho\theta \Longrightarrow \theta = \frac{s}{\rho}$$

$$\Longrightarrow \vec{r}'(s) = \left[-\rho.\frac{1}{\rho}.Sin\left(\frac{s}{\rho}\right),\ \rho.\frac{1}{\rho}Cos\left(\frac{s}{\rho}\right)\right] \Longrightarrow \vec{r}''(s) = \left[-\frac{1}{\rho}Cos\left(\frac{s}{\rho}\right),\ -\frac{1}{\rho}Sin\left(\frac{s}{\rho}\right)\right]$$

$$\Longrightarrow K = \|\vec{r}''(s)\| = \sqrt{\frac{1}{\rho^2}Cos^2\left(\frac{s}{\rho}\right) + \frac{1}{\rho^2}Sin^2\left(\frac{s}{\rho}\right)} = \sqrt{\frac{1}{\rho^2}\left[Cos^2\left(\frac{s}{\rho}\right) + Sin^2\left(\frac{s}{\rho}\right)\right]} = \frac{1}{\rho} \Longrightarrow \boldsymbol{K} = \frac{\mathbf{1}}{\boldsymbol{\rho}}$$

PREPOSITION (1st method) (BS;2018):

let $r(t)$ be a regular curve in R^3 then its curvature is $K = \frac{\|\dot{r}\times\ddot{r}\|}{\|\dot{r}\|^3}$ or $K = \frac{\|r''\times r'\|}{\|r'\|^3}$. **OR** let $\gamma(t)$ be a regular curve in R^3 then its curvature is $K = \frac{\|\ddot{\gamma}\times\dot{\gamma}\|}{\|\dot{\gamma}\|^3}$

PROOF: Let $\vec{r}$ $be\ a\ regular\ curve \Rightarrow \vec{r}' = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}.\frac{ds}{dt} \Rightarrow \vec{r}' = \frac{d\vec{r}}{dt}.\frac{ds}{d\vec{r}} = \frac{ds}{dt} \Rightarrow \vec{r}' = \frac{ds}{dt}$

Now $K = \|\vec{r}''\| = \left\|\frac{d^2\vec{r}}{ds^2}\right\| = \left\|\frac{d}{ds}\left(\frac{d\vec{r}}{ds}\right)\right\| = \left\|\frac{d}{ds}\left(\frac{d\vec{r}/dt}{ds/dt}\right)\right\| = \left\|\frac{d}{dt}\left(\frac{d\vec{r}/ds}{ds/dt}\right)\right\| = \left\|\frac{\frac{ds}{dt}\frac{d}{dt}\left(\frac{d\vec{r}}{ds}\right)-\frac{d\vec{r}}{ds}\frac{d}{dt}\left(\frac{ds}{dt}\right)}{\left(\frac{ds}{dt}\right)^2}\right\|$

$$K = \left\|\frac{\frac{ds}{dt}\cdot\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\cdot\frac{dt}{ds}\right)-\frac{d\vec{r}}{dt}\cdot\frac{dt}{ds}\left(\frac{d^2s}{dt^2}\right)}{\left(\frac{ds}{dt}\right)^2}\right\| = \left\|\frac{\frac{ds}{dt}\cdot\frac{d^2\vec{r}}{dt^2}\cdot\frac{dt}{ds}-\vec{r}'\cdot\frac{dt}{ds}\cdot\left(\frac{d^2s}{dt^2}\right)}{\left(\frac{ds}{dt}\right)^2}\right\| = \left\|\frac{\frac{ds}{dt}\cdot\vec{r}''-\vec{r}'\cdot\left(\frac{d^2s}{dt^2}\right)}{\left(\frac{ds}{dt}\right)^3}\right\| \ldots\ldots\ldots(i)$$

Since $\vec{r}' = \frac{ds}{dt} \Rightarrow (\vec{r}')^2 = \left(\frac{ds}{dt}\right)^2 \Rightarrow \vec{r}'.\vec{r}' = \left(\frac{ds}{dt}\right)^2 \Rightarrow 2\vec{r}''.\vec{r}' = 2\frac{ds}{dt}\left(\frac{d^2s}{dt^2}\right)$

$\Rightarrow \vec{r}''.\vec{r}' = \frac{ds}{dt}\left(\frac{d^2s}{dt^2}\right)$

$$(i) \Rightarrow K = \left\|\frac{\frac{ds}{dt}\left[\frac{ds}{dt}\vec{r}''-\vec{r}'.\left(\frac{d^2s}{dt^2}\right)\right]}{\left(\frac{ds}{dt}\right)^4}\right\| = \left\|\frac{\left(\frac{ds}{dt}\right)^2.\vec{r}''-\vec{r}'.\frac{ds}{dt}\left(\frac{d^2s}{dt^2}\right)}{\left(\frac{ds}{dt}\right)^4}\right\| = \frac{\|(\vec{r}'.\vec{r}').\vec{r}''-\vec{r}'(\vec{r}''.\vec{r}')\|}{\|\vec{r}'\|^4} \ldots\ldots\ldots(ii)$$

Now $\because$ $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c} \Rightarrow \vec{r}'\times(\vec{r}''\times\vec{r}') = (\vec{r}'.\vec{r}')\vec{r}''-(\vec{r}''.\vec{r}')\vec{r}'$

$(ii) \Rightarrow K = \frac{\|(\vec{r}'.\vec{r}').\vec{r}''-\vec{r}'(\vec{r}''.\vec{r}')\|}{\|\vec{r}'\|^4} = \frac{\|\vec{r}'\times(\vec{r}''\times\vec{r}')\|}{\|\vec{r}'\|^4}$

Since $\vec{r}' \perp (\vec{r}''\times\vec{r}')$ therefore

$$K = \frac{\|\vec{r}'\times(\vec{r}''\times\vec{r}')\|}{\|\vec{r}'\|^4} = \frac{\|\vec{r}'\|\|(\vec{r}''\times\vec{r}')\|Sin90^\circ}{\|\vec{r}'\|^4} = \frac{\|(\vec{r}''\times\vec{r}')\|}{\|\vec{r}'\|^3} \Rightarrow \boldsymbol{K} = \frac{\|(\vec{\boldsymbol{r}}''\times\vec{\boldsymbol{r}}')\|}{\|\vec{\boldsymbol{r}}'\|^3}$$

REMARK: if $\vec{r}' \neq 0$ then we can find the curvature of the curve by the above formula.

PREPOSITION (2nd method) (BS;2018):

let $r(t)$ be a regular curve in R^3 then its curvature is $K = \frac{\|\dot{r}\times\ddot{r}\|}{\|\dot{r}\|^3}$ or $K = \frac{\|r''\times r'\|}{\|r'\|^3}$. **OR** let $\gamma(t)$ be a regular curve in R^3 then its curvature is $K = \frac{\|\ddot{\gamma}\times\dot{\gamma}\|}{\|\dot{\gamma}\|^3}$

PROOF: Since $\vec{T} = \frac{\vec{r}'}{|\vec{r}'|} \Rightarrow \vec{r}' = |\vec{r}'|\vec{T}$

$\Rightarrow \vec{r}' = \frac{ds}{dt}\vec{T} \Rightarrow \vec{r}'' = \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\vec{T}'$

$\Rightarrow \vec{r}'\times\vec{r}'' = \left(\frac{ds}{dt}\right)^2(\vec{T}\times\vec{T}')$

$\Rightarrow |\vec{r}'\times\vec{r}''| = \left(\frac{ds}{dt}\right)^2|\vec{T}\times\vec{T}'| \Rightarrow |\vec{r}'\times\vec{r}''| = \left(\frac{ds}{dt}\right)^2|\vec{T}'| \qquad \because |\vec{T}\times\vec{T}'| = |\vec{T}||\vec{T}'|Sin90^\circ = |\vec{T}'|$

$\Rightarrow |\vec{T}'| = \frac{|\vec{r}'\times\vec{r}''|}{\left(\frac{ds}{dt}\right)^2} \Rightarrow |\vec{T}'| = \frac{|\vec{r}'\times\vec{r}''|}{|\vec{r}'|^2}$

$\Rightarrow K|\vec{r}'| = \frac{|\vec{r}'\times\vec{r}''|}{|\vec{r}'|^2}$

$\Rightarrow K = \frac{\|r''\times r'\|}{\|r'\|^3}$

EXAMPLE: (Circular helix with z – axis) if $\vec{r}(\theta) = (aCos\theta, aSin\theta, b\theta)$ then find the curvature of the curve.

SOLUTION: Given $\vec{r}(\theta) = (aCos\theta, aSin\theta, b\theta)$

$\Rightarrow \vec{r}'(\theta) = (-aSin\theta, aCos\theta, b) \Rightarrow \vec{r}''(\theta) = (-aCos\theta, -aSin\theta, 0)$

Then $(\vec{\boldsymbol{r}}'' \times \vec{\boldsymbol{r}}') = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -aCos\theta & -aSin\theta & 0 \\ -aSin\theta & aCos\theta & b \end{vmatrix}$

$\Rightarrow (\vec{\boldsymbol{r}}'' \times \vec{\boldsymbol{r}}') = (-abSin\theta - 0)\hat{\imath} + (0 + abCos\theta)\hat{\jmath} + (-a^2Cos^2\theta - a^2Sin^2\theta)\hat{k}$

$\Rightarrow (\vec{\boldsymbol{r}}'' \times \vec{\boldsymbol{r}}') = (-abSin\theta)\hat{\imath} + (abCos\theta)\hat{\jmath} - a^2\hat{k}$

$\Rightarrow \|(\vec{\boldsymbol{r}}'' \times \vec{\boldsymbol{r}}')\| = \sqrt{a^2b^2Sin^2\theta + a^2b^2Cos^2\theta + a^4} = \sqrt{a^2b^2 + a^4} = \boldsymbol{a}\sqrt{b^2 + a^2}$

Also $\Rightarrow \|\vec{\boldsymbol{r}}'\| = \sqrt{a^2Sin^2\theta + a^2Cos^2\theta + b^2} = \sqrt{a^2 + b^2}$ Then $\boldsymbol{K} = \frac{\|(\vec{\boldsymbol{r}}'' \times \vec{\boldsymbol{r}}')\|}{\|\vec{\boldsymbol{r}}'\|^3} = \frac{\boldsymbol{a}\sqrt{b^2+a^2}}{(\sqrt{a^2+b^2})^3} = \frac{\boldsymbol{a}}{a^2+b^2}$

EXERCISE: Find K where $\boldsymbol{K} = \frac{\|(\vec{\boldsymbol{r}}'' \times \vec{\boldsymbol{r}}')\|}{\|\vec{\boldsymbol{r}}'\|^3}$ or $K = \|\vec{r}''(s)\|$

i. $\vec{r}(t) = (1, Cosht)$

ii. $\vec{r}(t) = (Cos^3t, Sin^3t)$

iii. $\vec{r}(t) = \left(\frac{4}{5}Cost, 1 - Sint, -\frac{3}{5}Cost\right)$

iv. $\vec{r}(t) = \left(\frac{1}{3}(1+t)^{\frac{3}{2}}, \frac{1}{3}(1-t)^{\frac{3}{2}}, \frac{t}{\sqrt{2}}\right)$

v. let $\vec{r} = (x(s), y(s), 0)$ be a unit speed curve then prove that $K = |x'y'' - x''y'|$

vi. $\vec{r}(\theta) = (aCos\theta, aSin\theta, a\theta Cot\beta)$

- **Symbol $\boldsymbol{\tau}$ used is called torsion, discussed later.**

QUESTION: if $\vec{r}(t)$ is a unit speed curve with $K(t) > 0$ and $\tau(t) \neq 0\ \forall t$ show that if $\vec{r}$ is a spherical then $\frac{\tau}{K} = \frac{d}{ds}\left(\frac{K'}{\tau K^2}\right)$ if and only if $\rho^2 + (\rho'\sigma)^2 = r^2$ where r is constant radius of sphere.

SOLUTION: Consider $\frac{\tau}{K} = \frac{d}{ds}\left(\frac{K'}{\tau K^2}\right)$ ……… … (i)

$\therefore \rho = \frac{1}{K} \Rightarrow \rho' = -\frac{K'}{K^2} \Rightarrow -\rho' = \frac{K'}{K^2}$ $also\ \sigma = \frac{1}{\tau} \Rightarrow \tau = \frac{1}{\sigma}$

$(i) \Rightarrow \frac{\tau}{K} = \frac{d}{ds}\left(\frac{K'}{\tau K^2}\right) \Rightarrow \frac{\rho}{\sigma} = -\frac{d}{ds}(\sigma\rho') \Rightarrow \rho = -\sigma\frac{d}{ds}(\sigma\rho') \Rightarrow \rho + \sigma\frac{d}{ds}(\sigma\rho') = 0$

$\Rightarrow 2\rho'\rho + 2\rho'\sigma\frac{d}{ds}(\sigma\rho') = 0 \Rightarrow 2\int \rho\rho'\,ds + 2\int(\sigma\rho')\frac{d}{ds}(\sigma\rho')ds = 0$

$\Rightarrow 2\frac{\rho^2}{2} + 2\frac{(\sigma\rho')^2}{2} = r$ $\quad \therefore r\ being\ constant$ $\qquad \Rightarrow \boldsymbol{\rho^2 + (\rho'\sigma)^2 = r^2}$

CONVERSLY: Suppose that $\rho^2 + (\rho'\sigma)^2 = r^2 \Rightarrow 2\rho'\rho + 2\rho'\sigma\frac{d}{ds}(\sigma\rho') = 0$ diff.w.r.to 's'

$\Rightarrow 2\rho'\left[\rho + \sigma\frac{d}{ds}(\sigma\rho')\right] = 0 \Rightarrow 2\rho' \neq 0\ then\ \left[\rho + \sigma\frac{d}{ds}(\sigma\rho')\right] = 0 \Rightarrow \rho = -\sigma\frac{d}{ds}(\sigma\rho')$

$\Rightarrow \frac{\rho}{\sigma} = -\frac{d}{ds}(\sigma\rho') \Rightarrow \boldsymbol{\frac{\tau}{K} = \frac{d}{ds}\left(\frac{K'}{\tau K^2}\right)}$ $\therefore \rho = \frac{1}{K} \Rightarrow \rho' = -\frac{K'}{K^2} \Rightarrow -\rho' = \frac{K'}{K^2}$ $also\ \sigma = \frac{1}{\tau} \Rightarrow \tau = \frac{1}{\sigma}$

UNIT TANGENT VECTOR: Let $\vec{r}(s)$ be a unit speed curve in R^3 then $\vec{r}' = \frac{d\vec{r}}{ds} = \vec{t}$ is a unit tangent vector at any point $\vec{r}(s)$.

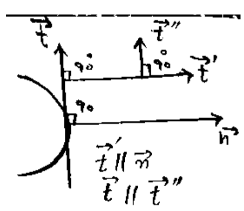
ALSO Let $\vec{r}(t)$ be a regular curve in R^3 then $\vec{t} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$ is a unit tangent vector at any point $\vec{r}(t)$.

TURNING ANGLE OF $\vec{\boldsymbol{r}}$: The smooth function $\varphi: (\alpha, \beta) \to R$ is called the turning angle of $\vec{r}$ determined by the condition $\varphi(s_0) = \varphi_0$

SIGNED CURVATURE: Arc rate of rotation of the tangent vector in clockwise or anti clockwise direction is called Signed curvature. It is the rate at which tangent vector of the curve rotates.
i.e. $K_s = \frac{d\theta}{ds}$
When the curve lies in a plane we may assign a sign of plus or minus to measure the arc rate of rotation the tangent clockwise or counter clockwise.

THEOREM: Suppose $\vec{r}(s)$ is a unit speed curve in R^2 then curvature of the curve is absolute valued of its signed curvature. i.e. $K = \|K_s\|$

PROOF:

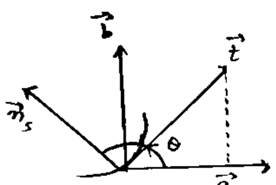


Suppose $\vec{r}(s)$ is a unit speed curve in R^2 . And let $\vec{r}' = \frac{d\vec{r}}{ds} = \vec{t}$ be the unit tangent vector of $\vec{r}(s)$ since $\vec{r}(s)$ is a unit speed curve.
Now there are two unit vectors perpendicular to $\vec{t}$ among them first is
$\vec{n}_s = signed\ unit\ normal\ to\ \vec{r}$ and second is $\vec{r}'' = \vec{t}'$ which is a unit vector obtained by rotating the tangent vector $\vec{t}$ in anti-clockwise direction by an angle $\frac{\pi}{2}$
Then $\vec{t}' \perp \vec{t} \implies \vec{t}' \parallel \vec{n}_s \implies \vec{t}' = K_s\vec{n}_s \implies \vec{r}'' = K_s\vec{n}_s \implies \|\vec{r}''\| = \|K_s\vec{n}_s\| \implies K = \|K_s\|\|\vec{n}_s\|$
$\implies K = \|K_s\|.1 \implies K = \|K_s\|$ hence the result.

OR let $\vec{r} = \vec{r}(s)$ is a unit speed curve and $\theta(s)$ be the angle through which a fixed vector must be rotate anticlockwise to bring it into coincidence with unit tangent vector $\vec{t}$ of $\vec{r}$ then
$K_s = \frac{d\theta}{ds} \implies K = \|K_s\|$

THEOREM: Let $\vec{r}(s)$ be a unit speed curve. Let $\theta(s)$ be the angle through which a fixed vector must be rotated anti – clockwise to bring it into coincidence with the unit tangent vector $\vec{t}$ of $\vec{r}$ then show that $K_s = \frac{d\theta}{ds}$

PROOF: let $\vec{a}$ be a fixed unit vector and $\vec{b}$ be the unit vector obtained by rotating $\vec{a}$ in an anti-clockwise direction by an angle $\frac{\pi}{2}$ then



$\vec{t} = \vec{a}Cos\theta + \vec{b}\,Sin\theta \implies \vec{t}' = \frac{d\hat{t}}{ds} = -\vec{a}Sin\theta\frac{d\theta}{ds} + \vec{b}Cos\theta\frac{d\theta}{ds} = \left(-\vec{a}Sin\theta + \vec{b}Cos\theta\right)\frac{d\theta}{ds}$
$\implies \vec{t}'.\vec{a} = \left[-(\vec{a}.\vec{a})Sin\theta + (\vec{b}.\vec{a})Cos\theta\right]\frac{d\theta}{ds} \implies \vec{t}'.\vec{a} = -Sin\theta\frac{d\theta}{ds}\ \ldots\ldots\ldots (i)$

Since we have $\vec{t}' \parallel \vec{n}_s \implies \vec{t}' = K_s\vec{n}_s$ then $(i) \implies K_s\vec{n}_s.\vec{a} = -Sin\theta\frac{d\theta}{ds}$
$\implies K_s|\vec{n}_s||\vec{a}|Cos\left(\theta + \frac{\pi}{2}\right) = -Sin\theta\frac{d\theta}{ds} \implies K_s(-Sin\theta) = -Sin\theta\frac{d\theta}{ds} \implies \boldsymbol{K_s} = \boldsymbol{\frac{d\theta}{ds}}$
$\implies$ result shows that signed curvature K_s is the arc rate of rotation of the tangent vector.

QUESTION: Find the signed curvature of $\vec{r}(t) = (t\,,\ Cosht)$

ANSWER:

Given $\vec{r}(t) = (t\,,\ Cosht) \Rightarrow \vec{r}'(t) = (1\,,\ Sinht) \Rightarrow \|\vec{r}'(t)\| = \sqrt{1 + Sinh^2 t} = \sqrt{Cosh^2 t}$

$\|\vec{r}'(t)\| = Cosht$

Then for arc length $s(t) = \int_{t_0}^{t} \|\vec{r}'(u)\| du \Rightarrow s(t) = \int_0^t Coshudu$

$\Rightarrow s(t) = |Sinhu|_0^t \Rightarrow s(t) = Sinht$

Now let φ is the angle between $\vec{r}'$ and x – axis then $Tan\varphi = Sinht = s$

$\Rightarrow Sec^2\varphi \frac{d\varphi}{ds} = 1 \Rightarrow \frac{d\varphi}{ds} = \frac{1}{Sec^2\varphi} \Rightarrow \boldsymbol{\frac{d\varphi}{ds}} = \frac{1}{1+Tan^2\varphi} = \frac{1}{1+s^2} \Rightarrow \boldsymbol{K_s} = \boldsymbol{\frac{1}{1+s^2}}$

RESULT: Let $\vec{r}(s)$ be a unit speed plane curve and $K_s = \frac{d\varphi}{ds}$ then $\varphi(l) - \varphi(0) = \int_0^l K_s(s)ds$ where l is the length of curve and $\varphi(l) - \varphi(0)$ is total signed curvature.

PROOF: Given that $\vec{r}(s)$ be a unit speed plane curve and $K_s = \frac{d\varphi}{ds}$ then

$K_s ds = d\varphi \Rightarrow \int_0^l K_s(s)ds = \int_0^l d\varphi \Rightarrow \varphi(l) - \varphi(0) = \int_0^l K_s(s)ds$ when l is the length of curve

COROLLARY: The total signed curvature of a closed plane curve is an integer multiple of $\pm 2\pi$

ROTATION THEOREM (Hopf's Umlaufsatz):

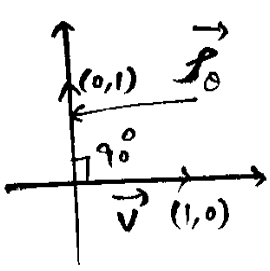
The total signed curvature of a simple closed curve in R^2 is $\pm 2\pi$

PROOF: let $\vec{r}(s)$ be a unit speed plane curve l is its length then the total signed curvature of $\vec{r}$ is $\varphi(l) - \varphi(0) = \int_0^l K_s(s)ds$ where φ is a turning angle of $\vec{r}$

Now $\vec{r}$ is $l = periodic$ then $\vec{r}(s + l) = \vec{r}(s) \Rightarrow \vec{r}'(s + l) = \vec{r}'(s) \Rightarrow \vec{r}'(l) = \vec{r}'(0)$ then by equation $[Cos\varphi(l)\,,\ Sin\varphi(l)\,] = [Cos\varphi(0)\,,\ Sin\varphi(0)]$
$\Rightarrow \varphi(l) - \varphi(0)$ is an integer multiple of 2π

RIGID MOTION: A rigid motion in R^2 is a mapping $M: R^2 \to R^2$ Of the form $M = T_{\vec{a}} \circ \vec{R}_\theta$ where $\vec{R}_\theta$ is an anticlockwise rotation by an angle θ with about the origin. And $\vec{R}_\theta(x, y) = (xCos\theta - ySin\theta,\ xSin\theta + yCos\theta)$ and $T_{\vec{a}}$ is the translation by the vector $\vec{a}$ and $T_{\vec{a}} = \vec{v} + \vec{a}$ for any $\vec{v}(x, y) \epsilon R^2$

EXAMPLE: For $\vec{R}_\theta(x, y) = (xCos\theta - ySin\theta,\ xSin\theta + yCos\theta)$
$\Rightarrow \vec{R}_\theta(1,0) = (1.Cos90° - 0.Sin90°,\ 1.Sin90° + 0.Cos90°) \Rightarrow \vec{R}_\theta(1,0) = (0,1)$



THEOREM: Let $K:(\alpha,\beta)\to R$ by any smooth function then there is a unit speed curve $\vec{r}:(\alpha,\beta)\to R^2$ whose signed curvature is K. further if $\tilde{r}:(\tilde{\alpha},\tilde{\beta})\to R^2$ is any other unit speed curve whose signed curvature is K, there is a rigid motion $\tilde{r}(s)=M[\vec{r}(s)]$; $\forall s\epsilon(\tilde{\alpha},\tilde{\beta})$

PROOF: For the first part fix $s_0\epsilon(\alpha,\beta)$ for any $s\epsilon(\alpha,\beta)$ then $\varphi(s)=\int_{s_0}^{s}K(u)du \therefore K_s=\frac{d\varphi}{ds}$

then $\vec{r}(s)=\left[\int_{s_0}^{s}Cos\varphi(t)dt\ ,\int_{s_0}^{s}Sin\varphi(t)dt\ \right]$

then tangent vector $\vec{r}'(s)=[Cos\varphi(s),Sin\varphi(s)\ ]$ is a unit vector making angle $\varphi(s)$ with the x – axis and thus $\vec{r}(s)$ is a unit speed plane curve and its signed curvature is $K(s)=\frac{d\varphi}{ds}$ or $\frac{d\varphi}{ds}=\frac{d}{ds}\int_{s_0}^{s}K(u)du=K(s)$

Now for the second part let $\tilde{\varphi}(s)$ be the angle between the x – axis and the unit tangent vector $\tilde{r}'$ of $\tilde{r}$ then $\tilde{r}'(s)=[Cos\tilde{\varphi}(s),Sin\tilde{\varphi}(s)\ ]$

$\Rightarrow\tilde{r}(s)=\left[\int_{s_0}^{s}Cos\tilde{\varphi}(t)dt\ ,\int_{s_0}^{s}Sin\tilde{\varphi}(t)dt\ \right]+\tilde{r}(s_0)$ ....(i)

now by using $\frac{d\tilde{\varphi}}{ds}=K(s)\Rightarrow\int_{s_0}^{s}d\tilde{\varphi}=\int_{s_0}^{s}K(u)du\Rightarrow\tilde{\varphi}(s)-\tilde{\varphi}(s_0)=\int_{s_0}^{s}K(u)du$

$\Rightarrow\tilde{\varphi}(s)=\int_{s_0}^{s}K(u)du+\tilde{\varphi}(s_0)\Rightarrow\tilde{\varphi}(s)=\varphi(s)+\tilde{\varphi}(s_0)$ also take $\tilde{r}(s_0)=\vec{a}$ and $\tilde{\varphi}(s_0)=\theta$

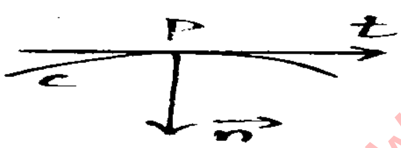
then $(i)\Rightarrow\tilde{r}(s)=\left[\int_{s_0}^{s}Cos[\varphi(t)+\theta]dt\ ,\int_{s_0}^{s}Sin[\varphi(t)+\theta]dt\ \right]+\vec{a}$

$\Rightarrow\tilde{r}(s)=\left[\int_{s_0}^{s}[Cos\varphi(t)Cos\theta-Sin\varphi(t)Sin\theta]dt\ ,\int_{s_0}^{s}[Sin\varphi(t)Cos\theta+Cos\varphi(t)Sin\theta]dt\ \right]+\vec{a}$

$\Rightarrow\boldsymbol{\tilde{r}(s)}=\left[\boldsymbol{Cos\theta\int_{s_0}^{s}Cos\varphi(t)dt-Sin\theta\int_{s_0}^{s}Sin\varphi(t)dt\ ,Cos\theta\int_{s_0}^{s}Sin\varphi(t)dt+Sin\theta\int_{s_0}^{s}Cos\varphi(t)dt}\ \right]+\boldsymbol{\vec{a}}$

$\Rightarrow\tilde{r}(s)=T_{\vec{a}}\circ\vec{R}_\theta(\vec{r}(s))\Rightarrow\tilde{r}(s)=M[\vec{r}(s)]$

UNIT PRINCIPAL NORMAL:

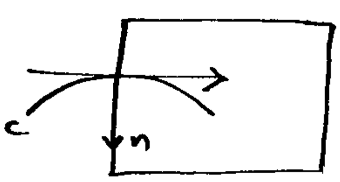


a vector perpendicular to the tangent at 'P' is called the unit principal normal at 'P' and it is denoted by $\vec{n}$. Where the straight line passing through 'P' on 'C' and parallel to the unit principal normal is called unit principal normal line at that point 'P'

PRINCIPAL NORMAL: if the curvature $K(s)$ is non – zero i.e. $K(s)\neq 0$ we define Principal normal of $\vec{r}$ at the point $\vec{r}(s)$ to be the vector $\vec{n}_s=\frac{1}{K(s)}\vec{t}'\Rightarrow\|\vec{n}_s\|=\left\|\frac{1}{K(s)}\right\|\,\|\vec{t}'\|$

$\Rightarrow\|\vec{n}_s\|=1$ which is our unit principal normal. $\therefore\|\vec{t}'\|=\|\vec{r}''\|=K$

OSCULATING PLANE OR PLANE OF CURVATURE:



Plane containing two consecutive tangents at a point P and three consecutive points at P
OR the plane parallel to the unit tangent and unit principal normal is called osculating plane at point 'P' to the curve 'C'
if $\vec{R}$ is any point on this plane then $\vec{R}-\vec{r},\vec{t}$ and $\vec{n}$ are Coplanar vectors so, $[\boldsymbol{\vec{R}-\vec{r},\vec{t},\vec{n}}]=\boldsymbol{0}$ this is the scalar triple product and this is the **equation of osculating plane**.

EQUATION OF OSCULATING PLANE (2[nd] form):

we know that c Since $\vec{r}' = \vec{t}$ and $\vec{r}'' = \vec{t}' = K\vec{n}$ $\Rightarrow \vec{n} = \frac{\vec{r}''}{K}$ $\Rightarrow \left[\vec{R} - \vec{r}, \vec{r}', \frac{\vec{r}''}{K}\right] = 0$
$\Rightarrow [\vec{R} - \vec{r}, \vec{t}, \rho\vec{r}''] = 0$ Also $\Rightarrow [\vec{R} - \vec{r}].\vec{b} = 0$

QUESTION: If $\vec{r}'$ *and* $\vec{r}''$ are linearly independent at a point $\vec{r_0}$ along the curve $\vec{r} = \vec{r}(t)$ then show that osculating plane is $[\vec{R} - \vec{r}, \vec{t}, \vec{n}] = 0$

Solution: given $\vec{r} = \vec{r}(t) \Rightarrow \vec{r}' = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}.\frac{ds}{dt} = \vec{t}s'$

$\Rightarrow \vec{r}'' = \frac{d}{dt}\vec{t}s' = \vec{t}s'' + \vec{t}'s' = \vec{t}s'' + K\vec{n}s' \Rightarrow \vec{r}' \times \vec{r}'' = (\vec{t}s') \times (\vec{t}s'' + K\vec{n}s') = (s')^3K\vec{b}$

At $\vec{r} = \vec{r_0} \Rightarrow \vec{r_0}' \times \vec{r_0}'' = (s_0')^3K_0\overrightarrow{b_0} \neq 0$ $\therefore \vec{r}'$ *and* $\vec{r}''$ are linearly independent at a point $\vec{r_0}$
$\Rightarrow \overrightarrow{b_0} = \frac{\vec{r_0}' \times \vec{r_0}''}{(s_0')^3K_0}$ also we know that equation of osculating plane at $\vec{r} = \vec{r_0}$ is $[\vec{R} - \vec{r_0}].\overrightarrow{b_0} = 0$
$\Rightarrow [\vec{R} - \vec{r_0}].\overrightarrow{b_0} = [\vec{R} - \vec{r_0}].\frac{\vec{r_0}' \times \vec{r_0}''}{(s_0')^3K_0} = 0 \Rightarrow [\vec{R} - \vec{r_0}].\vec{r_0}' \times \vec{r_0}'' = 0 \Rightarrow [\vec{R} - \vec{r_0}, \vec{r_0}', \vec{r_0}''] = 0$
$\Rightarrow [\vec{R} - \vec{r}, \vec{t}, \vec{n}] = 0$ $\qquad \therefore \vec{r} = \vec{r_0}, \vec{t} = \vec{r}'$ *and* $\vec{n} = \vec{r}''$

- **Possible Q:** If $\dot{x}$ *and* $\ddot{x}$ are linearly independent at a point x_0 along the curve $x = x(t)$ then show that osculating plane is $[Y - x_0, \dot{x_0}, \ddot{x_0}] = 0$; $Y = (x_1, x_2, x_3)$

Question: Find the equation of the osculating plane of the given curve
$\vec{r}(t) = (t, t^2, t^3)$ *at* $t = 1$

Solution: given curve $\vec{r}(t) = (t, t^2, t^3)$ $\Rightarrow \vec{r}(t = 1) = (1,1,1)$

$\Rightarrow \vec{r}'(t) = (1,2t, 3t^2)$ $\Rightarrow \vec{r}'(t = 1) = (1,2,3) \Rightarrow \vec{r}''(t) = (0,2,6t)$ $\Rightarrow \vec{r}''(t = 1) = (0,2,6)$

$[\vec{R} - \vec{r}, \vec{t}, \vec{n}] = 0 \qquad \Rightarrow \begin{vmatrix} x_1 - 1 & x_2 - 1 & x_3 - 1 \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} = 0$
$\Rightarrow 6(x_1 - 1) + 6(x_2 - 1) + 2(x_3 - 1) = 0 \Rightarrow 2(3x_1 + 3x_2 + x_3) - 14 = 0$

$\Rightarrow (3x_1 + 3x_2 + x_3) - 7 = 0$ this is equation of osculating plane for given curve.

Question: Find the equation of the osculating plane of the given curve
$\vec{r}(\theta) = aCos\theta\hat{\imath} + aSin\theta\hat{\jmath} + 0.\hat{k}$

Solution: given curve $\vec{r}(\theta) = aCos\theta\hat{\imath} + aSin\theta\hat{\jmath} + 0.\hat{k}$ then comparing it with
$\vec{r}(\theta) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ $\qquad \Rightarrow x = aCos\theta, y = aSin\theta, z = 0$

now $\left[\vec{R} - \vec{r}, \vec{r}', \frac{\vec{r}''}{K}\right] = 0 \qquad \Rightarrow \begin{vmatrix} X - x & Y - y & Z - z \\ x' & y' & z' \\ \frac{x''}{K} & \frac{y''}{K} & \frac{z''}{K} \end{vmatrix} = 0$

$\Rightarrow \frac{1}{K}\begin{vmatrix} X - x & Y - y & Z - z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} X - x & Y - y & Z - z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = 0$
now as $x = aCos\theta, y = aSin\theta, z = 0$
$\Rightarrow x' = -aSin\theta, y' = aCsos\theta, z' = 0 \Rightarrow x'' = -aCos\theta, y'' = -aSin\theta, z'' = 0$
$\Rightarrow \begin{vmatrix} X - aCos\theta & Y - aSin\theta & Z - 0 \\ -aSin\theta & aCsos\theta & 0 \\ -aCos\theta & -aSin\theta & 0 \end{vmatrix} = 0$
$\Rightarrow z(a^2Cos^2\theta + a^2Sin^2\theta) = 0 \Rightarrow za^2(Cos^2\theta + Sin^2\theta) = 0 \Rightarrow za^2 = 0$
$\Rightarrow a^2 \neq 0 \Rightarrow z = 0$ this is equation of osculating plane for given curve.

BINORMAL AT 'P':

the normal at 'P' which is perpendicular to the osculating plane is called binormal at 'P'

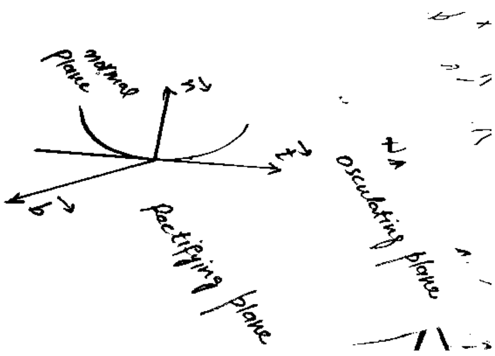
the vectors $\vec{b}, \vec{t}$ and $\vec{n}$ are perpendicular to each other. A straight line passing through 'P' and is parallel to the binormal at 'P' is called **binormal line**.

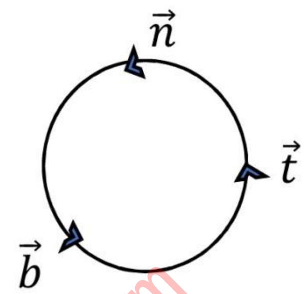
EQUATION OF BINORMAL:

We know that $\vec{R} = \vec{r} + u\vec{b}$ and Since $\vec{b} = \vec{t} \times \vec{n}$, $\vec{t} = \vec{r}\,'$, $\vec{n} = \frac{\vec{r}\,''}{\kappa}$

$\Rightarrow \vec{R} = \vec{r} + u\vec{b} \Rightarrow \vec{R} = \vec{r} + u(\vec{t} \times \vec{n}) \Rightarrow \boldsymbol{\vec{R} = \vec{r} + u(\vec{r}\,' \times \frac{\vec{r}\,''}{\kappa})}$ Which is required equation

RECTIFYING PLANE:





the plane through 'P' which is parallel to the unit tangent $\vec{t}$ and unit binormal $\vec{b}$ is called Rectifying plane. so

- $\vec{t}.\vec{t} = \vec{n}.\vec{n} = \vec{b}.\vec{b} = 1$
- $\vec{t}.\vec{b} = \vec{b}.\vec{n} = \vec{n}.\vec{t} = 0$
- $\hat{\vec{t}} \times \vec{n} = \vec{b}, \quad \vec{b} \times \vec{t} = \vec{n}, \quad \vec{n} \times \vec{b} = \vec{t}$
- $\vec{n} \times \vec{t} = -\vec{b}, \quad \vec{t} \times \vec{b} = -\vec{n}, \quad \vec{b} \times \vec{n} = -\vec{t}$
- The triplet $(\hat{\vec{t}}, \vec{n}, \vec{b})$ of unit tangent, unit principal normal and unit binormal is called **moving trihedran**. This is the set of orthonormal vectors of R^3 and is Right handed.

INTERESTING FACT:

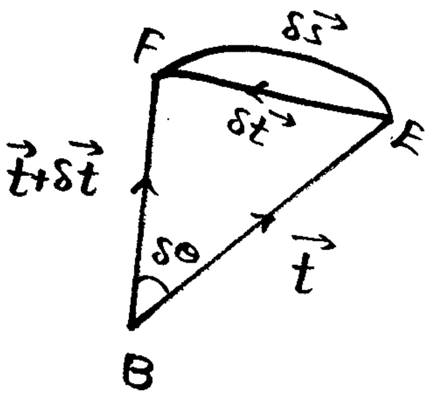
The plane determined by the normal and binormal vectors $\vec{n}$ and $\vec{b}$ at a point P on a curve C is called the **Normal Plane** of C at P. it consists of all lines that are orthogonal to the tangent vector $\vec{t}$. The plane determined by the vectors $\vec{t}$ and $\vec{n}$ is called **Osculating Plane** of C at P. it is the plane that comes closest to containing the part of the curve near P. (for a plane curve, the osculating plane is simply the plane that contains the curve).

SERRET FRENET FORMULAE (PP, 2019):

the Serret Frenet Formulae are derived from the fact that the frame vectors are mutually perpendicular and that they have unit length. Formulae are as follows;

i. $\vec{t}' = \frac{d\vec{t}}{ds} = K\vec{n}$ ii. $\vec{b}' = \frac{d\vec{b}}{ds} = -\tau\vec{n}$ iii. $\vec{n}' = \frac{d\vec{n}}{ds} = \tau\vec{b} - K\vec{t}$

PROOF (i): $\boldsymbol{\vec{t}' = \frac{d\vec{t}}{ds} = K\vec{n}}$



The unit tangent is not a constant vector as its direction varies from point to point of the curve. Let $\vec{t}$ and $\vec{t} + \overrightarrow{\delta t}$ are its values at two different points 'E' and 'F' respectively. The vectors $\overrightarrow{BE}$ and $\overrightarrow{BF}$ are respectively equal to these. Then

$\overrightarrow{\delta t} = \overrightarrow{EF}$ and $\overrightarrow{\delta\theta} = m\angle EBF$ also $\frac{d\vec{t}}{ds} = \lim_{\delta s \to 0} \overrightarrow{EF}$ and its direction is perpendicular to tangent $\vec{t}$ moreover $\left|\overrightarrow{BE}\right| = \left|\overrightarrow{BF}\right| = 1$ also the modulus value of $\frac{d\vec{t}}{ds}$ is the limiting value of $\frac{\delta\theta}{\delta s}$ which is 'K' hence $\frac{d\vec{t}}{ds} = \lim_{\delta s \to 0} \frac{\delta\vec{t}}{\delta s} = K\vec{n}$ where $\vec{n}$ is the unit vector perpendicular to $\vec{t}$ and in the plane of tangent at 'P' and at a consecutive point 'P' so $\vec{t}' = \frac{d\vec{t}}{ds} = K\vec{n}$

OR we know that

$K_s = \frac{d\theta}{ds} \Longrightarrow \|K_s\| = \left\|\frac{d\theta}{ds}\right\| \Longrightarrow K = \left\|\frac{d\theta}{d\vec{t}}.\frac{d\vec{t}}{ds}\right\| = \left\|\frac{d\theta}{d\vec{t}}\right\| \left\|\frac{d\vec{t}}{ds}\right\| = 1.\left\|\frac{d\vec{t}}{ds}\right\| = \left\|\vec{t}'\right\| \dots\dots (i)$

also $K = \left\|\vec{t}'\right\| = \|\vec{r}''\| \dots\dots (ii)$ and since $\|\vec{r}'\|^2 = 1 \Longrightarrow \vec{r}'.\vec{r}' = 1$

Diff.w.r.to 's' $\Longrightarrow 2\vec{r}'.\vec{r}'' = 0 \Longrightarrow \vec{r}' \perp \vec{r}'' \Longrightarrow \vec{r}'' \parallel \vec{n} \Longrightarrow \vec{r}'' = K\vec{n} \Longrightarrow \vec{t}' = K\vec{n}$ from (ii)

PROOF(Annual;2018) (ii): $\boldsymbol{\vec{b}' = \frac{d\vec{b}}{ds} = -\tau\vec{n}}$

Consider $b^2 = \vec{b}.\vec{b} = 1$

Diff. w.r.to 's'

$\vec{b}\frac{d\vec{b}}{ds} + \frac{d\vec{b}}{ds}\vec{b} = 0 \Longrightarrow 2\vec{b}\frac{d\vec{b}}{ds} = 0 \Longrightarrow \vec{b}\frac{d\vec{b}}{ds} = 0 \Longrightarrow \vec{b}.\vec{b}' = 0 \Longrightarrow \vec{b}'$ is perpendicular to $\vec{b}$ ....(i)

Again Consider $\vec{t}.\vec{b} = 1$

Diff. w.r.to 's'

$\frac{d\vec{t}}{ds}\vec{b} + \vec{t}\frac{d\vec{b}}{ds} = 0 \Longrightarrow \vec{t}'.\vec{b} + \vec{t}.\vec{b}' = 0 \Longrightarrow K\vec{n}.\vec{b} + \vec{t}.\vec{b}' = 0 \Longrightarrow \vec{t}.\vec{b}' = 0 \quad \therefore \vec{n}.\vec{b}$

$\Longrightarrow \vec{b}'$ is perpendicular to $\vec{t}$ ....(ii)

From (i) and (ii) $\Longrightarrow \vec{b}'$ is parallel to $\vec{n}$ $\Longrightarrow \vec{b}' = -\tau\vec{n}$ $\Longrightarrow \vec{b}' = \frac{d\vec{b}}{ds} = -\tau\vec{n}$

where τ measure the arc rate of rotation of binormal.

PROOF (iii): $\vec{\boldsymbol{n}}' = \frac{\boldsymbol{d\vec{n}}}{\boldsymbol{ds}} = \boldsymbol{\tau\vec{b} - K\vec{t}}$

Consider $\vec{n} = \vec{b} \times \vec{t} \implies$ Diff. w.r.to 's' $\frac{d\vec{n}}{ds} = \vec{b} \times \frac{d\vec{t}}{ds} + \frac{d\vec{b}}{ds} \times \vec{t} = \vec{b} \times \vec{t}' + \vec{b}' \times \vec{t}$

$\vec{n}' = \vec{b} \times K\vec{n} + (-\tau\vec{n}) \times \vec{t} = K(\vec{b} \times \vec{n}) + \tau(-\vec{n} \times \vec{t}) = K(-\vec{t}) + \tau(\vec{b}) = -K\vec{t} + \tau\vec{b}$

$\implies \vec{n}' = \frac{d\vec{n}}{ds} = \tau\vec{b} - K\vec{t}$

THE SERRET FRENET TRANSFORMATION (MATRIX FORM):

Since

i. $\vec{t}' = \frac{d\vec{t}}{ds} = K\vec{n}$ ii. $\vec{b}' = \frac{d\vec{b}}{ds} = -\tau\vec{n}$ iii. $\vec{n}' = \frac{d\vec{n}}{ds} = \tau\vec{b} - K\vec{t}$

Then the matrix $\begin{bmatrix} \vec{t}' \\ \vec{n}' \\ \vec{b}' \end{bmatrix} = \begin{bmatrix} 0 & K & 0 \\ -K & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix}$ is called matrix form of Serret Frenet Formulae.

Let $A = \begin{bmatrix} 0 & K & 0 \\ -K & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \implies A^T = \begin{bmatrix} 0 & -K & 0 \\ K & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} = -\begin{bmatrix} 0 & K & 0 \\ -K & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \implies A^T = -A$

$\implies$matirx is Skew Symmetric.

THE SERRET FRENET APPARATUS:

the Serret Frenet apparatus of a unit speed curve $\vec{r}(s)$ are $[K(s), \tau(s), \vec{t}(s), \vec{n}(s), \vec{b}(s)]$

Question: Show that $\vec{r}(s) = \left(\frac{5}{13}Cos(s),\ \frac{8}{13} - Sin(s),\ \frac{12}{13}Cos(s)\right)$ is a unit speed curve and find its Serret Frenet Apparatus.

Solution: Since $\vec{r}(s) = \left(\frac{5}{13}Cos(s),\ \frac{8}{13} - Sin(s),\ \frac{12}{13}Cos(s)\right)$

$\implies \vec{r}'(s) = \left(-\frac{5}{13}Sin(s),\ -Cos(s), -\frac{12}{13}Sin(s)\right)$

$\implies \|\vec{r}'(s)\|^2 = \frac{25}{169}Sin^2(s) + Cos^2(s) + \frac{144}{169}Sin^2(s) = Sin^2(s) + Cos^2(s) \implies \|\vec{r}'(s)\|^2 = 1$

$\implies \|\vec{r}'(s)\| = 1 \implies$ given curve is unit speed curve.

Now we can find the Serret Frenet Apparatus

$\implies \vec{r}''(s) = \left(-\frac{5}{13}Cos(s),\ Sin(s),\ -\frac{12}{13}Cos(s)\right) \implies \vec{r}'''(s) = \left(\frac{5}{13}Sin(s), Cos(s), \frac{12}{13}Sin(s)\right)$

$$\implies \vec{r}' \times \vec{r}'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\frac{5}{13}Sin(s) & -Cos(s) & -\frac{12}{13}Sin(s) \\ -\frac{5}{13}Cos(s) & Sin(s) & -\frac{12}{13}Cos(s) \end{vmatrix} = \left(\frac{12}{13},\ 0,\ \frac{5}{13}\right)$$

$\implies \|\vec{r}' \times \vec{r}''\|^2 = \frac{144}{169} + 0 + \frac{25}{169} = 1 \implies \|\vec{r}' \times \vec{r}''\| = 1$

Now $\implies \boldsymbol{\vec{r}' \times \vec{r}''.\vec{r}'''} = \frac{60}{169}Sin(s) + 0 - \frac{60}{169}Sin(s) \implies \boldsymbol{\vec{r}' \times \vec{r}''.\vec{r}'''} = 0$

For curvature $\implies K = \frac{\|\vec{r}' \times \vec{r}''\|}{\|\vec{r}'\|^3} = \frac{1}{1} = 1$ and for torsion $\implies \tau = \frac{\vec{r}'.\vec{r}'' \times \vec{r}'''}{\|\vec{r}' \times \vec{r}''\|^2} = \frac{0}{1} = 0$

Also since $\vec{t} = \vec{r}' = \left(-\frac{5}{13}Sin(s),\ -Cos(s), -\frac{12}{13}Sin(s)\right)$

$\vec{t}' = K\vec{n} = 1.\vec{n} = \vec{r}'' = \left(-\frac{5}{13}Cos(s),\ Sin(s),\ -\frac{12}{13}Cos(s)\right)$ and

$\vec{b} = \vec{t} \times \vec{n} = \vec{r}' \times \vec{r}'' = \left(\frac{12}{13},\ 0,\ \frac{5}{13}\right)$

PRACTICE: CALCULATE SERRET FRENET APPARATUS OF THE FOLLOWINGS;

i. $\vec{r}(s) = (1+t^2,\ t,\ t^3)$

ii. $\vec{r}(s) = (Cosbt,\ Sinbt,\ t)$

iii. $\vec{r}(s) = (t - Cost,\ Sint,\ t)$

iv. $\vec{r}(s) = (e^2Cost,\ e^2Sint,\ e^2Sint)$

v. $\vec{r}(s) = \left(rCos\left(\frac{s}{r}\right),\ rSin\left(\frac{s}{r}\right),\ 0\right)$

vi. Show that $\vec{r}(s) = \left(\frac{1}{3}(1+t)^{\frac{3}{2}},\ \frac{1}{3}(1-t)^{\frac{3}{2}},\ \frac{3}{\sqrt{2}}\right)$ is a unit speed curve and find its Serret Frenet Apparatus.

vii. **(BS;2016)** Show that $\vec{r}(s) = \frac{1}{2}\left(Cos^{-1}(s) - s\sqrt{1-s^2},\ 1-s^2,\ 0\right)$ is a unit speed curve and find its Serret Frenet Apparatus.

Question: **(BS;2018)** Show that $\vec{r}(t) = \left(\frac{4}{5}Cost,\ 1 - Sint, -\frac{3}{5}Cost\right)$ is a circle and find its radius, centre and plane in which it lies.

Solution: Since $\vec{r}(t) = \left(\frac{4}{5}Cost,\ 1 - Sint, -\frac{3}{5}Cost\right) \Rightarrow \vec{r}'(t) = \left(-\frac{4}{5}Sint,\ -Cost,\ \frac{3}{5}Sint\right)$

$\Rightarrow \|\vec{r}'(t)\|^2 = \frac{16}{25}Sin^2t + Cos^2t + \frac{9}{25}Sin^2t = Sin^2t + Cos^2t \Rightarrow \|\vec{r}'(t)\|^2 = 1$

$\Rightarrow \|\vec{r}'(t)\| = 1 \Rightarrow$ given curve is unit speed curve.

Now we can find the Serret Frenet Apparatus

$\Rightarrow \vec{r}''(t) = \left(-\frac{4}{5}Cost,\ Sint, \frac{3}{5}Cost\right) \Rightarrow \vec{r}'''(t) = \left(\frac{4}{5}Sint,\ Cost, -\frac{3}{5}Sint\right)$

$$\Rightarrow \vec{r}' \times \vec{r}'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\frac{4}{5}Sint & -Cost & \frac{3}{5}Sint \\ -\frac{4}{5}Cost & Sint & \frac{3}{5}Cost \end{vmatrix} = \left(-\frac{3}{5}, 0, -\frac{4}{5}\right)$$

$\Rightarrow \|\vec{r}' \times \vec{r}''\|^2 = \frac{9}{25} + 0 + \frac{16}{25} = 1 \Rightarrow \|\vec{r}' \times \vec{r}''\| = 1$

Now $\Rightarrow \boldsymbol{\vec{r}' \times \vec{r}''.\vec{r}'''} = -\frac{12}{25}Sint + 0 + \frac{12}{25}Sint \Rightarrow \boldsymbol{\vec{r}' \times \vec{r}''.\vec{r}'''} = 0$

For curvature $\Rightarrow K = \frac{\|\vec{r}' \times \vec{r}''\|}{\|\vec{r}'\|^3} = \frac{1}{1} = 1$ and for torsion $\Rightarrow \tau = \frac{\vec{r}' \times \vec{r}''.\vec{r}'''}{\|\vec{r}' \times \vec{r}''\|^2} = \frac{0}{1} = 0$

Also since $\vec{t} = \vec{r}' = \left(-\frac{4}{5}Sint,\ -Cost,\ \frac{3}{5}Sint\right)$

$\vec{t}' = K\vec{n} = 1.\vec{n} = \vec{r}'' = \left(-\frac{4}{5}Cost,\ Sint, \frac{3}{5}Cost\right) \Rightarrow \vec{n} = \left(-\frac{4}{5}Cost,\ Sint, \frac{3}{5}Cost\right)$

$\Rightarrow -\tau\vec{n} = -\tau\left(-\frac{4}{5}Cost,\ Sint, \frac{3}{5}Cost\right) \Rightarrow \vec{b}' = (0,\ 0, 0)\ \therefore \vec{b}' = -\tau\vec{n}\ \ and\ \tau = 0$

$\Rightarrow \vec{b}' = (0,0,0) \Rightarrow \vec{b} = (c_1, c_2, c_3) = constant\ unit\ vector$

Now let $x = \frac{4}{5}Cost,\ y = 1 - Sint,\ z = -\frac{3}{5}Cost \Rightarrow x^2 + y^2 + z^2 = 2y$ (after solving)

$\Rightarrow x^2 + y^2 - 2y + 1 + z^2 = 1 \Rightarrow \boldsymbol{x^2 + (y-1)^2 + z^2 = 1}$

Now if z = 0 then $(x-0)^2 + (y-1)^2 = (1)^2$ is an equation of circle centered at $(0,1)$ having radius '1' lies in xy – plane.

Or if x = 0 then $(y-1)^2 + (z-0)^2 = (1)^2$ is an equation of circle centered at $(1,0)$ having radius '1' lies in yz – plane.

Or if $y - 1 = 0 \Rightarrow y = 1$ then $(x-0)^2 + (z-0)^2 = (1)^2$ is an equation of circle centered at $(0,0)$ having radius '1' lies in xz – plane.

Question: if the nth derivative of $\vec{r}$ w.r.to 's' is given by $\vec{r}^n = a_n\vec{t} + b_n\vec{n} + c_n\vec{b}$ then prove the reduction formulae $a_{n+1} = a'_n - Kb_n$, $b_{n+1} = b'_n + Ka_n - \tau c_n$, $c_{n+1} = c'_n + \tau b_n$

Solution: Since given $\vec{r}^n = a_n\vec{t} + b_n\vec{n} + c_n\vec{b} \ldots\ldots\ldots (i)$

$\Rightarrow \vec{r}^{n+1} = a_{n+1}\vec{t} + b_{n+1}\vec{n} + c_{n+1}\vec{b} \ldots\ldots (A)$

Diff (i) w.r.to 's' $\vec{r}^{n+1} = a'_n\vec{t} + a_n\vec{t}' + b'_n\vec{n} + b_n\vec{n}' + c'_n\vec{b} + c_n\vec{b}'$

$\Rightarrow \vec{r}^{n+1} = a'_n\vec{t} + a_n(K\vec{n}) + b'_n\vec{n} + b_n(\tau\vec{b} - K\vec{t}) + c'_n\vec{b} + c_n(-\tau\vec{n})$

$\Rightarrow \vec{r}^{n+1} = (a'_n - Kb_n)\vec{t} + (a_nK + b'_n - \tau c_n)\vec{n} + (\tau b_n + c'_n)\vec{b} \ldots\ldots (B)$

Comparing $(A)$and(B) we get

$a_{n+1} = a'_n - Kb_n$, $b_{n+1} = b'_n + Ka_n - \tau c_n$, $c_{n+1} = c'_n + \tau b_n$

Question: Prove that $\vec{r}''' = K'\vec{n} - K^2\vec{t} + K\tau\vec{b}$

Solution: Since $\vec{r} = \vec{r}(s)$ $\quad\Rightarrow \vec{r}' = \frac{d\vec{r}}{ds} = \vec{t}$ $\quad\Rightarrow \vec{r}'' = \frac{d\vec{r'}}{ds} = \frac{d^2\vec{r}}{ds^2} = \frac{d}{ds}\overrightarrow{(t)} = \frac{d\vec{t}}{ds}$

$\Rightarrow \vec{r}'' = K\vec{n}$

$\Rightarrow \vec{r}''' = K\vec{n}' + K'\vec{n}$ $\quad\Rightarrow \vec{r}''' = K(\tau\vec{b} - K\vec{t}) + K'\vec{n}$ $\quad\Rightarrow \boldsymbol{\vec{r}''' = K'\vec{n} - K^2\vec{t} + K\tau\vec{b}}$

Question: Prove that $\vec{r}^{iv} = (K'' - K^3 - K\tau^2)\vec{n} - 3KK'\vec{t} + (2K'\tau + \tau'K)\vec{b}$

Solution: Since $\vec{r} = \vec{r}(s)$ and $\Rightarrow \vec{r}''' = K'\vec{n} - K^2\vec{t} + K\tau\vec{b}$

$\Rightarrow \vec{r}^{iv} = \frac{d}{ds}(K'\vec{n} - K^2\vec{t} + K\tau\vec{b})$

$\Rightarrow \vec{r}^{iv} = K''\vec{n} + K'\vec{n}' - 2KK'\vec{t} - K^2\vec{t}' + K'\tau\vec{b} + K\tau'\vec{b} + K\tau\vec{b}'$

$\Rightarrow \vec{r}^{iv} = K''\vec{n} + K'(\tau\vec{b} - K\vec{t}) - 2KK'\vec{t} - K^2(K\vec{n}) + K'\tau\vec{b} + K\tau'\vec{b} + K\tau(-\tau\vec{n})$

$\Rightarrow \vec{r}^{iv} = K''\vec{n} + K'\tau\vec{b} - K'K\vec{t} - 2KK'\vec{t} - K^3\vec{n} + K'\tau\vec{b} + K\tau'\vec{b} - K\tau^2\vec{n}$

$\Rightarrow \vec{r}^{iv} = (K'' - K^3 - K\tau^2)\vec{n} + (K'\tau + K'\tau + K\tau')\vec{b} - 3K'K\vec{t}$

$\boldsymbol{\vec{r}^{iv} = (K'' - K^3 - K\tau^2)\vec{n} - 3KK'\vec{t} + (2K'\tau + \tau'K)\vec{b}}$

QUESTION(BS;2019): if $\propto = \propto(s)$ is a unit speed curve whose image lies on a sphere of radius 'R' and centre 'C' then $K \neq 0$ if $\tau \neq 0$ then $C = \propto + \rho\vec{n} + \rho'\sigma\vec{b}$ and $\rho^2 + (\rho'\sigma)^2 = R^2$ where R is constant radius of sphere.

SOLUTION: Let $R \epsilon R^3$ be a vector in 3D, so we have $R = \propto(s) - C \Rightarrow R.R = (\propto(s) - C).(\propto(s) - C)$

$\Rightarrow R^2 = (\propto(s) - C).(\propto(s) - C) \Rightarrow 0 = 2(\propto(s) - C).\vec{t} \Rightarrow (\propto(s) - C).\vec{t} = 0$

$\therefore$ ***R being constant and differentiating***

$\Rightarrow (\propto(s) - C).\vec{t}' + (\propto(s) - C)'.\vec{t} = 0 \Rightarrow (\propto(s) - C).K\vec{n} + \vec{t}.\vec{t} = 0$

$\Rightarrow (\propto(s) - C).\vec{n} = -\frac{1}{K} = -\rho \Rightarrow (\propto(s) - C).\vec{n}' + (\propto(s) - C)'.\vec{n} = -\rho'$

$\Rightarrow (\propto(s) - C).(\tau\vec{b} - K\vec{t}) + \vec{t}.\vec{n} = -\rho' \Rightarrow \tau(\propto(s) - C).\vec{b} - K(\propto(s) - C).\vec{t} + 0 = -\rho'$

$\Rightarrow \tau(\propto(s) - C).\vec{b} - K(\propto(s) - C).\vec{t} + 0 = -\rho'$

$\Rightarrow \tau(\propto(s) - C).\vec{b} - 0 = -\rho' \Rightarrow (\propto(s) - C).\vec{b} = -\frac{\rho'}{\tau}$

Since $\vec{t},\vec{n},\vec{b}$ is an orthonormal set of three vectors, then every vector in R^3 can be expressed as linear combination of these basis sets, thus we can write $R = (\propto(s) - C) = \propto \vec{t} + b\vec{n} + c\vec{b}$ ……..(I)

$\Rightarrow R.\vec{t} = (\propto(s) - C).\vec{t} = \propto \vec{t}.\vec{t} + b\vec{n}.\vec{t} + c\vec{b}.\vec{t} = a \Rightarrow (\propto(s) - C).\vec{t} = \propto = 0$

Similarly $\Rightarrow (\propto(s) - C).\vec{n} = b = -\rho \Rightarrow (\propto(s) - C).\vec{b} = c = \rho'\sigma$

$(I) \Rightarrow R = (\propto(s) - C) = \propto \vec{t} + b\vec{n} + c\vec{b} \Rightarrow R = (0)\vec{t} + (-\rho)\vec{n} + (\rho'\sigma)\vec{b}$

$\Rightarrow R = (-\rho)\vec{n} + (\rho'\sigma)\vec{b}$

$\Rightarrow \boldsymbol{\rho^2 + (\rho'\sigma)^2 = R^2}$ (taking self-product)

QUESTION(BS;2016):

if s_1 is the arc length of centre of spherical curve of a regular curve $\propto = \propto(s) \epsilon C^n$; $n \geq 3$ having arc length 's' determine the relationship between 's' and 's_1' And also find Serret Frenet apparatus of the curve.

SOLUTION: Since the equation of center of spherical curve $\propto(s)$ is given by $C = \propto + \rho\vec{n} + \rho'\sigma\vec{b}$ where ρ *and* σ are the radius and torsion of curvature of curve respectively.

$$\Rightarrow \frac{dC}{ds} = \frac{dC}{ds_1}\frac{ds_1}{ds} = \propto'(s) + \rho'\vec{n} + \rho\vec{n}' + \rho''\sigma\vec{b} + \rho'\sigma'\vec{b} + \rho'\sigma\vec{b}'$$

$$\Rightarrow \vec{t}_1\frac{ds_1}{ds} = \vec{t} + \rho'\vec{n} + \rho(\tau\vec{b} - K\vec{t}) + \rho''\sigma\vec{b} + \rho'\sigma'\vec{b} + \rho'\sigma(-\tau\vec{n})$$

$$\Rightarrow \vec{t}_1\frac{ds_1}{ds} = (\rho\tau + \rho''\sigma + \rho'\sigma')\vec{b} = \left(\frac{\rho}{\sigma} + (\rho'\sigma)'\right)\vec{b} \quad \ldots\ldots\ldots(i)$$

$$\Rightarrow \vec{t}_1.\vec{t}_1\frac{ds_1}{ds}\frac{ds_1}{ds} = \left(\frac{\rho}{\sigma} + (\rho'\sigma)'\right)\left(\frac{\rho}{\sigma} + (\rho'\sigma)'\right)\vec{b}.\vec{b} \Rightarrow \frac{ds_1}{ds} = \left(\frac{\rho}{\sigma} + (\rho'\sigma)'\right) \Rightarrow \frac{ds}{ds_1} = \left(\frac{\rho}{\sigma} + (\rho'\sigma)'\right)^{-1}$$

Now $(i) \Rightarrow \vec{t}_1 = \frac{ds}{ds_1}\left(\frac{\rho}{\sigma} + (\rho'\sigma)'\right)\vec{b} = \left(\frac{\rho}{\sigma} + (\rho'\sigma)'\right)^{-1}\left(\frac{\rho}{\sigma} + (\rho'\sigma)'\right)\vec{b} \Rightarrow \vec{t}_1 = \vec{b}$

$$\Rightarrow \frac{d\vec{t}_1}{ds_1} = \frac{d\vec{b}}{ds}\frac{ds}{ds_1} = \vec{b}'\left(\frac{\rho}{\sigma} + (\rho'\sigma)'\right)^{-1} = (-\tau\vec{n})\left(\frac{\rho}{\sigma} + (\rho'\sigma)'\right)^{-1} \Rightarrow \vec{t}_1{}' = (-\tau\vec{n})\left(\frac{\rho}{\sigma} + (\rho'\sigma)'\right)^{-1}$$

$\Rightarrow \vec{n}_1 = -\vec{n}$ and $K_1 = (\rho + \sigma(\rho'\sigma)')^{-1}$

$\Rightarrow \vec{b}_1 = \vec{t}_1 \times \vec{n}_1 = -(\vec{b} \times \vec{n}) \Rightarrow \vec{b}_1 = \vec{t}$

$$\Rightarrow \frac{d\vec{b}_1}{ds_1} = \frac{d\vec{t}}{ds}\frac{ds}{ds_1} = \vec{t}'\left(\frac{\rho}{\sigma} + (\rho'\sigma)'\right)^{-1} = (K\vec{n})\left(\frac{\rho}{\sigma} + (\rho'\sigma)'\right)^{-1} \Rightarrow \vec{b}_1{}' = \left(\frac{\vec{n}}{\rho}\right)\left(\frac{\rho}{\sigma} + (\rho'\sigma)'\right)^{-1}$$

$$\Rightarrow (-\tau_1\vec{n}_1) = \left(\frac{\vec{n}}{\rho}\right)\left(\frac{\rho}{\sigma} + (\rho'\sigma)'\right)^{-1} \Rightarrow \tau_1 = \left[\rho\left(\frac{\rho}{\sigma} + (\rho'\sigma)'\right)\right]^{-1}$$

Where the Serret Frenet apparatus are $K_1, \tau_1, \vec{t}_1, \vec{n}_1, \vec{b}_1$

QUESTION: Prove that the rectifying line is parallel to the vector $\tau\vec{t} + K\vec{b}$

PROOF: Since the equation of rectifying plane is $(\vec{R} - \vec{r}).\vec{n} = 0$ ………. (i)

$(\vec{R} - \vec{r}).\vec{n}' + (-\vec{r}').\vec{n} = 0 \Rightarrow (\vec{R} - \vec{r}).(\tau\vec{b} - K\vec{t}) + \vec{t}.\vec{n} = 0 \Rightarrow (\vec{R} - \vec{r}).(\tau\vec{b} - K\vec{t})$ …….(ii) from (i) and (ii) $(\vec{R} - \vec{r})$ is perpendicular to both $\vec{n}$ *and* $(\tau\vec{b} - K\vec{t})$

Now $\vec{n} \times (\tau\vec{b} - K\vec{t}) = [\tau(\vec{n} \times \vec{b}) - K(\vec{n} \times \vec{t})] = [\tau(\vec{t}) - K(-\vec{b})] = \tau\vec{t} + K\vec{b}$

$\Rightarrow (\vec{R} - \vec{r})$ is parallel to $\tau\vec{t} + K\vec{b}$ and hence rectifying lines are parallel to the vector $\tau\vec{t} + K\vec{b}$

QUESTION: Suppose the path traced by the particle is $\vec{x} = \vec{x}(t)$ then prove that
(i) Acceleration vector lies in the osculating plane.
(ii)Find the tangential and normal component of the acceleration.

Solution: Since $\vec{x} = \vec{x}(t)$ and $\Rightarrow \vec{x}' = \frac{d\vec{x}}{dt} = \frac{d\vec{x}}{ds}.\frac{ds}{dt} = \vec{x}'.\frac{ds}{dt} = \vec{t}.\frac{ds}{dt}$

$$\Rightarrow \vec{x}'' = \frac{d^2\vec{x}}{dt^2} = \frac{d}{dt}\left(\vec{t}.\frac{ds}{dt}\right) = \frac{d}{ds}\left(\vec{t}.\frac{ds}{dt}\right).\frac{ds}{dt} = \left[\frac{d\vec{t}}{ds}.\frac{ds}{dt} + \vec{t}.\frac{d^2s}{dt^2}.\frac{d\vec{t}}{ds}\right]\frac{ds}{dt}$$

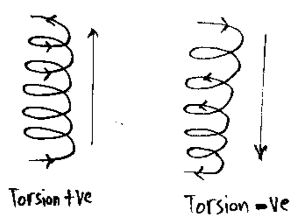
$$\Rightarrow \vec{x}'' = \vec{t}'\left(\frac{ds}{dt}\right)^2 + \frac{d^2s}{dt^2}.\vec{t} \qquad \Rightarrow \boldsymbol{\vec{x}'' = \left(\frac{ds}{dt}\right)^2 K\vec{n} + \frac{d^2s}{dt^2}.\vec{t}}$$

$\Rightarrow$ This shows that acceleration vector is in plane formed by $\vec{t}$ and $\vec{n}$ which is the osculating plane.

Tangential component of the acceleration = $\frac{d^2s}{dt^2}.\vec{t}$

normal component of the acceleration = $\left(\frac{ds}{dt}\right)^2 K\vec{n}$

TORSION: the rate of turning of binomial is called the torsion of the curve at the point 'P' . it is denoted by $'\boldsymbol{\tau}'$ Torsion can also be defined as **"the rate of rotation of the osculating plane"** this is also called **Second Curvature**.



- Curvature is always positive but torsion may be positive or negative.
- Torsion is regarded as positive if rotation of binormal as 's' increases is in same sense of right handed screw.

RADIUS OF TORSION: reciprocal of torsion is called radius of torsion and it is denoted by $'\delta'$ thus $\delta = \frac{1}{\tau}$ and $\tau = \frac{1}{\delta}$. It is important to note that there is no circle of torsion or centre of torsion associated with the curve in the same way as circle.

PREPOSITION(BS;2018): Let $\vec{r}(t)$ be a regular curve in R^3 with nowhere vanishing curvature then prove that $\tau = \frac{1}{K^2}[\vec{r}',\vec{r}'',\vec{r}'''] = \frac{[\vec{r}',\vec{r}'',\vec{r}''']}{\|\vec{r}'\times\vec{r}''\|^2}$ or $\tau = \frac{\vec{r}'.\vec{r}''\times\vec{r}'''}{\|\vec{r}'\times\vec{r}''\|^2} = \frac{\vec{r}'\times\vec{r}''.\vec{r}'''}{\|\vec{r}'\times\vec{r}''\|^2}$

PROOF : we know that $\vec{r}' = \vec{t}$,, $\vec{r}'' = \vec{t}' = K\vec{n}$ and $\vec{r}''' = K'\vec{n} + K\vec{n}' = K'\vec{n} + K(\tau\vec{b} - K\vec{t})$

then $[\vec{r}',\vec{r}'',\vec{r}'''] = \vec{r}'.\vec{r}''\times\vec{r}''' = \vec{t}.[K\vec{n}\times(K'\vec{n} + K\tau\vec{b} - K^2\vec{t})]$

$$\Rightarrow [\vec{r}',\vec{r}'',\vec{r}'''] = \vec{t}.\left[KK'(\vec{n}\times\vec{n}) + K^2\tau(\vec{n}\times\vec{b}) - K^3(\vec{n}\times\vec{t})\right]$$

$$\Rightarrow [\vec{r}',\vec{r}'',\vec{r}'''] = \vec{t}.\left[KK'(0) + K^2\tau(\vec{t}) - K^3(-\vec{b})\right] = K^2\tau(\vec{t}.\vec{t}) + K^3(\vec{t}.\vec{b}) = K^2\tau(1) + K^3(0)$$

$$\Rightarrow [\vec{r}',\vec{r}'',\vec{r}'''] = K^2\tau \qquad \Rightarrow \boldsymbol{\tau = \frac{1}{K^2}[\vec{r}',\vec{r}'',\vec{r}''']} \quad \ldots\ldots\ldots\ldots(i)$$

Now since $\vec{r}$ is a unit speed curve so $\vec{r}' \perp \vec{r}'' \Rightarrow \|\vec{r}'\times\vec{r}''\|^2 = \|\vec{r}'\|^2\|\vec{r}''\|^2 Sin^2 90° = 1.K^2.1$

$\Rightarrow \|\vec{r}'\times\vec{r}''\|^2 = K^2$ then $(i) \Rightarrow \boldsymbol{\tau = \frac{\vec{r}'.\vec{r}''\times\vec{r}'''}{\|\vec{r}'\times\vec{r}''\|^2}}$

We may replace $\vec{r}'.\vec{r}''\times\vec{r}'''$ with $\vec{r}'\times\vec{r}''.\vec{r}'''$ thus also $\tau = \frac{\vec{r}'\times\vec{r}''.\vec{r}'''}{\|\vec{r}'\times\vec{r}''\|^2}$

EXAMPLE (Annual;2019):

Find torsion of the circular helix $\vec{r}(\theta) = (aCos\theta\,, aSin\theta,\ b\theta)$ also show that circular helix is just a circle in xy – plane. **Or** if b = 0 then $\tau = 0$ then the curve will be straight plane. Also find its curvature.

Given $\vec{r}(\theta) = (aCos\theta\,, aSin\theta,\ b\theta) \Longrightarrow\ \vec{r}'(\theta) = (-aSin\theta\,, aCos\theta,\ b)$
$\Longrightarrow \vec{r}''(\theta) = (-aCos\theta\,, -\,aSin\theta,\ 0) \Longrightarrow \vec{r}'''(\theta) = (aSin\theta\,, -aCos\theta,\ 0)$

$$\Longrightarrow \vec{r}' \times \vec{r}'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -aSin\theta & aCos\theta & b \\ -aCos\theta & -aSin\theta & 0 \end{vmatrix} = (ab\mathrm{Sin}\theta)\hat{\imath} - (ab\mathrm{Cos}\theta)\hat{\jmath} + (a^2\mathrm{Sin}^2\theta + a^2\mathrm{Cos}^2\theta)\hat{k}$$

$\Longrightarrow \vec{r}' \times \vec{r}'' = (ab\mathrm{Sin}\theta\ ,\ ab\mathrm{Cos}\theta\,,\ a^2)$
$\Longrightarrow \|\vec{r}' \times \vec{r}''\|^2 = a^2b^2\mathrm{Sin}^2\theta + a^2b^2\mathrm{Cos}^2\theta + a^4 = a^2b^2(\mathrm{Sin}^2\theta + \mathrm{Cos}^2\theta) + a^4 = a^2b^2 + a^4$
$\Longrightarrow \|\vec{r}' \times \vec{r}''\|^2 = a^2(a^2 + b^2)$
Also $\Longrightarrow \boldsymbol{\vec{r}' \times \vec{r}''.\vec{r}'''} = (ab\mathrm{Sin}\theta\ ,\ ab\mathrm{Cos}\theta\,,\ a^2).(aSin\theta\,, -aCos\theta,\ 0)$
$\Longrightarrow \boldsymbol{\vec{r}' \times \vec{r}''.\vec{r}'''} = a^2b\mathrm{Sin}^2\theta + a^2b\mathrm{Cos}^2\theta + 0 = a^2b(\mathrm{Sin}^2\theta + \mathrm{Cos}^2\theta) = a^2b$
$\Longrightarrow \tau = \frac{\vec{r}'\times\vec{r}''.\vec{r}'''}{\|\vec{r}'\times\vec{r}''\|^2} = \frac{a^2b}{a^2(a^2+b^2)} = \frac{\boldsymbol{b}}{(\boldsymbol{a^2+b^2})}$ and if b = 0 then $\tau = 0$ then the curve will be straight plane. Or circular helix is just a circle in xy – plane.

EXAMPLE: Find torsion of the circular helix $\vec{r} = (aCos\theta\,, aSin\theta,\ a\theta Cot\beta)$

Given $\vec{r} = (aCos\theta\,, aSin\theta,\ a\theta Cot\beta)$ this is the curve drawn on the surface of circular cylinder cutting the generators at the constant angle β.

$\Longrightarrow \frac{d\vec{r}}{ds} = \vec{r}' = \vec{t} = (-aSin\theta\,, aCos\theta, a\ Cot\beta)\theta'$ But this is the unit vector so that its square its unity and therefore $|\vec{t}|^2 = 1 \Longrightarrow a^2(\theta')^2 = Sin^2\beta \Longrightarrow (\theta')^2 = \frac{Sin^2\beta}{a^2} \Longrightarrow (\theta')^2 = cosntant$
$\Longrightarrow \vec{r}'' = (-aCos\theta\,, -aSin\theta, 0)(\theta')^2 + 0 \Longrightarrow \vec{r}'' = \vec{t}' = K\vec{n} = -a(Cos\theta\,, Sin\theta, 0)(\theta')^2$

Then the PRINCIPAL normal is the unit vector $\vec{n} = (Cos\theta\,, Sin\theta, 0)$ and $K = a(\theta')^2 = \frac{Sin^2\beta}{a}$

$\Longrightarrow \vec{r}''' = a(Sin\theta\,, -Cos\theta,\ 0)(\theta')^3$

$$\Longrightarrow \vec{r}'' \times \vec{r}''' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -aCos\theta(\theta')^2 & -aSin\theta(\theta')^2 & 0 \\ aSin\theta(\theta')^3 & -aCos\theta(\theta')^3 & 0 \end{vmatrix} = a^2(0\,, 0{,}1)(\theta')^5$$

Hence $\Longrightarrow K^2\tau = \vec{r}' \times \vec{r}''.\vec{r}''' = [\vec{r}', \vec{r}'', \vec{r}'''] = a^3Cot\beta(\theta')^6$

$$\Longrightarrow \tau = \frac{a^3Cot\beta(\theta')^6}{K^2} = \frac{a^3Cot\beta\left(\sqrt{\frac{Sin^2\beta}{a^2}}\right)^6}{\left(\frac{Sin^2\beta}{a}\right)^2} \Longrightarrow \boldsymbol{\tau = \frac{1}{a}Sin\beta Cos\beta}$$

QUESTION: Prove that **(i)** If $K = 0$ at all the points then curve is a straight line.
(ii) If $\tau = 0$ at all the points then curve is a plane.

Solution:

Proof (i)

Using Serret Frenet formulae $\vec{t}' = K\vec{n}$

Now if $K = 0$ then $\vec{t}' = 0$

$\Rightarrow \vec{t} = cosntant$

$\Rightarrow unit\ tangent\ is\ cosntant$
i.e. tangent is fixed

This is possible only if curve is a straight line.

Proof (ii)

Again Using Serret Frenet formulae
$\vec{b}' = -\tau\vec{n}$

Now if $\tau = 0$ then $\vec{b}' = 0$

$\Rightarrow \vec{b} = cosntant$

This is possible only if curve is a plane.

***POSSIBLE Q* (PP;2016,18)**: *Prove that if $K = 0$ at all the points then curve will be straight line. Also show the behavior of the curve if $\tau = 0$*

PREPOSTION: Let $\vec{r}$ be a regular curve in R^3 i.e. curve of class ≥ 3 with nowhere vanishing curvature then then image of $\vec{r}$ is contained in a plane if and only if $\tau = 0$ at every point of the curve.

PROOF: We can assume that $\vec{r}$ is a unit speed curve and $\vec{r} = \vec{r}(s)$ where's' a parameter. And suppose first the image of $\vec{r}$ contained in plane $\vec{V}.\vec{N} = d$ where $\vec{N}$ is a fixed vector and $\vec{V} \epsilon R^3$ and we can assume that $\vec{N}$ is a unit vector

Now since $\vec{r}$ lies on plane $\vec{V}.\vec{N} = d$ therefore $\vec{r}.\vec{N} = d \Rightarrow \vec{r}'.\vec{N} + \vec{r}.\vec{N}' = 0$
$\Rightarrow \vec{r}'.\vec{N} = 0 \dots\dots(i)$ $\therefore \vec{N}$ is a fixed also $\Rightarrow \vec{t}.\vec{N} = 0 \dots\dots\dots(ii) \therefore \vec{r}' = \vec{t}$
$(ii) \Rightarrow \vec{t}'.\vec{N} + \vec{t}.\vec{N}' = 0 \Rightarrow \vec{t}'.\vec{N} = 0 \Rightarrow K\vec{n}.\vec{N} = 0 \Rightarrow K \neq 0 \Rightarrow \vec{n}.\vec{N} = 0 \dots\dots\dots(iii)$
equation $(ii) and\ (iii)$ show that $\vec{N}$ is perpendicular to both $\vec{t}\ and\ \vec{n} \Rightarrow (\vec{b} = \vec{t} \times \vec{n}) \parallel \vec{N}$
now since $\vec{b}\ and\ \vec{N}$ are unit vectors and $\vec{b}(s)$ is smooth function of 's' we must have
$\vec{b}(s) = \pm\vec{N} \Rightarrow \vec{b}$ is constant $\Rightarrow \vec{b}' = 0 \Rightarrow -\tau\vec{n} = 0 \Rightarrow -\vec{n} \neq 0$
$\Rightarrow \tau = 0$ for every point of the curve
CONVERSLY: Using Serret Frenet formulae $\vec{b}' = -\tau\vec{n}$
Now if $\tau = 0$ then $\vec{b}' = 0 \Rightarrow \vec{b} = cosntant$ this is possible only if curve is a plane.

- **Possible Question**: Show that a curve is a plane curve iff all osculating planes have a common point of intersections.

PREPOSTION (Annual;2015,19):

Show that the Principal normals at consecutive points do not intersect unless $\tau = 0$

PROOF: Suppose 'P" and 'Q' are two consecutive points with position vectors $\vec{r}\ and\ \vec{r} + d\vec{r}$ and unit PRINCIPAL normals be $\vec{n}\ and\ \vec{n} + d\vec{n}$. For intersection of the PRINCIPAL normals the necessary condition is that the three vectors $d\vec{r}$, $\vec{n}\ and\ d\vec{n}$ be coplanar. i.e. $\vec{r}'$, $\vec{n}\ and\ \vec{n}'$ be coplanar

This requires $[\vec{r}', \vec{n}, \vec{n}'] = 0 \Rightarrow [\vec{t},\ \vec{n},\ \tau\vec{b} - K\vec{t}] = 0 \Rightarrow \tau[\vec{t},\ \vec{n},\ \vec{b}] = 0 \Rightarrow \boldsymbol{\tau = 0}, [\vec{t},\ \vec{n},\ \vec{b}] \neq 0$
thus the PRINCIPAL normals at consecutive points do not intersect unless $\tau = 0$

QUESTION: Prove that the necessary and sufficient condition for a curve to be plane is $[\vec{r}',\vec{r}'',\vec{r}'''] = 0$.

PROOF: $(\boldsymbol{NECESSARY\ CONDITION})$**:** If the curve is plane then $\tau = 0$

Since we prove that $\tau = \frac{1}{K^2}[\vec{r}',\vec{r}'',\vec{r}'''] \Rightarrow \frac{1}{K^2}[\vec{r}',\vec{r}'',\vec{r}'''] = 0 \Rightarrow \frac{1}{K^2} \neq 0 \Rightarrow [\vec{r}',\vec{r}'',\vec{r}'''] = 0$

$(\boldsymbol{SUFFICIENT\ CONDITION})$:

Given that $[\vec{r}',\vec{r}'',\vec{r}'''] = 0 \Rightarrow \frac{1}{K^2}[\vec{r}',\vec{r}'',\vec{r}'''] = 0 \Rightarrow \tau = 0 \Rightarrow$ Curve is plane.

PREPOSTION (Annual;2018,19):

For $\vec{r} = \vec{r}(s)$ find the curvature and torsion of its centre of spherical curve.

Solution: Let $\vec{r} = \vec{r}(s) \Rightarrow \vec{r}' = \frac{d\vec{r}}{dt} = \frac{d\vec{r}(s)}{dt} = \frac{d\vec{r}(s)}{ds}.\frac{ds}{dt} = \vec{r}'(s).s' \Rightarrow \vec{r}' = \vec{r}'(s).s' \ldots\ldots(i)$

$\Rightarrow \vec{r}'' = \frac{d\vec{r}'}{dt} = \frac{d}{dt}[\vec{r}'(s).s'] = \frac{d\vec{r}'(s)}{dt}s' + \vec{r}'(s)\frac{ds'}{dt} = \frac{d\vec{r}'(s)}{ds}.\frac{ds}{dt}s' + \vec{r}'(s)s''$

$\Rightarrow \vec{r}'' = \vec{r}''(s)(s')^2 + \vec{r}'(s)s'' \ldots\ldots(ii)$

Taking cross product of $(i) and\ (ii) \Rightarrow \vec{r}' \times \vec{r}'' = [\vec{r}'(s).s'] \times [\vec{r}''(s)(s')^2 + \vec{r}'(s)s'']$

$\Rightarrow \vec{r}'' \times \vec{r}' = [\vec{r}'(s) \times \vec{r}''(s)](s')^3 + [\vec{r}'(s) \times \vec{r}'(s)]s''s' = [\vec{r}'(s) \times \vec{r}''(s)](s')^3 + 0$

$\Rightarrow \vec{r}' \times \vec{r}'' = [\vec{r}'(s) \times \vec{r}''(s)](s')^3 \ldots\ldots(iii)$

$(iii) \Rightarrow \frac{\|\vec{r}'\times\vec{r}''\|}{\|\vec{r}'\|^3} = \frac{\|\vec{r}'(s)\times\vec{r}''(s)\|\|s'\|^3}{\|\vec{r}'(s)\|^3\|s'\|^3} = \frac{\|\vec{r}'(s)\times\vec{r}''(s)\|}{\|\vec{r}'(s)\|^3} = \frac{\|\vec{r}'(s)\|\|\vec{r}''(s)\|Sin90^\circ}{\|\vec{r}'(s)\|^3}$

$\Rightarrow \frac{\|\vec{r}'\times\vec{r}''\|}{\|\vec{r}'\|^3} = \|\vec{r}''(s)\| = K \qquad\qquad \therefore \|\vec{r}'(s)\| = 1\ being\ unit\ speed\ curve.$

FOR TORTION:

Now since $\vec{r} = \vec{r}(s) \Rightarrow \vec{r}' = \vec{r}'(s).s' \ldots\ldots(i)$ and $\Rightarrow \vec{r}'' = \vec{r}''(s)(s')^2 + \vec{r}'(s)s'' \ldots\ldots(ii)$

Then $\Rightarrow \vec{r}''' = \frac{d\vec{r}''}{dt} = \frac{d\vec{r}''(s)}{ds}.\frac{ds}{dt}.(s')^2 + \vec{r}''(s)2s's'' + \frac{d\vec{r}'(s)}{ds}.\frac{ds}{dt}.s'' + \vec{r}'(s)s'''$

$\Rightarrow \vec{r}''' = \vec{r}'''(s).(s')^3 + 2\vec{r}''(s)s's'' + \vec{r}''(s)s's'' + \vec{r}'(s)s'''$

$\Rightarrow \vec{r}''' = \vec{r}'''(s).(s')^3 + 3\vec{r}''(s)s's'' + \vec{r}'(s)s''' \ldots\ldots(iii)$

Now $\Rightarrow \vec{r}' \times \vec{r}'' = [\vec{r}'(s) \times \vec{r}''(s)](s')^3$

Then $\Rightarrow (\vec{r}' \times \vec{r}'').\vec{r}''' = \{[\vec{r}'(s) \times \vec{r}''(s)](s')^3\}.\{\vec{r}'''(s).(s')^3 + 3\vec{r}''(s)s's'' + \vec{r}'(s)s'''\}$

$$\Rightarrow (\vec{r}' \times \vec{r}'').\vec{r}''' = \{[\vec{r}'(s) \times \vec{r}''(s)].\vec{r}'''(s)\}(s')^6 + 3\{[\vec{r}'(s) \times \vec{r}''(s)].\vec{r}''(s)\}(s')^4s'' + \{[\vec{r}'(s) \times \vec{r}''(s)].\vec{r}'(s)(s')^3s'''\}$$

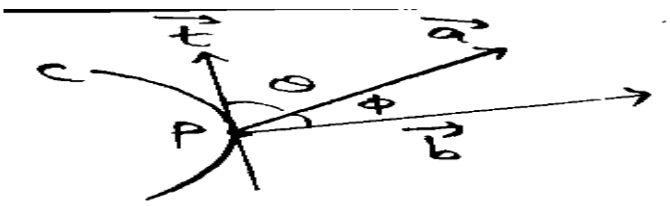
$\Rightarrow (\vec{r}' \times \vec{r}'').\vec{r}''' = \{[\vec{r}'(s) \times \vec{r}''(s)].\vec{r}'''(s)\}(s')^6 + 0 + 0$

$\Rightarrow (\vec{r}' \times \vec{r}'').\vec{r}''' = \{[\vec{r}'(s) \times \vec{r}''(s)].\vec{r}'''(s)\}(s')^6$ Also $\|\vec{r}' \times \vec{r}''\|^2 = \|\vec{r}'(s) \times \vec{r}''(s)\|^2(s')^6$

Then $\Rightarrow \frac{(\vec{r}'\times\vec{r}'').\vec{r}'''}{\|\vec{r}'\times\vec{r}''\|^2} = \frac{\{[\vec{r}'(s)\times\vec{r}''(s)].\vec{r}'''(s)\}(s')^6}{\|\vec{r}'(s)\times\vec{r}''(s)\|^2(s')^6} = \frac{\{[\vec{r}'(s)\times\vec{r}''(s)].\vec{r}'''(s)\}}{\|\vec{r}'(s)\times\vec{r}''(s)\|^2} = \tau$

QUESTION: if tangent and binormal at a point of curve make angle 'θ' and 'φ' respectively with the fixed direction then show that $\frac{Sin\theta d\theta}{Sin\varphi d\varphi} = -\frac{K}{\tau}$

Solution:



Let C be a given curve with a point 'P' on it and 'a' is any unit vector along the fixed direction making angles 'θ' and 'φ' with tangent and binormal respectively. Then

$\vec{t}.\vec{a} = |\vec{t}|.|\vec{a}|\, \mathrm{Cos}\theta = \mathrm{Cos}\theta$

$\Rightarrow \vec{t}.\vec{a} = \mathrm{Cos}\theta$

Differentiate with respect to 's'

$\Rightarrow \vec{t}'.\vec{a} + 0 = -\mathrm{Sin}\theta \frac{d\theta}{ds}$

$\Rightarrow K\vec{n}.\vec{a} = -\mathrm{Sin}\theta \frac{d\theta}{ds}$ ....(i)

Similarly

$\vec{b}.\vec{a} = |\vec{b}|.|\vec{a}|\, \mathrm{Cos}\varphi = \mathrm{Cos}\varphi$

$\Rightarrow \vec{b}.\vec{a} = \mathrm{Cos}\varphi$

Differentiate with respect to 's'

$\Rightarrow \vec{b}'.\vec{a} + 0 = -\mathrm{Sin}\varphi \frac{d\varphi}{ds}$

$\Rightarrow -\tau\vec{n}.\vec{a} = -\mathrm{Sin}\varphi \frac{d\varphi}{ds}$ ....(ii)

Dividing eq (i) by (ii) $\frac{K\vec{n}.\vec{a}}{-\tau\vec{n}.\vec{a}} = \frac{-\mathrm{Sin}\theta\frac{d\theta}{ds}}{-\mathrm{Sin}\varphi\frac{d\varphi}{ds}}$ $\Rightarrow -\frac{K}{\tau} = \frac{\boldsymbol{Sin\theta d\theta}}{\boldsymbol{Sin\varphi d\varphi}}$

Question: For any curve $\vec{r} = (4aCos^3u, 4aSin^3u, 3cCos2u)$ prove that $\vec{n} = (Sinu, Cosu, 0)$ and $K = \frac{a}{6(a^2+c^2)Sin2u}$

Solution: Since $\vec{r} = (4aCos^3u, 4aSin^3u, 3cCos2u)$

$\Rightarrow \vec{r}' = \vec{t} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{du}.\frac{du}{ds} = (-12aCos^2u\, Sinu, 12aSin^2u\, Cosu, -6cSin2u)u'$

$\Rightarrow \vec{t} = (-12aCos^2u\, Sinu, 12aSin^2u\, Cosu, -6cSin2u)u'$

$\Rightarrow \vec{t} = (-6aCosu\, Sin2u, 12aSin^2u\, Cosu, -6cSin2u)u'$

$\Rightarrow \vec{t} = 6Sin2u(-aCosu, aSinu, -c)u'$ ....(i)

$\Rightarrow |\vec{t}| = |6Sin2u(-aCosu, aSinu, -c)u'|$

$\Rightarrow 1 = \sqrt{(6Sin2u)^2[(-aCosu)^2 + (aSinu)^2 + (-c)^2](u')^2}$

$\Rightarrow 1 = (6Sin2u)\sqrt{[a^2Cos^2u + a^2Sin^2u + c^2]}.(u')$

$\Rightarrow 1 = (6Sin2u)\sqrt{[a^2(Cos^2u + Sin^2u) + c^2]}.(u')$

$\Rightarrow 1 = (6Sin2u)\sqrt{[a^2 + c^2]}.(u')$ $\Rightarrow \boldsymbol{u' = \frac{1}{(6Sin2u)\sqrt{a^2+c^2}}}$

$(i) \Rightarrow \vec{t} = 6Sin2u(-aCosu, aSinu, -c).\frac{1}{(6Sin2u)\sqrt{a^2+c^2}}$ $\Rightarrow \vec{t} = \frac{(-aCosu, aSinu, -c)}{\sqrt{a^2+c^2}}$

Differentiate w.r.to 's' $\Rightarrow \vec{t}' = \frac{d\vec{t}}{ds} = \frac{d\vec{t}}{du}.\frac{du}{ds} = \frac{(aSinu, aCosu, 0)}{\sqrt{a^2+c^2}}u'$

$\Longrightarrow\ K\vec{n} = \frac{(aSinu, aCosu, 0)}{\sqrt{a^2+c^2}}.\frac{1}{(6Sin2u)\sqrt{a^2+c^2}}$

$\Longrightarrow\ K\vec{n} = \frac{(aSinu, aCosu, 0)}{(6Sin2u)(a^2+c^2)} = \frac{a}{(6Sin2u)(a^2+c^2)}.(Sinu, Cosu, 0)$ ....(ii)

$\Longrightarrow\ |K\vec{n}| = \left|\frac{a}{(6Sin2u)(a^2+c^2)}.(Sinu, Cosu, 0)\right|$

$\Longrightarrow\ |K\vec{n}| = \sqrt{\left[\frac{a}{(6Sin2u)(a^2+c^2)}\right]^2.[(Sinu)^2+(Cosu)^2]}$

$\Longrightarrow\ |K\vec{n}| = \sqrt{\left[\frac{a}{(6Sin2u)(a^2+c^2)}\right]^2} = \left[\frac{a}{(6Sin2u)(a^2+c^2)}\right] \Longrightarrow\ |\boldsymbol{K}| = \left[\frac{\boldsymbol{a}}{(\boldsymbol{6Sin2u})(\boldsymbol{a^2+c^2})}\right]$

$\Longrightarrow\ K\vec{n} = K.(Sinu, Cosu, 0)\ \Longrightarrow\ \vec{\boldsymbol{n}} = (\boldsymbol{Sinu}, \boldsymbol{Cosu}, \boldsymbol{0})$ Hence proved

Question: Prove that the position vector of the current point on a curve satisfied the differential equation $\frac{d}{ds}\left\{\delta\frac{d}{ds}\left(\rho\frac{d^2\vec{r}}{ds^2}\right)\right\}+\frac{d}{ds}\left(\frac{\delta}{\rho}\frac{d\vec{r}}{ds}\right)+\frac{\rho}{\delta}\frac{d^2\vec{r}}{ds^2} = 0$

Solution:

$L.H.S = \frac{d}{ds}\left\{\delta\frac{d}{ds}\left(\rho\frac{d^2\vec{r}}{ds^2}\right)\right\}+\frac{d}{ds}\left(\frac{\delta}{\rho}\frac{d\vec{r}}{ds}\right)+\frac{\rho}{\delta}\frac{d^2\vec{r}}{ds^2}$

$= \frac{d}{ds}\left\{\delta\frac{d}{ds}(\rho K\vec{n})\right\}+\frac{d}{ds}\left(\frac{\delta}{\rho}\vec{t}\right)+\frac{\rho}{\delta}K\vec{n}$

$= \frac{d}{ds}\left\{\delta\frac{d}{ds}(\rho K\vec{n})\right\}+\frac{d}{ds}(\delta K\vec{t})+\rho\tau K\vec{n}$ $\qquad \therefore \frac{1}{\delta} = \tau\ \&\ \frac{1}{\rho} = \mathrm{K}$

$= \frac{d}{ds}\left\{\delta\frac{d}{ds}(\vec{n})\right\}+\frac{d}{ds}(\delta K\vec{t})+\tau\vec{n}$ $\qquad \therefore \frac{1}{\delta} = \tau\ \&\ \frac{1}{\rho} = \mathrm{K}$

$= \frac{d}{ds}\{\delta(\tau\vec{b}-K\vec{t})\}+\delta' K\vec{t}+\delta K'\vec{t}+\delta K\vec{t'}+\tau\vec{n} = \frac{d}{ds}\{\delta\tau\vec{b}-\delta K\vec{t}\}+\delta' K\vec{t}+\delta K'\vec{t}+\delta K\vec{t'}+\tau\vec{n}$

$= \frac{d}{ds}\{\vec{b}-\delta K\vec{t}\}+\delta' K\vec{t}+\delta K'\vec{t}+\delta K\vec{t'}+\tau\vec{n}$ $\qquad \therefore \frac{1}{\delta} = \tau\ \&\ \frac{1}{\rho} = \mathrm{K}$

$= \vec{b}'-\delta' K\vec{t}-\delta K'\vec{t}-\delta K\vec{t'}+\delta' K\vec{t}+\delta K'\vec{t}+\delta K\vec{t'}+\tau\vec{n} = -\tau\vec{n}+\tau\vec{n} = 0 = \mathrm{R.H.S}$

PARAMETER OTHER THAN 's': these are as follows;

- $K = \frac{|\vec{r}'\times\vec{r}''|}{|s'|^3}$
- $\vec{b} = \frac{\vec{r}'\times\vec{r}''}{K(s')^3}$
- $\tau = \frac{[\vec{r}'\times\vec{r}''.\vec{r}''']}{K^2(s')^6} = \frac{[\vec{r}'\times\vec{r}''.\vec{r}''']}{K^2|\vec{r}'|^6}$
- $\vec{n} = \frac{[s'\vec{r}''-s''\vec{r}']}{K(s')^3}$
- $K^2 = \frac{(\vec{r}'')^2-(s'')^2}{(s')^4}$

PROVE THAT $\boldsymbol{K} = \frac{|\vec{r}'\times\vec{r}''|}{|s'|^3}$ ***and*** $\vec{\boldsymbol{b}} = \frac{\vec{r}'\times\vec{r}''}{K(s')^3}$

Since $\vec{r} = \vec{r}(u)$ and $\Longrightarrow\ \vec{r}' = \frac{d\vec{r}}{du} = \frac{d\vec{r}}{ds}.\frac{ds}{du} = \vec{r}'.\frac{ds}{dt} = \vec{t}.s'$ ....(i)

$\Longrightarrow\ \vec{r}'' = \frac{d^2\vec{r}}{du^2} = \frac{d}{du}(\vec{t}.s') = \left[\frac{d\vec{t}}{du}.s'+\vec{t}.\frac{ds'}{ds}\right] = \left(\frac{d\vec{t}}{ds}.\frac{ds}{du}\right).s'+\vec{t}.\left(\frac{ds'}{ds}.\frac{ds}{du}\right)$

$\Longrightarrow\ \vec{r}'' = \left(\frac{d\vec{t}}{ds}.\frac{ds}{du}\right).s'+\vec{t}.s'' = \vec{t}'.s'.s'+\vec{t}.s'' = K\vec{n}(s')^2+\vec{t}.s''$

$\Longrightarrow\ \vec{r}'' = K\vec{n}(s')^2+\vec{t}.s''$ .... (ii)

$$\Rightarrow\ \vec{r}''' = K'\vec{n}(s')^2 + K\vec{n}'(s')^2 + K\vec{n}2s's'' + \frac{d\vec{t}}{du}.s'' + \vec{t}.\frac{ds''}{du}$$

$$\Rightarrow\ \vec{r}''' = K'\vec{n}(s')^2 + K\left(\frac{d\vec{n}}{ds}.\frac{ds}{du}\right)(s')^2 + K\vec{n}2s's'' + \left(\frac{d\vec{t}}{ds}.\frac{ds}{du}\right).s'' + \vec{t}.\frac{ds''}{du}$$

$$\Rightarrow\ \vec{r}''' = K'\vec{n}(s')^2 + K\vec{n}'s'(s')^2 + 2K\vec{n}s's'' + \vec{t}'s's'' + \vec{t}s'''$$

$$\Rightarrow\ \vec{r}''' = K'\vec{n}(s')^2 + K(\tau\vec{b} - K\vec{t})(s')^3 + 2K\vec{n}s's'' + K\vec{n}s's'' + \vec{t}s'''$$

$$\Rightarrow\ \vec{r}''' = K'\vec{n}(s')^2 + K\tau\vec{b}(s')^3 - K^2\vec{t}(s')^3 + 2K\vec{n}s's'' + K\vec{n}s's'' + \vec{t}s'''$$

$$\Rightarrow\ \vec{r}''' = K'\vec{n}(s')^2 + K\tau\vec{b}(s')^3 - K^2\vec{t}(s')^3 + 3K\vec{n}s's'' + \vec{t}s''' \quad \text{..............(iii)}$$

From (i) and (ii) we have

$$\vec{r}' \times \vec{r}'' = \vec{t}.s' \times (K\vec{n}(s')^2 + \vec{t}.s'') = K(s')^3(\vec{t}\times\vec{n}) + s's''(\vec{t}\times\vec{t}) = K(s')^3(\vec{b}) + s's''(0)$$

$$\Rightarrow \vec{r}' \times \vec{r}'' = K(s')^3(\vec{b}) \Rightarrow |\vec{r}' \times \vec{r}''| = |K(s')^3(\vec{b})| = |K(s')^3||\vec{b}| = \mathrm{K}|s'|^3.1$$

$\Rightarrow \boldsymbol{K} = \frac{|\vec{r}'\times\vec{r}''|}{|s'|^3}$ **and** $\Rightarrow \vec{\boldsymbol{b}} = \frac{\vec{r}'\times\vec{r}''}{K(s')^3}$

REMARK:

- Arc rate to turning of tangent is called curvature.
- Arc rate to turning of binormal is called tortion.
- Arc rate to turning of Principal normal is called screw curvature.

PROVE THAT $\boldsymbol{\tau} = \frac{[\vec{r}'\times\vec{r}''.\vec{r}''']}{K^2(s')^6} = \frac{[\vec{r}'\times\vec{r}''.\vec{r}''']}{K^2|\vec{r}'|^6}$

Since $\Rightarrow \vec{r}' \times \vec{r}'' = K(s')^3(\vec{b})$

$$\Rightarrow \vec{r}' \times \vec{r}''.\vec{r}''' = K(s')^3(\vec{b}).\{K'\vec{n}(s')^2 + K\tau\vec{b}(s')^3 - K^2\vec{t}(s')^3 + 3K\vec{n}s's'' + \vec{t}s'''\}$$

$$\Rightarrow \vec{r}' \times \vec{r}''.\vec{r}''' = \mathrm{K}K'(s')^5(\vec{b}.\vec{n}) + K^2\tau(s')^6(\vec{b}.\vec{b}) - KK^2(s')^6(\vec{b}.\vec{t}) + 3K^2(s')^4s''(\vec{b}.\vec{n}) + K(s')^3s'''(\vec{b}.\vec{t})$$

$$\Rightarrow \vec{r}' \times \vec{r}''.\vec{r}''' = \mathrm{K}K'(s')^5(0) + K^2\tau(s')^6(1) - KK^2(s')^6(0) + 3K^2(s')^4s''(0) + K(s')^3s'''(0)$$

$$\Rightarrow \vec{r}' \times \vec{r}''.\vec{r}''' = K^2\tau(s')^6 \Rightarrow \boldsymbol{\tau} = \frac{[\vec{r}'\times\vec{r}''.\vec{r}''']}{K^2(s')^6} = \frac{[\vec{r}'\times\vec{r}''.\vec{r}''']}{K^2|\vec{r}'|^6} \quad \therefore\ \vec{r}' = \vec{t}.s' \Rightarrow |\vec{r}'| = |\vec{t}|.|s'| = s'$$

PROVE THAT $\vec{\boldsymbol{n}} = \frac{[s'\vec{r}''-s''\vec{r}']}{K(s')^3}$

PROOF

Since $\vec{r}' = \vec{t}.s'$ and $\vec{r}'' = K\vec{n}(s')^2 + \vec{t}.s''$

$$\Rightarrow s''\vec{r}' = s''(\vec{t}.s') = \vec{t}s's'' \quad \text{................(i)}$$

$$s'\vec{r}'' = s'(K\vec{n}(s')^2 + \vec{t}.s'') = K\vec{n}(s')^3 + \vec{t}.s's'' \quad \text{..............(ii)}$$

Subtracting equation (i) from (ii) $\Rightarrow s'\vec{r}'' - s''\vec{r}' = K\vec{n}(s')^3 + \vec{t}.s's'' - \vec{t}s's'' = K\vec{n}(s')^3$

$$\vec{n} = \frac{[s'\vec{r}''-s''\vec{r}']}{K(s')^3}$$

PROVE THAT $\quad K^2 = \frac{(\vec{r}'')^2-(s'')^2}{(s')^4}$

PROOF: Since $\quad \vec{r}'' = \vec{t}'(s')^2 + \vec{t}.s''$

$\Rightarrow (\vec{r}'')^2 = \vec{r}''.\vec{r}'' = \left(\vec{t}'(s')^2 + \vec{t}.s''\right).\left(\vec{t}'(s')^2 + \vec{t}.s''\right)$

$\Rightarrow (\vec{r}'')^2 = \vec{t}'.\vec{t}'(s')^4 + \vec{t}'.\vec{t}(s')^2s'' + \vec{t}'.\vec{t}.s''(s')^2 + \vec{t}.\vec{t}.(s'')^2$

$\Rightarrow (\vec{r}'')^2 = K\vec{n}.K\vec{n}(s')^4 + K\vec{n}.\vec{t}(s')^2s'' + K\vec{n}.\vec{t}.s''(s')^2 + (1).(s'')^2$

$\Rightarrow (\vec{r}'')^2 = K^2\vec{n}.\vec{n}(s')^4 + K\vec{n}.\vec{t}(s')^2s'' + K\vec{n}.\vec{t}.s''(s')^2 + (1).(s'')^2$

$\Rightarrow (\vec{r}'')^2 = K^2(1)(s')^4 + K\vec{n}.\vec{t}(s')^2s'' + K\vec{n}.\vec{t}.s''(s')^2 + (1).(s'')^2$

$\Rightarrow (\vec{r}'')^2 = K^2(s')^4 + (s'')^2 \qquad\qquad \therefore \vec{n}.\vec{t} = 0$

$\Rightarrow (\vec{r}'')^2 - (s'')^2 = K^2(s')^4$

$\Rightarrow \boldsymbol{K^2} = \frac{(\vec{r}'')^2-(s'')^2}{(s')^4} \Rightarrow \boldsymbol{K^2} = \frac{(\vec{r}'')^2-(s'')^2}{(|\vec{r}'|)^4} \qquad \vec{r}' = \vec{t}.s' \Rightarrow |\vec{r}'| = |\vec{t}|.|s'| = s'$

Question: for the curve $x = a(3u - u^3), y = 3au^2, z = a(3u + u^3)$ prove that $\boldsymbol{K = \tau}$

Solution:

Since $K = \frac{|\vec{r}'\times\vec{r}''|}{|s'|^3} = \frac{|\vec{r}'\times\vec{r}''|}{|\vec{r}'|^3}$ and $\quad \tau = \frac{[\vec{r}'\times\vec{r}''.\vec{r}''']}{K^2(s')^6} = \frac{[\vec{r}'\times\vec{r}''.\vec{r}''']}{K^2(\vec{r}')^6}$

Now $\vec{r} = a(3u - u^3), 3au^2, a(3u + u^3)$

$\Rightarrow \overrightarrow{r'} = (3a - 3au^2), 6au, (3a - 3au^2) \qquad$ ....(i)

$\Rightarrow |\overrightarrow{r'}| = \sqrt{(3a - 3au^2)^2, +(6au)^2 + (3a - 3au^2)^2}$

$\Rightarrow |\overrightarrow{r'}| = \sqrt{9a^2 + 9a^2u^4 - 18a^2u^2 + 36a^2u^2 + 9a^2 + 9a^2u^4 + 18a^2u^2}$

$\Rightarrow |\overrightarrow{r'}| = \sqrt{18a^2 + 18a^2u^4 + 36a^2u^2} = \sqrt{18(a^2 + a^2u^4 + 2a^2u^2)}$

$\Rightarrow |\overrightarrow{r'}| = \sqrt{18(a + au^2)^2} = 3a\sqrt{2}(1 + u^2)$

$(i) \Rightarrow \overrightarrow{r''} = -6au, 6a, 6au$ and $\overrightarrow{r'''} = -6a, 0, 6a$

$\vec{r}' \times \vec{r}'' = [(3a - 3au^2), 6au, (3a - 3au^2)] \times [-6au, 6a, 6au]$

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a(3 - 3u^2) & 6au & a(3 - 3u^2) \\ -6au & 6a & 6au \end{vmatrix}$$

$\Rightarrow \vec{r}' \times \vec{r}'' = \left(36a^2u^2 - 6a^2(3 + 3u^2)\right)\hat{\imath} - \left(6a^2u(3 - 3u^2) + 6a^2u(3 + 3u^2)\right)\hat{\jmath}$
$\quad + (6a^2(3 - 3u^2) + 6a^2u^2)\hat{k}$

$\Rightarrow \vec{r}' \times \vec{r}'' = (36a^2u^2 - 18a^2 - 18a^2u^2)\hat{\imath} - (18a^2u - 18a^2u^3 + 18a^2u + 18a^2u^3)\hat{\jmath}$
$\quad + (18a^2 - 18a^2u^2 + 36a^2u^2)\hat{k}$

$\Rightarrow \vec{r}' \times \vec{r}'' = (36a^2u^2 - 18a^2 - 18a^2u^2)\hat{\imath} - (36a^2u)\hat{\jmath} + (18a^2 - 18a^2u^2 + 36a^2u^2)\hat{k}$

$\Rightarrow \vec{r}' \times \vec{r}'' = 18a^2\left[(2u^2 - 1 - u^2)\hat{\imath} - (2u)\hat{\jmath} + (1 - u^2 + 2u^2)\hat{k}\right]$

$\Rightarrow \vec{r}' \times \vec{r}'' = 18a^2\left[(u^2 - 1)\hat{\imath} + (2u)\hat{\jmath} + (1 + u^2)\hat{k}\right]$

$\Longrightarrow |\vec{r}' \times \vec{r}''| = \sqrt{(18a^2)^2[(u^2-1)^2+(2u)^2+(1+u^2)^2]}$

$\Longrightarrow |\vec{r}' \times \vec{r}''| = 18a^2\sqrt{u^4-2u^2+1+4u^2+1+u^4+2u^2} = 18a^2\sqrt{2u^4+4u^2+2}$

$\Longrightarrow |\vec{r}' \times \vec{r}''| = 18a^2\sqrt{2}\sqrt{u^4+2u^2+1} = 18a^2\sqrt{2}\sqrt{(u^2+1)^2} = 18a^2\sqrt{2}(u^2+1)$

Now $K = \frac{|\vec{r}'\times\vec{r}''|}{|\vec{r}'|^3}$

$\Longrightarrow K = \frac{18a^2\sqrt{2}(u^2+1)}{\left|3a\sqrt{2}(1+u^2)\right|^3} = \frac{18a^2\sqrt{2}(u^2+1)}{27.a^3.2.\sqrt{2}.(1+u^2)^3} = \frac{1}{3a(1+u^2)^2}$ ....(A)

Now for τ we have

$\vec{r}'\times\vec{r}''.\vec{r}''' = 18a^2\left[(u^2-1)\hat{\imath}+(2u)\hat{\jmath}+(1+u^2)\hat{k}\right].[-6a,0,6a]$

$\Longrightarrow \vec{r}'\times\vec{r}''.\vec{r}''' = 18a^2[-6au^2+6a+0+6a+6au^2] = 18a^2[12a] = 216a^3$

Now $\tau = \frac{[\vec{r}'\times\vec{r}''.\vec{r}''']}{K^2|\vec{r}'|^6} = \frac{216a^3}{[\frac{1}{3a(1+u^2)^2}]^2\left(3a\sqrt{2}(1+u^2)\right)^6} = \frac{216a^3}{\frac{1}{9a^2(1+u^2)^4}[3^6.8.a^6\sqrt{2}(1+u^2)^6}$

$\Longrightarrow \tau = \frac{216a^3.9a^2(1+u^2)^4}{[3^6.8.a^6\sqrt{2}(1+u^2)^6} = \frac{27}{[3^4a(1+u^2)^2} = \frac{3^3}{[3^4a(1+u^2)^2} = \frac{1}{3a(1+u^2)^2}$

$\Longrightarrow \tau = \frac{216a^3.9a^2(1+u^2)^4}{[3^6.8.a^6\sqrt{2}(1+u^2)^6} = \frac{27}{[3^4a(1+u^2)^2} = \frac{3^3}{[3^4a(1+u^2)^2} = \frac{1}{3a(1+u^2)^2} \Longrightarrow \tau = \frac{1}{3a(1+u^2)^2}$ ....(B)

From (A) and (B) $\boldsymbol{K = \tau}$

NOTE: when two planes intersected then we get a line as a result of intersection and when three planes intersect then we get a point as a result of intersection. When two surfaces intersect each other we get a curve.

Question: for the curve $x = a(u - Sinu), y = a(1 - Cosu), z = bu$ find $\boldsymbol{K\&\tau}$

Solution:

Since $K = \frac{|\vec{r}'\times\vec{r}''|}{|s'|^3} = \frac{|\vec{r}'\times\vec{r}''|}{|\vec{r}'|^3}$ and $\tau = \frac{[\vec{r}'\times\vec{r}''.\vec{r}''']}{K^2(s')^6} = \frac{[\vec{r}'\times\vec{r}''.\vec{r}''']}{K^2(\vec{r}')^6}$

Now $\vec{r} = a(u - Sinu), a(1 - Cosu), bu \quad \Longrightarrow \overrightarrow{r'} = a(1 - Cosu), aSinu, b$ ....(i)

$\Longrightarrow \left|\overrightarrow{r'}\right| = \sqrt{[a(1-Cosu)]^2, +(aSinu)^2+(b)^2}$

$\Longrightarrow \left|\overrightarrow{r'}\right| = \sqrt{a^2+a^2Cos^2u-2a^2Cosu+a^2Sin^2u+b^2}$

$\Longrightarrow \left|\overrightarrow{r'}\right| = \sqrt{a^2+a^2-2a^2Cosu+b^2} = \sqrt{2a^2-2a^2Cosu+b^2} = \sqrt{2a^2(1-Cosu)+b^2}$

$(i) \Longrightarrow \overrightarrow{r''} = aSinu, aCosu, 0$ and $\overrightarrow{r'''} = aCosu, -aSinu, 0$

$\Longrightarrow \vec{r}'\times\vec{r}'' = [a(1-Cosu), aSinu, b] \times [aSinu, aCosu, 0]$

$$\Longrightarrow \vec{r}'\times\vec{r}'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a(1-Cosu) & aSinu & b \\ aSinu & aCosu & 0 \end{vmatrix} = (-abCosu)\hat{\imath} - (-abSinu)\hat{\jmath} + (a^2Cosu - a^2)\hat{k}$$

$\Longrightarrow |\vec{r}'\times\vec{r}''| = \sqrt{(-abCosu)^2+(-abSinu)^2+(a^2Cosu-a^2)^2}$

$\Longrightarrow |\vec{r}'\times\vec{r}''| = \sqrt{a^2b^2Cos^2u+a^2b^2Sin^2u+a^4(Cosu-1)^2}$

$\Longrightarrow |\vec{r}'\times\vec{r}''| = \sqrt{a^2b^2+a^4(Cosu-1)^2} = a\sqrt{b^2+a^2(Cosu-1)^2}$

Now $K = \frac{|\vec{r}'\times\vec{r}''|}{|\vec{r}'|^3} \implies K = \frac{a\sqrt{b^2+a^2(Cosu-1)^2}}{\left|\sqrt{2a^2(1-Cosu)+b^2}\right|^3}$

Now for τ we have

$\vec{r}'\times\vec{r}''.\vec{r}''' = \left[(-abCosu)\hat{\imath} - (-abSinu)\hat{\jmath} + (a^2Cosu - a^2)\hat{k}\right].[aCosu, -aSinu, 0]$

$\implies \vec{r}'\times\vec{r}''.\vec{r}''' = -a^2bCos^2u - a^2bSin^2u + 0 = -a^2b$

Now $\tau = \frac{[\vec{r}'\times\vec{r}''.\vec{r}''']}{K^2|\vec{r}'|^6} = \frac{-a^2b}{\left[\frac{a\sqrt{b^2+a^2(Cosu-1)^2}}{\left|\sqrt{2a^2(1-Cosu)+b^2}\right|^3}\right]^2\left(\sqrt{2a^2(1-Cosu)+b^2}\right)^6}$

$\implies \tau = \frac{-a^2b}{a^2(b^2+a^2(Cosu-1)^2)}.\frac{\left[\sqrt{2a^2(1-Cosu)+b^2}\right]^6}{[2a^2(1-Cosu)+b^2]^3}$

$\implies \tau = \frac{-b}{(b^2+a^2(Cosu-1)^2)}.\frac{[2a^2(1-Cosu)+b^2]^3}{[2a^2(1-Cosu)+b^2]^3} \implies \tau = \frac{-b}{(b^2+a^2(Cosu-1)^2)}$

Question: for a point of the curve of intersection of the surfaces $x^2 - y^2 = c^2, y = xTanh\frac{z}{c}$ prove that $\rho = \delta = \frac{2x^2}{c}$ where $\delta = \frac{1}{\tau}$ & $\rho = \frac{1}{K}$

Solution:

Since $x^2 - y^2 = c^2 \implies x^2 - y^2 = c^2(Cosh^2\theta - Sinh^2\theta) = (c^2Cosh^2\theta - c^2Sinh^2\theta)$

$\implies x^2 - y^2 = (c^2Cosh^2\theta - c^2Sinh^2\theta)$

$\implies x^2 = c^2Cosh^2\theta,\ y^2 = c^2Sinh^2\theta \qquad \implies x = cCosh\theta \ldots..(i),\ y = cSinh\theta \ldots\ldots(ii)$

From (i) and (ii) $\quad \frac{y}{x} = Tanh\theta$

But we are given that $\frac{y}{x} = Tanh\frac{z}{c} \implies Tanh\theta = Tanh\frac{z}{c} \implies \theta = \frac{z}{c} \implies z = c\theta$

So $\vec{r} = (x,y,z) = (cCosh\theta, cSinh\theta, c\theta) \implies \vec{r}' = (cSinh\theta, cCosh\theta, c)$

$\implies |\vec{r}'| = \sqrt{(cSinh\theta)^2 + (cCosh\theta)^2 + c^2} = \sqrt{c^2(Sinh^2\theta + Cosh^2\theta + 1)}$

$\implies |\vec{r}'| = c\sqrt{(Cosh^2\theta + Cosh^2\theta)} = c\sqrt{2Cosh^2\theta} = \sqrt{2}cCosh\theta$

Now as $\implies \vec{r}' = (cSinh\theta, cCosh\theta, c)$

$\implies \vec{r}'' = (cCosh\theta, cSinh\theta, 0) \qquad \implies \vec{r}''' = (cSinh\theta, cCosh\theta, 0)$

Now $\vec{r}'\times\vec{r}'' = ? \implies \vec{r}'\times\vec{r}'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ cSinh\theta & cCosh\theta & c \\ cCosh\theta & cSinh\theta & 0 \end{vmatrix}$

$\implies \vec{r}'\times\vec{r}'' = (-c^2Sinh\theta)\hat{\imath} - (-c^2Cosh\theta)\hat{\jmath} + (c^2Sinh^2\theta - c^2Cosh^2\theta)\hat{k}$

$\implies \vec{r}'\times\vec{r}'' = (-c^2Sinh\theta)\hat{\imath} + (c^2Cosh\theta)\hat{\jmath} + c^2(-1)\hat{k} = (-c^2Sinh\theta)\hat{\imath} + (c^2Cosh\theta)\hat{\jmath} - c^2\hat{k}$

$\implies |\vec{r}'\times\vec{r}''| = \sqrt{(-c^2Sinh\theta)^2 + (c^2Cosh\theta)^2 + (-c^2)^2} = \sqrt{c^4Sinh^2\theta + c^4Cosh^2\theta + c^4}$

$\implies |\vec{r}'\times\vec{r}''| = c^2\sqrt{Sinh^2\theta + Cosh^2\theta + 1} = c^2\sqrt{Cosh^2\theta + Cosh^2\theta} = c^2\sqrt{2Cosh^2\theta}$

$\implies |\vec{r}'\times\vec{r}''| = c^2\sqrt{2}Cosh\theta$

$\Rightarrow \vec{r}' \times \vec{r}''.\vec{r}''' = \left[(-c^2\text{Sinh}\theta)\hat{\imath} + (c^2\text{Cosh}\theta)\hat{\jmath} - c^2\hat{k}\right].(c\text{Sinh}\theta)\hat{\imath} + (c\text{Cosh}\theta)\hat{\jmath} + 0\hat{k}$

$\Rightarrow \vec{r}' \times \vec{r}''.\vec{r}''' = -c^3\text{Sinh}^2\theta + c^3\text{Cosh}^2\theta + 0$

Now $K = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} \quad \Rightarrow K = \frac{c^2\sqrt{2}\text{Cosh}\theta}{(\sqrt{2}c\text{Cosh}\theta)^3} = \frac{c(c\sqrt{2}\text{Cosh}\theta)}{(\sqrt{2}c\text{Cosh}\theta)^3} = \frac{c}{(\sqrt{2}c\text{Cosh}\theta)^2} = \frac{c}{2c^2\text{Cosh}^2\theta}$

$\Rightarrow K = \frac{c}{2x^2} = \frac{1}{\rho} \qquad\qquad \Rightarrow \rho = \frac{2x^2}{c}$ .... (iii)

Also $\tau = \frac{[\vec{r}' \times \vec{r}''.\vec{r}''']}{K^2|\vec{r}'|^6} = \frac{-c^3\text{Sinh}^2\theta + c^3\text{Cosh}^2\theta}{K^2|\vec{r}'|^6} = \frac{c(-c^2\text{Sinh}^2\theta + c^2\text{Cosh}^2\theta)}{K^2|\vec{r}'|^6} = \frac{c(x^2-y^2)}{(\frac{c}{2x^2})^2(\sqrt{2}c\text{Cosh}\theta)^6}$

$\Rightarrow \tau = \frac{c(c^2)}{\frac{c^2}{4x^4}.2c^6\text{Cosh}^6\theta} = \frac{c^3.4x^4}{c^2.8x^6} = \frac{c}{2x^2} = \frac{1}{\delta} \qquad \Rightarrow \delta = \frac{2x^2}{c}$ .... (iv)

From (iii) and (iv) $\qquad \rho = \delta = \frac{2x^2}{c}$

REMARK:

A curve is defined uniquely by its curvature and torsion as functions of natural parameters. i.e. $K = K(s)\ \ and\ \tau = \tau(s)$

NATURAL REPRESENTATION OR INTRINSIC EQUATIONS:

The equations $K = K(s)\ \ and\ \tau = \tau(s)$ which give the curvature and torsion of the curve as functions of natural parameter ‘s’ are called the natural or intrinsic equations of a curve, for they completely define the curve.

QUESTION (BS;2017):

Find the natural representation or intrinsic equation of the curve

$$x = e^t\{aCoste_1 + aSinte_2 + be_3\}\ ;\ -\infty < t < \infty$$

Solution: Given $x = e^t\{aCoste_1 + aSinte_2 + be_3\}\ ;\ -\infty < t < \infty$

$\Rightarrow \dot{x} = e^t\{a(Cost - Sint)e_1 + a(Sint + Cost)e_2 + be_3\}$

$\Rightarrow \ddot{x} = e^t\{a(Cost - Sint - Sint - Cost)e_1 + a(Sint + Cost + Cost - Sint)e_2 + be_3\}$

$\Rightarrow \ddot{x} = e^t\{-2a(Sint)e_1 + 2a(Cost)e_2 + be_3\}$

$\Rightarrow \dddot{x} = e^t\{-2a(Sint + Cost)e_1 + 2a(Cost - Sint)e_2 + be_3\}$

$\Rightarrow \dot{x} \times \ddot{x} = e^{2t}\{ab(Sint - Cost)e_1 - ab(Cost + Sint)e_2 + 2a^2e_3\}$

$\Rightarrow |\dot{x}| = e^t\sqrt{2a^2 + b^2}$ and $\quad \Rightarrow |\dot{x} \times \ddot{x}| = \sqrt{2}ae^{2t}\sqrt{2a^2 + b^2}$

Using above values find $K(s), \tau(s), \vec{t}(s), \vec{n}(s), \vec{b}(s)$

Question: Prove that the curve $x = aSin^2u$, $y = aSinuCosu$, $z = aCosu$ lies on a sphere and also verify that all the normal planes pass through origin.

Solution: Let $x = aSin^2u$, $y = aSinuCosu$, $z = aCosu$

$\Rightarrow x^2 + y^2 + z^2 = (aSin^2u)^2 + (aSinuCosu)^2 + (aCosu)^2$

$\Rightarrow x^2 + y^2 + z^2 = a^2Sin^4u + a^2Sin^2uCos^2u + a^2Cos^2u$

$\Rightarrow x^2 + y^2 + z^2 = a^2Sin^2u(Sin^2u + Cos^2u) + a^2Cos^2u = a^2Sin^2u + a^2Cos^2u$

$\Rightarrow x^2 + y^2 + z^2 = a^2(Sin^2u + Cos^2u) \quad \Rightarrow \boldsymbol{x^2 + y^2 + z^2 = a^2}$

Which is the equation of sphere with centre at origin and radius 'a'

Now equation of normal plane is $(\vec{R} - \vec{r}).\vec{t} = 0 \quad \Rightarrow \vec{R}.\vec{t} - \vec{r}.\vec{t} = 0$

Now if the plane pass through origin then $\vec{R} = 0 \quad \Rightarrow \vec{r}.\vec{t} = 0$

Thus all the normal planes will pass through origin if $\vec{r}.\vec{t} = 0$

Now $\vec{r} = aSin^2u,\ aSinuCosu,\ aCosu$

$\vec{t} = \vec{r}' = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{du}.\frac{du}{ds} = (2aSinuCosu, \quad -aSin^2u, \quad -aSinu)\,u'$

$\Rightarrow \vec{r}.\vec{t} = (aSin^2u, \quad aSinuCosu,\ aCosu).[(2aSinuCosu, \quad -aSin^2u, \quad -aSinu)\,u']$

$\Rightarrow \vec{r}.\vec{t} = (2a^2Sin^3uCosu - a^2Sin^3uCosu + a^2Cos^3uSinu - a^2SinuCosu)\,.u'$

$\Rightarrow \vec{r}.\vec{t} = (a^2Sin^3uCosu + a^2Cos^3uSinu - a^2SinuCosu)\,.u'$

$\Rightarrow \vec{r}.\vec{t} = [a^2SinuCosu(Sin^2u + Cos^2u) - a^2SinuCosu]\,.u'$

$\Rightarrow \vec{r}.\vec{t} = [a^2SinuCosu - a^2SinuCosu]\,.u' = [0]\,.u' = 0 \qquad \Rightarrow \vec{r}.\vec{t} = 0$

Thus all the normal planes will pass through origin

Question: for any binormal unit vector $\vec{b}$ find $\vec{b}''$ and $\vec{b}'''$

Solution: Since $\vec{b}' = -\tau\vec{n} \Rightarrow \vec{b}'' = \frac{d\vec{b}'}{ds} = -\tau'\vec{n} - \tau\vec{n}' \Rightarrow \vec{b}'' = -\tau'\vec{n} - \tau(\tau\vec{b} - K\vec{t})$

$\Rightarrow \boldsymbol{\vec{b}'' = -\tau'\vec{n} - \tau^2\vec{b} + \tau K\vec{t}}$

$\Rightarrow \vec{b}''' = \frac{d\vec{b}''}{ds} = -\tau''\vec{n} - \tau'\vec{n}' - 2\tau\tau'\vec{b} - \tau^2\vec{b}' + \tau'K\vec{t} + \tau K'\vec{t} + \tau K\vec{t}'$

$\Rightarrow \vec{b}''' = -\tau''\vec{n} - \tau'(\tau\vec{b} - K\vec{t}) - 2\tau\tau'\vec{b} - \tau^2(-\tau\vec{n}) + \tau'K\vec{t} + \tau K'\vec{t} + \tau K(K\vec{n})$

$\Rightarrow \vec{b}''' = -\tau''\vec{n} - \tau\tau'\vec{b} + \tau'K\vec{t} - 2\tau\tau'\vec{b} + \tau^3\vec{n} + \tau'K\vec{t} + \tau K'\vec{t} + \tau K^2\vec{n}$

$\Rightarrow \vec{b}''' = \vec{t}(\tau'K + \tau'K + \tau K') + (-\tau'' + \tau^3 + \tau K^2)\vec{n} - \tau\tau'\vec{b} - 2\tau\tau'\vec{b}$

$\Rightarrow \boldsymbol{\vec{b}''' = \vec{t}(2\tau'K + \tau K') + (-\tau'' + \tau^3 + \tau K^2)\vec{n} - 3\tau\tau'\vec{b}}$

Question: for any curvature evaluate $\vec{b}''.\vec{b}'$ where $\vec{b}$ is the binormal unit vector.

Solution: Since $\vec{b}' = -\tau\vec{n}$ and $\vec{b}'' = -\tau'\vec{n} - \tau^2\vec{b} + \tau K\vec{t}$

$\Rightarrow \vec{b}''.\vec{b}' = \left(-\tau'\vec{n} - \tau^2\vec{b} + \tau K\vec{t}\right).(-\tau\vec{n})$

$\Rightarrow \vec{b}''.\vec{b}' = \tau\tau'(\vec{n}.\vec{n}) + \tau^3\left(\vec{b}.\vec{n}\right) - \tau^2 K\left(\vec{t}.\vec{n}\right)$
$\Rightarrow \boldsymbol{\vec{b}''.\vec{b}' = \tau\tau'}$

MOMENTS:

if a vector $\vec{d}$ is localized in a line through the point P whose position vector is $\vec{r}$ relative to origin O then the moment of $\vec{d}$ about O is $\vec{r} \times \vec{d}$. It is a scalar quantity. Moment of a vector about a point is a vector, called , Moment vector.

Let $\vec{r} = \vec{r}(s)$ be a regular curve of class ≥ 3 with $\vec{t}, \vec{n}, \vec{b}$ then the moments of $\vec{t}, \vec{n}, \vec{b}$ at a point on $\vec{r}(s)$ about the origin are defined as $m_1 = \vec{r} \times \vec{t}$, $m_2 = \vec{r} \times \vec{n}$, $m_3 = \vec{r} \times \vec{b}$

QUESTION: if m_1, m_2, m_3 are the moments about the origin of unit vectors $\vec{t}, \vec{n}, \vec{b}$ localized in the tangent , normal and binormal and dashes denote differentiation w.r.to ‘s’ then show that
$m_1' = Km_2$, $m_2' = \vec{b} - Km_1 + \tau m_3$, $m_3' = -\vec{n} - \tau m_2$

Proof:

if $\vec{r}$ is a current point then by definition of moment of forces about a point
$m_1 = \vec{r} \times \vec{t}$, $m_2 = \vec{r} \times \vec{n}$, $m_3 = \vec{r} \times \vec{b}$

then differentiating above w.r.to ‘s’ $\Rightarrow m_1' = \vec{r}' \times \vec{t} + \vec{r} \times \vec{t}' = \vec{t} \times \vec{t} + \vec{r} \times (K\vec{n})$
$\Rightarrow m_1' = 0 + K(\vec{r} \times \vec{n}) \Rightarrow m_1' = Km_2$

also

$\Rightarrow m_2' = \vec{r}' \times \vec{n} + \vec{r} \times \vec{n}' = \vec{t} \times \vec{n} + \vec{r} \times \left(\tau\vec{b} - K\vec{t}\right) = \vec{b} + \tau\left(\vec{r} \times \vec{b}\right) - K\left(\vec{r} \times \vec{t}\right)$
$\Rightarrow m_2' = \vec{b} - Km_1 + \tau m_3$

Similarly

$\Rightarrow m_3' = \vec{r}' \times \vec{b} + \vec{r} \times \vec{b}' = \vec{t} \times \vec{b} + \vec{r} \times (-\tau\vec{n}) = -\vec{n} - \tau(\vec{r} \times \vec{n})$

$\Rightarrow m_3' = -\vec{n} - \tau m_2$

THEOREM (BS;2018): if plane of curvature at every point of the curve passes through the fixed point then prove that curve is plane.

Proof:

Let $\vec{R}$ be the position vector of the current point lying in the osculating plane (plane of curvature) and $\vec{r}$ be the position vector of the point 'P' on the curve, then the equation of osculating plane is

$[\vec{R}-\vec{r},\vec{t},\vec{n}]=0 \Rightarrow \vec{R}-\vec{r}.\vec{t}\times\vec{n}=0 \Rightarrow (\vec{R}-\vec{r}).\vec{b}=0$ ....(i)

Let $\vec{R}_1$ be the fixed point then $\vec{R}_1$ must satisfy the equation (i) since plane of curvature is passing through fixed point.

$\Rightarrow (\vec{R}_1-\vec{r}).\vec{b}=0$ ....(ii)

Diff. w.r.to 's' $\Rightarrow \left(\frac{d\vec{R}_1}{ds}-\frac{d\vec{r}}{ds}\right).\vec{b}+(\vec{R}_1-\vec{r}).\frac{d\vec{b}}{ds}=0$

$\Rightarrow (0-\vec{t}).\vec{b}+(\vec{R}_1-\vec{r}).(-\tau\vec{n})=0 \quad \Rightarrow \vec{t}.\vec{b}+(\vec{R}_1-\vec{r}).(\tau\vec{n})=0$
$\Rightarrow \tau(\vec{R}_1-\vec{r}).\vec{n}=0 \quad (\vec{R}_1-\vec{r}).\vec{n}=0$

If $\boldsymbol{\tau}=\mathbf{0}$ then curve is plane.

Now we prove that $(\vec{R}_1-\vec{r}).\vec{n}\neq 0$

Let $(\vec{R}_1-\vec{r}).\vec{n}=0 \Rightarrow (\vec{R}_1-\vec{r})\ is\ \perp\ to\ \vec{n}$ and also from (ii) $(\vec{R}_1-\vec{r})\ is\ \perp\ to\ \vec{b}$
$\Rightarrow (\vec{R}_1-\vec{r})\ is\ parallel\ to\ \vec{t}$

So $\Rightarrow (\vec{R}_1-\vec{r})=\lambda\,\vec{t} \Rightarrow \boldsymbol{\vec{R}_1=\vec{r}+\lambda\,\vec{t}}$ which is the equation of the tangent.

$\Rightarrow$ tangent is fixed . since $\vec{R}$ is fixed $\Rightarrow$ curve is straight line. Which is not possible.

Hence $(\vec{R}_1-\vec{r}).\vec{n}\neq 0 \quad \Rightarrow \boldsymbol{\tau=0}$ **and curve is plane.**

SKEW CURVATURE: the arc rate of rotation of normal is called skew curvature. i.e.
$\frac{d\vec{n}}{ds}=\vec{n}'=(\tau\vec{b}-K\vec{t}) \quad \Rightarrow \left|\frac{d\vec{n}}{ds}\right|=|\vec{n}'|=\sqrt{\tau^2+K^2}$ and this term $|\vec{n}'|$ is called centre of circle of curvature.

Question: If $\vec{n}'=(\tau\vec{b}-K\vec{t})$ then find $\vec{n}''$ and $\vec{n}'''$

Solution: Since $\vec{n}'=(\tau\vec{b}-K\vec{t}) \Rightarrow \vec{n}''=\frac{d\vec{n}'}{ds}=\tau'\vec{b}+\tau\vec{b}'-K'\vec{t}-K\vec{t}'$
$\Rightarrow \vec{n}''=\tau'\vec{b}+\tau(-\tau\vec{n})-K'\vec{t}-K(K\vec{n}) \qquad \Rightarrow \vec{n}''=\tau'\vec{b}-\tau^2\vec{n}-K'\vec{t}-K^2\vec{n}$
$\Rightarrow \boldsymbol{\vec{n}''=(-\tau^2-K^2)\vec{n}+\tau'\vec{b}-K'\vec{t}}$

$$\Rightarrow \vec{n}'''=\frac{d\vec{n}''}{ds}=(-2\tau\tau'-2KK')\vec{n}+(-\tau^2-K^2)\vec{n}'+\tau''\vec{b}+\tau'\vec{b}'-K''\vec{t}-K'\vec{t}'$$

$$\Rightarrow \vec{n}'''=-2\tau\tau'\vec{n}-2KK'\vec{n}+(-\tau^2-K^2)(\tau\vec{b}-K\vec{t})+\tau''\vec{b}+\tau'(-\tau\vec{n})-K''\vec{t}-K'(K\vec{n})$$

$$\Rightarrow \vec{n}'''=-2\tau\tau'\vec{n}-2KK'\vec{n}-\tau^3\vec{b}+K\tau^2\vec{t}-K^2\tau\vec{b}+K^3\vec{t}+\tau''\vec{b}-\tau\tau'\vec{n}-K''\vec{t}-KK'\vec{n}$$

$$\Rightarrow \boldsymbol{\vec{n}'''=(K\tau^2+K^3-K'')\vec{t}+(-3\tau\tau'-3KK')\vec{n}+(\tau''-\tau^3-K^2\tau)\vec{b}}$$

SOME USEFUL RESULTS

For $\vec{r} = \vec{r}(s)$ $\quad \Rightarrow \vec{r}' = \vec{t} \quad \Rightarrow \vec{r}'' = K\vec{n} \quad \Rightarrow \vec{r}''' = K'\vec{n} - K^2\vec{t} + K\tau\vec{b}$

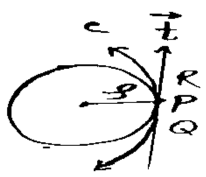
$\Rightarrow \vec{r}^{iv} = (K'' - K^3 - K\tau^2)\vec{n} - 3KK'\vec{t} + ((2K'\tau + \tau'K))\vec{b}$

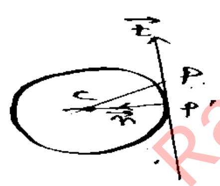
Then

- $\vec{r}'.\vec{r}'' = \vec{t}.K\vec{n} = K(\vec{t}.\vec{n}) = 0 \Rightarrow \boldsymbol{\vec{r}'.\vec{r}'' = 0}$
- $\vec{r}'.\vec{r}^{iv} = \vec{t}.\left[(K'' - K^3 - K\tau^2)\vec{n} - 3KK'\vec{t} + ((2K'\tau + \tau'K))\vec{b}\right] = -3KK' \quad \because \vec{t}.\vec{t} = 1$
- $\vec{r}'.\vec{r}''' = \vec{t}.\left[K'\vec{n} - K^2\vec{t} + K\tau\vec{b}\right] = -K^2 \quad \because \vec{t}.\vec{t} = 1$
- $\vec{r}''.\vec{r}''' = K\vec{n}.\left[K'\vec{n} - K^2\vec{t} + K\tau\vec{b}\right] = KK' \quad \because \vec{n}.\vec{n} = 1$
- $\vec{r}''.\vec{r}^{iv} = K\vec{n}.\left[(K'' - K^3 - K\tau^2)\vec{n} - 3KK'\vec{t} + ((2K'\tau + \tau'K))\vec{b}\right] = K(K'' - K^3 - K\tau^2)$
- $\vec{r}'''.\vec{r}^{iv} = \left[K'\vec{n} - K^2\vec{t} + K\tau\vec{b}\right].\left[(K'' - K^3 - K\tau^2)\vec{n} - 3KK'\vec{t} + ((2K'\tau + \tau'K))\vec{b}\right]$
- $\vec{r}'''.\vec{r}^{iv} = \left[K'\vec{n} - K^2\vec{t} + K\tau\vec{b}\right].\left[(K'' - K^3 - K\tau^2)\vec{n} - 3KK'\vec{t} + ((2K'\tau + \tau'K))\vec{b}\right]$
- $\Rightarrow \vec{r}'''.\vec{r}^{iv} = K'K'' - K'K^3 - KK'\tau^2 + 3K^3K' + 2KK'\tau^2 + K^2K'\tau' + K^2\tau\tau'$

CIRCLE OF CURVATURE: the circle of curvature at ‘P’ is the circle passing through three consecutive points on the curve ultimately coincident at ‘P’ then centre ‘C’ of such circle is called **Centre of Curvature**. And its radius is called **Radius of Curvature**. And it is denoted by $'\rho'$ the circle of curvature lies in osculating plane at ‘P’ and its curvature is same as that of curvature at ‘P’ so it has two consecutive tangents PQ and PR

ALTERNATIVELY:





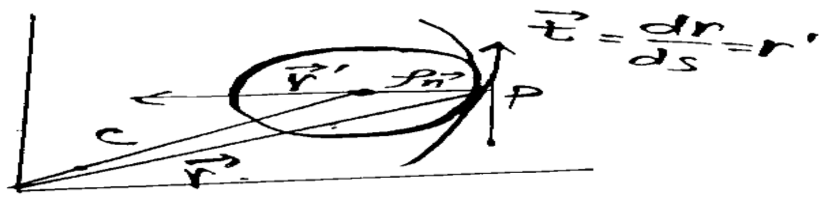
the centre of curvature of a point ‘P’ is the point of intersection of Principal Normal at ‘P’ or with a Principal Normal and a Principal Normal at a consecutive point P’ which lies in the osculating plane.

EQUATION OF CENTRE OF CURVATURE:

let $\vec{C}$ be the position vector of centre of curvature from the origin and $\vec{r}$ be the position vector of point ‘P’ then $\boxed{\vec{C} = \vec{r} + \rho\vec{n}}$ where $'\rho'$ is the radius of curvature and its direction is always along the normal at ‘P’

THEOREM: the tangent to its locus is parallel to tangent to the curve. Prove that it lies on normal plane.

PROOF:



Since $\vec{C} = \vec{r} + \rho\vec{n} \quad \Rightarrow \frac{d\vec{C}}{ds} = \vec{r'} + \rho'\vec{n} + \rho\vec{n}' \quad \Rightarrow \frac{d\vec{C}}{ds} = \vec{t} + \rho'\vec{n} + \rho(\tau\vec{b} - K\vec{t})$

$\Rightarrow \frac{d\vec{C}}{ds} = \vec{t} + \rho'\vec{n} + \rho\tau\vec{b} - \rho K\vec{t} \quad \Rightarrow \frac{d\vec{C}}{ds} = \vec{t} + \rho'\vec{n} + \rho\tau\vec{b} - \vec{t} \qquad \therefore \rho = \frac{1}{K}$

$\Rightarrow \frac{d\vec{C}}{ds} = \rho'\vec{n} + \rho\tau\vec{b} \quad \Rightarrow$ tangent to its locus lies on normal plane.

THEOREM: if the radius of curvature is constant for the given curve C then prove that the tangent to its locus of the centre of curvature is parallel to the binormal at point 'P' to C.

PROOF: Since $\vec{C} = \vec{r} + \rho\vec{n} \quad \Rightarrow \frac{d\vec{C}}{ds} = \vec{r'} + \rho'\vec{n} + \rho\vec{n}' \quad \Rightarrow \frac{d\vec{C}}{ds} = \vec{t} + \rho'\vec{n} + \rho(\tau\vec{b} - K\vec{t})$

$\Rightarrow \frac{d\vec{C}}{ds} = \vec{t} + \rho'\vec{n} + \rho\tau\vec{b} - \rho K\vec{t} \quad \Rightarrow \frac{d\vec{C}}{ds} = \vec{t} + \rho'\vec{n} + \rho\tau\vec{b} - \vec{t} \qquad \therefore \rho = \frac{1}{K}$

$\Rightarrow \frac{d\vec{C}}{ds} = \rho\tau\vec{b} \quad \therefore \rho = constant \text{ then } \rho' = 0$

$\Rightarrow \frac{d\vec{C}}{ds_1}.\frac{ds_1}{ds} = \frac{ds_1}{ds}\vec{t_1} = \rho\tau\vec{b} \qquad \Rightarrow \frac{ds_1}{ds}\vec{t_1} = \rho\tau\vec{b}$ ....(i)

$\Rightarrow \left(\frac{ds_1}{ds}\vec{t_1}\right).\left(\frac{ds_1}{ds}\vec{t_1}\right) = (\rho\tau\vec{b}).(\rho\tau\vec{b})$

$\Rightarrow \left(\frac{ds_1}{ds}\right)^2.(\vec{t_1}.\vec{t_1}) = (\rho\tau)^2.(\vec{b}.\vec{b}) \Rightarrow \left(\frac{ds_1}{ds}\right) = (\rho\tau) \qquad (i) \Rightarrow \vec{t_1} = \vec{b}$

Which show that the tangent to locus of the centre of curvature is parallel to the binormal at point 'P' to C.

THEOREM: If the radius of curvature is constant for the given curve C then prove that the curvature of locus C_1 is same as the curvature of given curve i.e. $K_1 = K$.

PROOF: Since $\vec{C} = \vec{r} + \rho\vec{n} \quad \Rightarrow \frac{d\vec{C}}{ds_1} = \frac{d}{ds}(\vec{r} + \rho\vec{n}).\frac{ds}{ds_1} = (\vec{r'} + \rho'\vec{n} + \rho\vec{n}').\frac{ds}{ds_1}$

$\Rightarrow \vec{t_1} = [\vec{t} + \rho'\vec{n} + \rho(\tau\vec{b} - K\vec{t})].\frac{ds}{ds_1} \quad \Rightarrow \vec{t_1} = [\vec{t} + \rho'\vec{n} + \rho\tau\vec{b} - \rho K\vec{t}].\frac{ds}{ds_1}$

$\Rightarrow \vec{t_1} = [\vec{t} + \rho'\vec{n} + \rho\tau\vec{b} - \vec{t}].\frac{ds}{ds_1} \qquad \therefore \rho = \frac{1}{K}$

$\Rightarrow \vec{t_1} = [\rho\tau\vec{b}].\frac{ds}{ds_1} \qquad \therefore \rho = constant \text{ then } \rho' = 0$

$\Rightarrow |\vec{t_1}| = \left|[\rho\tau\vec{b}].\frac{ds}{ds_1}\right|$

$\Rightarrow 1 = [\rho\tau].\frac{ds}{ds_1} \qquad \therefore |\vec{t_1}| = |\vec{b}| = 1 \qquad \Rightarrow \frac{ds}{ds_1} = \frac{1}{\rho\tau}$

Now since $\vec{t_1} = \vec{b} \quad \Rightarrow \frac{d\vec{t_1}}{ds_1} = \frac{d}{ds}(\vec{b}).\frac{ds}{ds_1} \quad \Rightarrow \vec{t_1}' = \vec{b}'.\frac{1}{\rho\tau} \Rightarrow K_1\vec{n_1} = -\tau\vec{n}.\frac{1}{\rho\tau}$

$\Rightarrow K_1\vec{n_1} = -\frac{\vec{n}}{\rho} \quad \Rightarrow K_1\vec{n_1} = -K\vec{n} \quad \therefore \rho = \frac{1}{K} \qquad \Rightarrow \boldsymbol{K_1 = K}$ and $\vec{n_1} = -\vec{n}$

THEOREM: If the radius of curvature is constant for the given curve C then prove that the torsion of locus of centre of curvature varies inversely as the torsion of the given curve i.e. $\boldsymbol{\tau_1 \propto \frac{1}{\tau}}$.

PROOF: Since $\vec{t_1} = \vec{b} \Rightarrow \frac{d\vec{t_1}}{ds_1} = \frac{d}{ds}(\vec{b}).\frac{ds}{ds_1} \Rightarrow \vec{t_1}' = \vec{b}'.\frac{1}{\rho\tau} \Rightarrow K_1\vec{n_1} = -\tau\vec{n}.\frac{1}{\rho\tau}$

$\Rightarrow K_1\vec{n_1} = -\frac{\vec{n}}{\rho} \Rightarrow K_1\vec{n_1} = -K\vec{n} \quad \therefore \rho = \frac{1}{K} \quad \Rightarrow \boldsymbol{K_1 = K}$ and $\vec{n_1} = -\vec{n}$

Then $\vec{t_1} \times \vec{n_1} = \vec{b} \times -\vec{n} \Rightarrow \vec{b_1} = \vec{t} \Rightarrow \frac{d\vec{b_1}}{ds_1} = \frac{d}{ds}(\vec{t}).\frac{ds}{ds_1} \Rightarrow \vec{b_1}' = \vec{t}'.\frac{1}{\rho\tau}$

$\Rightarrow -\tau_1\vec{n_1} = K\vec{n}.\frac{1}{\rho\tau} \Rightarrow -\tau_1\vec{n_1} = K\vec{n}.K\frac{1}{\tau} \Rightarrow \tau_1 = -K^2\frac{1}{\tau} \Rightarrow \boldsymbol{\tau_1 \propto \frac{1}{\tau}} \quad \therefore K^2$ is constant

QUESTION: if S_1 is the arc length of the locus of centre of curvature then show that

$$\frac{ds_1}{ds} = \frac{1}{K^2}\sqrt{K^2\tau^2 + (K')^2} = \sqrt{\left(\frac{\rho}{\delta}\right)^2 + (\rho')^2}$$

SOLUTION: Since $\vec{t} = \frac{d\vec{r}}{ds}$, $\vec{n}$, $\vec{b}$ be the tangent , normal and binormal to the given curve C. Similarly $\vec{t_1} = \frac{d\vec{r}}{ds_1}$, $\vec{n_1}$, $\vec{b_1}$ are the tangent , normal and binormal to the curve formed by the locus of centre of curvature.

Since $\vec{C} = \vec{r} + \rho\vec{n} \quad \Rightarrow \frac{d\vec{C}}{ds_1} = \frac{d}{ds}(\vec{r} + \rho\vec{n}).\frac{ds}{ds_1} = (\vec{r'} + \rho'\vec{n} + \rho\vec{n'}).\frac{ds}{ds_1}$

$\Rightarrow \vec{t_1} = [\vec{t} + \rho'\vec{n} + \rho(\tau\vec{b} - K\vec{t})].\frac{ds}{ds_1}$

$\Rightarrow \vec{t_1} = [\vec{t} + \rho'\vec{n} + \rho\tau\vec{b} - \rho K\vec{t}].\frac{ds}{ds_1} \qquad \Rightarrow \vec{t_1} = [\vec{t} + \rho'\vec{n} + \rho\tau\vec{b} - \vec{t}].\frac{ds}{ds_1} \quad \therefore \rho = \frac{1}{K}$

$\Rightarrow \vec{t_1} = [\rho'\vec{n} + \rho\tau\vec{b}].\frac{ds}{ds_1} \quad \Rightarrow |\vec{t_1}| = |[\rho'\vec{n} + \rho\tau\vec{b}]|.\left|\frac{ds}{ds_1}\right|$

$\Rightarrow 1 = \sqrt{(\rho')^2 + (\rho\tau)^2}.\frac{ds}{ds_1} \Rightarrow 1 = \sqrt{(\rho')^2 + \left(\frac{\rho}{\delta}\right)^2}.\frac{ds}{ds_1} \Rightarrow \frac{ds_1}{ds} = \sqrt{(\rho')^2 + \left(\frac{\rho}{\delta}\right)^2}$

Now $\therefore \rho = \frac{1}{K} \quad \Rightarrow \rho' = -\frac{1}{K^2}K' \quad \Rightarrow \frac{ds_1}{ds} = \sqrt{\left(-\frac{1}{K^2}K'\right)^2 + \left(\frac{1}{K}\tau\right)^2}$

$\Rightarrow \frac{ds_1}{ds} = \sqrt{\frac{(K')^2}{K^4} + \frac{\tau^2}{K^2}} \qquad \Rightarrow \boldsymbol{\frac{ds_1}{ds} = \frac{1}{K^2}\sqrt{K^2\tau^2 + (K')^2}}$

QUESTION (Annual;2018)

Prove that for any curve C $[\vec{t}', \vec{t}'', \vec{t}'''] = [\vec{r}'', \vec{r}''', \vec{r}^{iv}] = K^3(K\tau' - K'\tau) = K^5\frac{d}{ds}\left(\frac{\tau}{K}\right)$

SOLUTION: Since For $\vec{r} = \vec{r}(s) \qquad \Rightarrow \vec{r}' = \vec{t} \qquad \Rightarrow \vec{r}'' = \vec{t}' = K\vec{n}$

$\Rightarrow \vec{r}''' = \vec{t}'' = K'\vec{n} - K^2\vec{t} + K\tau\vec{b} \Rightarrow \vec{r}^{iv} = \vec{t}''' = (K'' - K^3 - K\tau^2)\vec{n} - 3KK'\vec{t} + (2K'\tau + \tau'K)\vec{b}$

Then $[\vec{t}', \vec{t}'', \vec{t}'''] = [\vec{r}'', \vec{r}''', \vec{r}^{iv}] = \begin{vmatrix} 0 & K & 0 \\ -K^2 & K' & K\tau \\ -3KK' & K'' - K^3 - K\tau^2 & 2K'\tau + \tau'K \end{vmatrix}$

$\Rightarrow [\vec{t}', \vec{t}'', \vec{t}'''] = [\vec{r}'', \vec{r}''', \vec{r}'^{v}] = 0 + K[-3K^2K'\tau + K^2(2K'\tau + \tau'K)] + 0$

$\Rightarrow [\vec{t}', \vec{t}'', \vec{t}'''] = [\vec{r}'', \vec{r}''', \vec{r}'^{v}] = -3K^3K'\tau + 2K^3K'\tau + \tau'^{K^4} = \; K^3[K\tau' - K'\tau] = \frac{K^5[K\tau' - K'\tau]}{K^2}$

$\Rightarrow [\vec{t}', \vec{t}'', \vec{t}'''] = [\vec{r}'', \vec{r}''', \vec{r}'^{v}] = K^5\frac{d}{ds}\left(\frac{\tau}{K}\right)$

QUESTION: Prove that for any curve C, $\left[\vec{b}',\vec{b}'',\vec{b}'''\right] = \tau^3(K'\tau - K\tau') = \tau^5 \frac{d}{ds}\left(\frac{K}{\tau}\right)$

SOLUTION: Since $\vec{b}' = -\tau\vec{n} \quad \Rightarrow \vec{b}'' = -\tau'\vec{n} - \tau^2\vec{b} + \tau K\vec{t}$

$$\Rightarrow \vec{b}''' = \vec{t}(2\tau' K + \tau K') + (-\tau'' + \tau^3 + \tau K^2)\vec{n} - 3\tau\tau'\vec{b}$$

Then $\left[\vec{b}',\vec{b}'',\vec{b}'''\right] = \begin{vmatrix} 0 & -\tau & 0 \\ \tau K & -\tau' & -\tau^2 \\ 2\tau' K + \tau K' & -\tau'' + \tau^3 + \tau K^2 & -3\tau\tau' \end{vmatrix}$

$$\Rightarrow \left[\vec{b}',\vec{b}'',\vec{b}'''\right] = 0 - \tau[-3\tau^2 K\tau' + 2K\tau'\tau^2 + K'\tau^3] + 0$$

$$\Rightarrow \left[\vec{b}',\vec{b}'',\vec{b}'''\right] = \tau^3[-3K\tau' + 2K\tau' + K'\tau] = \tau^3[-K\tau' + K'\tau]$$

$$\Rightarrow \left[\vec{b}',\vec{b}'',\vec{b}'''\right] = \frac{\tau^3.\tau^2[K'\tau - K\tau']}{\tau^2} = \tau^5 \frac{d}{ds}\left(\frac{K}{\tau}\right)$$

QUESTION (BS;2019): Show that the shortest distance between the Principal normals at consecutive points distance is $\frac{\rho ds}{\sqrt{\rho^2+\delta^2}}$

SOLUTION: let $P(\vec{r})$ and $Q(\vec{r} + d\vec{r})$ be two consecutive points and $\vec{n}$ and $\vec{n} + d\vec{n}$ are unit principal normal at P and Q respectively. We have to find the shortest distance between $\vec{n}$ and $\vec{n} + d\vec{n}$ and this the perpendicular distance.

The perpendicular vector to both $\vec{n}$ and $\vec{n} + d\vec{n}$ is $\vec{n} \times (\vec{n} + d\vec{n})$ then

$$\vec{n} \times (\vec{n} + d\vec{n}) = \vec{n} \times \vec{n} + \vec{n} \times d\vec{n} = 0 + \vec{n} \times \frac{d\vec{n}}{ds} ds = \vec{n} \times \vec{n}' ds = \vec{n} \times \left(\tau\vec{b} - K\vec{t}\right) ds$$

$$\Rightarrow \vec{n} \times (\vec{n} + d\vec{n}) = \left[\tau\left(\vec{n} \times \vec{b}\right) - K\left(\vec{n} \times \vec{t}\right)\right] ds = \left[\tau\left(\vec{t}\right) - K\left(-\vec{b}\right)\right] ds = \left[\tau\vec{t} + K\vec{b}\right] ds$$

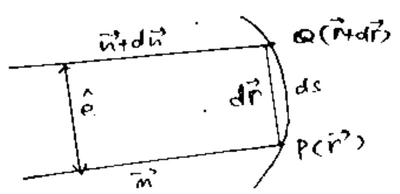
This is perpendicular vector to both $\vec{n}$ and $\vec{n} + d\vec{n}$

Now we will find its unit vector.

Let $\hat{e}$ be its unit vector in the direction of $\vec{n} \times (\vec{n} + d\vec{n})$ or $\vec{n} \times d\vec{n}$

$\hat{e} = \frac{[\tau\vec{t} + K\vec{b}]\, ds}{\sqrt{\tau^2+K^2}\, ds} = \frac{[\tau\vec{t} + K\vec{b}]}{\sqrt{\tau^2+K^2}}$ This is unit vector perpendicular vector to both $\vec{n}$ and $\vec{n} + d\vec{n}$

Now the shortest distance between the two principal normals at P and Q is given by



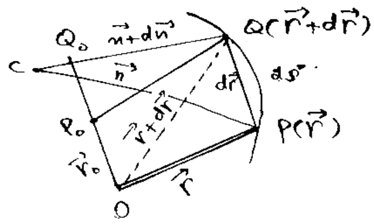
Shortest distance = projection of $d\vec{r}$ upon $\hat{e} = \hat{e}.d\vec{r} = \frac{[\tau\vec{t} + K\vec{b}]}{\sqrt{\tau^2+K^2}}.d\vec{r} = \frac{[\tau\vec{t} + K\vec{b}]}{\sqrt{\tau^2+K^2}}.\frac{d\vec{r}}{ds} ds$

$$shortest\ disance = \left[\frac{[\tau\vec{t} + K\vec{b}]}{\sqrt{\tau^2+K^2}}.\vec{t}\right] ds = \left[\frac{\tau}{\sqrt{\tau^2+K^2}}\right] ds \quad \because \vec{t}.\vec{t} = 1\ and\ \vec{t}.\vec{b} = 0$$

$$\Rightarrow S.D = \left[\frac{1/\delta}{\sqrt{\left(1/\delta\right)^2 + \left(1/\rho\right)^2}}\right] ds \quad \Rightarrow S.D = \frac{\rho ds}{\sqrt{\rho^2+\delta^2}}$$

QUESTION (Annual;2018): Show that the shortest distance between the Principal normals at consecutive points distance divides the radius of curvature in the ratio $\rho^2 : \delta^2$

SOLUTION:



let the shortest distance line meet the unit principal $\vec{n}$ at 'P' $\vec{n}+d\vec{n}$ at 'Q' then the vectors $\overrightarrow{QP_o}$, $\overrightarrow{QQ_o}$, $\overrightarrow{P_oQ_o}$ are Coplanar then $\left[\overrightarrow{QP_o}\ ,\ \overrightarrow{QQ_o}\ ,\ \overrightarrow{P_oQ_o}\right] = 0$

As C is the circle of curvature at 'P' then we have to prove that $\frac{|CP_o|}{|P_oP|} = \frac{\tau^2}{K^2} = \frac{\rho^2}{\delta^2} = \rho^2 : \delta^2$

$\overrightarrow{QQ_o}$ is a vector parallel to $\vec{n}+d\vec{n}$ and $\overrightarrow{P_oQ_o}$ is a vector parallel to the vector which is perpendicular to the both $\vec{n}$ and $\vec{n}+d\vec{n}$. Let $\vec{r_o}$ be the position vector of the point P_o then $\overrightarrow{QP_o} = \vec{r_o} - (\vec{r}+d\vec{r})$

So $\left[\overrightarrow{QP_o}\ ,\ \overrightarrow{QQ_o}\ ,\ \overrightarrow{P_oQ_o}\right] = 0 \implies \left[\vec{r_o} - (\vec{r}+d\vec{r})\ ,\ \vec{n}+d\vec{n}\ ,\ [\tau\vec{t}+K\vec{b}]\,ds\right] = 0$ .... (i)

Now equation of principal normal is $\vec{R} = \vec{r} + \mu_o\vec{n}$

Since $P_o(\vec{r_o})$ is lying on normal $\vec{n}$ So it must satisfied equation of normal $\vec{r_o} = \vec{r} + \mu_o\vec{n}$

Hence $(i) \implies \left[\vec{r}+\mu_o\vec{n}-\vec{r}-d\vec{r}\ ,\ \vec{n}+d\vec{n}\ ,\ [\tau\vec{t}+K\vec{b}]\,ds\right] = 0$

$\implies \left[\mu_o\vec{n} - \frac{d\vec{r}}{ds}ds\ ,\ \vec{n}+\frac{d\vec{n}}{ds}ds\ ,\ [\tau\vec{t}+K\vec{b}]\,ds\right] = 0$

$\implies \left[\mu_o\vec{n} - \vec{t}ds\ ,\ \vec{n}+[\tau\vec{b}-K\vec{t}]ds\ ,\ [\tau\vec{t}+K\vec{b}]\,ds\right] = 0$

$\implies \begin{vmatrix} -ds & \mu_o & 0 \\ -Kds & 1 & \tau ds \\ \tau ds & 0 & Kds \end{vmatrix} = 0 \implies -ds(Kds) - \mu_o[-(Kds)^2 - (\tau ds)^2] + 0 = 0$

$\implies -Kds^2 + \mu_o[K^2ds^2 + \tau^2ds^2] = 0 \implies [-K + \mu_oK^2 + \mu_o\tau^2]ds^2 = 0$

$\implies ds^2 \neq 0 \implies [-K + \mu_oK^2 + \mu_o\tau^2] = 0 \implies \mu_o = \frac{k}{\tau^2+K^2} = |P_oP|$

Now Since $|CP| = |CP_o| + |P_oP| \implies |CP_o| = |CP| - |P_oP| = \rho - \frac{k}{\tau^2+K^2}$

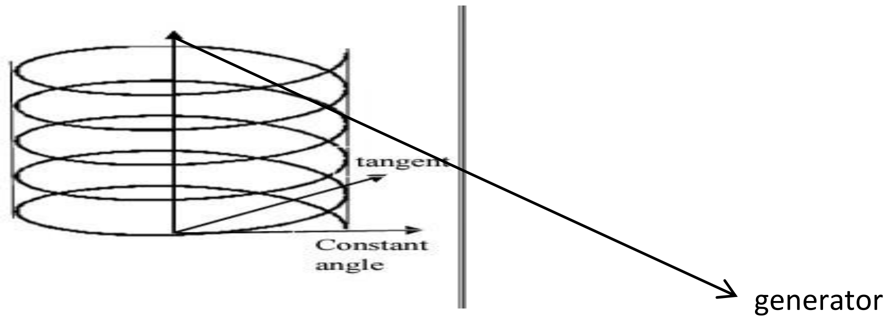
$\implies |CP_o| = \frac{1}{K} - \frac{k}{\tau^2+K^2} = \frac{\tau^2}{K(\tau^2+K^2)}$

Now $\frac{|CP_o|}{|P_oP|} = \frac{\frac{\tau^2}{K(\tau^2+K^2)}}{\frac{k}{\tau^2+K^2}} = \frac{\frac{\tau^2}{K}}{K} = \frac{\tau^2}{K^2} = \frac{\rho^2}{\delta^2} = \rho^2 : \delta^2$

$\implies$ the shortest distance divides the radius of curvature in the ratio $\rho^2 : \delta^2$

HELIX: a curve traced on the surface of a cylinder and cutting the generator with constant angle is called Helix. Thus the tangent to the Helix is inclined at a constant angle with direction of the cylinder. Thus of $\vec{t}$ is the unit tangent to the Helix and $\vec{a}$ is the constant vector parallel to the generator of the helix then $\vec{t}.\vec{a} = |\vec{t}||\vec{a}|Cos \propto = 1.1. Cos \propto = Cos \propto = constant$

$\Rightarrow \vec{t}.\vec{a} = constant$ $\quad\quad$ $\therefore \propto$ is fixed and it is the angle between $\vec{t}$ and $\vec{a}$



> **THEOREM (BS;2018):** prove that necessary and sufficient condition for a curve to be helix is
> $$\frac{K}{\tau} = constant$$

PROOF: (***NECESSARY CONDITION***):

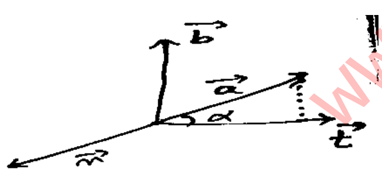
Let the curve be helix then we have to prove that $\frac{K}{\tau} = constant$. As the curve is helix so the unit tangent $\vec{t}$ at any point to the curve makes a constant angle with the fixed direction of cylinder. Let $\vec{a}$ be the unit constant vector along that parallel to the generator then

$\vec{t}.\vec{a} = |\vec{t}||\vec{a}|Cos \propto = 1.1. Cos \propto = Cos \propto = constant \Rightarrow \vec{t}.\vec{a} = constant$

Diff. w.t. s $\Rightarrow \vec{t}'.\vec{a} + 0 = 0 \Rightarrow K\vec{n}.\vec{a} = 0 \Rightarrow K \neq 0 \Rightarrow \vec{n}.\vec{a} = 0$

If $K = 0$ then curve is straight line that is not possible. Since curve is a Helix. So $\vec{n}.\vec{a} = 0$

$\Rightarrow \vec{n} \perp \vec{a} = 0 \Rightarrow \vec{a}$ will lies in the plane determined by $\vec{t}$ and $\vec{b}$ which is rectifying plane.



$\Rightarrow \vec{a} = Cos \propto \vec{t} + Sin \propto \vec{b}$

Diff. w.t. s $\Rightarrow 0 = Cos \propto \vec{t}' + Sin \propto \vec{b}' \Rightarrow 0 = Cos \propto (K\vec{n}) + Sin \propto (-\tau\vec{n})$

$\Rightarrow \vec{n}[Cos \propto K - Sin \propto \tau] = 0 \Rightarrow [Cos \propto K - Sin \propto \tau] = 0 \Rightarrow \frac{K}{\tau} = \frac{Sin\propto}{Cos\propto} = Tan \propto$

$\Rightarrow \frac{K}{\tau} = constant$ $\quad$ $\therefore \propto$ is fixed

(***SUFFICIENT CONDITION***): Let $\frac{K}{\tau} = constant$ then we have to prove that a curve is a helix for this . it is sufficient to prove that $\vec{t}.\vec{a} = constant$

Let $\frac{K}{\tau} = \frac{1}{C}$ …………………(i) where C is constant.

Now consider $\frac{d\vec{t}}{ds} = \vec{t}' = K\vec{n} \Rightarrow \frac{d\vec{t}}{ds} = \frac{\tau}{C}\vec{n}$ …………………(ii)

Similarly $\frac{d\vec{b}}{ds} = \vec{b}' = -\tau\vec{n} \Rightarrow \frac{1}{C}\frac{d\vec{b}}{ds} = -\frac{\tau}{C}\vec{n}$ …………………(iii)

Adding (ii) and (iii) $\Longrightarrow \frac{d\vec{t}}{ds} + \frac{1}{C}\frac{d\vec{b}}{ds} = \frac{\tau}{C}\vec{n} - \frac{\tau}{C}\vec{n} = 0$

$\Longrightarrow \frac{d}{ds}\left(\vec{t} + \frac{\vec{b}}{C}\right) = 0 \Longrightarrow \frac{d}{ds}(C\vec{t} + \vec{b}) = 0$ Integrating w.r.to 's' $\Longrightarrow (C\vec{t} + \vec{b}) = \vec{a} = constant$

$\Longrightarrow \vec{t}.(C\vec{t} + \vec{b}) = \vec{t}.\vec{a} \Longrightarrow C(\vec{t}.\vec{t}) + (\vec{b}.\vec{t}) = \vec{t}.\vec{a} \Longrightarrow C(1) + (0) = \vec{t}.\vec{a}$

$\Longrightarrow C = \vec{t}.\vec{a} \quad \Longrightarrow \vec{t}.\vec{a} = constant \Longrightarrow$ given curve is helix.

- Possible Question: A unit speed curve $\propto (s)$ with $K \neq 0$is a helix iff there exists a constant 'c' such that $\boldsymbol{\tau} = \boldsymbol{cK} \quad \forall \boldsymbol{s}$

QUESTION (BS;2019): prove that for a curve $\frac{K}{\tau} = \boldsymbol{constant}$ whose position vector is given by $\vec{r} = (aCos\theta, aSin\theta, a\theta Cot\beta)$ where β is constant angle. Find its curvature.

SOLUTION: given that $\vec{r} = (aCos\theta, aSin\theta, a\theta Cot\beta) \Longrightarrow \vec{r}' = (-aSin\theta, aCos\theta, aCot\beta)$
$\Longrightarrow \vec{r}'' = (-aCos\theta, -aSin\theta, 0) \Longrightarrow \vec{r}''' = (aSin\theta, aCos\theta, 0)$

$\Longrightarrow |\vec{r}'| = \sqrt{(-a\text{Sin}\theta)^2 + (a\text{Cos}\theta)^2 + (aCot\beta)^2} = \sqrt{a^2(\text{Sin}^2\theta + \text{Cos}^2\theta) + (aCot\beta)^2}$

$\Longrightarrow |\vec{r}'| = \sqrt{a^2 + a^2Cot^2\beta} \Longrightarrow |\vec{r}'| = \text{a}\sqrt{1 + Cot^2\beta} = \text{a}\sqrt{Cosec^2\beta} = \text{a}Cosec\beta$

$$\Longrightarrow \vec{r}' \times \vec{r}'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -aSin\theta & aCos\theta & aCot\beta \\ -aCos\theta & -aSin\theta & 0 \end{vmatrix}$$

$\Longrightarrow \vec{r}' \times \vec{r}'' = (a^2\text{Sin}\theta Cot\beta)\hat{\imath} - (a^2\text{Cos}\theta Cot\beta)\hat{\jmath} + (a^2\text{Sin}^2\theta + a^2\text{Cos}^2\theta)\hat{k}$

$\Longrightarrow \vec{r}' \times \vec{r}'' = (a^2\text{Sin}\theta Cot\beta)\hat{\imath} - (a^2\text{Cos}\theta Cot\beta)\hat{\jmath} + a^2\hat{k}$

$\Longrightarrow |\vec{r}' \times \vec{r}''| = \sqrt{(a^2\text{Sin}\theta Cot\beta)^2 + (-a^2\text{Cos}\theta Cot\beta)^2 + (a^2)^2}$

$\Longrightarrow |\vec{r}' \times \vec{r}''| = \sqrt{a^4\text{Sin}^2\theta Cot^2\beta + a^4\text{Cos}^2\theta Cot^2\beta + a^4}$

$\Longrightarrow |\vec{r}' \times \vec{r}''| = \sqrt{a^4Cot^2\beta(\text{Sin}^2\theta + \text{Cos}^2\theta) + c^4} = \sqrt{a^4Cot^2\beta + c^4} = \sqrt{a^4(Cot^2\beta + 1)}$

$\Longrightarrow |\vec{r}' \times \vec{r}''| = \sqrt{a^4(Cosec^2\beta)} = a^2Cosec\beta$

Now $K = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} \Longrightarrow K = \frac{a^2Cosec\beta}{(\text{a}Cosec\beta)^3} = \frac{a^2Cosec\beta}{a^3Cosec^3\beta} = \frac{1}{a}\text{Sin}^2\beta$

Now $\vec{r}' \times \vec{r}''.\vec{r}''' = \left[(a^2\text{Sin}\theta Cot\beta)\hat{\imath} - (a^2\text{Cos}\theta Cot\beta)\hat{\jmath} + a^2\hat{k}\right].\left(aSin\theta\hat{\imath} - aCos\theta\hat{\jmath} + 0\hat{k}\right)]$

$\Longrightarrow \vec{r}' \times \vec{r}''.\vec{r}''' = a^3Sin^2\theta Cot\beta + a^3Cos^2\theta Cot\beta + 0 = a^3Cot\beta(\text{Sin}^2\theta + \text{Cos}^2\theta) = a^3Cot\beta$

Now $\tau = \frac{[\vec{r}' \times \vec{r}''.\vec{r}''']}{K^2|\vec{r}'|^6} = \frac{a^3Cot\beta}{[\frac{1}{a}\text{Sin}^2\beta]^2(aCosec\beta)^6} = \frac{a^3Cot\beta}{\frac{1}{a^2}\text{Sin}^4\beta.a^6Cosec^6\beta} = \frac{1}{a}\text{Sin}\beta Cos\beta$

So $\frac{K}{\tau} = \frac{\frac{1}{a}\text{Sin}^2\beta}{\frac{1}{a}\text{Sin}\beta Cos\beta} = Tan\beta = constant$

REMARKS:

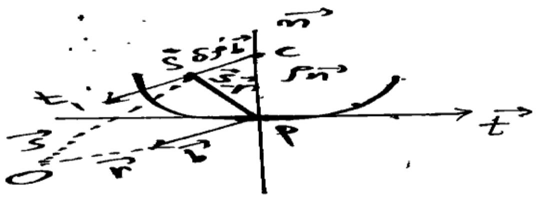
- If $\frac{K}{\tau} = 0$ then curve is a straight line.
- If $\frac{K}{\tau} = \propto$ then curve is a plane where $\propto$ is angle between $\vec{t}$ and $\vec{a}$.

SPHERE OF CURVATURE OR OSCULATING SPHERE:

The sphere which passes through four consecutive points on the curve alternately coincident at point 'P' is called the osculating sphere. The centre of sphere of curvature is denoted by $\vec{s}$ and its radius is denoted by $\vec{R}$

THEOREM: derive an expression for the radius of spherical curvature and position vector of the centre of spherical curvature.

PROOF: Let $\vec{R}$ be the position vector of the point 'P' on the curve and $\vec{s}$ be the position vector of centre of spherical curvature. The centre of spherical curvature is limiting position of intersection of three normal planes at consecutive points.



Now equation of normal plane $\Rightarrow (\vec{s}-\vec{r}).\vec{t} = 0 \quad \dots\dots\dots\dots (i)$

$$\Rightarrow \left(\frac{d\vec{s}}{ds} - \vec{r}'\right).\vec{t} + (\vec{s}-\vec{r})\vec{t}' = 0 \Rightarrow \left(\frac{d\vec{s}}{ds} - \vec{t}\right).\vec{t} + (\vec{s}-\vec{r}).(K\vec{n}) = 0$$

$$\Rightarrow \left(\frac{d\vec{s}}{ds}.\vec{t}\right) - (\vec{t}.\vec{t}) + (\vec{s}-\vec{r})(K\vec{n}) = 0 \Rightarrow 0 - 1 + (\vec{s}-\vec{r}).(K\vec{n}) = 0 \Rightarrow (\vec{s}-\vec{r}).K\vec{n} = 1$$

$$\Rightarrow (\vec{s}-\vec{r}).\vec{n} = \frac{1}{K} \Rightarrow (\vec{s}-\vec{r})\,\vec{n} = \rho \dots\dots\dots\dots (ii)$$

Diff. w.r.to arc length $\Rightarrow \left(\frac{d\vec{s}}{ds} - \frac{d\vec{r}}{ds}\right)\vec{n} + (\vec{s}-\vec{r})\,\vec{n}' = \rho'$

$$\Rightarrow \frac{d\vec{s}}{ds}.\vec{n} - \vec{t}.\vec{n} + (\vec{s}-\vec{r})\left(\tau\vec{b} - K\vec{t}\right) = \rho'$$

$$\Rightarrow 0 - 0 + (\vec{s}-\vec{r})\left(\tau\vec{b} - K\vec{t}\right) = \rho' \Rightarrow (\vec{s}-\vec{r})\left(\tau\vec{b} - K\vec{t}\right) = \rho'$$

$$\Rightarrow \tau(\vec{s}-\vec{r}).\vec{b} - K(\vec{s}-\vec{r}).\vec{t} = \rho' \Rightarrow \tau(\vec{s}-\vec{r}).\vec{b} = \rho' \qquad \therefore (\vec{s}-\vec{r}).\vec{t} = 0$$

$$\Rightarrow (\vec{s}-\vec{r}).\vec{b} = \frac{1}{\tau}\rho' \Rightarrow (\vec{s}-\vec{r}).\vec{b} = \delta\rho' \dots\dots\dots\dots (iii)$$

Now the vector $(\vec{s}-\vec{r})$ satisfies the equations (i),(ii) and (iii)

That is $(\vec{s}-\vec{r}).\vec{t} = 0$,, $(\vec{s}-\vec{r})\,\vec{n} = \rho$,, $(\vec{s}-\vec{r}).\vec{b} = \delta\rho'$

$(i) \Rightarrow (\vec{s}-\vec{r})\vec{t}.\vec{t} = 0 \Rightarrow (\vec{s}-\vec{r}) = 0 \quad (ii) \Rightarrow (\vec{s}-\vec{r})\,\vec{n}.\vec{n} = \rho.\vec{n} \Rightarrow (\vec{s}-\vec{r}) = \rho\vec{n}$

$(iii) \Rightarrow (\vec{s}-\vec{r})\vec{b}.\vec{b} = \delta\rho'.\vec{b} \Rightarrow (\vec{s}-\vec{r}) = \delta\rho'\vec{b}$

$$\Rightarrow (\vec{s}-\vec{r}) = 0 + \rho\vec{n} + \delta\rho'\vec{b} \Rightarrow \vec{s} = \vec{r} + \rho\vec{n} + \delta\rho'\vec{b}$$

Where $\vec{s}$ is the position vector of centre of spherical curvature. Now $\rho\vec{n}$ is the vector $\overrightarrow{PC}$ and $\delta\rho'\vec{b}$ is the position vector $\overrightarrow{CS}$. So the centre of spherical curvature lies on the axis of centre of curvature and it is at distance $\delta\rho'$

Now radius of spherical curvature is given by $\vec{R} = (\vec{s}-\vec{r}) = \rho\vec{n} + \delta\rho'\vec{b}$
$\Rightarrow R = \sqrt{(\rho)^2 + (\delta\rho')^2}$

REMARK: for the curve of constant curvature the radius is equal to curvature.

PROOF: Let $\rho = cosntant \Longrightarrow \rho' = 0 \Longrightarrow R = \sqrt{(\rho)^2 + 0} \Longrightarrow R = \rho$ required.

THEOREM: prove that the tangent, principal normal and binormal to the locus of centre of osculating sphere C is parallel to the binormal, principal normal and tangent to the given curve.

PROOF:

Let position vector of centre of osculating sphere is $\vec{s} = \vec{r} + \rho\vec{n} + \delta\rho'\vec{b}$

Diff.w.r.to arc length 's_1' $\Longrightarrow \vec{t}_1 = \frac{d\vec{s}}{ds_1} = \frac{d}{ds}\left(\vec{r} + \rho\vec{n} + \delta\rho'\vec{b}\right)\frac{d\vec{s}}{ds_1}$

$$\Longrightarrow \vec{t}_1 = \frac{d\vec{s}}{ds_1} = \left(\vec{r}' + \rho'\vec{n} + \rho\vec{n}' + \delta'\rho'\vec{b} + \delta\rho''\vec{b} + \delta\rho'\vec{b}'\right)\frac{d\vec{s}}{ds_1}$$

$$\Longrightarrow \vec{t}_1 = \left(\vec{t} + \rho'\vec{n} + \rho\left(\tau\vec{b} - K\vec{t}\right) + \delta'\rho'\vec{b} + \delta\rho''\vec{b} + \delta\rho'(-\tau\vec{n})\right)\frac{d\vec{s}}{ds_1}$$

$$\Longrightarrow \vec{t}_1 = \left(\vec{t} + \rho'\vec{n} + \rho\tau\vec{b} - \rho K\vec{t} + \delta'\rho'\vec{b} + \delta\rho''\vec{b} - \tau\delta\rho'\vec{n}\right)\frac{d\vec{s}}{ds_1}$$

$$\Longrightarrow \vec{t}_1 = \left(\vec{t} + \rho'\vec{n} + \rho\tau\vec{b} - \vec{t} + \delta'\rho'\vec{b} + \delta\rho''\vec{b} - \rho'\vec{n}\right)\frac{d\vec{s}}{ds_1}$$

$\Longrightarrow \vec{t}_1 = \left(\rho\tau\vec{b} + \delta'\rho'\vec{b} + \delta\rho''\vec{b}\right)\frac{d\vec{s}}{ds_1} \Longrightarrow \vec{t}_1 = (\rho\tau + \delta'\rho' + \delta\rho'').\vec{b}\frac{d\vec{s}}{ds_1}$ … … … … (i)

$(i) \Longrightarrow \left|\vec{t}_1\right| = \left|(\rho\tau + \delta'\rho' + \delta\rho'').\vec{b}\frac{d\vec{s}}{ds_1}\right|$

$\Longrightarrow 1 = \sqrt{(\rho\tau + \delta'\rho' + \delta\rho'')^2}\left(\frac{d\vec{s}}{ds_1}\right) \Longrightarrow \frac{d\vec{s}}{ds_1} = \frac{1}{\rho\tau + \delta'\rho' + \delta\rho''}$

$(i) \Longrightarrow \vec{t}_1 = (\rho\tau + \delta'\rho' + \delta\rho'').\vec{b}\left(\frac{1}{\rho\tau + \delta'\rho' + \delta\rho''}\right) \Longrightarrow \boldsymbol{\vec{t}_1 = \vec{b}}$

$\Longrightarrow$ **tangent is parallel to the binormal of the given curve.**

Now $\vec{t}_1 = \vec{b} \Longrightarrow \frac{d\vec{t}_1}{ds_1} = \frac{d\vec{b}}{ds}.\frac{ds}{ds_1} \Longrightarrow \vec{t}'_1 = \vec{b}'.\frac{ds}{ds_1} \Longrightarrow K_1\vec{n}_1 = (-\tau\vec{n}).\frac{ds}{ds_1}$

$$\Longrightarrow \vec{n}_1 = -\vec{n} \;\; and \;\; K_1 = \tau\frac{ds}{ds_1}$$

$\Longrightarrow$ **$\boldsymbol{\vec{n}_1}$ of the locus is parallel to $\boldsymbol{\vec{n}}$ of the curve but direction is opposite.**

Now $\vec{t}_1 = \vec{b}$ and $\vec{n}_1 = -\vec{n} \Longrightarrow \vec{t}_1 \times \vec{n}_1 = -\vec{b} \times \vec{n} \Longrightarrow \vec{b}_1 = \vec{t}$

$\Longrightarrow$ **binormal of locus is parallel to the tangent to the curve.**

THEOREM (PP): prove that the product of the curvature of C and C_1 at the corresponding points is equal to the product of the (curvature) torsion of these points.i.e. $\boldsymbol{KK_1 = \tau\tau_1}$

PROOF:

Consider $\vec{t}_1 = \vec{b}$ $\Rightarrow \frac{d\vec{t}_1}{ds_1} = \frac{d\vec{b}}{ds}.\frac{ds}{ds_1} \Rightarrow \vec{t'}_1 = \vec{b}'.\frac{ds}{ds_1} \Rightarrow K_1\vec{n}_1 = (-\tau\vec{n}).\frac{ds}{ds_1}$

$\Rightarrow \vec{n}_1 = -\vec{n}$ and $K_1 = \tau\frac{ds}{ds_1} \Rightarrow \boldsymbol{\frac{ds}{ds_1} = \frac{K_1}{\tau}}$ …………….. (i)

Again Consider $\vec{b}_1 = \vec{t}$ $\Rightarrow \frac{d\vec{b}_1}{ds_1} = \frac{d\vec{t}}{ds}.\frac{ds}{ds_1} \Rightarrow \vec{b'}_1 = \vec{t}'.\frac{ds}{ds_1} \Rightarrow -\tau_1\vec{n}_1 = (K\vec{n}).\frac{ds}{ds_1}$

$\Rightarrow -\tau_1\vec{n}_1 = (K\vec{n}).\left(\frac{K_1}{\tau}\right) \Rightarrow -\tau_1(-\vec{n}) = (K\vec{n}).\left(\frac{K_1}{\tau}\right) \Rightarrow \tau_1\vec{n} = (K\vec{n}).\left(\frac{K_1}{\tau}\right) \Rightarrow \boldsymbol{KK_1 = \tau\tau_1}$

NOTE: for a curve to be spherical curve the centre and radius of osculating sphere must be fixed. Or centre and radius are independent of the point on the curve.

THEOREM: necessary and sufficient condition for a curve to be spherical curve is that $\boldsymbol{\frac{\rho}{\delta} + \frac{d}{ds}\left(\frac{\rho'}{\tau}\right) = 0}$ for every point of curve.

PROOF: (***NECESSARY CONDITION***):

Let the curve be spherical curve then we have to prove that $\frac{\rho}{\delta} + \frac{d}{ds}\left(\frac{\rho'}{\tau}\right) = 0$

As the curve is spherical that is the curve lies in the sphere, therefore the sphere is osculating and so its centre and radius is fixed.

The radius of osculating plane is given by $\Rightarrow R = \sqrt{(\rho)^2 + (\delta\rho')^2}$ $\Rightarrow R^2 = (\rho)^2 + (\delta\rho')^2$

$\Rightarrow 0 = 2\rho\rho' + 2(\delta\rho')\frac{d}{ds}(\delta\rho')$ diff.w.r.to 's' and R is fixed.

$\Rightarrow 2\rho'\left[\rho + (\delta)\frac{d}{ds}(\delta\rho')\right] = 0 \Rightarrow 2\rho' \neq 0, \left[\rho + (\delta)\frac{d}{ds}(\delta\rho')\right] = 0 \Rightarrow \frac{\rho}{\delta} + \frac{d}{ds}\left(\frac{\rho'}{\tau}\right) = 0$

(***SUFFICIENT CONDITON***): Let $\frac{\rho}{\delta} + \frac{d}{ds}\left(\frac{\rho'}{\tau}\right) = 0$ then we have to show that curve is spherical curve. For this we have to prove that the radius and centre of osculating sphere are fixed.

As $\frac{\rho}{\delta} + \frac{d}{ds}\left(\frac{\rho'}{\tau}\right) = 0 \Rightarrow \rho + (\delta)\frac{d}{ds}(\delta\rho') = 0$

$\Rightarrow 2\rho'\left[\rho + (\delta)\frac{d}{ds}(\delta\rho')\right] = 0 \Rightarrow 2\rho\rho' + 2(\delta\rho')\frac{d}{ds}(\delta\rho') = 0$

Integrating w.r.to 's' $\Rightarrow (\rho)^2 + (\delta\rho')^2 = C^2$ where C is constant.

To find C we will compare it with Radius of osculating plane i.e.

$\Rightarrow R = \sqrt{(\rho)^2 + (\delta\rho')^2} \Rightarrow R^2 = (\rho)^2 + (\delta\rho')^2 \Rightarrow R^2 = C^2 \Rightarrow R = C$

$\Rightarrow R$ is fixed and is constant. $\Rightarrow$ radius is independent of the point of the curve

Now position vector of the centre of osculating sphere is given by $\vec{s} = \vec{r} + \rho\vec{n} + \delta\rho'\vec{b}$

Diff.w.r.to arc length 's' $\Rightarrow \frac{d\vec{s}}{ds} = \frac{d}{ds}(\vec{r} + \rho\vec{n} + \delta\rho'\vec{b})$

$$\Rightarrow \frac{d\vec{s}}{ds} = (\vec{r}' + \rho'\vec{n} + \rho\vec{n}' + \delta'\rho'\vec{b} + \delta\rho''\vec{b} + \delta\rho'\vec{b}')$$

$$\Rightarrow \frac{d\vec{s}}{ds} = \left(\vec{t} + \rho'\vec{n} + \rho\left(\tau\vec{b} - K\vec{t}\right) + \delta'\rho'\vec{b} + \delta\rho''\vec{b} + \delta\rho'(-\tau\vec{n})\right)$$

$$\Rightarrow \frac{d\vec{s}}{ds} = \left(\vec{t} + \rho'\vec{n} + \rho\tau\vec{b} - \rho K\vec{t} + \delta'\rho'\vec{b} + \delta\rho''\vec{b} - \tau\delta\rho'\vec{n}\right)$$

$$\Rightarrow \frac{d\vec{s}}{ds} = \left(\vec{t} + \rho'\vec{n} + \rho\tau\vec{b} - \vec{t} + \delta'\rho'\vec{b} + \delta\rho''\vec{b} - \rho'\vec{n}\right) \Rightarrow \frac{d\vec{s}}{ds} = (\rho\tau + \delta'\rho' + \delta\rho'')\vec{b}$$

$$\Rightarrow \frac{d\vec{s}}{ds} = \left[\frac{\rho}{\delta} + \frac{d}{ds}(\delta\rho')\right]\vec{b} \Rightarrow \frac{d\vec{s}}{ds} = [0]\vec{b} \Rightarrow \frac{d\vec{s}}{ds} = 0 \Rightarrow \vec{s} = constant$$

$\Rightarrow$ Centre of osculating sphere is fixed or it is independent of the point to the curve. Hence the curve is spherical curve.

SPHERICAL INDI CATRIX: it is defined as **"the locus of the point whose position vector is equal or parallel to the unit tangent $\vec{t}$ of the given curve C is called the spherical indicatrix"** Of the tangent to the curve. it is denoted by $\vec{r}_1 = \vec{t}$

Spherical indicatrix always lies on the surface of the unit sphere i.e. having radius R = 1

THEOREM (PP): prove that the curvature of the spherical indicatrix of tangent is the ratio of skew curvature to circular curvature. i.e. $K_1 = \frac{\sqrt{K^2+\tau^2}}{K}$

PROOF: As the given indicatrix is spherical of tangent so it can be written as $\vec{r}_1 = \vec{t}$

$$\Rightarrow \frac{d\vec{r}_1}{ds_1} = \frac{d\vec{t}}{ds}.\frac{ds}{ds_1} \Rightarrow \frac{d\vec{r}_1}{ds_1} = \vec{t}_1 = \vec{t}'.\frac{ds}{ds_1} \Rightarrow \vec{t}_1 = (K\vec{n}).\frac{ds}{ds_1} \Rightarrow \vec{t}_1 = \vec{n} \; and \; 1 = K\frac{ds}{ds_1}$$

$$\Rightarrow \frac{ds}{ds_1} = \frac{1}{K} \ldots\ldots\ldots\ldots(i)$$

Now $\vec{t}_1 = \vec{n}$

$$\Rightarrow \frac{d\vec{t}_1}{ds_1} = \frac{d\vec{n}}{ds}.\frac{ds}{ds_1} \Rightarrow \vec{t}'_1 = \vec{n}'.\frac{ds}{ds_1} \Rightarrow K_1\vec{n}_1 = \left(\tau\vec{b} - K\vec{t}\right).\frac{1}{K} \Rightarrow |K_1\vec{n}_1| = \left|\left(\tau\vec{b} - K\vec{t}\right).\frac{1}{K}\right|$$

$$\Rightarrow {K_1}^2 = \frac{K^2+\tau^2}{K^2} \Rightarrow \boldsymbol{K_1} = \frac{\sqrt{\boldsymbol{K^2+\tau^2}}}{\boldsymbol{K}}$$

THEOREM(PP): Prove that the Torsion of the spherical indicatrix of tangent is $\boldsymbol{\tau_1} = \frac{\boldsymbol{K\tau' - \tau K'}}{\boldsymbol{K(K^2+\tau^2)}}$

PROOF: As equation of the radius of osculating sphere is $R^2 = (\rho)^2 + (\delta\rho')^2$ and as the indicatrix lies on the sphere of the unit radius and it is the locus of the curve so we have,

$$1 = {\rho_1}^2 + (\delta_1{\rho_1}')^2 \Rightarrow 1 = \frac{1}{{K_1}^2} + \left(\frac{{\rho_1}'}{\tau_1}\right)^2 \ldots\ldots\ldots\ldots(i)$$

Now as $\rho_1 = \frac{1}{K_1} \Rightarrow {\rho_1}' = \frac{-1}{{K_1}^2}{K_1}'$ then $(i) \Rightarrow 1 = \frac{1}{{K_1}^2} + \left(\frac{\frac{-1}{{K_1}^2}{K_1}'}{\tau_1}\right)^2 \Rightarrow 1 = \frac{1}{{K_1}^2} + \frac{1}{{\tau_1}^2}\left(\frac{{K_1}'^2}{{K_1}^4}\right)$

$$\Rightarrow 1 - \frac{1}{{K_1}^2} = \frac{1}{{\tau_1}^2}\left(\frac{{K_1}'^2}{{K_1}^4}\right) \Rightarrow \frac{{K_1}^2-1}{{K_1}^2} = \frac{{K_1}'^2}{{\tau_1}^2{K_1}^4} \Rightarrow {K_1}^2 - 1 = \frac{{K_1}'^2}{{\tau_1}^2{K_1}^2} \Rightarrow {\tau_1}^2 = \frac{1}{{K_1}^2-1}.\frac{{K_1}'^2}{{K_1}^2}$$

$$\Rightarrow \tau_1 = \frac{{K_1}'}{K_1\sqrt{{K_1}^2-1}} \ldots\ldots\ldots\ldots(ii)$$

Now since we know that $K_1 = \frac{\sqrt{K^2+\tau^2}}{K} \Rightarrow {K_1}' = \frac{d}{ds}\left(\frac{\sqrt{K^2+\tau^2}}{K}\right)\frac{ds}{ds_1}$

$$\Rightarrow K_1{}' = \left[\frac{K\frac{1}{2\sqrt{K^2+\tau^2}}(2KK'+2\tau\tau\prime)-\sqrt{K^2+\tau^2}K\prime}{K^2}\right]\left(\frac{1}{K}\right)$$

$$\Rightarrow K_1' = \frac{K(KK'+\tau\tau')-(K^2+\tau^2)K'}{K^3\sqrt{K^2+\tau^2}} = \frac{K^2K'+K\tau\tau'-K^2K'-\tau^2K'}{K^3\sqrt{K^2+\tau^2}} = \frac{K\tau\tau'-\tau^2K'}{K^3\sqrt{K^2+\tau^2}}$$

So now using the values of K_1 and $K_1{}'$ in (ii)

$$(ii) \Rightarrow \tau_1 = \frac{\frac{K\tau\tau'-\tau^2K'}{K^3\sqrt{K^2+\tau^2}}}{\frac{\sqrt{K^2+\tau^2}}{K}\sqrt{\frac{K^2+\tau^2}{K^2}-1}} = \frac{K\tau\tau'-\tau^2K'}{K^3\sqrt{K^2+\tau^2}}\frac{1}{\frac{\sqrt{K^2+\tau^2}}{K}\sqrt{\frac{K^2+\tau^2-K^2}{K^2}}}$$

$$\Rightarrow \tau_1 = \frac{K\tau\tau'-\tau^2K'}{K^3\sqrt{K^2+\tau^2}}\frac{1}{\frac{\sqrt{K^2+\tau^2}}{K}\sqrt{\frac{\tau^2}{K^2}}} = \frac{K\tau\tau'-\tau^2K'}{K^3\sqrt{K^2+\tau^2}}\frac{1}{\frac{\sqrt{K^2+\tau^2}}{K}.\left(\frac{\tau}{K}\right)}$$

$$\Rightarrow \tau_1 = \frac{K\tau\tau'-\tau^2K'}{K^3\sqrt{K^2+\tau^2}}\frac{1}{\frac{\sqrt{K^2+\tau^2}.\tau}{K^2}} = \frac{\tau(K\tau'-\tau K')}{K^3\sqrt{K^2+\tau^2}}.\frac{K^2}{\sqrt{K^2+\tau^2}.\tau} \Rightarrow \boldsymbol{\tau_1} = \frac{\boldsymbol{K\tau'-\tau K'}}{\boldsymbol{K(K^2+\tau^2)}}$$

THEOREM: Prove that the Torsion of the spherical indicatrix of binormal is $\boldsymbol{\tau_1} = \frac{\boldsymbol{\tau K'-K\tau'}}{\boldsymbol{\tau(K^2+\tau^2)}}$

PROOF: As equation of the radius of osculating sphere is $R^2 = (\rho)^2 + (\delta\rho')^2$ and as the indicatrix lies on the sphere of the unit radius and it is the locus of the curve so we have,

$$1 = \rho_1{}^2 + (\delta_1\rho_1{}')^2 \Rightarrow 1 = \frac{1}{K_1{}^2} + \left(\frac{\rho_1{}'}{\tau_1}\right)^2 \dots\dots\dots\dots..(i)$$

Now as $\rho_1 = \frac{1}{K_1} \Rightarrow \rho_1{}' = \frac{-1}{K_1{}^2}K_1{}'$

$$(i) \Rightarrow 1 = \frac{1}{K_1{}^2} + \left(\frac{\frac{-1}{K_1{}^2}K_1{}'}{\tau_1}\right)^2 \Rightarrow 1 = \frac{1}{K_1{}^2} + \frac{1}{\tau_1{}^2}\left(\frac{K_1{}'^2}{K_1{}^4}\right)$$

$$\Rightarrow 1 - \frac{1}{K_1{}^2} = \frac{1}{\tau_1{}^2}\left(\frac{K_1{}'^2}{K_1{}^4}\right) \Rightarrow \frac{K_1{}^2-1}{K_1{}^2} = \frac{K_1{}'^2}{\tau_1{}^2K_1{}^4} \Rightarrow K_1{}^2 - 1 = \frac{K_1{}'^2}{\tau_1{}^2K_1{}^2} \Rightarrow \tau_1{}^2 = \frac{1}{K_1{}^2-1}.\frac{K_1{}'^2}{K_1{}^2}$$

$$\Rightarrow \tau_1 = \frac{K_1{}'}{K_1\sqrt{K_1{}^2-1}} \dots\dots\dots\dots(ii)$$

Now since we know that $K_1 = \frac{\sqrt{K^2+\tau^2}}{\tau} \Rightarrow K_1{}' = \frac{d}{ds}\left(\frac{\sqrt{K^2+\tau^2}}{\tau}\right)\frac{ds}{ds_1}$

$$\Rightarrow K_1{}' = \left[\frac{\tau\frac{1}{2\sqrt{K^2+\tau^2}}(2KK'+2\tau\tau\prime)-\sqrt{K^2+\tau^2}\tau\prime}{\tau^2}\right]\left(\frac{1}{\tau}\right)$$

$$\Rightarrow K_1' = \frac{\tau(KK'+\tau\tau')-(K^2+\tau^2)\tau'}{\tau^3\sqrt{K^2+\tau^2}} = \frac{\tau KK'+\tau^2\tau'-K^2\tau'-\tau^2\tau'}{\tau^3\sqrt{K^2+\tau^2}} = \frac{\tau KK'-K^2\tau'}{\tau^3\sqrt{K^2+\tau^2}}$$

So now using the values of K_1 and $K_1{}'$ in (ii)

$$(ii) \Rightarrow \tau_1 = \frac{\frac{\tau KK'-K^2\tau'}{\tau^3\sqrt{K^2+\tau^2}}}{\frac{\sqrt{K^2+\tau^2}}{\tau}\sqrt{\frac{K^2+\tau^2}{\tau^2}-1}} = \frac{\tau KK'-K^2\tau'}{\tau^3\sqrt{K^2+\tau^2}}\frac{1}{\frac{\sqrt{K^2+\tau^2}}{\tau}\sqrt{\frac{K^2+\tau^2-\tau^2}{\tau^2}}}$$

$$\Rightarrow \tau_1 = \frac{\tau KK'-K^2\tau'}{\tau^3\sqrt{K^2+\tau^2}}\frac{1}{\frac{\sqrt{K^2+\tau^2}}{\tau}\sqrt{\frac{K^2}{\tau^2}}} = \frac{\tau KK'-K^2\tau'}{\tau^3\sqrt{K^2+\tau^2}}\frac{1}{\frac{\sqrt{K^2+\tau^2}}{\tau}\left(\frac{K}{\tau}\right)}$$

$$\Rightarrow \tau_1 = \frac{\tau KK'-K^2\tau'}{\tau^3\sqrt{K^2+\tau^2}}\frac{1}{\frac{\sqrt{K^2+\tau^2}.K}{\tau^2}} = \frac{K(\tau K'-K\tau')}{\tau^3\sqrt{K^2+\tau^2}}.\frac{\tau^2}{\sqrt{K^2+\tau^2}.K} \Rightarrow \boldsymbol{\tau_1} = \frac{\boldsymbol{\tau K'-K\tau'}}{\boldsymbol{\tau(K^2+\tau^2)}}$$

WORKING RULE TO FIND OUT SPHERICAL INDICATRIX (IMAGES):

Given a space curve whose position vector is $\vec{r}$ then

- Find out the unit tangent $\vec{t}$
- Find out the unit principal normal $\vec{n}$
- Find out the unit binormal $\vec{b}$
- Equate $\vec{t} = \vec{r}$
- Equate $\vec{n} = \vec{r}$
- Equate $\vec{b} = \vec{r}$

QUESTION: find out the spherical images of the circular helix $\vec{r} = (aCos\theta, aSin\theta, c\theta)$; $c \neq 0$

SOLUTION: Given $\vec{r} = (aCos\theta, aSin\theta, c\theta)$; $c \neq 0$

Diff. w.r.to 's' $\Rightarrow \vec{r}' = \vec{t} = (-\,aSin\theta, aCos\theta, c)\frac{d\theta}{ds}$ ....(i)

Squaring both sides $\Rightarrow \vec{t}.\vec{t} = 1 = (\,a^2Sin^2\theta + a^2Cos^2\theta + c^2)\left(\frac{d\theta}{ds}\right)^2$

$\Rightarrow 1 = (\,a^2 + c^2)\left(\frac{d\theta}{ds}\right)^2 \Rightarrow \left(\frac{d\theta}{ds}\right)^2 = \frac{1}{(\,a^2+c^2)} \Rightarrow \left(\frac{d\theta}{ds}\right) = \frac{1}{\sqrt{(\,a^2+c^2)}}$

$\Rightarrow \left(\frac{ds}{d\theta}\right) = \sqrt{(\,a^2 + c^2)} = \lambda = constant\ quantity\ (say)$ ....(ii)

$(i) \Rightarrow \vec{t} = (-\,aSin\theta, aCos\theta, c)\frac{1}{\lambda}$ ....(iii)

Diff. w.r.to 's' $(i) \Rightarrow \vec{t}' = \frac{d}{d\theta}(-\,aSin\theta, aCos\theta, c)\frac{1}{\lambda}\frac{d\theta}{ds} \Rightarrow \vec{t}' = (-aCos\theta\,, -aSin\theta, 0)\frac{1}{\lambda}\frac{d\theta}{ds}$

Using (ii) $\Rightarrow K\vec{n} = (-aCos\theta\,, -aSin\theta, 0)\frac{1}{\lambda^2}$ ....(iv)

Squaring both sides $\Rightarrow K^2 = a^2(\,Cos^2\theta + Sin^2\theta + 0)\frac{1}{\lambda^4} \Rightarrow K^2 = \frac{a^2}{\lambda^4} \Rightarrow K = \frac{a}{\lambda^2}$ ....(v)

$(iv) \Rightarrow K\vec{n} = (-Cos\theta\,, -Sin\theta, 0)K \Rightarrow \vec{n} = (-Cos\theta\,, -Sin\theta, 0)$ ....(vi)

Now

$$\vec{b} = \vec{t} \times \vec{n} = \frac{1}{\lambda}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -aSin\theta & aCos\theta & c \\ -Cos\theta & -Sin\theta & 0 \end{vmatrix} = \frac{1}{\lambda}\left[(cSin\theta)\hat{\imath} - (cCos\theta)\hat{\jmath} + (aSin^2\theta + aCos^2\theta)\hat{k}\right]$$

$\Rightarrow \vec{b} = \frac{1}{\lambda}\left[(cSin\theta)\hat{\imath} - (cCos\theta)\hat{\jmath} + a\hat{k}\right]$.............(vii)

From (iii),(vi) and (vii) we can obtain the spherical indicatrix.

The spherical indicatrix of the tangent are $\vec{r} = \vec{t} \Rightarrow x = \frac{-aSin\theta}{\lambda}, y = \frac{aCos\theta}{\lambda}, z = \frac{c}{\lambda}$

The spherical indicatrix of the principal normal are $\vec{r} = \vec{n} \Rightarrow x = -Cos\theta, y = -Sin\theta, z = 0$

The spherical indicatrix of the binormal are $\vec{r} = \vec{b} \Rightarrow x = \frac{cSin\theta}{\lambda}, y = \frac{-cCos\theta}{\lambda}, z = \frac{a}{\lambda}$

INVOLUTES AND EVOLUTES: when the tangent to a curve C is normal to another curve C_1 then C_1 is called involute of C and C is called evolute of C_1

QUESTION: Determine the curvature of the involute of a curve $\propto (s)$

SOLUTION:

Since we know that $\vec{t}_1(s_1) = \vec{n}(s) \Longrightarrow \vec{t}_1{}'(s_1) = \vec{n}'(s)s' \Longrightarrow K_1\vec{n}_1 = (\tau\vec{b} - K\vec{t})\frac{1}{(c-s)K}$

$\Longrightarrow (K_1)^2(\vec{n}_1.\vec{n}_1) = \frac{(\tau\vec{b}-K\vec{t})(\tau\vec{b}-K\vec{t})}{(c-s)^2K^2} \Longrightarrow (K_1)^2(\vec{n}_1.\vec{n}_1) = \frac{(\tau\vec{b}-K\vec{t})(\tau\vec{b}-K\vec{t})}{(c-s)^2K^2} = \frac{\tau^2+K^2}{(c-s)^2K^2}$

$\Longrightarrow (K_1)^2 = \frac{\tau^2+K^2}{(c-s)^2K^2} \Longrightarrow K_1 = \left|\frac{\tau^2+K^2}{(c-s)^2K^2}\right|$; $K_1, K \neq 0$ required curvature of the involute of a curve $\propto (s)$

QUESTION: Determine the equation in involutes of the circular helix $\vec{r} = (aCost, aSint, bt)$; $a > 0$, $b \neq 0$

SOLUTION: Given $\vec{r} = (aCost, aSint, bt)$; $a > 0$, $b \neq 0$

$\Longrightarrow \vec{r}'(t) = (-aSint, aCost, b) \Longrightarrow |\vec{r}'(t)| = \sqrt{a^2 + b^2}$

Since $T = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{(-aSint,aCost,b)}{\sqrt{a^2+b^2}}$ also $s = \int_0^t |\vec{r}'(t)|dt = (\sqrt{a^2 + b^2})t$

Thus the equation of involute of the curve is

$$B(t) = \vec{r}(t) + (c - s)T(t) = (aCost, aSint, bt) + \left(c - \left(\sqrt{a^2 + b^2}\right)t\right)\frac{(-aSint, aCost, b)}{\sqrt{a^2 + b^2}}$$

$$B(t) = \left(aCost - a\frac{(c-(\sqrt{a^2+b^2})t)Sint}{\sqrt{a^2+b^2}}, aSint + a\frac{(c-(\sqrt{a^2+b^2})t)Cost}{\sqrt{a^2+b^2}}, bt\right) + c\frac{(c-(\sqrt{a^2+b^2})t)t}{\sqrt{a^2+b^2}}$$

Put $r = \frac{c}{\sqrt{a^2+b^2}} \Longrightarrow B(t) = (aCost - arSint + atSint, aSint + arCost - atCost, br)$

$\Longrightarrow B(t) = [a(Cost + tSint) - arSint , \quad a(Sint - tCost) + arCost \ , br]$

$\Longrightarrow$ Involute is a plane curve. $\therefore x_3 = br$

THEOREM: prove that there exist infinitely many involutes to the given space curve C.

PROOF:

Let the given space curve be $\vec{r} = \vec{r}(s)$

Now the position vector of the point $P_1(\vec{r}_1)$ to the curve C_1 can be written as $\overrightarrow{PP_1} = \lambda\vec{t}$

Then $\overrightarrow{OP_1} = \overrightarrow{OP} + \overrightarrow{PP_1} \Longrightarrow \vec{r}_1 = \vec{r} + \lambda\vec{t}$ … … … ….(i)

Diff. w.r.to 's_1'$\Longrightarrow \frac{d\vec{r}_1}{ds_1} = \frac{d}{ds}(\vec{r} + \lambda\vec{t}).\frac{ds}{ds_1}$

$$\Longrightarrow \vec{t}_1 = (\vec{r}' + \lambda'\vec{t} + \lambda\vec{t}').\frac{ds}{ds_1} \Longrightarrow \vec{t}_1 = (\vec{t} + \lambda'\vec{t} + \lambda K\vec{n}).\frac{ds}{ds_1}$$

Taking dot product with $\vec{t} \Longrightarrow \vec{t}_1.\vec{t} = (\vec{t}.\vec{t} + \lambda'\vec{t}.\vec{t} + \lambda K\vec{n}.\vec{t}).\frac{ds}{ds_1} \Longrightarrow 0 = (1 + \lambda' + 0).\frac{ds}{ds_1}$

$\Longrightarrow (1 + \lambda').\frac{ds}{ds_1} = 0 \Longrightarrow \frac{ds}{ds_1} \neq 0,, (1 + \lambda') = 0 \Longrightarrow \lambda' = -1 \Longrightarrow \lambda = -s + c$ where c is constant.

$(i) \Longrightarrow \vec{r}_1 = \vec{r} + (-s + c)\vec{t} \Longrightarrow \boldsymbol{\vec{r}_1 = \vec{r} + (c - s)\vec{t}}$

This equation signifies that due to presence of an arbitrary constant C there exist an infinite number of involutes for the given curve C.

THEOREM: prove that tangent at any point $P_1$of the involute C_1 is parallel to the normal at a corresponding point to the curve C.

PROOF: Since we know that $\vec{r}_1 = \vec{r} + \lambda\vec{t}$

Diff. w.r.to 's_1'$\Longrightarrow \frac{d\vec{r}_1}{ds_1} = \frac{d}{ds}(\vec{r} + \lambda\vec{t}).\frac{ds}{ds_1}$

$\Longrightarrow \vec{t}_1 = (\vec{r}' + \lambda'\vec{t} + \lambda\vec{t}').\frac{ds}{ds_1} \Longrightarrow \vec{t}_1 = (\vec{t} - \vec{t} + (c - s)K\vec{n}).\frac{ds}{ds_1} \Longrightarrow \vec{t}_1 = ((c - s)K\vec{n}).\frac{ds}{ds_1}$

$\Longrightarrow \vec{t}_1$ is parallel to $\vec{n} \Longrightarrow \vec{t}_1 = \vec{n} \Longrightarrow \vec{n} = ((c - s)K\vec{n}).\frac{ds}{ds_1} \Longrightarrow \frac{ds}{ds_1} = \frac{1}{(c-s)K} \Longrightarrow \boldsymbol{\frac{ds_1}{ds} = (c - s)K}$

This implies that tangent at any point $P_1$of the involute C_1 is parallel to the normal at a corresponding point to the curve C.

- **Possible Question**: Prove that the tangents of involute and normals of its evolutes are parallel at corresponding points.

QUESTION: obtain expression for the curvature to the involute C_1

SOLUTION: Since $\vec{t}_1$ is parallel to $\vec{n}$

Diff. w.r.to 's_1'$\Longrightarrow \frac{d\vec{t}_1}{ds_1} = \frac{d\vec{n}}{ds}.\frac{ds}{ds_1} \Longrightarrow \vec{t}'_1 = \vec{n}'.\frac{ds}{ds_1} \Longrightarrow K_1\vec{n}_1 = (\tau\vec{b} - K\vec{t}).\frac{1}{(c-s)K}$

Squaring both sides $\Longrightarrow {K_1}^2 = \frac{K^2+\tau^2}{(c-s)^2K^2} \Longrightarrow \boldsymbol{K_1 = \frac{\sqrt{K^2+\tau^2}}{(c-s)K}}$

THEOREM: Prove that there are an infinite families of evolutes for the given space curve.

PROOF: Let the given space curve be $\vec{r} = \vec{r}(s)$

Since the tangent at P_1 of C_1 is normal to the curve C at corresponding point P i.e. the tangent at P_1 of C_1 lies in the normal plane to the given curve at $\vec{r}$

Now $\vec{r}_1 = \vec{r} + \lambda\vec{n} + \mu\vec{b}$ ………….(i) where λ and μ are to be determined and both are functions of arc length 's'

Diff. (i) w.r.to 's' $\Longrightarrow \frac{d\vec{r}_1}{ds} = \frac{d}{ds}\left(\vec{r} + \lambda\vec{n} + \mu\vec{b}\right) \Longrightarrow \frac{d\vec{r}_1}{ds} = \left(\vec{r}' + \lambda'\vec{n} + \lambda\vec{n}' + +\mu'\vec{b} + +\mu\vec{b}'\right)$

$\Longrightarrow \frac{d\vec{r}_1}{ds} = \left(\vec{t} + \lambda'\vec{n} + \lambda\left(\tau\vec{b} - K\vec{t}\right) + \mu'\vec{b} + +\mu(-\tau\vec{n})\right)$

$$\Longrightarrow \frac{d\vec{r}_1}{ds} = \left(\vec{t} + \lambda'\vec{n} + \lambda\tau\vec{b} - \lambda K\vec{t} + \mu'\vec{b} - \mu\tau\vec{n}\right)$$

$\Longrightarrow \frac{d\vec{r}_1}{ds} = (1 - \lambda K)\vec{t} + (\lambda' - \mu\tau)\vec{n} + (\lambda\tau + \mu')\vec{b}$ ………….(ii)

As $\frac{d\vec{r}_1}{ds}$ lies in the normal plane so $\frac{d\vec{r}_1}{ds} = \lambda\vec{n} + \mu\vec{b}$ ………….(iii)

Now comparing (ii) and (iii) we get

$$\Longrightarrow (1 - \lambda K) = 0 \Longrightarrow 1 = \lambda K \Longrightarrow \lambda = \frac{1}{K} \Longrightarrow \lambda = \rho$$

$\Longrightarrow (\lambda' - \mu\tau) = \lambda \Longrightarrow 1 = \frac{(\lambda' - \mu\tau)}{\lambda}$ and $\Longrightarrow (\lambda\tau + \mu') = \mu \Longrightarrow 1 = \frac{(\lambda\tau + \mu')}{\mu}$

$$\Longrightarrow \frac{(\lambda' - \mu\tau)}{\lambda} = \frac{(\lambda\tau + \mu')}{\mu} \Longrightarrow \frac{\lambda'}{\lambda} - \frac{\mu\tau}{\lambda} = \frac{\lambda\tau}{\mu} + \frac{\mu'}{\mu} \Longrightarrow \frac{\lambda\tau}{\mu} + \frac{\mu\tau}{\lambda} = \frac{\lambda'}{\lambda} - \frac{\mu'}{\mu}$$

$$\Longrightarrow \tau\left(\frac{\lambda^2 + \mu^2}{\lambda\mu}\right) = \frac{\lambda'\mu - \mu'\lambda}{\lambda\mu} \Longrightarrow \tau = \frac{\lambda'\mu - \mu'\lambda}{\lambda^2 + \mu^2}$$

Since $\int_0^s \tau ds = \psi + c \Longrightarrow \int_0^s \left(\frac{\lambda'\mu - \mu'\lambda}{\lambda^2 + \mu^2}\right) ds = \psi + c \Longrightarrow -\frac{1}{\lambda^2}\int_0^s \left(\frac{\mu'\lambda - \lambda'\mu}{1 + \left(\frac{\mu}{\lambda}\right)^2}\right) ds = \psi + c$

$$\Longrightarrow \psi + c = -tan^{-1}\left(\frac{\mu}{\lambda}\right) \Longrightarrow Tan(\psi + c) = -\left(\frac{\mu}{\lambda}\right) \Longrightarrow Tan(\psi + c) = -\left(\frac{\mu}{\rho}\right) \quad \therefore \lambda = \rho$$

$$\Longrightarrow \mu = -\rho Tan(\psi + c) \quad (i) \Longrightarrow \vec{r}_1 = \vec{r} + \lambda\vec{n} + \mu\vec{b} \Longrightarrow \vec{r}_1 = \vec{r} + \rho\vec{n} + [-\rho Tan(\psi + c)]\vec{b}$$

$\Longrightarrow \boldsymbol{\vec{r}_1 = \vec{r} + \rho\vec{n} - \rho Tan(\psi + c)\vec{b}}$ where C is any arbitrary constant so due to this we say that there are infinite families of evolutes for the given space curve C.

THEOREM: prove that the locus of centre of curvature is an evolute only when the curve is plane curve.

PROOF: The equation of evolute is $\vec{r}_1 = \vec{r} + \rho\vec{n} - \rho Tan(\psi + c)\vec{b}$ ………….(i) where C is any arbitrary constant and for the different values of C we have different evolutes.

Now the equation of locus of centre of curvature is $\vec{c} = \vec{r} + \rho\vec{n}$ ………….(ii)

Equations (i) and (ii) are identical when $\rho Tan(\psi + c)\vec{b}$

$\Rightarrow \rho \neq 0, \vec{b} \neq 0 \Rightarrow Tan(\psi + c) = 0 \Rightarrow \psi + c = n\pi \quad ; n = 0,1,2,3 \ldots\ldots$

$\Rightarrow \psi = n\pi - c \Rightarrow \psi' = 0 \Rightarrow \tau = 0 \qquad \therefore \int \tau\, ds = \psi \Rightarrow \psi' = \tau$

$\Rightarrow$ curve is plane curve.

THEOREM: prove that the ratio of the torsion and curvature of an evolute of a space curve (involute) is $\frac{\tau_1}{K_1} = -Tan(\psi + a)$

PROOF: The equation of evolute is $\vec{r}_1 = \vec{r} + \rho\vec{n} - \rho Tan(\psi + a)\vec{b}$

Diff. w.r.to 's_1'$\Rightarrow \frac{d\vec{r}_1}{ds_1} = \frac{d}{ds}\big(\vec{r} + \rho\vec{n} - \rho Tan(\psi + a)\vec{b}\big).\frac{ds}{ds_1}$

$\Rightarrow \vec{t}_1 = \big(\vec{r}' + \rho'\vec{n} + \rho\vec{n}' - \rho' Tan(\psi + a)\vec{b} - \rho Sec^2(\psi + a)\psi'\vec{b} - \rho Tan(\psi + a)\vec{b}'\big).\frac{ds}{ds_1}$

$\Rightarrow \vec{t}_1 = \Big(\vec{t} + \rho'\vec{n} + \rho(\tau\vec{b} - K\vec{t}) - \rho' Tan(\psi + a)\vec{b} - \rho Sec^2(\psi + a)\tau\vec{b} - \rho Tan(\psi + a)(-\tau\vec{n})\Big).\frac{ds}{ds_1}$

$\Rightarrow \vec{t}_1 = \big(\vec{t} + \rho'\vec{n} + \rho\tau\vec{b} - \rho K\vec{t} - \rho' Tan(\psi + a)\vec{b} - \rho Sec^2(\psi + a)\tau\vec{b} + \rho Tan(\psi + a)\tau\vec{n}\big).\frac{ds}{ds_1}$

$\Rightarrow \vec{t}_1 = \big(\vec{t} + \rho'\vec{n} + \rho\tau\vec{b} - \vec{t} - \rho' Tan(\psi + a)\vec{b} - \rho Sec^2(\psi + a)\tau\vec{b} + \rho Tan(\psi + a)\tau\vec{n}\big).\frac{ds}{ds_1}$

$\therefore \frac{1}{\rho} = K$

$\Rightarrow \vec{t}_1 = \big(\rho'\vec{n} + \rho\tau\vec{b} - \rho' Tan(\psi + a)\vec{b} - \rho[1 + Tan^2(\psi + a)]\tau\vec{b} + \rho Tan(\psi + a)\tau\vec{n}\big).\frac{ds}{ds_1}$

$\Rightarrow \vec{t}_1 = \big(\rho'\vec{n} + \rho\tau\vec{b} - \rho' Tan(\psi + a)\vec{b} - \rho\tau\vec{b} - \rho\tau\vec{b}Tan^2(\psi + a) + \rho Tan(\psi + a)\tau\vec{n}\big).\frac{ds}{ds_1}$

$\Rightarrow \vec{t}_1 = \big[\big(\rho' + \rho\tau Tan(\psi + a)\big)\vec{n} - Tan(\psi + a)\vec{b}\big(\rho' + \rho\tau Tan(\psi + a)\big)\big].\frac{ds}{ds_1}$

$\Rightarrow \vec{t}_1 = \big(\rho' + \rho\tau Tan(\psi + a)\big)\big[\vec{n} - Tan(\psi + a)\vec{b}\big].\frac{ds}{ds_1}$ ………….(i)

Squaring both sides $\Rightarrow 1 = \big(\rho' + \rho\tau Tan(\psi + a)\big)^2\big[\vec{n} - Tan(\psi + a)\vec{b}\big]^2.\left(\frac{ds}{ds_1}\right)^2$

$\Rightarrow 1 = \big(\rho' + \rho\tau Tan(\psi + a)\big)^2[1 + Tan^2(\psi + a)].\left(\frac{ds}{ds_1}\right)^2$

$\Rightarrow 1 = \big(\rho' + \rho\tau Tan(\psi + a)\big)^2 Sec^2(\psi + a).\left(\frac{ds}{ds_1}\right)^2$

$\Rightarrow \left(\frac{ds}{ds_1}\right)^2 = \frac{1}{\big(\rho' + \rho\tau Tan(\psi + a)\big)^2 Sec^2(\psi + a)} \Rightarrow \frac{ds}{ds_1} = \frac{1}{\big(\rho' + \rho\tau Tan(\psi + a)\big)Sec(\psi + a)}$

$(i) \Rightarrow \vec{t}_1 = \big(\rho' + \rho\tau Tan(\psi + a)\big)\big[\vec{n} - Tan(\psi + a)\vec{b}\big].\frac{1}{\big(\rho' + \rho\tau Tan(\psi + a)\big)Sec(\psi + a)}$

$\Rightarrow \vec{t}_1 = \frac{\vec{n} - Tan(\psi + a)\vec{b}}{Sec(\psi + a)} \Rightarrow \vec{t}_1 = \left[\vec{n} - \frac{Sin(\psi + a)\vec{b}}{Cos(\psi + a)}\right].Cos(\psi + a)$

$\Rightarrow \vec{t}_1 = \vec{n}Cos(\psi + a) - Sin(\psi + a)\vec{b}$

Diff. w.r.to 's_1'$\Rightarrow \frac{d\vec{t}_1}{ds_1} = \frac{d}{ds}\left(\vec{n}Cos(\psi + a) - Sin(\psi + a)\vec{b}\right).\frac{ds}{ds_1}$

$$\Rightarrow \frac{d\vec{t}_1}{ds_1} = \left(\vec{n}'Cos(\psi + a) - \vec{n}Sin(\psi + a)\psi' - Cos(\psi + a)\psi'\vec{b} - Sin(\psi + a)\vec{b}'\right).\frac{ds}{ds_1}$$

$$\Rightarrow \frac{d\vec{t}_1}{ds_1} = \left((\tau\vec{b} - K\vec{t})Cos(\psi + a) - \vec{n}Sin(\psi + a)\tau - Cos(\psi + a)\tau\vec{b} - Sin(\psi + a)(-\tau\vec{n})\right).\frac{ds}{ds_1}$$

$$\Rightarrow \vec{t}_1{}' = \left(\tau Cos(\psi + a)\vec{b} - KCos(\psi + a)\vec{t} - \tau Sin(\psi + a)\vec{n} - \tau Cos(\psi + a)\vec{b} + \tau Sin(\psi + a)\vec{n}\right).\frac{ds}{ds_1}$$

$$\Rightarrow \vec{t}_1{}' = \left(-KCos(\psi + a)\vec{t}\right).\frac{ds}{ds_1}$$

$$\Rightarrow K_1\vec{n}_1 = \left(-KCos(\psi + a)\vec{t}\right).\frac{ds}{ds_1} \Rightarrow K_1 = KCos(\psi + a)\frac{ds}{ds_1} \ldots\ldots\ldots\ldots..(ii)$$

And $\Rightarrow \vec{n}_1 = -\vec{t} \Rightarrow \vec{n}_1 \parallel \vec{t}$ but in opposite direction.

Now for $\vec{b}_1$ we have $\vec{b}_1 = \vec{t}_1 \times \vec{n}_1 = \left[\vec{n}Cos(\psi + a) - Sin(\psi + a)\vec{b}\right] \times (-\vec{t})$

$$\Rightarrow \vec{b}_1 = Cos(\psi + a)(-\vec{n} \times \vec{t}) - Sin(\psi + a)(-\vec{b} \times \vec{t}) = Cos(\psi + a)\vec{b} + Sin(\psi + a)\vec{n}$$

Diff. w.r.to 's_1'$\Rightarrow \frac{d\vec{b}_1}{ds_1} = \frac{d}{ds}\left(Cos(\psi + a)\vec{b} + Sin(\psi + a)\vec{n}\right).\frac{ds}{ds_1}$

$$\Rightarrow \vec{b}_1{}' = \left(-Sin(\psi + a)\psi'\vec{b} + Cos(\psi + a)\vec{b}' + Cos(\psi + a)\psi'\vec{n} + Sin(\psi + a)\vec{n}'\right).\frac{ds}{ds_1}$$

$$\Rightarrow -\tau_1\vec{n}_1 = \left(-Sin(\psi + a)\tau\vec{b} + Cos(\psi + a)(-\tau\vec{n}) + Cos(\psi + a)\tau\vec{n} + Sin(\psi + a)(\tau\vec{b} - K\vec{t})\right).\frac{ds}{ds_1}$$

$$\Rightarrow -\boldsymbol{\tau_1\vec{n}_1} = \left(-\boldsymbol{Sin(\psi + a)\tau\vec{b}} - \boldsymbol{Cos(\psi + a)\tau\vec{n}} + \boldsymbol{Cos(\psi + a)\tau\vec{n}} + \boldsymbol{\tau\vec{b}Sin(\psi + a)} - \boldsymbol{K\vec{t}Sin(\psi + a)}\right).\frac{ds}{ds_1}$$

$$\Rightarrow -\tau_1\vec{n}_1 = \left(-K\vec{t}Sin(\psi + a)\right).\frac{ds}{ds_1} \Rightarrow \tau_1\vec{n}_1 = \left(K\vec{t}Sin(\psi + a)\right).\frac{ds}{ds_1}$$

$$\Rightarrow -\tau_1\vec{t} = \left(K\vec{t}Sin(\psi + a)\right).\frac{ds}{ds_1} \qquad \therefore \vec{n}_1 = -\vec{t}$$

$$\Rightarrow \tau_1 = -\left(KSin(\psi + a)\right).\frac{ds}{ds_1} \ldots\ldots\ldots\ldots..(iii)$$

Dividing $(iii)$by $(ii) \Rightarrow \frac{\tau_1}{K_1} = \frac{-(KSin(\psi+a)).\frac{ds}{ds_1}}{KCos(\psi+a)\frac{ds}{ds_1}} \Rightarrow \boldsymbol{\frac{\tau_1}{K_1} = -Tan(\psi + a)}$

NOTE: when curve is specified by the equation giving the curvature and torsion as function of 's' (arc length) i.e. $K = f(s)$ and $\tau = g(s)$ then these are called the **intrinsic equations** of the curve.

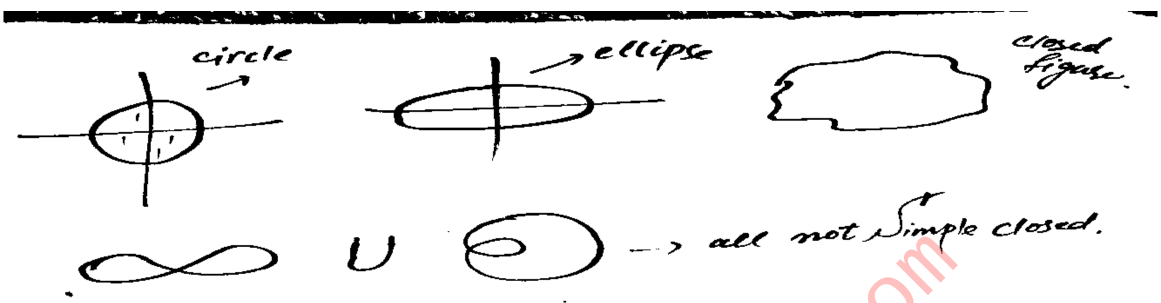
GLOBAL PROPERTIES OF CURVES

CLOSED CURVE: A regular curve $\vec{r}: R \to R^n$ is closed if $\vec{r}$ is a periodic i.e.$\vec{r}(t+a) = \vec{r}(t) \;\; \forall t$ and $a \neq 0$ then the period of $\vec{r}$ is the last such number a.

REMARK:

- If $\vec{r}$ is a periodic a point moving around $\vec{r}$ returns to its starting point after time a, of course every is o – periodic.
- If $\vec{r}$ is a periodic a point i.e. $\vec{r}(t+a) = \vec{r}(t) \;\Rightarrow\; \vec{r}(t-a+a) = \vec{r}(t-a)$ $\Rightarrow\; \vec{r}(t) = \vec{r}\big(t+(-a)\big) \Rightarrow\; \vec{r}$ is (–a) – periodic.

SIMPLE CLOSED CURVE: A simple closed curve in R^2 is a closed curve in R^2 that has no self-intersection.



LENGTH OF SIMPLE CLOSED CURVE: The length of simple closed curve $\vec{r}$ of a period a is defined as $L = \int_0^a \|\vec{r}'(t)\| dt$

LEMMA: If $\vec{r}(t)$ is a closed curve with period 'a' and let $\propto(s)$ be its unit speed reparameterization by arc length 's' then $\propto(s)$ is closed with period $L = \int_0^a \|\vec{r}'(t)\| dt$

PROOF: Consider $S(t) = \int_0^a \|\vec{r}'(u)\| du \ldots\ldots\ldots\ldots (i)$

$\Rightarrow\; S(t+a) = \int_0^{t+a} \|\vec{r}'(u)\| du = \int_0^a \|\vec{r}'(u)\| du + \int_a^{t+a} \|\vec{r}'(u)\| du = L + \int_a^{t+a} \|\vec{r}'(u)\| du$

$\Rightarrow\; S(t+a) = L + I \;\ldots\ldots\ldots\ldots (ii)$

Now consider $I = \int_a^{t+a} \|\vec{r}'(u)\| du$ and put $u = v + a \Rightarrow v = u - a \Rightarrow dv = du$ also change limits then $I = \int_0^t \|\vec{r}'(v+a)\| dv \ldots\ldots\ldots\ldots (iii)$

Now since $\vec{r}(t)$ is a closed curve with period 'a' $\vec{r}(v+a) = \vec{r}(v) \Rightarrow \vec{r}'(v+a) = \vec{r}'(v)$ then $(iii) \Rightarrow I = \int_0^t \|\vec{r}'(v)\| dv = S(t)$ then $(ii) \Rightarrow\; S(t+a) = L + S(t) \Rightarrow\; S + L = S$

let $\propto(S+L) = \propto(S(t)+L) = \propto\big(S(t+a)\big) = \vec{r}(t+a) = \vec{r}(t) = \propto\big(S(t)\big) = \propto(S)$

$\Rightarrow \propto(S+L) = \propto(S) \Rightarrow \propto(S)$ is closed curve.

Since 'a' is a lest number such that $\vec{r}(t+a) = \vec{r}(t) \;\; \forall t$ therefore 'L' must be the least positive integer such that $\propto(S+L) = \propto(S) \Rightarrow \propto(S)$ is closed with period 'L'.

REMARK: The total signed curvature if a simple closed curve in R^2 is $\pm 2\pi$

QUESTION: If $\vec{r}(t)$ is a simple closed curve with period 'a' and $\vec{t}, \vec{n}_s, K_s$ are its unit tangent vector, signed unit normal and signed curvature respectively then show that

$$\vec{t}(t+a) = \vec{t}(t)\ ,\ \vec{n}_s(t+a) = \vec{n}_s(t) \text{ also } K_s(t+a) = K_s(t)$$

SOLUTION: since $\vec{r}(t)$ is a simple closed curve with period 'a' i.e.

$\vec{r}(t+a) = \vec{r}(t) \Longrightarrow \vec{r}'(t+a) = \vec{r}'(t) \Longrightarrow \boldsymbol{\vec{t}(t+a) = \vec{t}(t)}$

Rotating $\vec{t}$ in anticlockwise by $\frac{\pi}{2}$ we obtain $\boldsymbol{\vec{n}_s(t+a) = \vec{n}_s(t)}$

Now $\vec{n}_s(t+a) = \vec{n}_s(t) \Longrightarrow \vec{n}'_s(t+a) = \vec{n}'_s(t)$

$\Longrightarrow -K_s(t+a)\vec{t}(t+a) = -K_s(t)\vec{t}(t) \quad \therefore \vec{n}' = \tau\vec{b} - K\vec{t} = \vec{n}' = 0 - K\vec{t} = -K\vec{t}$

$\Longrightarrow -K_s(t+a)\vec{t}(t) = -K_s(t)\vec{t}(t) \Longrightarrow \boldsymbol{K_s(t+a) = K_s(t)} \qquad \therefore \vec{t}(t+a) = \vec{t}(t)$

JORDAN CURVE THEOREM (Just Statement): Any simple closed curve in the plane has an 'interior' and 'exterior' more precisely, the set of points of R^2 that are not the point of curve is the disjoint union of two subsets of R^2 denoted by $int(\vec{r})$ and $ext(\vec{r})$ with the following properties

(i) $int(\vec{r})$ is bounded. i.e it is contained inside the circle of sufficiently large radius.

(ii) $ext(\vec{r})$ is unbounded

(iii) Both of the regions $int(\vec{r})$ and $ext(\vec{r})$ are connected. i.e. they have the property that any two points can be joined by a curve contained entirely in the region. (But any curve joining a point of $int(\vec{r})$ to a point of $ext(\vec{r})$ must be cross the curve)

AREA OF SIMPLE CLOSED CURVE: The area contained by a simple closed curve $\vec{r}$ is $A(\vec{r}) = \iint_{int(\vec{r})} dxdy$

these type of integral can be computed using Green's theorem which relates line integral with double integral.

GREEN'S THEOREM: If $\vec{r}$ is a simple closed curve which bounded the region $int(\vec{r})$ and which is transversal counter clockwise then $\int_{\vec{r}} fdx + gdy = \iint_{int(\vec{r})}(g_x - f_y)dxdy$ where $f(x,y)$ and $g(x,y)$ are smooth functions.

(i.e. functions with continuous partial derivatives of all orders)

AREA OF CLOSED CURVE: If $\vec{r}(t) = [x(t), y(t)]$ is a closed curve then $A(\vec{r}) = \frac{1}{2}\int_{\vec{r}} xdy - ydx$

EXAMPLE: Find area of circle

SOLUTION: let $\vec{r}(t) = [aCost, aSint]$ be a curve of circle with $0 \le t \le 2\pi$ then using $\boldsymbol{A(\vec{r}) = \frac{1}{2}\int_{\vec{r}} xdy - ydx = \frac{1}{2}\int_0^{2\pi}[(aCost)(aCost)dt - (aSint)(-aSint)dt] = \frac{1}{2}\int_0^{2\pi} a^2 dt = \pi a^2}$

- **PRACTICE:** Find area of ellipse (**HINT:** use $\vec{r}(t) = [aCost, bSint]$; $0 \le t \le 2\pi$)

RESULT: If $\vec{r}(t) = [x(t), y(t)]$ is positively oriented simple closed curve in R^2 with period 'T' then $A\big(int(\vec{r})\big) = \frac{1}{2}\int_0^T (xy' - yx')\,dt$

PROOF: From Green's theorem we have $\iint_{int(\vec{r})}(g_x - f_y)dxdy = \int_{\vec{r}} fdx + gdy$ ……….(i)

taking $g = \frac{1}{2}x, f = -\frac{1}{2}y \Rightarrow g_x = \frac{1}{2}, f_y = -\frac{1}{2}$ where $0 \le t \le T$

$(i) \Rightarrow \iint_{int(\vec{r})}\left(\frac{1}{2}+\frac{1}{2}\right)dxdy = \int_0^T -\frac{1}{2}ydx + \frac{1}{2}xdy \Rightarrow \iint_{int(\vec{r})} dxdy = \frac{1}{2}\int_0^T \frac{xdy - ydx}{dt}dt$

$\Rightarrow \iint_{int(\vec{r})} dxdy = \frac{1}{2}\int_0^T \left[x\frac{dy}{dt} - y\frac{dx}{dt}\right]dt \Rightarrow A\big(int(\vec{r})\big) = \frac{1}{2}\int_0^T (xy' - yx')\,dt$

ALTERNATIVE METHOD: Since area of simple closed curve in R^2 is $A(\vec{r}) = \frac{1}{2}\int_{\vec{r}} xdy - ydx$ then since $\vec{r}$ is defined on $[0, T]$ $\Rightarrow A(\vec{r}) = \frac{1}{2}\int_0^T \frac{xdy - ydx}{dt}dt = \frac{1}{2}\int_0^T \left[x\frac{dy}{dt} - y\frac{dx}{dt}\right]dt$

$\Rightarrow A\big(int(\vec{r})\big) = \frac{1}{2}\int_0^T (xy' - yx')\,dt$

WIRTINGER'S INEQUALITY: Let $F; [0, \pi] \to R$ be a smooth function for which $F(0) = F(\pi) = 0$ then $\int_0^\pi \left(\frac{dF}{dt}\right)^2 dt \ge \int_0^t [F(t)]^2 dt$ where equality holds iff $F = DSint$; $\forall\ t\epsilon[0, \pi]$ where D is constant.

PROOF: Let $F(t) = G(t)Sint \Rightarrow \frac{dF}{dt} = \frac{dG}{dt}Sint + G(t)Cost$

$\Rightarrow \int_0^\pi \left(\frac{dF}{dt}\right)^2 dt = \int_0^\pi \left(\frac{dG}{dt}Sint + G(t)Cost\right)^2 dt$

$\Rightarrow \int_0^\pi \left(\frac{dF}{dt}\right)^2 dt = \int_0^\pi \left(\frac{dG}{dt}\right)^2 Sin^2tdt + \int_0^\pi G^2Cos^2tdt + 2\int_0^\pi \left(G\frac{dG}{dt}\right)(SintCost)dt$

$\Rightarrow \int_0^\pi \left(\frac{dF}{dt}\right)^2 dt = \int_0^\pi \left(\frac{dG}{dt}\right)^2 Sin^2tdt + \int_0^\pi G^2Cos^2tdt + I$ ……….(i)

$I = 2\int_0^\pi \left(G\frac{dG}{dt}\right)(SintCost)dt = 2(SintCost)\int_0^\pi \left(G\frac{dG}{dt}\right)dt - \int_0^\pi \int \left(G\frac{dG}{dt}\right)dt\left[\frac{d}{dt}(2SintCost)\right]dt$

Now since $F(0) = F(\pi) = 0 \Rightarrow G(0) = G(\pi) = 0$ $because G(t) = \frac{F(t)}{Sint}$

$I = (SintCost)\left|\frac{G^2}{2}\right|_0^\pi - \int_0^\pi \left(\frac{G^2}{2}\right)\left[\frac{d}{dt}(Sin2t)\right]dt = 0 - \int_0^\pi \left(\frac{G^2}{2}\right)(2Cos2t)dt$

$I = 0 - \int_0^\pi G^2\,(Cos^2t - Sin^2t)dt = \int_0^\pi G^2Sin^2tdt - \int_0^\pi G^2Cos^2tdt$

$(i) \Rightarrow \int_0^\pi \left(\frac{dF}{dt}\right)^2 dt = \int_0^\pi \left(\frac{dG}{dt}\right)^2 Sin^2tdt + \int_0^\pi G^2Cos^2tdt + \int_0^\pi G^2Sin^2tdt - \int_0^\pi G^2Cos^2tdt$

$\Rightarrow \int_0^\pi \left(\frac{dF}{dt}\right)^2 dt = \int_0^\pi \left(\frac{dG}{dt}\right)^2 Sin^2tdt + \int_0^\pi G^2Sin^2tdt = \int_0^\pi \left(\frac{dG}{dt}\right)^2 Sin^2tdt + \int_0^\pi [F(t)]^2dt$

$\Rightarrow \int_0^\pi \left(\frac{dF}{dt}\right)^2 dt - \int_0^\pi [F(t)]^2dt = \int_0^\pi \left(\frac{dG}{dt}\right)^2 Sin^2tdt \ge 0$ $\quad \because \left(\frac{dG}{dt}\right)^2 Sin^2t \ge 0$ ……….(ii)

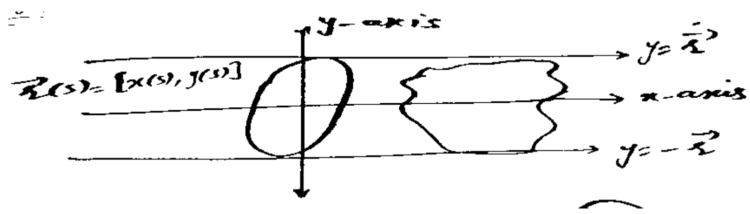
$\Rightarrow \int_0^\pi \left(\frac{dF}{dt}\right)^2 dt - \int_0^\pi [F(t)]^2dt \ge 0 \Rightarrow \int_0^\pi \left(\frac{dF}{dt}\right)^2 dt \ge \int_0^t [F(t)]^2dt$ ……….(iii)

also right side of (ii) is $Zero \Leftrightarrow \frac{dG}{dt} = 0$; $\forall\ t$

$\Leftrightarrow G = Cosntant \Leftrightarrow G = D \Leftrightarrow F(t) = G(t)Sint \Leftrightarrow F = DSint$

ISOPERIMETRIC INEQUALITY: Let $\vec{r}$ be a simple closed curve , $l(\vec{r})$ be its length , $A(\vec{r})$ be the area contained by it, then $A(\vec{r}) \leq \frac{1}{4\pi} l^2(\vec{r})$ and equality holds iff $\vec{r}$ is a circle.

PROOF: Let $\vec{r}(s) = [x(s), y(s)]$ be an arc length parameterization of $\vec{r}$ after a translation and rotation (rigid motion), we can assume that the curve $\vec{r}$ is contained in a slab $|y| \leq \vec{r}$ and it touches the line $y = \pm\vec{r}$ at exactly one point.



Let $\tilde{r}$ be the circle of radius of r centered at the origin, notice that the arc length parameterization of the $\vec{r}$ indices a parameterization. (Not necessary arc length parameterization) of the circle $\tilde{r}$ by a projection perpendicular to y – axis , therefore $\tilde{r}(s) = [\tilde{x}(s), y(s)]$; $s\epsilon[0, l]$ and $A(\tilde{r}) = \pi r^2$

Now by Green's theorem $\iint_{int(\vec{r})}(g_x - f_y)dxdy = \int_{\vec{r}} fdx + gdy$

taking $g = x, f = 0;$ $A(\vec{r}) = \iint_{int(\vec{r})} dxdy = \int_0^l xdy = \int_0^l \frac{xdy}{dt} dt = \int_0^l xy'\, dt \ldots\ldots\ldots(i)$

Also taking $g = 0, f = -y$; $A(\vec{r}) = \int_0^l -ydx = \int_0^l \frac{-ydx}{dt} dt = \int_0^l -yx'\, dt \ldots\ldots\ldots(ii)$

Adding (i) and (ii) $\Rightarrow 2A(\vec{r}) = \int_0^l (xy' - yx')\, dt \Rightarrow A(\vec{r}) = \frac{1}{2}\int_0^l (xy' - yx')\, dt$

$(ii) \Rightarrow A(\vec{r}) = \int_0^l -y(s)x'(s)\, d(s) \ldots\ldots(iii)$ and $(i) \Rightarrow A(\tilde{r}) = \int_0^l \tilde{x}(s)y'(s)\, d(s) \ldots\ldots(iv)$
Adding (iii) and (iv)

$\Rightarrow A(\vec{r}) + A(\tilde{r}) = \int_0^l (\tilde{x}(s)y'(s) - y(s)x'(s))\, ds = \int_0^l < \tilde{x}(s), y(s) >< y'(s)x'(s) > ds$

$\Rightarrow A(\vec{r}) + A(\tilde{r}) \leq \int_0^l \|\tilde{x}(s) - y(s)\|\|y'(s) - x'(s)\|\, ds \leq rl \Rightarrow A(\vec{r}) + A(\tilde{r}) \leq rl$ **by shwartz's**

Now by using the fact $GM \leq AM \Rightarrow \sqrt{ab} \leq \frac{a+b}{2} \Rightarrow \sqrt{A(\vec{r})A(\tilde{r})} \leq \frac{A(\vec{r})+A(\tilde{r})}{2} \Rightarrow A(\vec{r})\pi r^2 \leq \left(\frac{rl}{2}\right)^2$

$\Rightarrow A(\vec{r}) \leq \frac{r^2 l^2}{4\pi r^2} \Rightarrow A(\vec{r}) \leq \frac{l^2(\vec{r})}{4\pi}$

It is obvious that equality holds for $A(\vec{r}) \leq \frac{l^2(\vec{r})}{4\pi}$ iff and only if $\vec{r}$ is a circle. In that case $A(\vec{r}) = \pi r^2$ and $\frac{l^2(\vec{r})}{4\pi} = \frac{4\pi^2 r^2}{4\pi} \Rightarrow l(\vec{r}) = 2\pi r$

CONVEX: A simple closed curve $\vec{r}$ is called convex if its $int(\vec{r})$ is convex, in usual sense that the straight line segment joining any two point of $int(\vec{r})$ is contained entirely in $int(\vec{r})$ otherwise it is not convex.

Common Parametric Curves in the Complex Plane

- A parameterization of the **line** containing the points z_0 and z_1 is $z(t) = z_0(1-t) + z_1 t$ where $-\infty < t < \infty$
- A parameterization of the **line segment** from the points z_0 and z_1 is
  $z(t) = z_0(1-t) + z_1 t$ where $0 \le t \le 1$
- A parameterization of the **ray e**manating from z_0 and containing z_1 is
  $z(t) = z_0(1-t) + z_1 t$ where $0 \le t < \infty$
- A parameterization of the **circle** centered at z_0 with radius r is
  $z(t) = z_0 + r(Cost + iSint) = z_0 + re^{it}$ where $0 \le t \le 2\pi$

REMARK: The following statements for a convex curve are equivalent.

- The simple closed curve $\vec{r}$ is convex.
- If a line segment meets the curve $\vec{r}$, the intersection is either a line segment (which could possibly degenerate to a single point) or exactly at two points.
- The curve $\vec{r}$ lies on one side of the tangent line at every point on$\vec{r}$.
- The signed curvature K_s of the curve does not change the sign.

EXAMPLE(BS;2018): The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is a convex.

SOLUTION: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be interior points of ellipse then $\frac{x_1{}^2}{a^2} + \frac{y_1{}^2}{b^2} < 1 \ldots \ldots (i)$ and $\frac{x_2{}^2}{a^2} + \frac{y_2{}^2}{b^2} < 1 \ldots \ldots (ii)$ then the line segment joining P and Q is given by $t(x_1, y_1) + (1-t)(x_2, y_2) = (tx_1 + (1-t)x_2, ty_1 + (1-t)y_2)$

Now $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{[tx_1 + (1-t)x_2]^2}{a^2} + \frac{[ty_1 + (1-t)y_2]^2}{b^2}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = t^2\left[\frac{x_1{}^2}{a^2} + \frac{y_1{}^2}{b^2}\right] + (1-t)^2\left[\frac{x_2{}^2}{a^2} + \frac{y_2{}^2}{b^2}\right] + \frac{t(1-t)}{a^2b^2}(2x_1x_2 + 2y_1y_2) \ldots \ldots (iii)$

Since $(x_1 - x_2)^2 \ge 0 \Rightarrow 2x_1x_2 \le x_1{}^2 + x_2{}^2 \ldots . (iv)$

and similarly $2y_1y_2 \le y_1{}^2 + y_2{}^2 \ldots . (v)$ then using all above equations in (iii)

$(iii) \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} < t^2[1] + (1-t)^2[1] + \frac{t(1-t)}{a^2b^2}[(x_1{}^2 + x_2{}^2)\,(y_1{}^2 + y_2{}^2)]$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} < t^2 + 1 + t^2 - 2t + t(1-t)\left[\frac{x_1{}^2 + y_1{}^2 + x_2{}^2 + y_2{}^2}{a^2b^2}\right]$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} < 2t^2 + 1 - 2t + t(1-t)[\frac{x_1{}^2 + y_1{}^2 + x_2{}^2 + y_2{}^2}{a^2b^2}]$

Now if $a = 1, b = 1$ $then$ $x_1{}^2 + x_2{}^2 < 1$ and $y_1{}^2 + y_2{}^2 < 1$

Thus $\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} < 2t^2 + 1 - 2t + t(1-t)[1+1] < 2t^2 + 1 - 2t + 2t - 2t^2 = 1$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$ and hence proved ellipse is convex.

VERTEX OF THE CURVE: A vertex of the curve $\vec{r}(t)$ in R^2 is a point where its signed curvature K_s has stationary point i.e. $\frac{dK_s}{dt} = 0$ **OR** a critical point of 'K' is called a vertex of curve $\vec{r}(t)$

- A closed curve must have at least two vertices the maximum and minimum point of 'K'
- Every point of circle is vertex.

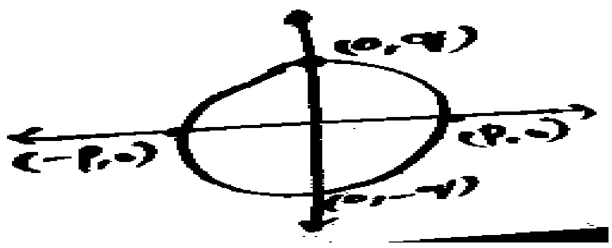
EXAMPLE: The ellipse $\vec{r}(t) = [aCost, bSint]$; $0 \le t \le 2\pi$ is a convex simple closed curve with period 2π , find the vertex of the curve.

SOLUTION:

Let $\vec{r}(t) = [aCost, bSint] \Rightarrow \vec{r}'(t) = [-aSint, bCost] \Rightarrow \vec{r}''(t) = [-aCost, -bSint]$

then $K_s = \|\vec{r}''(t)\| = \sqrt{a^2Cos^2t + b^2Sin^2t}$

For vertex put $\frac{dK_s}{dt} = 0 \Rightarrow \frac{d}{dt}\sqrt{a^2Cos^2t + b^2Sin^2t} = 0 \Rightarrow Sin2t\,(-p+q) = 0$ after solving $\Rightarrow Sin2t = 0,\ (-p+q) \neq 0 \Rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, so K_s vanishes exactly at 4 points. Hence proved the curve is simple convex closed curve having vertices $\vec{r}(t) = [aCost, bSint]$ then at $\boldsymbol{t = 0 \Rightarrow \vec{r}(t) = [0, b], t = \frac{\pi}{2} \Rightarrow \vec{r}(t) = [a, 0], t = \pi \Rightarrow \vec{r}(t) = [-a, 0], t = \frac{3\pi}{2} \Rightarrow \vec{r}(t) = [0, -b]}$

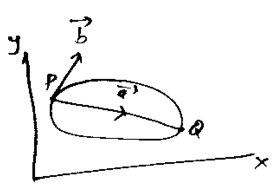


- Signed curvature K_s has relation with vertices so far ellipse we only four points. (next theorem illustrate this)

FOUR VERTEX THEOREM (PP): A convex simple closed curve in R^2 has at least four vertices.

PROOF: Assume that $\vec{r}(t)$ is a unit speed closed curve , so that its period is the length 'l' of $\vec{r}$ then consider $I = \int_0^l K'_s(t)\vec{r}(t)dt = |\vec{r}(t)K_s(t)|_0^l - \int_0^l K_s(t)\vec{r}'(t)dt = 0 - \int_0^l K_s(t)\vec{t}dt$ $\Rightarrow I = \int_0^l \vec{n}'_s(t)dt = |\vec{n}_s(t)|_0^l = \vec{n}_s(l) - \vec{n}_s(0) = 0 \Rightarrow I = 0$ $\vec{r}$ is closed.

Since $K_s(t)$ is a continuous function on the closed interval $[0, l]$ it attain maximum and minimum points (say) 'P' and 'Q'



we can assume $P \neq Q$ since otherwise $K_s(t)$ would be constant. Now $\vec{r}$ is a closed curve and every point of $\vec{r}$ is a vertex. let $\vec{a}$ be a unit vector parallel to $\overline{PQ}$ and $\vec{b}$ be a vector obtained by rotating $\vec{a}$ in anticlockwise by making an angle of $\frac{\pi}{2}$

Now $I = \int_0^l K'_s(t)\vec{r}(t)dt = 0 \Rightarrow \vec{b}.I = \int_0^l K'_s(\vec{b}.\vec{r})dt = 0 \Rightarrow \int_0^l K'_s(\vec{b}.\vec{r})dt = 0 \ldots\ldots\ldots (i)$

now suppose 'P' and 'Q' are vertices of $\vec{r}$ and since $\vec{r}$ is convex then straight line joining 'P' and 'Q' divides the curve into two segments. And since there are no other vertices therefore we have $K'_s > 0$ on one segment and $K'_s < 0$ on other segment.

Then $(i) \Rightarrow either \int_0^l K'_s(\vec{b}.\vec{r})dt < 0\ or\ \int_0^l K'_s(\vec{b}.\vec{r})dt > 0$ ('P' and 'Q' where it vanishes) so it its contradiction.

Hence there must be one more vertex say 'R'. if there are no other vertices then point 'P,Q and R' divide the curve into three segments on each of which always either$K'_s > 0$ or $K'_s < 0$but K'_s must have the same sign on the adjacent segments then there is a straight line which divides $\vec{r}$ again into two segments on which K'_s is always positive and negative. And this is impossible. Hence there must be a fourth vertex. And this complete the proof.

Theorem (Four Vertex Theorem)

A convex simple closed curve γ in $\mathbb{R}^2$ has atleast Four vertices.

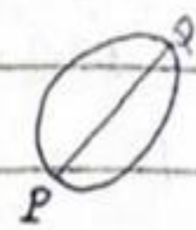
Proof:

Note that at each vertex $\dot{k}_s = 0$
If k_s is contant on any line segment of γ, then every point on line segment is vertex, So we are don.
Now also for circles $k_s = \frac{1}{R} \Rightarrow \dot{k}_s = 0$.
So every point on circle is also a vertex. Therefore we can assume that γ contains neither straight line nor circuler arcs. We have to show that γ contains atleast Four vertices. Suppose on contrary that P and Q are only two vertices of γ and suppose that at P, k_s attains it's global max and at Q, k_s attains it's global min. We shall prove this assumption leads to contradiction because vertices come

in pairs and this will complete the proof.

Note that $P \neq Q$ because otherwise k_s should be constant.

Since γ is convex, so straight line joining P and Q divides γ into two segments.



Since there are no other vertices so we must have $k_s > 0$ on one segment and $k_s < 0$ on other segment.

Let $\gamma : [0,1] \to \mathbb{R}^2$ be unit speed reparametrization of γ s.t.
$\gamma(0) = \gamma(l) = P$ and $\gamma(a) = Q$; $a \in (0,l)$.

We can assume without loss of generality (by applying rigid motion) that P is at origin and Q is at x-axis.

Then integral

$$\int_0^l \dot{k}_s \, (\gamma . \hat{j}) \, dt \neq 0 \quad \text{--- (1)}$$

because $\dot{k}_s \, (\gamma . j)$ never changes sign $\qquad \|\gamma . j\| = \|\gamma\| \|j\| \cos\theta$

It means > 0 on right curve and remain < 0 on left curve.

Now Consider

$$\int_0^l \dot{k}_s \, \gamma \, dt = \int_0^l \gamma \, \dot{k}_s \, dt = \gamma k_s \Big|_0^l - \int_0^l \dot{\gamma} \, k_s \, dl$$

$$= k_s [\gamma(l) - \gamma(0)] - \int_0^l \dot{\gamma} k_s \, dt$$

$$= k_s(0) - \int_0^l \vec{t} \, k_s \, dt \qquad \because \dot{\gamma} = k, \ \gamma(l) = \gamma(0)$$

$$= -\int_0^l \vec{t} \, k_s \, dt$$

We have

$$\dot{\vec{n}}_s = \tau b - k_s \vec{t}$$

Since γ lies in R^2 $\Rightarrow$ $\tau = 0$

$$\therefore \dot{\vec{n}}_s = -k_s \vec{t}$$

$$\therefore \int_0^l k_s \gamma \, dt = \int_0^l \dot{\vec{n}}_s \, dt$$

$$= n_s \Big|_0^l = 0 \quad \because n_s(l) = n_s(0)$$

$$\therefore \int_0^l k_s \gamma \, dt = 0$$

$$\Rightarrow \int_0^l \dot{k}_s (\gamma . j) \, dt = 0$$

which is contradiction to (1)

This contradiction proves that γ does not have atleast two vertices. Since vertices come in pairs, So γ must have atleast four vertices

This completes the proof

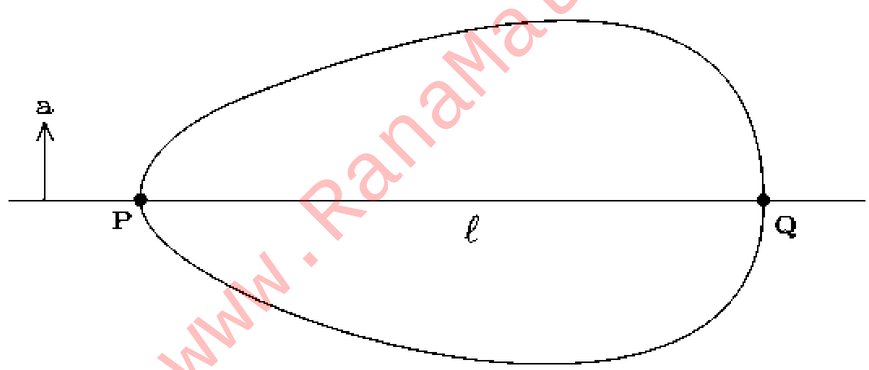
Theorem 3.3.3 (Four Vertex Theorem)

Every convex simple closed curve in $\mathbb{R}^2$ has at least four vertices.

The conclusion of this theorem actually remains true without the assumption of convexity, but the proof is then more difficult than the one we are about to give.

Proof

Let γ be a parametrization of a convex simple closed curve in $\mathbb{R}^2$, and let ℓ be its length. Assume for a contradiction that γ has fewer than four vertices. We show first that there is a straight line L that divides γ into two segments, in one of which $\dot{\kappa}_s > 0$ and in the other $\dot{\kappa}_s \leq 0$ (or possibly $\dot{\kappa}_s \geq 0$ on one and $\dot{\kappa}_s < 0$ on the other). Indeed, κ_s attains all of its values on the closed interval $[0, \ell]$, so κ_s must attain its maximum and minimum values at some points $\mathbf{p}$ and $\mathbf{q}$ of γ. We can assume that $\mathbf{p} \neq \mathbf{q}$, since otherwise κ_s would be constant, γ would be a circle (by Example 2.2.7), and every point of γ would be a vertex. If $\mathbf{p}$ and $\mathbf{q}$ were the only vertices of γ, we would have $\dot{\kappa}_s > 0$ on one of the segments into which the line through $\mathbf{p}$ and $\mathbf{q}$ divides γ and $\dot{\kappa}_s < 0$ on the other. Suppose now that there is just one more vertex, say $\mathbf{r}$. Then, $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ divide γ into three segments, on each of which either $\dot{\kappa}_s > 0$ or $\dot{\kappa}_s < 0$. It follows that there are two adjacent segments on which $\dot{\kappa}_s > 0$ or two on which $\dot{\kappa}_s < 0$ (except at the point at which the two segments meet). This proves our assertion. $\square$



Let $\mathbf{a}$ be a unit vector perpendicular to L, so that $\gamma \cdot \mathbf{a} > 0$ on one side of L and $\gamma \cdot \mathbf{a} < 0$ on the other. Then, the quantity $\dot{\kappa}_s(\gamma \cdot \mathbf{a})$ is either always > 0 or always < 0, except at the two points in which L intersects the curve. It follows that

$$\int_0^{\ell} \dot{\kappa}_s(\gamma \cdot \mathbf{a}) \, dt \neq 0, \tag{3.8}$$

as this integral is definitely > 0 in the first case and < 0 in the second. But, using the equation $\dot{\mathbf{n}}_s = -\kappa_s \mathbf{t}$ (see Exercise 2.2.1), we get

$$\dot{\kappa}_s \gamma = (\kappa_s \gamma)^{\cdot} - \kappa_s \dot{\gamma} = (\kappa_s \gamma + \mathbf{n}_s)^{\cdot},$$

so the integrand on the left-hand side of (3.8) is the derivative of $(\kappa_s \gamma + \mathbf{n}_s) \cdot \mathbf{a} = \lambda$, say. Since γ is ℓ-periodic,

$$\gamma(t + \ell) = \gamma(t) \quad \text{for all} \quad t,$$

differentiating with respect to t shows that the tangent vector $\mathbf{t}$ of γ is also ℓ-periodic:

$$\mathbf{t}(t + \ell) = \dot{\gamma}(t + \ell) = \dot{\gamma}(t) = \mathbf{t}(t).$$

Rotating by $\pi/2$ gives

$$\mathbf{n}_s(t+\ell) = \mathbf{n}_s(t),$$

and hence $\kappa_s(t+\ell) = \kappa_s(t)$. It follows that $\lambda(t+\ell) = \lambda(t)$ for all t, so the integral in (3.8) is equal to

$$\int_0^\ell \dot{\lambda}(t)\,dt = \lambda(\ell) - \lambda(0) = 0.$$

This contradiction proves that γ must have at least four vertices. $\square$

EXERCISE: Show that the length $l(\vec{r})$ and $A(int(\vec{r}))$ are unchanged by applying the rigid motion to $\vec{r}$

SOLUTION: Since $\vec{r}(t) = [x(t), y(t)]$ then we have for a curve $\vec{r}(t)$ of period 'a'

$l = \int_0^a \|\vec{r}'(t)\| dt = \int_0^a \sqrt{x'^2 + y'^2} dt \ldots\ldots (i)$ and $A(int(\vec{r})) = \frac{1}{2}\int_0^a (xy' - yx')\, dt \ldots\ldots (ii)$

if $\tilde{r}$ is obtained from $\vec{r}$ by a translation then $\tilde{r}' = \vec{r}'$ and if $\tilde{r}$ is obtained from $\vec{r}$ by a rotation of and angle θ about the origin then $R_\theta(\vec{r}) = R_\theta(x, y) = (xCos\theta - ySin\theta\, , xSin\theta + yCos\theta)$ then $\tilde{x} = xCos\theta - ySin\theta \Longrightarrow \tilde{x}' = x'Cos\theta - y'Sin\theta$ also $\tilde{y} = xSin\theta + yCos\theta \Longrightarrow \tilde{y}' = x'Sin\theta + y'Cos\theta$

now consider $\tilde{x}'^2 + \tilde{y}'^2 = (x'Cos\theta - y'Sin\theta)^2 + (x'Sin\theta + y'Cos\theta)^2 = x'^2 + y'^2$ after solving.

So $l(\tilde{r}) = \int_0^a \|\tilde{r}'(t)\| dt = \int_0^a \sqrt{\tilde{x}'^2 + \tilde{y}'^2} dt = \int_0^a \sqrt{x'^2 + y'^2} dt = \int_0^a \|\vec{r}'(t)\| dt = l(\vec{r})$

$\Longrightarrow l(\tilde{r}) = l(\vec{r}) \ \ldots\ldots (iii)$

Now for area

$\tilde{x}\tilde{y}' - \tilde{y}\tilde{x}' = (xCos\theta - ySin\theta)(x'Sin\theta + y'Cos\theta) - (xSin\theta + yCos\theta)(x'Cos\theta - y'Sin\theta)$

$\tilde{x}\tilde{y}' - \tilde{y}\tilde{x}' = xy'(Cos^2\theta + Sin^2\theta) + yx'(Cos^2\theta + Sin^2\theta) = xy' - yx'$

$\Longrightarrow A(int(\tilde{r})) = \frac{1}{2}\int_0^a (\tilde{x}\tilde{y}' - \tilde{y}\tilde{x}')\, dt = \frac{1}{2}\int_0^l (xy' - yx')\, dt = A(int(\vec{r})))$

$\Longrightarrow A(int(\tilde{r})) = l(\vec{r}) \ \ldots\ldots (iv)$

Hence from (iii) and (iv) proved that length and area remains unchanged.

EXERCISE (PP): Show that ellipse $\vec{r}(t) = [aCost, bSint]$; $0 \le t \le 2\pi$ is a simple closed curve and compute area of its interior and also length of curve.

SOLUTION: Suppose $\vec{r}(t') = \vec{r}(t) \Longrightarrow [aCost', bSint'] = [aCost, bSint] \Longrightarrow t' = t$

$\Longrightarrow t' = t + 2k\pi \quad \therefore t' - t = 2k\pi$ Then $\vec{r}$ is a simple closed curve. Now for period 2π we have $A(int(\vec{r})) = \frac{1}{2}\int_0^{2\pi} (xy' - yx')\, dt = \pi ab$ where $x = aCost, y = bSint$ also $l(\vec{r}) = \int_0^{2\pi} \|\vec{r}'(t)\| dt = \int_0^{2\pi} \sqrt{x'^2 + y'^2} dt = \int_0^{2\pi} \sqrt{a^2Sin^2t + b^2Cos^2t} dt$

EXAMPLE: By applying isoperimetric inequality to the ellipse prove that then $\int_0^{2\pi} \sqrt{a^2Sin^2t + b^2Cos^2t} dt \ge 2\pi\sqrt{ab}$ and equality holds iff $a = b$

SOLUTION: Since by isoperimetric inequality $A(\vec{r}) \le \frac{1}{4\pi} l^2(\vec{r}) \ \ldots\ldots\ldots (i)$ and for ellipse we have $A(int(\vec{r})) = \pi ab$ and $l(\vec{r}) = \int_0^{2\pi} \sqrt{a^2Sin^2t + b^2Cos^2t} dt$ using these in (i)

$\Longrightarrow \pi ab \le \frac{1}{4\pi} l^2(\vec{r}) \ \Longrightarrow 4\pi^2 ab \le l^2(\vec{r}) \Longrightarrow \sqrt{4\pi^2 ab} \le \sqrt{l^2(\vec{r})} \Longrightarrow l(\vec{r}) \ge 2\pi\sqrt{ab}$

$\Longrightarrow \int_0^{2\pi} \sqrt{a^2Sin^2t + b^2Cos^2t} dt \ge 2\pi\sqrt{ab}$

Now for holding above inequality put $a = b$

$\Longleftrightarrow \int_0^{2\pi} \sqrt{a^2Sin^2t + a^2Cos^2t} dt = 2\pi\sqrt{aa} \Longleftrightarrow \int_0^{2\pi} a\sqrt{Sin^2t + Cos^2t} dt = 2\pi a$

$\Longleftrightarrow a\int_0^{2\pi} dt = 2\pi a \Longleftrightarrow a|t|_0^{2\pi} = 2\pi a \Longleftrightarrow 2\pi a = 2\pi a$

DIFFERENTIAL GEOMETRY OF SURFACES

OPEN SET: A subset U of R^n is called open if for any $a \epsilon U \exists \in > 0$ such that for every point $u\epsilon R^n$ we have $\|u - a\| < \in \Longrightarrow u\epsilon U$

EXAMPLES: The whole of R^n is an open set.

i. The set $D_r(a) = \{u\epsilon R^n : \|u - a\| < r\}$ is an open set and is called open ball with centre 'a' and radius 'r'
   - If $n = 1 \Longrightarrow D_r(a)$ is an open interval.
   - If $n = 2 \Longrightarrow D_r(a)$ is an open disk.
   - If $n = 3 \Longrightarrow D_r(a)$ is an open spheres.

ii. The set $\overline{D_r(a)} = \{u\epsilon R^n : \|u - a\| \leq r\}$ is not an open set and is called closed ball with centre 'a' and radius 'r'

CONTINUOUS MAPPING: Suppose $X \subseteq R^m$ and $Y \subseteq R^n$ then a mapping $f: X \to Y$ is said to be a continuous at $a\epsilon X$ if for any $\in > 0 \exists \delta > 0$ s.that $\|f(x) - f(a)\| < \in$ whenever $|x - a| < \delta$

HOMEOMORPHISM: A mapping $f: X \to Y$ is said to be Homeomorphism if 'f' is continuous and bijective and $f^{-1}: Y \to X$ is also continuous, then X is said to be Homeomorphic to Y.

SURFACE: A surface $S\ of\ R^3$ is the locus of the point whose coordinates are functions of two independent parameters 'u' and 'v'

Thus $x = f_1(u, v), y = f_2(u, v), z = f_3(u, v)$ are parametric equations of surfaces.

OR: A surface in $R^3$is a set of all points whose coordinates satisfy a single equation $f(x, y, z) = 0$ for example sphere, paraboliods, hyperboloid etc. (discussed in start)

OR: A surface may be regarded as the locus of a point whose position vector $\vec{r}$ is a function of two independent parameters u and v.

We not that any relation between the parameters $(f\ (u, v) = 0)$ represents a curve on the surface, because $\vec{r}$ than becomes a function of only one independent parameter.

In particular, the curve on the surface, along which one of the parameters remains constant are called **<u>parametric curves</u>**.

Position of any point on the surface is uniquely determined by the values of 'u' and 'v'. So that the parameters 'u' and 'v' constitute a system of coordinates which are called **<u>curvilinear coordinates</u>**.

REMARKS:

1) Elimination of u and v from the parametric equations will give rise to the equation of the form $f(x, y, z) = c$ which is called the implicit form of the surface.
2) An equation of the form $z = f(x, y)$ or $x_3 = f(x_1, x_2)$ is called **monge's form**.

ONE PARAMETER FAMILY OF SURFACE: an equation of the form $F(x, y, z, a) = 0$ where 'a' is constant represents a surface. It the value of constant 'a' is changed then we get another surface.

The set of all surfaces corresponding to different values of 'a' is called one parameter family of surface with parameter 'a' .<u>This parameter 'a' has different significance from that of 'u' and 'v'</u> (u and v relate a single surface while 'a' remains constant for single surface) these relate to a single surface and vary from point to point of that surface. These are curvilinear coordinates of a point on the single surface (the parameter 'a' however determines the particular member of the family of surfaces and have the some value at all points of that member.

e.g. $F(x,y,z) = x^2 + y^2 + z^2 - a^2 = 0$ Represent a sphere with centre at origin and radius 'a' where 'a' is that parameter.

CHARACTERISTICS OF A SURFACE

The curve of intersection of two surfaces of the family corresponding to the parameters value of 'a' and $a + \delta a$ is determined by the equation $F(x,y,z,a) = 0$ and $F(x,y,z,a+\delta a) = 0$

And therefore by equations $F(a) = 0$ and $F(a+\delta a) - F(a) = 0$

(for the sake of similarity we write $F(x,y,z,a) = F(a)$)

Now if $\delta a \to 0$ the curve becomes the curve of intersection of the consecutive members of the family with parameter value and its defining equation are $\boldsymbol{F(a) = 0}$ and $\boldsymbol{\frac{\partial}{\partial a}F(a) = 0}$

This curve is called characteristic of the surface for the parametric value 'a'

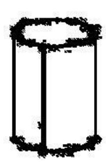
LEVEL SURFACE: A level surface 'S' is a function $\varphi(x,y,z)$ define by the locus of a point $P(x,y,z)$ in domain 'D' such that $\varphi(x,y,z) = c$ where 'c' is constant.

OR: A level surface is a surface $S = \{(x,y,z)\epsilon R^3 : f(x,y,z) = 0\}$ where f is smooth function. So surface is also smooth.

PARAMETERIZED SURFACE OR SURFACE PATCH: A subset $S\ of\ R^3$ is a surface if for every point $P \epsilon S$ $\exists$ an open set U in R^2 and an open set W in R^3 containing P such that a Homeomorphism $\sigma : U \to S \cap W$ is called a **surface patch or parameterization** of 'S'. Where $S \cap W \subseteq S$ and since 'W' is open then $S \cap W$ is also open in 'S'

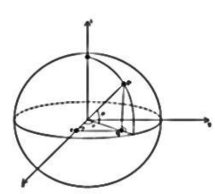
A collection of such surfaces whose image cover the whole of 'S' is called an **Atlas** of S.

EXAMPLES: **(i)** Circular cylinder of radius 'a' and axis is z − axis and $S = \{(x,y,z)\epsilon R^3 : x^2 + y^2 = a^2\}$ and the parameterization of 'S' is



$\sigma(u,v) = (aCosu, aSinu, v)$

(ii) A sphere of radius 'a' and $S = \{(x,y,z)\epsilon R^3 : x^2 + y^2 + z^2 = a^2\}$ and the parameterization



of 'S' is $\sigma(\theta,\varphi) = (aSin\theta Sin\varphi, aSin\theta Cos\varphi, aCos\theta)$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $0 \le \varphi \le 2\pi$

(iii) A circular cone i.e. $S = \{(x,y,z)\epsilon R^3 : x^2 + y^2 = z^2\}$ and the parameterization of 'S' is

$\sigma(u,v) = \left(u.v, \pm\sqrt{u^2+v^2}\right)$

SMOOTH SURFACE: If U is an open set i.e. $U\epsilon R^3$ a mapping $\sigma: U \to R^3$ is said to be a smooth surface if each of the n – components of 'σ' which are functions $U \to R$ have continuous partial derivatives of all orders.

REMARK:

i. **(Classical result of surfaces)** One of the classical result of the differential geometry is that if $\sigma: U \to R^3$ of two components then $\frac{\partial\sigma}{\partial x\partial y} = \frac{\partial\sigma}{\partial y\partial x}$ where $U \subseteq R^2$

ii. A mapping $\sigma: U \to R^3$ is smooth surface if σ is of two components 'u' and 'v' i.e. $\sigma(u, v) = (\sigma_1(u, v), \sigma_2(u, v), \sigma_3(u, v))$ then $\frac{\partial\sigma}{\partial u} = \left(\frac{\partial\sigma_1}{\partial u}, \frac{\partial\sigma_2}{\partial u}, \frac{\partial\sigma_3}{\partial u}\right)$ and $\frac{\partial\sigma}{\partial v} = \left(\frac{\partial\sigma_1}{\partial v}, \frac{\partial\sigma_2}{\partial v}, \frac{\partial\sigma_3}{\partial v}\right)$

iii. We often use the following abbreviations;

$$\frac{\partial\sigma}{\partial u} = \sigma_u , \vec{r}_2 = \vec{r}_v = \frac{\partial\vec{r}}{\partial v} = \frac{\partial\sigma}{\partial v} = \sigma_v$$

$$\vec{r}_{11} = \vec{r}_{uu} = \frac{\partial^2\vec{r}}{\partial u^2} = \frac{\partial^2\sigma}{\partial u^2} = \sigma_{uu}, , \vec{r}_{22} = \vec{r}_{vv} = \frac{\partial^2\vec{r}}{\partial v^2} = \frac{\partial^2\sigma}{\partial v^2} = \sigma_{vv}, , \vec{r}_{12} = \vec{r}_{uv} = \frac{\partial^2\vec{r}}{\partial u\partial v} = \frac{\partial^2\sigma}{\partial u\partial v} = \sigma_{uv}$$

iv. In general $\sigma_{vu} \neq \sigma_{uv}$ but if $\sigma: U \to R^3$ is smooth then $\sigma_{vu} = \sigma_{uv}$

REGULAR PARAMETERIZATION OF SURFACE: A surface $\sigma: U \to R^3$ is called regular if it is smooth and the vectors σ_u *and* σ_v are linearly independent at all points$(u, v)\epsilon U$, equivalently, σ should be smooth and $\sigma_u \times \sigma_v \neq \vec{0}$

QUESTION: Let $U \subseteq R^2$ and $f: U \to R^3$ then the surface $z = f(x, y)$ is parameterized by $\sigma(u, v) = (u. v, f(u, v))$ prove that it is regular.

Solution: Given $\sigma(u, v) = (u. v, f(u, v)) \Rightarrow \sigma_u = (1,0, f_u), \sigma_v = (0,1, f_u)$

$$\Rightarrow \sigma_u \times \sigma_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} = (f_u, f_u, 1) \neq \vec{0} \Rightarrow surface\ is\ regular$$

QUESTION: The parameterization of surface is $\sigma(u, v) = (u + v, u - v, u^2 + v^2)$ show that this surface represents the elliptical paraboloid $z = \frac{1}{2}(x^2 + y^2)$ and it is regular.

Solution: Suppose from given $\sigma(u, v)$ we have
$x = u + v \dots\dots (i), y = u - v \dots\dots (ii), z = u^2 + v^2 \dots\dots (iii)$

Adding (i) *and* $(ii) \Rightarrow u = \frac{1}{2}(x + y)$ and Subtracting (i) *and* $(ii) \Rightarrow v = \frac{1}{2}(x - y)$

$$(iii) \Rightarrow z = \left(\frac{1}{2}(x + y)\right)^2 + \left(\frac{1}{2}(x - y)\right)^2 \Rightarrow z = \frac{1}{2}(x^2 + y^2)$$

Now $\sigma(u, v) = (u + v, u - v, u^2 + v^2) \Rightarrow \sigma_u = (1,1,2u), \sigma_v = (1, -1,2v)$

$$\Rightarrow \sigma_u \times \sigma_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 2u \\ 1 & -1 & 2v \end{vmatrix} = 2(u + v, u - v, -1) \neq \vec{0} \Rightarrow surface\ is\ regular$$

QUESTION: The mapping $\sigma(u,v) = (CosuSinv, SinuSinv, Cosv)$ defines a mapping of the uv – plane onto a unit sphere. Then prove that $x^2 + y^2 + z^2 = 1$ and examine it is regular or not?

Solution: Suppose from Given $\sigma(u,v)$ we have $x = CosuSinv, y = SinuSinv, z = Cosv$ then $\Longrightarrow x^2 + y^2 + z^2 = 1$

Now $\sigma_u = (-SinuSinv, CosuSinv, 0), \sigma_v = (CosuCosv, SinuCosv, -Sinv)$

$$\Longrightarrow \sigma_u \times \sigma_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -SinuSinv & CosuSinv & 0 \\ CosuCosv & SinuCosv & -Sinv \end{vmatrix} = Sinv(-CosuSinv, -SinuSinv, -Cosv)$$

$\Longrightarrow \sigma_u \times \sigma_v \neq \vec{0}$ $when\ Sinv \neq 0\ i.e. v = (2n+1)\frac{\pi}{2} \Longrightarrow surface\ is\ regular\ for v = (2n+1)\frac{\pi}{2}$

$\Longrightarrow \sigma_u \times \sigma_v = \vec{0}$ $when\ Sinv = 0\ i.e. v = n\pi \Longrightarrow surface\ is\ not\ regular\ for v = n\pi$

REMARK: Another parameterization of surface (unit sphere) can be express as $\sigma(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right)$

EXAMPLE: The parameterization of surface $\sigma(u,v) = (\rho Cosu, \rho Sinu, v)$ where $\rho > 0$ show that it represent the right circular cylinder with radius ρ and axis is z – axis. And show that it is regular.

Solution: Suppose from given $\sigma(u,v)$ we have $x = \rho Cosu, y = \rho\, Sinu, z = v$ then $\Longrightarrow x^2 + y^2 = \rho^2$ which is circular cylinder.

Now $\sigma_u = (-\rho Sinu, \rho Cosu, 0), \sigma_v = (0,0,1)$

$$\Longrightarrow \sigma_u \times \sigma_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\rho Sinu & \rho\, Cosu & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\rho Cosu, \rho Sinv, 0) \neq \vec{0} \Longrightarrow surface\ is\ regular$$

NOTE: We have two methods of reparameterization of a surface, one is global reparameterization by using the constraint equation $F(x,y,z) = 0$ and other parametrically by expressing (x,y,z) in terms of (u,v) varying over a domain as $\sigma(u,v) = (f(u,v), g(u,v), h(u,v))$

PRACTICE: Show that following are regular or not?

i. $\sigma(u,v) = (u, v, vu)$
ii. $\sigma(u,v) = (u, v^2, v^3)$
iii. $\sigma(u,v) = (u + u^2, v, v^2)$

EXAMPLE: Show that $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1$ where p,q,r are non – zero constants, is a smooth surface of the form $\sigma(u,v) = \left(u, v, \pm r\sqrt{1 - \frac{u^2}{p^2} - \frac{v^2}{q^2}}\right)$

Solution: Suppose $x = u, y = v$ then given as $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1 \Longrightarrow \frac{u^2}{p^2} + \frac{v^2}{q^2} + \frac{z^2}{r^2} = 1$

$\Longrightarrow \frac{z^2}{r^2} = 1 - \frac{u^2}{p^2} - \frac{v^2}{q^2} \Longrightarrow z^2 = r^2\left(1 - \frac{u^2}{p^2} - \frac{v^2}{q^2}\right) \Longrightarrow z = \pm r\sqrt{1 - \frac{u^2}{p^2} - \frac{v^2}{q^2}}$

therefor $\Longrightarrow \sigma(u,v) = \left(u, v, \pm r\sqrt{1 - \frac{u^2}{p^2} - \frac{v^2}{q^2}}\right)$

SECANT LINE: The line which cut the curve at two points.

TANGENT LINE: tangent to any curve drawn at a space is called a tangent line to the surface.

Or The line which cut the curve at one point.

TANGENT PLANE: tangent plane to a surface at a point 'P' is the plane containing all the tangent lines to the surface at that point.

OR A tangent plane to a surface 'S' at point $\vec{P}\epsilon S$ is a tangent vector at $\vec{P}$ of a curve in 'S' passing through $\vec{P}$

Now suppose $\vec{r}(t) = \sigma(u,v) \Longrightarrow \vec{r}'(t) = \frac{\partial\sigma}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial\sigma}{\partial v}\frac{\partial v}{\partial t} = \lambda\sigma_u + \lambda\sigma_v$ this is the equation of tangent plane at $\vec{P}\epsilon S$ of a surface $\sigma: S \to R^3$

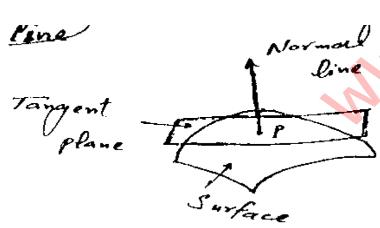
TANGENT SPACE: Set of all tangent plane vector to surface 'S' at $\vec{P} = \sigma(u_0, v_0)$ is called tangent space and it can be expressed as $T_p(S) = \{\lambda\sigma_u + \lambda\sigma_v : \lambda, u\epsilon R\}$

CURVE ON THE SURFACE: If $\vec{r}: (\propto, \beta) \to R^3$ is a parameterized curve contained in the image of a surface patch $\sigma: U \to R^3$ in the Atlas of 'S' then there is a mapping $\vec{r}: (\propto, \beta) \to U$ such that $\vec{r}(t) = \sigma\big(u(t), v(t)\big)$ where $u(t), v(t)$ are necessarily smooth functions.

STANDARD UNIT NORMAL TO THE SURFACE: A surface 'S' defined by a mapping $\sigma: U \to R^3$ where $U \subseteq R^2$ containing a point $\vec{P}$ then standard unit normal to the surface is defined as $\vec{N}_\sigma = \vec{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$

OR The normal to the surface at any point is perpendicular to every tangent line $(\vec{r}_1 \; and \; \vec{r}_2)$ and its direction is along $|\vec{r}_1 \times \vec{r}_2|$ Thus $\Longrightarrow \vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} = \frac{\vec{r}_1 \times \vec{r}_2}{H}$

NORMAL LINE TO THE SURFACE: The line which is perpendicular to the tangent line.
OR The straight line passes through 'P' perpendicular to the tangent plane at 'P' is called normal line to the surface at 'P' and equation of normal line at 'P' is given by $\vec{y} = \vec{x} + t\vec{N}_\sigma$



EXAMPLE: Find the equation of tangent plane and normal line to the surface represented by $\sigma(u,v) = (u, v, u^2 - v^2)$ at $\vec{P}(1,1,0)$

SOLUTION: Given $\sigma(u,v) = (u, v, u^2 - v^2)$ Then $\sigma_u = (1,0,2u)$, $\sigma_v = (0,1,-2v)$ and $\sigma(1,1) = (1,1,0)$ is a given point 'P' and equation of tangent plane at $\vec{P}(1,1,0)$ is $\vec{r}'(t) = \lambda\sigma_u + \lambda\sigma_v = \lambda(1,0,2u) + \lambda(0,1,-2v)$

Or $T_p(S) = \{\lambda\lambda(1,0,2u) + \lambda(0,1,-2v) : \lambda, u\epsilon R\} = \{(\lambda, u, 2\lambda - 2u) : \lambda, u\epsilon R\}$

For normal line we find $\vec{N}_\sigma = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{(-2,2,1)}{3} = \left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

Now by using equation of normal line $\vec{y} = \vec{x} + t\vec{N}_\sigma = (1,1,0) + t\left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

$\vec{y} = 1 - \frac{2}{3}t, 1 + \frac{2}{3}t, \frac{1}{3}t$

- **Practice:** Find the equation of tangent plane and normal line to the surface represented by $\sigma(r,\theta) = (rCosh\theta, rSinh\theta, r^2)$ at $\vec{P}(1,01)$

LEVEL SURFACE: A level surface 'S' is a function $\varphi(x,y,z)$ define by the locus of a point $P(x,y,z)$ in domain 'D' such that $\varphi(x,y,z) = c$ where 'c' is constant.

OR: A level surface is a surface $S = \{(x,y,z)\epsilon R^3 : f(x,y,z) = 0\}$ where f is smooth function. So surface is also smooth.

CONDITON TO PROVE SMOOTH LEVEL SURFACES: A surface is smooth if $\nabla f = (f_x, f_y, f_z)$ of $f(x,y,z)$ does not vanishes at 'P' i.e. $\|\nabla f\| \neq 0$

EXAMPLE: Show that surface $x^2 + y^2 + z^2 = 1$ is smooth surface.

SOLUTION: Suppose $f(x,y,z) = x^2 + y^2 + z^2 - 1 \Longrightarrow f_x = 2x, f_y = 2y, f_z = 2z$

Now $\nabla f = (f_x, f_y, f_z) = 2(x,y,z) \Longrightarrow \|\nabla f\| = 2\sqrt{x^2 + y^2 + z^2} = 2\sqrt{1} = 2 \neq 0$
$\Longrightarrow$ given surface is smooth.

PRACTICE: Show that following surface are smooth or not?

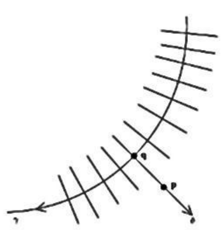
i. $z^2 = x^2 + y^2$

ii. $x^2 + y^2 + z^4 = 1$

iii. $x^2 + y^2 + z^2 + a^2 - b^2 = 4a^2(x^2 + y^2)$ *where* $a > b > 0$ *are constants*

RULED SURFACE: A ruled surface is a surface which is the union of straight lines and it is also called ruling of the surfaces.

OR a surface which is generated by the motion of one parameter family of straight lines is called a ruled surface.



TYPES OF RULED SURFACE: (i) Developable Surface (ii) Skew Surface

DEVELOPABLE SURFACE: If consecutive generators intersect, then the ruled surface is called developable. e.g. Cones and Cylinders.

SKEW SURFACE: If consecutive generators do not intersect, then the ruled surface is called developable. e.g. Hyperboloid of one sheet and Hyperbolic Paraboloid.

SURFACE OF REVOLUTION: A surface generated by the rotation of the plane curve about an axis in its plane is called a **surface of revolution**.

If z – axis is taken as the axis of revolution and 'u' denotes the distance of a point from z – axis then surface may be expressed as $x = uCos\varphi, \quad y = uSin\varphi, \quad z = f(u)$

We may also use $\sigma(u,v) = (uCosv, \quad uSinv, \quad f(u))$

THEOREM: Derive an equation of the tangent plane and equation of the normal at a point 'P' to the surface $F(x,y,z)=0$

PROOF: Let $F(x,y,z)=0 \dots\dots\dots\dots(i)$ be the equation of surface. Let's' be the length of the curve measured from a point 'A' to the point $'(x,y,z)'$
Diff. (i) w.r.to 's' $\Longrightarrow F_x x' + F_y y' + F_z z' = 0$

$\Longrightarrow (F_x,F_y,F_z)(x',y',z')=0 \dots\dots\dots\dots(ii)$ Where the vector (x',y',z') is a unit tangent $\vec{t}$ to the curve on the surface at point $'(x,y,z)'$

Equation (ii) shows that $\vec{t}$ is perpendicular to (F_x,F_y,F_z) which is $\nabla F = grad\ F = \left(\frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z}\right)$ it is clear that all the tangent lines to the surface at the point $'(x,y,z)'$ are perpendicular to (F_x,F_y,F_z) and hence lies in the plane through (x,y,z) perpendicular to this vector and this plane is called tangent plane to the surface at that point and normal to the plane at that point of contact is called normal to the surface.

Since the line joining the point $R(X,Y,Z)$ on the tangent plane to the point of contact is perpendicular to the normal so we have

$$(\vec{R}-\vec{r}).\nabla F = 0 \Longrightarrow (X-x,Y-y,Z-z)\left(\frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z}\right)=0$$
$$\Longrightarrow (X-x,Y-y,Z-z)(F_x,F_y,F_z)=0 \Longrightarrow \boldsymbol{(X-x)F_x+(Y-y)F_y+(Z-z)F_z=0}$$

Which is the equation of tangent plane and equation of normal will be

$$\vec{R}=\vec{r}+\mu\nabla F \Longrightarrow \vec{R}-\vec{r}=\mu\nabla F \Longrightarrow (X-x,Y-y,Z-z)=\mu(F_x,F_y,F_z)$$

$$\Longrightarrow \frac{X-x}{F_x}=\frac{Y-y}{F_y}=\frac{Z-z}{F_z}=\mu$$

If we eliminate μ then we have $\boldsymbol{\frac{X-x}{F_x}=\frac{Y-y}{F_y}=\frac{Z-z}{F_z}}$ which is equation of normal to the surface.

THEOREM: Prove that sum of square of the intercepts made by the tangent plane to the surface is constant. i.e. $x^{2/3}+y^{2/3}+z^{2/3}=a^{2/3}$

PROOF: Given surface is $F(x,y,z)=x^{2/3}+y^{2/3}+z^{2/3}-a^{2/3}=0 \dots\dots\dots(i)$

$$F_x=\frac{\partial F}{\partial x}=\frac{2}{3}x^{-1/3}, F_y=\frac{\partial F}{\partial y}=\frac{2}{3}y^{-1/3}, F_z=\frac{\partial F}{\partial z}=\frac{2}{3}z^{-1/3}$$

Now the equation of tangent plane is $(X-x)F_x+(Y-y)F_y+(Z-z)F_z=0$

$$\Longrightarrow (X-x)\frac{2}{3x^{\frac{1}{3}}}+(Y-y)\frac{2}{3y^{\frac{1}{3}}}+(Z-z)\frac{2}{3z^{\frac{1}{3}}}=0$$

$$\Longrightarrow \frac{2}{3}\left[\frac{X-x}{x^{\frac{1}{3}}}+\frac{Y-y}{y^{\frac{1}{3}}}+\frac{Z-z}{z^{\frac{1}{3}}}\right]=0 \Longrightarrow \frac{2}{3}\neq 0,, \left[\frac{X-x}{x^{\frac{1}{3}}}+\frac{Y-y}{y^{\frac{1}{3}}}+\frac{Z-z}{z^{\frac{1}{3}}}\right]=0$$

$$\Longrightarrow \frac{X}{x^{\frac{1}{3}}}-\frac{x}{x^{\frac{1}{3}}}+\frac{Y}{y^{\frac{1}{3}}}-\frac{y}{y^{\frac{1}{3}}}+\frac{Z}{z^{\frac{1}{3}}}-\frac{z}{z^{\frac{1}{3}}}=0 \Longrightarrow \frac{X}{x^{\frac{1}{3}}}-x^{2/3}+\frac{Y}{y^{\frac{1}{3}}}-y^{2/3}+\frac{Z}{z^{\frac{1}{3}}}-z^{2/3}=0$$

$$\Longrightarrow \frac{X}{x^{\frac{1}{3}}}+\frac{Y}{y^{\frac{1}{3}}}+\frac{Z}{z^{\frac{1}{3}}}=x^{2/3}+y^{2/3}+z^{2/3}=a^{2/3} \text{ by (i)} \quad \Longrightarrow \frac{X}{x^{\frac{1}{3}}a^{2/3}}+\frac{Y}{y^{\frac{1}{3}}a^{2/3}}+\frac{Z}{z^{\frac{1}{3}}a^{2/3}}=1$$

Thus the intercept with coordinates axis are $\left(x^{\frac{1}{3}}a^{2/3},0,0\right),\left(0,y^{\frac{1}{3}}a^{2/3},0\right),\left(0,0,z^{\frac{1}{3}}a^{2/3}\right)$

$$Then\ sum\ of\ square\ of\ intercept = \left(x^{\frac{1}{3}}a^{2/3}\right)^2+\left(y^{\frac{1}{3}}a^{2/3}\right)^2+\left(z^{\frac{1}{3}}a^{2/3}\right)^2$$

$$\Longrightarrow\ sum\ of\ square\ of\ intercept\ = a^{4/3}\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}+y^{\frac{2}{3}}\right) = a^{4/3}\left(a^{\frac{2}{3}}\right)=a^{6/3}=a^2=cosntant$$

THEOREM: Prove that the tangent plane at any point to the surface is constant. i.e. $xyz = a^3$ and to the coordinate plane form a tetrahedron of constant volume.

PROOF: Since $V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$ and given surface is $F(x,y,z) = xyz - a^3 = 0$

$\Longrightarrow F_x = yz, F_y = xz, F_z = xy$

Now the equation of tangent plane is $(X-x)F_x + (Y-y)F_y + (Z-z)F_z = 0$

$\Longrightarrow (X-x)yz + (Y-y)xz + (Z-z)xy = 0$

$\Longrightarrow Xyz - xyz + Yxz - xyz + Zxy - xyz = 0 \Longrightarrow Xyz + Yxz + Zxy = 3xyz$

$$\Longrightarrow \frac{Xyz}{3xyz} + \frac{Yxz}{3xyz} + \frac{Zxy}{3xyz} = 1 \Longrightarrow \frac{X}{3x} + \frac{Y}{3y} + \frac{Z}{3z} = 1$$

Now the point of intersection are $(0,0,0), (3x,0,0), (0,3y,0), (0,0,3z)$

$$\Longrightarrow V = \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ 3x & 0 & 0 & 1 \\ 0 & 3y & 0 & 1 \\ 0 & 0 & 3z & 1 \end{vmatrix} \Longrightarrow V = \frac{1}{6}\begin{vmatrix} 3x & 0 & 0 \\ 0 & 3y & 0 \\ 0 & 0 & 3z \end{vmatrix} \Longrightarrow V = \frac{1}{6}(27xyz) = \frac{9}{2}a^3 = constant$$

QUESTION(PP): The normal at a point 'P' of the ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ meet the coordinate planes in the points G_1, G_2, G_3 respectively. Prove that the ratios $\frac{|PG_1|}{|PG_2|}, \frac{|PG_2|}{|PG_3|}, \frac{|PG_3|}{|PG_1|}$ are constant.

SOLUTION: Given surface is $F(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \Longrightarrow F_x = \frac{2x}{a^2}, F_y = \frac{2y}{b^2}, F_z = \frac{2z}{c^2}$

Now the equation of tangent plane at any point $P(x,y,z)$ to the surface is $\Longrightarrow \frac{X-x}{F_x} = \frac{Y-y}{F_y} = \frac{Z-z}{F_z}$

$$\Longrightarrow \frac{X-x}{\frac{2x}{a^2}} = \frac{Y-y}{\frac{2y}{b^2}} = \frac{Z-z}{\frac{2z}{c^2}} \Longrightarrow \frac{X-x}{\frac{x}{a^2}} = \frac{Y-y}{\frac{y}{b^2}} = \frac{Z-z}{\frac{z}{c^2}}$$

Now the normal meet at YZ – Plane at G_1 then $X = 0$

$$\Longrightarrow \frac{-x}{\frac{x}{a^2}} = \frac{Y-y}{\frac{y}{b^2}} = \frac{Z-z}{\frac{z}{c^2}} \Longrightarrow -a^2 = \frac{Y-y}{\frac{y}{b^2}} = \frac{Z-z}{\frac{z}{c^2}}$$

$$\Longrightarrow -a^2 = \frac{Y-y}{\frac{y}{b^2}} \Longrightarrow Y - y = -\frac{a^2y}{b^2} = -\frac{a^2y}{b^2} + y = y\left(-\frac{a^2}{b^2} + 1\right) \Longrightarrow Y = y\left(\frac{b^2-a^2}{b^2}\right)$$

$$and \Longrightarrow -a^2 = \frac{Z-z}{\frac{z}{c^2}} \Longrightarrow Z - z = -\frac{a^2z}{c^2} = -\frac{a^2z}{c^2} + z = z\left(-\frac{a^2}{c^2} + 1\right) \Longrightarrow Z = z\left(\frac{c^2-a^2}{c^2}\right)$$

Thus a point lying in YZ – Plane which is G_1 have the coordinates $\left[0, y\left(\frac{b^2-a^2}{b^2}\right), z\left(\frac{c^2-a^2}{c^2}\right)\right]$

Similarly a point lying in XZ – Plane which is G_2 have the coordinates $\left[x\left(\frac{a^2-b^2}{a^2}\right), 0, z\left(\frac{c^2-a^2}{c^2}\right)\right]$

Thus a point lying in XY – Plane which is G_3 have the coordinates $\left[x\left(\frac{a^2-b^2}{a^2}\right), y\left(\frac{b^2-a^2}{b^2}\right), 0\right]$

$$|PG_1| = \sqrt{x^2 + \left[y\left(\frac{b^2-a^2}{b^2}\right) - y\right]^2 + \left[z\left(\frac{c^2-a^2}{c^2}\right) - z\right]^2}$$

$$\Longrightarrow |PG_1| = \sqrt{x^2 + y^2\left[\left(\frac{b^2-a^2-b^2}{b^2}\right)\right]^2 + z^2\left[\left(\frac{c^2-a^2-c^2}{c^2}\right)\right]^2}$$

$$\Longrightarrow |PG_1| = \sqrt{x^2 + y^2\frac{a^4}{b^4} + z^2\frac{a^4}{c^4}} \Longrightarrow |PG_1| = a^2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$$

Similarly $\Longrightarrow |PG_2| = b^2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$ and $\Longrightarrow |PG_3| = c^2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$

So $\frac{|PG_1|}{|PG_2|} = \frac{a^2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}{b^2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}} = \frac{a^2}{b^2} = constant$

Similarly $\frac{|PG_2|}{|PG_3|} = \frac{b^2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}{c^2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}} = \frac{b^2}{c^2} = constant$ Also $\frac{|PG_3|}{|PG_1|} = \frac{c^2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}{a^2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}} = \frac{c^2}{a^2} = constant$

QUESTION: the normal at a point 'P' of the ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ meet the coordinate planes in the points G_1, G_2, G_3 respectively. Prove that the ratios $|PG_1|:|PG_2|:|PG_3|$ are constant.

SOLUTION: Given surface is $F(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \Longrightarrow F_x = \frac{2x}{a^2}, F_y = \frac{2y}{b^2}, F_z = \frac{2z}{c^2}$

Now the equation of tangent plane at any point $P(x,y,z)$ to the surface is $\Longrightarrow \frac{X-x}{F_x} = \frac{Y-y}{F_y} = \frac{Z-z}{F_z}$

$$\Longrightarrow \frac{X-x}{\frac{2x}{a^2}} = \frac{Y-y}{\frac{2y}{b^2}} = \frac{Z-z}{\frac{2z}{c^2}} \Longrightarrow \frac{X-x}{\frac{x}{a^2}} = \frac{Y-y}{\frac{y}{b^2}} = \frac{Z-z}{\frac{z}{c^2}}$$

Now the normal meet at YZ – Plane at G_1 then $X = 0$

$$\Longrightarrow \frac{-x}{\frac{x}{a^2}} = \frac{Y-y}{\frac{y}{b^2}} = \frac{Z-z}{\frac{z}{c^2}} \Longrightarrow -a^2 = \frac{Y-y}{\frac{y}{b^2}} = \frac{Z-z}{\frac{z}{c^2}}$$

$$\Longrightarrow -a^2 = \frac{Y-y}{\frac{y}{b^2}} \Longrightarrow Y - y = -\frac{a^2y}{b^2} = -\frac{a^2y}{b^2} + y = y\left(-\frac{a^2}{b^2} + 1\right) \Longrightarrow Y = y\left(\frac{b^2-a^2}{b^2}\right)$$

$$and \ \Longrightarrow -a^2 = \frac{Z-z}{\frac{z}{c^2}} \Longrightarrow Z - z = -\frac{a^2z}{c^2} = -\frac{a^2z}{c^2} + z = z\left(-\frac{a^2}{c^2} + 1\right) \Longrightarrow Y = z\left(\frac{c^2-a^2}{c^2}\right)$$

Thus a point lying in YZ – Plane which is G_1 have the coordinates $\left[0, y\left(\frac{b^2-a^2}{b^2}\right), z\left(\frac{c^2-a^2}{c^2}\right)\right]$

Similarly a point lying in XZ – Plane which is G_2 have the coordinates $\left[x\left(\frac{a^2-b^2}{a^2}\right), 0, z\left(\frac{c^2-a^2}{c^2}\right)\right]$

Thus a point lying in XY – Plane which is G_3 have the coordinates $\left[x\left(\frac{a^2-b^2}{a^2}\right), y\left(\frac{b^2-a^2}{b^2}\right), 0\right]$

$$|PG_1| = \sqrt{x^2 + \left[y\left(\frac{b^2-a^2}{b^2}\right) - y\right]^2 + \left[z\left(\frac{c^2-a^2}{c^2}\right) - z\right]^2}$$

$$\Longrightarrow |PG_1| = \sqrt{x^2 + y^2\left[\left(\frac{b^2-a^2-b^2}{b^2}\right)\right]^2 + z^2\left[\left(\frac{c^2-a^2-c^2}{c^2}\right)\right]^2}$$

$$\Longrightarrow |PG_1| = \sqrt{x^2 + y^2\frac{a^4}{b^4} + z^2\frac{a^4}{c^4}} \Longrightarrow |PG_1| = a^2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$$

Similarly $\Longrightarrow |PG_2| = b^2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$ and $\Longrightarrow |PG_3| = c^2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$

Hence $|PG_1|:|PG_2|:|PG_3| = a^2:b^2:c^2$ is constant.

QUESTION: Show that the tangent plane at the point common to the surface $a(xy + yz + zx) = xyz$ and a sphere whose centre is at origin and radius 'b' $x^2 + y^2 + z^2 = b^2$ makes intercept on the axis whose sum is constant.

SOLUTION:

Given surface is $F(x, y, z) = a(xy + yz + zx) - xyz = 0$

$$\Rightarrow (axy + ayz + azx) - xyz = 0 \Rightarrow F(x, y, z) = \frac{a}{x} + \frac{a}{y} + \frac{a}{z} - 1 = 0 \qquad \because \div\, ing\ by\ xyz$$

$$F_x = -\frac{a}{x^2}, F_y = -\frac{a}{y^2}, F_z = -\frac{a}{z^2}$$

Now the equation of tangent plane at any point $P(x, y, z)$ to the surface is

$$\Rightarrow (X - x)F_x + (Y - y)F_y + (Z - z)F_z = 0$$

$$\Rightarrow (X - x)\left(-\frac{a}{x^2}\right) + (Y - y)\left(-\frac{a}{y^2}\right) + (Z - z)\left(-\frac{a}{z^2}\right) = 0$$

$$\Rightarrow \frac{X}{x^2} - \frac{x}{x^2} + \frac{Y}{y^2} - \frac{y}{y^2} + \frac{Z}{z^2} - \frac{z}{z^2} = 0 \Rightarrow \frac{X}{x^2} - \frac{1}{x} + \frac{Y}{y^2} - \frac{1}{y} + \frac{Z}{z^2} - \frac{1}{z} = 0$$

$$\Rightarrow \frac{X}{x^2} + \frac{Y}{y^2} + \frac{Z}{z^2} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \Rightarrow \frac{X}{x^2} + \frac{Y}{y^2} + \frac{Z}{z^2} = \frac{1}{a} \qquad \therefore \frac{(xy+yz+zx)}{xyz} = \frac{1}{a}$$

$$\Rightarrow \frac{X}{\frac{x^2}{a}} + \frac{Y}{\frac{y^2}{a}} + \frac{Z}{\frac{z^2}{a}} = 1$$

Thus the intercepts of the coordinates axis are $\left(\frac{x^2}{a}, 0,0\right), \left(0, \frac{y^2}{a}, 0\right), \left(0,0, \frac{z^2}{a}\right)$ and

$$\Rightarrow \frac{x^2}{a} + \frac{y^2}{a} + \frac{z^2}{a} = \frac{x^2+y^2+z^2}{a} = \frac{b^2}{a} = constant$$

QUESTION: Find equation of tangent plane to the surface $z = x^2 + y^2$ at $P(1, -1, 2)$

SOLUTION:

Given surface is $F(x, y, z) = x^2 + y^2 - z = 0$

$$F_x = 2x, F_y = 2y, F_z = -1 \Rightarrow F_x = 2, F_y = -2, F_z = -1 \qquad \therefore x = 1, y = -1$$

Now the equation of tangent plane at any point $P(x, y, z)$ to the surface is

$$\Rightarrow (X - x)F_x + (Y - y)F_y + (Z - z)F_z = 0 \Rightarrow (X - 1)2 - (Y - y)2 - (Z - z)1 = 0$$

$$\Rightarrow (2X - 2) - (Y + 1)2 - (Z - z)1 = 0 \Rightarrow 2X - 2 - 2Y - 2 - Z + z = 0$$

$$\Rightarrow 2X - 2Y - Z + z - 4 = 0$$

ENVELOPS

The locus of characteristics of $F(a) = 0$ as 'a' varies is called envelope of the family of surface with parametric value 'a' . it is determined by eliminating 'a' from $F(a) = 0$ and $\frac{\partial}{\partial a}F(a) = 0$

QUESTION: Find the envelope of the family of paraboloids $x^2 + y^2 = 4a(z-a)$ where 'a ' is parameter.

SOLUTION:

Given surface is $F(a) = x^2 + y^2 - 4a(z-a) = 0 \ldots\ldots\ldots\ldots.(i)$

$$\Rightarrow \frac{\partial}{\partial a}F(a) = -4z + 8a = 0 \Rightarrow -4z + 8a = 0 \Rightarrow a = \frac{1}{2}z$$

$$\Rightarrow x^2 + y^2 - 4\left(\frac{1}{2}z\right)\left(z - \frac{1}{2}z\right) = 0 \Rightarrow x^2 + y^2 - 2z\left(\frac{2z-z}{2}\right) = 0$$

$$\Rightarrow 2x^2 + 2y^2 - 2z^2 + z^2 = 0 \Rightarrow 2x^2 + 2y^2 - z^2 = 0 \Rightarrow 2x^2 + 2y^2 = z^2$$

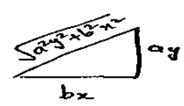
QUESTION: show that the envelope of surfaces $\frac{x}{a}Cos\theta Sin\varphi + \frac{y}{b}Sin\theta Sin\varphi + \frac{z}{c}Cos\varphi = 1$ is the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ where θ and φ are independent parameters.

SOLUTION

Given surface is $F(x,y,z,\theta,\varphi) = \frac{x}{a}Cos\theta Sin\varphi + \frac{y}{b}Sin\theta Sin\varphi + \frac{z}{c}Cos\varphi - 1 = 0 \ \ldots\ldots\ldots..(i)$

$$\Rightarrow \frac{\partial}{\partial a}F(\theta) = -\frac{x}{a}Sin\theta Sin\varphi + \frac{y}{b}Cos\theta Sin\varphi = 0 \Rightarrow Sin\varphi\left(-\frac{x}{a}Sin\theta + \frac{y}{b}Cos\theta\right) = 0$$

$$\Rightarrow Sin\varphi \neq 0,,\left(-\frac{x}{a}Sin\theta + \frac{y}{b}Cos\theta\right) = 0 \Rightarrow \frac{x}{a}Sin\theta = \frac{y}{b}Cos\theta \Rightarrow \frac{Sin\theta}{Cos\theta} = \frac{ya}{bx}$$



$$\Rightarrow Tan\theta = \frac{ya}{bx} \Rightarrow Cos\theta = \frac{bx}{\sqrt{a^2y^2+b^2x^2}} \Rightarrow Sin\theta = \frac{ay}{\sqrt{a^2y^2+b^2x^2}}$$

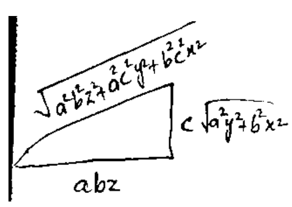
$$\Rightarrow \frac{\partial}{\partial a}F(\varphi) = \frac{x}{a}Cos\theta Cos\varphi + \frac{y}{b}Sin\theta Cos\varphi - \frac{z}{c}Sin\varphi = 0$$

$$\Rightarrow Cos\varphi\left(\frac{x}{a}Cos\theta + \frac{y}{b}Sin\theta\right) - \frac{z}{c}Sin\varphi = 0 \Rightarrow \frac{z}{c}Sin\varphi = Cos\varphi\left(\frac{x}{a}Cos\theta + \frac{y}{b}Sin\theta\right)$$

$$\Rightarrow \frac{z}{c}\frac{Sin\varphi}{Cos\varphi} = \left(\frac{x}{a}Cos\theta + \frac{y}{b}Sin\theta\right) \Rightarrow \frac{z}{c}Tan\varphi = \left(\frac{x}{a}Cos\theta + \frac{y}{b}Sin\theta\right)$$

$$\Rightarrow \frac{z}{c}Tan\varphi = \left(\frac{x}{a}.\frac{bx}{\sqrt{a^2y^2+b^2x^2}} + \frac{y}{b}.\frac{ay}{\sqrt{a^2y^2+b^2x^2}}\right)$$

$$\Rightarrow \frac{z}{c}Tan\varphi = \frac{b^2x^2+a^2y^2}{ab\sqrt{a^2y^2+b^2x^2}} = \frac{\sqrt{a^2y^2+b^2x^2}}{ab} \Rightarrow Tan\varphi = \frac{c\sqrt{a^2y^2+b^2x^2}}{abz}$$



$$\Rightarrow Sin\varphi = \frac{c\sqrt{a^2y^2+b^2x^2}}{\sqrt{b^2c^2x^2+a^2c^2y^2+a^2b^2z^2}} \Rightarrow Cos\varphi = \frac{abz}{\sqrt{b^2c^2x^2+a^2c^2y^2+a^2b^2z^2}}$$

$$(i) \Rightarrow \frac{x}{a}.\frac{bx}{\sqrt{a^2y^2+b^2x^2}}.\frac{c\sqrt{a^2y^2+b^2x^2}}{\sqrt{b^2c^2x^2+a^2c^2y^2+a^2b^2z^2}} + \frac{y}{b}.\frac{ay}{\sqrt{a^2y^2+b^2x^2}}.\frac{c\sqrt{a^2y^2+b^2x^2}}{\sqrt{b^2c^2x^2+a^2c^2y^2+a^2b^2z^2}} +$$
$$\frac{z}{c}\frac{abz}{\sqrt{b^2c^2x^2+a^2c^2y^2+a^2b^2z^2}} - 1 = 0$$

$$\Rightarrow \frac{cbx^2}{a\sqrt{b^2c^2x^2+a^2c^2y^2+a^2b^2z^2}} + \frac{acy^2}{b\sqrt{b^2c^2x^2+a^2c^2y^2+a^2b^2z^2}} + \frac{abz^2}{c\sqrt{b^2c^2x^2+a^2c^2y^2+a^2b^2z^2}} - 1 = 0$$

$$\Rightarrow \frac{cbx^2+acy^2+abz^2}{abc\sqrt{b^2c^2x^2+a^2c^2y^2+a^2b^2z^2}} = 1 \Rightarrow \frac{\sqrt{b^2c^2x^2+a^2c^2y^2+a^2b^2z^2}}{abc} = 1$$

$$\Rightarrow \sqrt{b^2c^2x^2 + a^2c^2y^2 + a^2b^2z^2} = abc \Rightarrow b^2c^2x^2 + a^2c^2y^2 + a^2b^2z^2 = a^2b^2c^2$$

$\Rightarrow \frac{b^2c^2x^2}{a^2b^2c^2} + \frac{a^2c^2y^2}{a^2b^2c^2} + \frac{a^2b^2z^2}{a^2b^2c^2} = 1 \Rightarrow \boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$ which is required

QUESTION: Find the envelope of the family of cones
$(ax + x + y + z - 1)(ay + z) = ax(x + y + z - 1)$ with 'a' being parameter.

SOLUTION:

Given surface is $F(a) = (ax + x + y + z - 1)(ay + z) - ax(x + y + z - 1) = 0$

$\Rightarrow \frac{\partial F}{\partial a} = (ax + x + y + z - 1)(y) + (x)(ay + z) - x(x + y + z - 1) = 0$
$\Rightarrow axy + xy + y^2 + zy - y + axy + xz - x^2 - xy - xz + x = 0$
$\Rightarrow 2axy + y^2 + zy - y - x^2 + x = 0 \Rightarrow 2axy = -y^2 - zy + y + x^2 - x$
$\Rightarrow a = \frac{x^2-y^2-x+y-zy}{2xy}$ Using this value in (i) we will get required eq. of envelope.

QUESTION (Annual;2016):

Sphere of constant radii 'b' having their Centre at fixed circle $x^2 + y^2 = a^2, z = 0$ prove that their envelope is the surface $(x^2 + y^2 + z^2 + a^2 - b^2)^2 = 4a^2(x^2 + y^2)$

SOLUTION

$x^2 + y^2 + z^2 = b^2$ is the equation of sphere with centre at origin

Now the centre of sphere will be $(aCos\theta, aSin\theta, 0)$

$$\Rightarrow (x - aCos\theta)^2 + (y - aSin\theta)^2 + (z - 0)^2 = b^2$$

$$\Rightarrow x^2 + a^2Cos^2\theta - 2axCos\theta + y^2 + a^2Sin^2\theta - 2aySin\theta + z^2 = b^2$$

$$\Rightarrow x^2 + y^2 + z^2 + a^2 - 2a(xCos\theta + ySin\theta) = b^2 \dots\dots\dots\dots(i)$$

$$\Rightarrow \frac{\partial F}{\partial \theta} = 2axSin\theta - 2ayCos\theta = 0 \Rightarrow 2axSin\theta = 2ayCos\theta \Rightarrow \frac{Sin\theta}{Cos\theta} = \frac{y}{x}$$

$$\Rightarrow Tan\theta = \frac{y}{x} \Rightarrow Sin\theta = \frac{y}{\sqrt{x^2+y^2}} \Rightarrow Cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$$(i) \Rightarrow x^2 + y^2 + z^2 + a^2 - 2a\left(x.\frac{x}{\sqrt{x^2+y^2}} + y.\frac{y}{\sqrt{x^2+y^2}}\right) = b^2$$

$$\Rightarrow x^2 + y^2 + z^2 + a^2 - 2a\left(\frac{x^2+y^2}{\sqrt{x^2+y^2}}\right) = b^2 \Rightarrow x^2 + y^2 + z^2 + a^2 - 2a\sqrt{x^2 + y^2} = b^2$$

$$\Rightarrow x^2 + y^2 + z^2 + a^2 - b^2 = 2a\sqrt{x^2 + y^2} \Rightarrow \boldsymbol{(x^2 + y^2 + z^2 + a^2 - b^2)^2 = 4a^2(x^2 + y^2)}$$

QUESTION: Find the envelope of family of the surface $F(x, y, z, a, b) = 0$ in which parameters 'a' and 'b' are connected by the equation $f(a, b) = 0$

SOLUTION

Given surface is $F(x, y, z, a, b) = 0 \dots\dots..(i)$ and $f(a, b) = 0 \dots\dots..(ii)$

Diff. (i) w.r.to 'a' and 'b' having total differential $(i) \Rightarrow \frac{\partial F}{\partial a}da + \frac{\partial F}{\partial b}db = 0 \dots\dots..(iii)$

Diff. (ii) w.r.to 'a' and 'b' having total differential $(ii) \Rightarrow \frac{\partial f}{\partial a}da + \frac{\partial f}{\partial b}db = 0 \dots\dots..(iv)$

Now

$(iv) \Longrightarrow \frac{\partial f}{\partial a}da + \frac{\partial f}{\partial b}db = 0 \Longrightarrow f_a da + f_b db = 0 \Longrightarrow f_a + f_b \frac{db}{da} = 0 \Longrightarrow \boldsymbol{\frac{db}{da}} = -\boldsymbol{\frac{f_a}{f_b}} \dots \dots .. (v)$

Also $(iii) \Longrightarrow \frac{\partial F}{\partial a}da + \frac{\partial F}{\partial b}db = 0 \Longrightarrow F_a da + F_b db = 0 \Longrightarrow F_a + F_b \frac{db}{da} = 0$

$\Longrightarrow F_a + F_b \left(-\frac{f_a}{f_b}\right) = 0$ $\quad\quad$ $\therefore$ Using (v) $\quad \Longrightarrow F_a = F_b \left(\frac{f_a}{f_b}\right) \Longrightarrow \boldsymbol{\frac{F_a}{f_a}} = \boldsymbol{\frac{F_b}{f_b}} \dots \dots .. (vi)$

Elimination of 'a' and 'b' from $(i), (ii) and\ (vi)$ will give us the required equation of envelope.

QUESTION: Prove that the envelope of the surface $F(x, y, z, a, b, c) = 0$ in which parameters 'a' and 'b' and 'c' are connected by the equation $f(a, b, c) = 0$ is obtain by eliminating 'a' , 'b', 'c' from $F = 0, f = 0, \frac{F_a}{f_a} = \frac{F_b}{f_b} = \frac{F_c}{f_c}$

SOLUTION

Given surface is $F(x, y, z, a, b, c) = 0 \dots \dots .. (i)$ and $f(a, b, c) = 0 \dots \dots .. (ii)$

Diff. (i) w.r.to 'a', 'b' and 'c' having total differential $\Longrightarrow \frac{\partial F}{\partial a}da + \frac{\partial F}{\partial b}db + \frac{\partial F}{\partial c}dc = 0 \dots \dots .. (iii)$

Diff. (ii) w.r.to 'a', 'b' and 'c' having total differential $\Longrightarrow \frac{\partial f}{\partial a}da + \frac{\partial f}{\partial b}db + \frac{\partial f}{\partial c}dc = 0 \dots \dots .. (iv)$

Now multiplying eq. (iii) by $\frac{\partial f}{\partial c}$ and (iv) by $\frac{\partial F}{\partial c}$ and subtracting

$\Longrightarrow f_c F_a da + f_c F_b db + f_c F_c dc = 0$ and $\Longrightarrow F_c f_a da + F_c f_b db + F_c f_c dc = 0$

Then required answer $\Longrightarrow (f_c F_a - F_c f_a)da + (f_c F_b - F_c f_b)db = 0$

$$\Longrightarrow (f_c F_a - F_c f_a) + (f_c F_b - F_c f_b)\frac{db}{da} = 0 \dots \dots .. (v)$$

Let $\frac{db}{da} = k$ where db and da are the changes in parameters 'a' and 'b'. Then for different values of db and da we have different values of 'k' and it will always non – zero. So eq. (v) will hold only when $\Longrightarrow (f_c F_a - F_c f_a) = 0\ and\ (f_c F_b - F_c f_b) = 0$

$$\Longrightarrow f_c F_a = F_c f_a\ and\ f_c F_b = F_c f_b \Longrightarrow \boldsymbol{\frac{F_a}{f_a}} = \boldsymbol{\frac{F_c}{f_c}} \dots \dots .. (vi) and \boldsymbol{\frac{F_b}{f_b}} = \boldsymbol{\frac{F_c}{f_c}} \dots \dots .. (vii)$$

From $(vi)\ and\ (vii) \Longrightarrow \boldsymbol{\frac{F_a}{f_a}} = \boldsymbol{\frac{F_b}{f_b}} = \boldsymbol{\frac{F_c}{f_c}}$ and also $\boldsymbol{F = 0, f = 0}$ are required.

QUESTION: Prove that the envelope of the plane $lx + my + nz = p$ is an ellipsoid where $p^2 = a^2l^2 + b^2m^2 + c^2n^2$

SOLUTION

Given $F(x, y, z, l, m, n) = lx + my + nz - p = 0 \dots \dots \dots \dots .. (i)$

$$\Longrightarrow p^2 = a^2l^2 + b^2m^2 + c^2n^2 \Longrightarrow p = \sqrt{a^2l^2 + b^2m^2 + c^2n^2}$$

$$(i) \Longrightarrow F(x, y, z, l, m, n) = lx + my + nz - \sqrt{a^2l^2 + b^2m^2 + c^2n^2} = 0$$

Now $\frac{\partial F}{\partial l} = x - \frac{2a^2l}{2\sqrt{a^2l^2+b^2m^2+c^2n^2}} = 0 \Longrightarrow x - \frac{a^2l}{p} = 0 \Longrightarrow x = \frac{a^2l}{p} \Longrightarrow \boldsymbol{l} = \boldsymbol{\frac{px}{a^2}}$

$$\frac{\partial F}{\partial m} = y - \frac{2b^2m}{2\sqrt{a^2l^2+b^2m^2+c^2n^2}} = 0 \Longrightarrow y - \frac{b^2m}{p} = 0 \Longrightarrow y = \frac{b^2m}{p} \Longrightarrow \boldsymbol{m} = \boldsymbol{\frac{py}{b^2}}$$

$$\frac{\partial F}{\partial n} = z - \frac{2c^2n}{2\sqrt{a^2l^2+b^2m^2+c^2n^2}} = 0 \Longrightarrow z - \frac{c^2n}{p} = 0 \Longrightarrow z = \frac{c^2n}{p} \Longrightarrow \boldsymbol{n} = \boldsymbol{\frac{pz}{c^2}}$$

$(i) \Rightarrow F(x,y,z,l,m,n) = \frac{px}{a^2}x + \frac{py}{b^2}y + \frac{pz}{c^2}z = p$

$\Rightarrow p\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right] = p \Rightarrow \left[\frac{\boldsymbol{x^2}}{\boldsymbol{a^2}} + \frac{\boldsymbol{y^2}}{\boldsymbol{b^2}} + \frac{\boldsymbol{z^2}}{\boldsymbol{c^2}}\right] = \mathbf{1}$ Which is required equation of ellipsoid.

NOTE: two surfaces are said to touch each other at a common point when they have same tangent plane and normal at that point.

THEOREM: Prove that the envelopes touché each member of the family of the surface at all points of its characteristics.

PROOF:

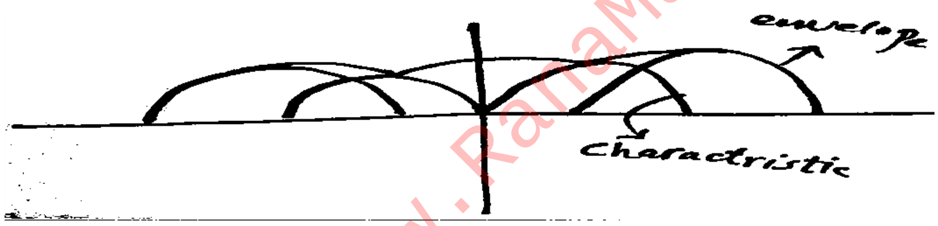
The characteristic corresponding to the parametric value 'a' lies both on the surface and envelope and hence have the same parametric value.

The normal to the surface $F(x,y,z,a) = 0$ is parallel to the vector $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$ and the equation of envelope is obtained by elimination from $\boldsymbol{F(a) = 0} \ldots\ldots (\boldsymbol{i})$ and $\frac{\boldsymbol{\partial}}{\boldsymbol{\partial a}}\boldsymbol{F(a) = 0} \ldots (\boldsymbol{ii})$
So the normal to the envelope of the surface $F(x,y,z,a) = 0$ is the vector

$$\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial a}\cdot\frac{\partial a}{\partial x}, \quad \frac{\partial F}{\partial y} + \frac{\partial F}{\partial a}\cdot\frac{\partial a}{\partial y}, \quad \frac{\partial F}{\partial z} + \frac{\partial F}{\partial a}\cdot\frac{\partial a}{\partial z}\right) \ldots\ldots\ldots\ldots\ldots (iii)$$

Using (ii) in (iii) $\Rightarrow \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$ $\qquad \therefore \frac{\partial}{\partial a}F(x,y,z,a) = 0$

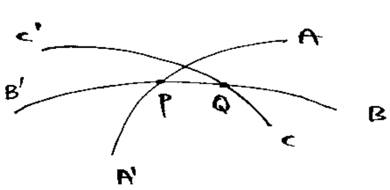
Hence this show that all common points the surface and envelope have the same normal $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$ therefore have the same tangent plane. So that surface and envelope touch each other at all points of characteristic.



EDGE OF REGRESSION: The locus of ultimate intersection of consecutive characteristics of one parameter family of surface is called Edge of Regression.

THEOREM: Prove that each characteristic touches the edges of regression.

PROOF:



If A,B,C are the three consecutive characteristics. And A and B intersect at 'P' and B and C intersect at 'Q'. Thus 'P' and 'Q' are consecutive points at characteristic B and on edge of regression.

Hence ultimately as A and C tends to coincide with B then the chord PQ will become the common tangent to the characteristic and edge of regression. Thus Surface or curve are said to touch each other at a common point if they have same tangent to the characteristic.

METHOD TO FINDING THE EDGE OF REGRESSION

Let $F(x,y,z,a)=0$ be the one parameter family of surface with parameter 'a' then the equations of characteristic corresponding to the parametric values 'a' and $'a+\delta a'$ are $F(x,y,z,a)=0$ & $\frac{\partial}{\partial a}F(x,y,z,a)=0$ also $F(x,y,z,a+\delta a)=0$ & $\frac{\partial}{\partial a}F(x,y,z,a+\delta a)=0$

It follows that $\lim_{\delta a\to 0}\frac{F_a(x,y,z,a+\delta a)-F_a(x,y,z,a)}{\delta a}=0$ where $F_a=\frac{\partial F}{\partial a}$ $\Longrightarrow \frac{\partial^2 F}{\partial a^2}=0 \Longrightarrow F_{aa}=0$

So we can find the edge of regression by eliminating 'a' from $F(a)=0, F_a(a)=0, F_{aa}(a)=0$

QUESTION: Find the edges of regression of the envelope of the planes $xSin\theta - yCos\theta + z = a\theta$ with θ being parameter.

SOLUTION:

Let $F(x,y,z,\theta)=xSin\theta - yCos\theta + z - a\theta = 0$ ……………..(i)

Now $\Longrightarrow F_\theta = \frac{\partial F}{\partial \theta} = xCos\theta + ySin\theta - a = 0$ ……………..(ii)

$$\Longrightarrow F_{\theta\theta}=\frac{\partial^2 F}{\partial \theta^2} = -xSin\theta + yCos\theta = 0 \text{ ……………..}(iii)$$

$(iii) \Longrightarrow xSin\theta = yCos\theta$ ……………..$(iv) \Longrightarrow \frac{Sin\theta}{Cos\theta}=\frac{y}{x} \Longrightarrow Tan\theta = \frac{y}{x} \Longrightarrow \theta = Tan^{-1}\left(\frac{y}{x}\right)$

Also $Sin\theta = \frac{y}{\sqrt{x^2+y^2}}$,,, $Cos\theta = \frac{x}{\sqrt{x^2+y^2}}$ ……………..(v)

Now using (iv) in (i)

$(i) \Longrightarrow yCos\theta - yCos\theta + z - a\theta = 0 \Longrightarrow z = a\theta \Longrightarrow z = aTan^{-1}\left(\frac{y}{x}\right)$ ……………..(vi)

Now using (v) in (ii) $(ii) \Longrightarrow x\frac{x}{\sqrt{x^2+y^2}} + y\frac{y}{\sqrt{x^2+y^2}} - a = 0 \Longrightarrow \frac{x^2+y^2}{\sqrt{x^2+y^2}} = a \Longrightarrow \sqrt{x^2+y^2}=a$

$\Longrightarrow x^2+y^2=a^2$ ……………..(vii)

Hence equation (vi) and (vii) are the equations of edges of regression.

QUESTION: Find the envelope and edges of regression of the family of ellipsoids $c^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)+\frac{z^2}{c^2}=1$ with c being parameter.

SOLUTION:

Let $F(x,y,z,c)=c^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)+\frac{z^2}{c^2}-1=0$ ……………..(i)

Now $\Longrightarrow F_c = \frac{\partial F}{\partial c} = 2c\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-\frac{2z^2}{c^3}=0$ ……………..(ii)

$$\Longrightarrow 2c\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=\frac{2z^2}{c^3} \Longrightarrow c\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=\frac{z^2}{c^3} \Longrightarrow c^4=\frac{z^2}{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)} \Longrightarrow c^2=\frac{z}{\sqrt{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}}$$

$$\Longrightarrow \frac{z}{\sqrt{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)+\frac{z^2}{\frac{z}{\sqrt{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}}}-1=0 \Longrightarrow \frac{z}{\sqrt{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)+z\sqrt{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}-1=0$$

$$\Longrightarrow z\sqrt{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}+z\sqrt{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}-1=0 \Longrightarrow 2z\sqrt{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}=1$$

$\Longrightarrow 4z^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=1$ Which is the equation of envelope

Now $(ii) \Longrightarrow \frac{\partial^2 F}{\partial c^2}=2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-\frac{6z^2}{c^2}=0 \ldots\ldots\ldots\ldots\ldots (iii)$

$(iii)\frac{c^2}{2} \Longrightarrow c^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-3z^2=0 \ldots\ldots\ldots\ldots\ldots (iv)$

$$\boldsymbol{c^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)+\frac{z^2}{c^2}-1=0}$$
$$c^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-3z^2=0$$

Now subtracting (iv) from (i)

$$\Longrightarrow \frac{z^2}{c^2}-1-3z^2=0$$

$$\Longrightarrow \frac{z^2-3z^2}{c^2}=1 \Longrightarrow -2z^2=c^2$$

$(i) \Longrightarrow -2z^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)+\frac{z^2}{-2z^2}-1=0 \Longrightarrow 4z^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)+1+2=0$

$$\Longrightarrow 4z^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)+3=0 \Longrightarrow z^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=-\frac{3}{4}\ldots\ldots\ldots\ldots\ldots (v)$$

Now $(ii)c \Longrightarrow 4c^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-\frac{2z^2}{c^2}=0 \Longrightarrow 4(-2z^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-\frac{2z^2}{(-2z^2)}=0$

$$\Longrightarrow -4z^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)+1=0 \Longrightarrow z^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=-\frac{1}{4}\ldots\ldots\ldots\ldots\ldots (vi)$$

Since from $(v) \Longrightarrow z^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=-\frac{3}{4}$ not possible

so required equation of regression is $z^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=-\frac{1}{4}$

DEVELOPABLE SURFACE:

In one parameter family of planes the characteristic being the intersection of consecutive plane are straight lines. These straight lines are called generator of the envelope and envelope is called Developable surface.

REMARK:

- The reason of name lies in the fact that plane is developed into a surface without stretching or tearing. Any surface which satisfies this property is called Developable surface.
- Tangent plane of Developable surface depends upon only one parameter.

QUESTION: Find the condition that the given surface is developable.

SOLUTION: Let $z=f(x,y)$ be an equation of surface then the equation of tangent plane at $P(x,y,z)$ will be $Z-z=(X-x)\frac{\partial F}{\partial x}+(Y-y)\frac{\partial F}{\partial y}$

Tangent plane of developable surface depends upon only one parameter. There must be some relation between $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ which may be written as

$\frac{\partial F}{\partial x}=\varphi\left(\frac{\partial F}{\partial y}\right)\ldots\ldots\ldots\ldots (i)$ where φ is constant.

Diff. (i) w.r.to 'x' $\Longrightarrow \frac{\partial^2 F}{\partial x^2}=\varphi\left(\frac{\partial^2 F}{\partial x\partial y}\right)\ldots\ldots\ldots\ldots (ii)$

Diff. (i) w.r.to 'y' $\Longrightarrow \frac{\partial^2 F}{\partial y\partial x}=\varphi\left(\frac{\partial^2 F}{\partial y^2}\right) \Longrightarrow \frac{\partial^2 F}{\partial y^2}=\frac{1}{\varphi}\left(\frac{\partial^2 F}{\partial x\partial y}\right)\ldots\ldots\ldots\ldots (iii)$ $\quad \therefore \frac{\partial^2 F}{\partial y\partial x}=\frac{\partial^2 F}{\partial x\partial y}$

Multiplying (ii) and (iii) $\Longrightarrow \frac{\partial^2 F}{\partial x^2}.\frac{\partial^2 F}{\partial y^2} = \varphi\left(\frac{\partial^2 F}{\partial x \partial y}\right).\frac{1}{\varphi}\left(\frac{\partial^2 F}{\partial x \partial y}\right)$

$\Longrightarrow \frac{\partial^2 F}{\partial x^2}.\frac{\partial^2 F}{\partial y^2} = \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2$ which is the condition for the given surface to be developable.

QUESTION: For the surface $xy = (z-c)^2$ prove it is developable.

SOLUTION: Given surface $xy = (z-c)^2$

$\Longrightarrow \sqrt{xy} = z - c \Longrightarrow z = \sqrt{xy} + c \Longrightarrow z = f(x,y) = \sqrt{xy} + c$

Now $\frac{\partial F}{\partial x} = \frac{1}{2\sqrt{xy}}.y$ similarly $\frac{\partial F}{\partial y} = \frac{1}{2\sqrt{xy}}.x \Longrightarrow \frac{\partial^2 F}{\partial x^2} = \frac{-y^2}{4(xy)^{\frac{3}{2}}}$ and $\Longrightarrow \frac{\partial^2 F}{\partial y^2} = \frac{-x^2}{4(xy)^{\frac{3}{2}}}$

Also $\frac{\partial^2 F}{\partial x \partial y} = \frac{-y}{4(xy)^{\frac{3}{2}}} + \frac{1}{2\sqrt{xy}} = \frac{-xy+2xy}{4(xy)^{\frac{3}{2}}} = \frac{xy}{4(xy)^{\frac{3}{2}}}$

Now to prove surface is developable use $\Longrightarrow \frac{\partial^2 F}{\partial x^2}.\frac{\partial^2 F}{\partial y^2} = \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2$

$\Longrightarrow \left(\frac{-y^2}{4(xy)^{\frac{3}{2}}}\right)\left(\frac{-x^2}{4(xy)^{\frac{3}{2}}}\right) = \left(\frac{xy}{4(xy)^{\frac{3}{2}}}\right)^2 \Longrightarrow \frac{x^2y^2}{16(xy)^3} = \frac{x^2y^2}{16(xy)^3} \Longrightarrow 1 = 1$

EDGE OF REGRESSION OF A DEVELOPABLE SURFACE: The edge or regression of a developable surface is the locus of intersection of consecutive generators and are touched by each generator.

OSCULATING DEVELOPABLE: The envelope of the osculating plane is called osculating developable.

QUESTION(PP): Fid the osculating developable of the circular helix $\vec{r} = (aCos\theta, aSin\theta, b\theta)$

SOLUTION: For osculating developable we have $[(\vec{R} - \vec{r}),\ \vec{t},\ \vec{n}] = 0 \ldots\ldots(i)$

Given $\vec{r} = (aCos\theta, aSin\theta, b\theta) \Longrightarrow \vec{t} = \vec{r}' = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{d\theta}.\frac{d\theta}{ds} = (-aSin\theta, aCos\theta, b)\theta'$

and $\vec{t}' = \frac{d\vec{t}}{ds} = \frac{d\vec{t}}{d\theta}.\frac{d\theta}{ds} = (-aCos\theta, -aSin\theta, 0)(\theta')^2 \Longrightarrow K\vec{n} = (-aCos\theta, -aSin\theta, 0)(\theta')^2$

$\Longrightarrow \vec{n} = \frac{1}{K}(-aCos\theta, -aSin\theta, 0)(\theta')^2$

$$(i) \Longrightarrow [(\vec{R} - \vec{r}),\ \vec{t},\ \vec{n}] = 0 \Longrightarrow \begin{vmatrix} X - aCos\theta & Y - bSin\theta & Z - b\theta \\ -aSin\theta\theta' & aCos\theta\theta' & b\theta' \\ -aCos\theta\frac{(\theta')^2}{K} & -aSin\theta\frac{(\theta')^2}{K} & 0 \end{vmatrix} = 0$$

$(X - aCos\theta)\left[0 + abSin\theta\frac{(\theta')^3}{K}\right] - (Y - bSin\theta)\left[0 + abCos\theta\frac{(\theta')^3}{K}\right] +$
$(Z - b\theta)\left[a^2Sin^2\theta\frac{(\theta')^3}{K} + a^2Cos^2\theta\frac{(\theta')^3}{K}\right] = 0$

$\Longrightarrow$
$XabSin\theta\frac{(\theta')^3}{K} - a^2bCos\theta Sin\theta\frac{(\theta')^3}{K} - YabCos\theta\frac{(\theta')^3}{K} + ab^2Sin\theta Cos\theta\frac{(\theta')^3}{K} +$
$a^2(Z - b\theta)\left[Sin^2\theta\frac{(\theta')^3}{K} + Cos^2\theta\frac{(\theta')^3}{K}\right] = 0$

$\Longrightarrow (X - aCos\theta)abSin\theta\frac{(\theta')^3}{K} - (Y - bSin\theta)abCos\theta\frac{(\theta')^3}{K} + a^2(Z - b\theta) = 0$

$\Longrightarrow (X - aCos\theta)bSin\theta\frac{(\theta')^3}{K} - (Y - bSin\theta)bCos\theta\frac{(\theta')^3}{K} + a(Z - b\theta) = 0$ is required equation.

THEOREM: Prove that the generator of the osculating developable of a twisted curve is tangent to the curve.

PROOF: The equation of the osculating plane at any point 'P' with $\vec{r}$ on the curve is plane

$(\vec{R}-\vec{r}).\vec{b}=0 \dots\dots (i)$ where $\vec{R}$ is fixed.

Diff. (i) w.r.to 's' $\Rightarrow (\vec{R}-\vec{r}).\vec{b}'+(0-\vec{r}').\vec{b}=0 \Rightarrow (\vec{R}-\vec{r}).(-\tau\vec{n})+\vec{t}.\vec{b}=0$

$\Rightarrow (-\tau)(\vec{R}-\vec{r}).\vec{n}+0=0 \Rightarrow \tau \neq 0 \Rightarrow (\vec{R}-\vec{r}).\vec{n}=0 \dots\dots\dots.\dots (ii)$

Now characteristic of surface is given by Equations (i) and (ii) is the of rectifying plane. The characteristic of generator is the intersection of osculating plane and rectifying plane and it is tangent to the curve at that point 'P'

Hence generator of osculating developable are tangent to the curve.

THEOREM(PP): Prove that the edge of regression for osculating developable is curve itself.

PROOF: For edge or regression $(\vec{R}-\vec{r}).\vec{n}=0$ where $\vec{R}$ is fixed.

Diff. w.r.to 's' $\Rightarrow (\vec{R}-\vec{r}).\vec{n}'+(0-\vec{r}').\vec{n}=0 \Rightarrow (\vec{R}-\vec{r}).(\tau\vec{b}-K\vec{t})+(0-\vec{t}).\vec{n}=0$

$\Rightarrow [\tau(\vec{R}-\vec{r}).\vec{b}-K(\vec{R}-\vec{r}).\vec{t}]-\vec{t}.\vec{n}=0 \Rightarrow [\tau(\vec{R}-\vec{r}).\vec{b}-K(\vec{R}-\vec{r}).\vec{t}]-0=0$

$\Rightarrow [0-K(\vec{R}-\vec{r}).\vec{t}]-0=0 \qquad \therefore (\vec{R}-\vec{r}).\vec{b}=0$

$K\neq 0 \Rightarrow (\vec{R}-\vec{r}).\vec{t}=0 \dots\dots\dots.\dots (iii)$ Also from previous result $(\vec{R}-\vec{r}).\vec{b}=0$ and $(\vec{R}-\vec{r}).\vec{n}=0$ shows that $(\vec{R}-\vec{r})$ is perpendicular to $\vec{t},\vec{n}$ *and* $\vec{b}$ which is not possible. Hence

$\Rightarrow (\vec{R}-\vec{r})=0 \Rightarrow \vec{R}=\vec{r} \Rightarrow$ Edge of regression is curve itself.

THEOREM: A point on the edge of regression corresponding to a point $P(\vec{r})$ on the given curve is given by $\vec{R}=\vec{r}+\frac{K(\tau\vec{t}+K\vec{b})}{K'\tau-K\tau'}$

PROOF: For edge or regression $(\vec{R}-\vec{r}).\vec{n}=0 \dots..(i)$ where $\vec{R}$ is fixed.

Diff. w.r.to 's' $\Rightarrow (\vec{R}-\vec{r}).\vec{n}'+(0-\vec{r}').\vec{n}=0 \Rightarrow (\vec{R}-\vec{r}).(\tau\vec{b}-K\vec{t})+(0-\vec{t}).\vec{n}=0$

$\Rightarrow (\vec{R}-\vec{r}).(\tau\vec{b}-K\vec{t})-\vec{t}.\vec{n}=0 \Rightarrow (\vec{R}-\vec{r}).(\tau\vec{b}-K\vec{t})-0=0$

$\Rightarrow (\vec{R}-\vec{r}).(\tau\vec{b}-K\vec{t})=0$

Diff. w.r.to 's' $\Rightarrow (\vec{R}-\vec{r}).(\tau\vec{b}'+\tau'\vec{b}-K'\vec{t}-K\vec{t}')+(0-\vec{r}').(\tau\vec{b}-K\vec{t})=0$

$\Rightarrow (\vec{R}-\vec{r}).(\tau\vec{b}'+\tau'\vec{b}-K'\vec{t}-K\vec{t}')-\tau\vec{t}.\vec{b}+K\vec{t}.\vec{t}=0 \Rightarrow (\vec{R}-\vec{r}).(\tau\vec{b}'+\tau'\vec{b}-K'\vec{t}-K\vec{t}')+K=0$

$\Rightarrow (\vec{R}-\vec{r}).(-\tau^2\vec{n}+\tau'\vec{b}-K'\vec{t}-K^2\vec{n})+K=0$

$\Rightarrow -\tau^2(\vec{R}-\vec{r}).\vec{n}+\tau'(\vec{R}-\vec{r}).\vec{b}-K'(\vec{R}-\vec{r}).\vec{t}-K^2(\vec{R}-\vec{r}).\vec{n}+K=0$

$\Rightarrow 0+\tau'(\vec{R}-\vec{r}).\vec{b}-K'(\vec{R}-\vec{r}).\vec{t}-0+K=0 \Rightarrow (\vec{R}-\vec{r})[\tau'\vec{b}-K'\vec{t}]+K=0 \dots..(ii)$

Since $(\vec{R}-\vec{r}) \parallel (\tau\vec{b}-K\vec{t}) \Rightarrow (\vec{R}-\vec{r})=l(\tau\vec{b}-K\vec{t}) \Rightarrow \vec{R}=\vec{r}+l(\tau\vec{b}-K\vec{t})$

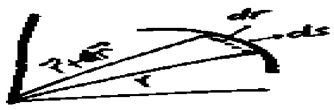
$(ii) \Rightarrow l(\tau\vec{b}-K\vec{t})[\tau'\vec{b}-K'\vec{t}]+K=0 \Rightarrow l[\tau'K-K'\tau]+K=0 \Rightarrow l=K/[\tau'K-K'\tau]$

$\Rightarrow \vec{R}=\vec{r}+\frac{K(\tau\vec{t}+K\vec{b})}{K'\tau-K\tau'}$

CURVILINEAR COORDINATES AND FUNDAMENTAL MAGNITUDE

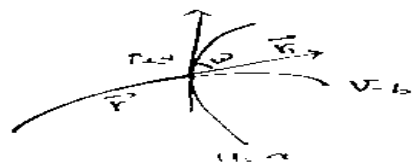
LENGTH OF SIMPLE CLOSED CURVE: The length of simple closed curve $\vec{r}$ of a period a is defined as $L = \int_0^a \|\vec{r}'(t)\| dt$

FIRST FUNDAMENTAL FORM OF THE SURFACES OR FIRST ORDER MAGNITUDES: Consider tow Neighboring points on the surface with position vectors $\vec{r}$ and $\vec{r} + d\vec{r}$ corresponding to the parameters u, v and $u + du$ $v + dv$ respectively Then $\vec{r} = \vec{r}(u, v) \Longrightarrow d\vec{r} = \frac{\partial \vec{r}}{\partial u} du + \frac{\partial \vec{r}}{\partial v} dv$



As the two points are adjacent on the curve passing through them, the length ds of the arc joining them is equal to their actual distance $|d\vec{r}|$ apart. Thus $ds^2 = |d\vec{r}|^2$

We shall use suffix "1" to indicate the partial differentiation w.r.to 'u' and suffix "2" to indicate the



particular differentiation w.r .to 'v' hence

$\vec{r}_1 = \vec{r}_u = \frac{\partial \vec{r}}{\partial u}, , \vec{r}_2 = \vec{r}_v = \frac{\partial \vec{r}}{\partial v}, , \vec{r}_{11} = \vec{r}_{uu} = \frac{\partial^2 \vec{r}}{\partial u^2}, , \vec{r}_{22} = \vec{r}_{vv} = \frac{\partial^2 \vec{r}}{\partial v^2}, , \vec{r}_{12} = \vec{r}_{uv} = \frac{\partial^2 \vec{r}}{\partial u \partial v}$ and so on.

We note that the vector $\vec{r}_1$ is along the tangent to the curve $v = constant$ at the point $\vec{r}$ and the vector $\vec{r}_2$ is along the tangent to the curve $u = constant$ at the point $\vec{r}$

Now $ds^2 = |d\vec{r}|^2 = \left(\frac{\partial \vec{r}}{\partial u} du + \frac{\partial \vec{r}}{\partial v} dv\right)^2 = (\vec{r}_1 du + \vec{r}_2 dv)^2 = \vec{r}_1{}^2 du^2 + \vec{r}_2{}^2 dv^2 + 2\vec{r}_1 \vec{r}_2 du dv$

Then we will write $\vec{r}_1{}^2 = E, , \vec{r}_2{}^2 = G$ and $F = \vec{r}_1 \vec{r}_2$

Then $\boldsymbol{ds^2} = |\boldsymbol{d\vec{r}}|^2 = \boldsymbol{Edu^2} + \boldsymbol{Gdv^2} + \boldsymbol{2Fdudv}$ this equation is called the **fundamental form of the first order**. Where E,F,G are called fundamental magnitudes of first order.

We may also use $\vec{r}_1 = \vec{r}_u = \frac{\partial \vec{r}}{\partial u} = \frac{\partial \sigma}{\partial u} = \sigma_u, \vec{r}_2 = \vec{r}_v = \frac{\partial \vec{r}}{\partial v} = \frac{\partial \sigma}{\partial v} = \sigma_v$
$\vec{r}_{11} = \vec{r}_{uu} = \frac{\partial^2 \vec{r}}{\partial u^2} = \frac{\partial^2 \sigma}{\partial u^2} = \sigma_{uu}, , \vec{r}_{22} = \vec{r}_{vv} = \frac{\partial^2 \vec{r}}{\partial v^2} = \frac{\partial^2 \sigma}{\partial v^2} = \sigma_{vv}, , \vec{r}_{12} = \vec{r}_{uv} = \frac{\partial^2 \vec{r}}{\partial u \partial v} = \frac{\partial^2 \sigma}{\partial u \partial v} = \sigma_{uv}$

Then $\vec{r}_1{}^2 = E = \sigma_u . \sigma_u = \|\sigma_u\|^2, , \vec{r}_2{}^2 = G = \sigma_v . \sigma_v = \|\sigma_v\|^2$ and $F = \vec{r}_1 \vec{r}_2 = \sigma_u . \sigma_v$

QUESTION: For $\sigma(u, v)$ find first fundamental form of the surface.

SOLUTION: Suppose $\vec{r}(t) = \sigma(u, v) \Longrightarrow \vec{r}'(t) = \lambda \sigma_u + u\sigma_v \ldots \ldots (i)$
$\Longrightarrow \vec{r}'.\vec{r}' = (\lambda\sigma_u + \lambda\sigma_v).(\lambda\sigma_u + \lambda\sigma_v) = \lambda^2(\sigma_u.\sigma_u) + 2\lambda u(\sigma_u.\sigma_v) + u^2(\sigma_v.\sigma_v) \ldots \ldots (A)$

Where $E = \sigma_u.\sigma_u = \|\sigma_u\|^2, , G = \sigma_v.\sigma_v = \|\sigma_v\|^2$ and $F = \sigma_u.\sigma_v$

And $d\vec{r} = \sigma_u du + \sigma_v dv$ $\ldots \ldots \ldots \ldots (ii)$ i.e. total differential

Comparing (i) and (ii) $\lambda = du$ and $u = dv$

$(A) \Longrightarrow \vec{r}'.\vec{r}' = Edu^2 + Gdv^2 + 2Fdudv$ this the 1st fundamental form of the surface.

Then length of curve of first fundamental form is $L = \int_0^a \|\vec{r}'(t)\| dt$

Since $\Longrightarrow \vec{r}'.\vec{r}' = \|\vec{r}'(t)\|^2 = Edu^2 + Gdv^2 + 2Fdudv$

$$\Longrightarrow L = \int_0^a \|\vec{r}'(t)\| dt = \int_0^a \sqrt{Edu^2 + Gdv^2 + 2Fdudv}\, dt$$

PROPERTIES:

i. $\boldsymbol{H = \sqrt{EG - F^2}}$

let $EG = F^2 = \vec{r_1}^2.\vec{r_2}^2 = (\vec{r_1}.\vec{r_2})^2 = (|\vec{r_1}||\vec{r_2}|Cosw)^2 = |\vec{r_1}|^2|\vec{r_2}|^2Cos^2w = \vec{r_1}^2\vec{r_2}^2Cos^2w$
where 'w' is the angle between $\vec{r_1}$ and $\vec{r_2}$

Now $EG - F^2 = \vec{r_1}^2\vec{r_2}^2 - \vec{r_1}^2\vec{r_2}^2Cos^2w = \vec{r_1}^2\vec{r_2}^2(1 - Cos^2w) = \vec{r_1}^2\vec{r_2}^2Sin^2w = H^2$ (Constant)
$\Rightarrow H = \sqrt{EG - F^2}$ where H is any arbitrary positive constant.

ii. $\boldsymbol{ds = \sqrt{E}du}$

Since $ds^2 = |d\vec{r}|^2 = Edu^2 + Gdv^2 + 2Fdudv$ then the length of the parametric curve $v = constant$ is obtained by putting $dv = 0$ so we get $ds^2 = Edu^2 \Rightarrow \boldsymbol{ds = \sqrt{E}du}$

iii. $\boldsymbol{ds = \sqrt{G}dv}$

Since $ds^2 = |d\vec{r}|^2 = Edu^2 + Gdv^2 + 2Fdudv$ then the length of the parametric curve $u = constant$ is obtained by putting $du = 0$ so we get $ds^2 = Gdv^2 \Rightarrow \boldsymbol{ds = \sqrt{G}dv}$

iv. Now we find the unit vectors in the direction of $\vec{r_1}$ adn $\vec{r_2}$. For this let $\vec{a}$ and $\vec{b}$ be unit tangents to the parametric curves $v = constant$ and $u = constant$ then
$\vec{a} = \frac{\vec{r_1}}{|\vec{r_1}|} = \frac{\vec{r_1}}{\sqrt{E}}$ and $\vec{b} = \frac{\vec{r_2}}{|\vec{r_2}|} = \frac{\vec{r_2}}{\sqrt{G}}$

v. the two parametric curves through any point of the surface cut an angle w such that

$$\vec{a}\,.\vec{b} = |\vec{a}||\vec{b}|Cosw \Rightarrow Cosw = \frac{\vec{a}\,.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\frac{\vec{r_1}}{\sqrt{E}}.\frac{\vec{r_2}}{\sqrt{G}}}{1.1} = \frac{\vec{r_1}.\vec{r_2}}{\sqrt{EG}} = \frac{F}{\sqrt{EG}} \Rightarrow \boldsymbol{Cosw = \frac{F}{\sqrt{EG}}}$$

now $Sinw = \sqrt{1 - Cos^2w} = \sqrt{1 - \frac{F^2}{EG}} = \sqrt{\frac{EG - F^2}{EG}} \Rightarrow \boldsymbol{Sinw = \frac{H}{\sqrt{EG}}}$

also the two parametric curves through any point of the surface cut an angle w such that

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|Sinw \Rightarrow Sinw = \frac{\vec{a}\times\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\frac{\vec{r_1}}{\sqrt{E}}\times\frac{\vec{r_2}}{\sqrt{G}}}{1.1} = \frac{\vec{r_1}\times\vec{r_2}}{\sqrt{EG}} = \frac{H}{\sqrt{EG}} \Rightarrow \boldsymbol{Sinw = \frac{H}{\sqrt{EG}}}$$

now $Tanw = \frac{Sinw}{Cosw} = \frac{\frac{H}{\sqrt{EG}}}{\frac{F}{\sqrt{EG}}} = \frac{H}{F} \Rightarrow \boldsymbol{Tanw = \frac{H}{F}}$

vi. For $Cosw = \frac{F}{\sqrt{EG}}$ that the parametric curves will be cut at right angle at any point if and only if $w = 90^0$
now if consider $F = 0 \Rightarrow Cosw = \frac{F}{\sqrt{EG}} = 0 \Rightarrow Cosw = 0 \Rightarrow w = 90^0$
hence the curves are orthogonal.
Now if the curves are orthogonal then $Cos90^0 = 0 = \frac{F}{\sqrt{EG}} \Rightarrow F = 0$

vii. **Surface is plane surface if $\boldsymbol{F = \sigma_u.\sigma_v = 0}$**

QUESTION: Find the first fundamental form of the surface for plane $\sigma(u, v) = \vec{a} + \vec{u}p + \vec{v}q$ where $p \perp q$ both are unit vectors.

SOLUTION: $\sigma(u, v) = \vec{a} + \vec{u}p + \vec{v}q \Rightarrow \sigma_u = p, \sigma_v = q$

Then $E = \sigma_u.\sigma_u = \|p\|^2 = 1,, G = \sigma_v.\sigma_v = \|q\|^2 = 1$ and $F = \sigma_u.\sigma_v = 0$ $\quad\therefore p \perp q$

For 1st fundamental form we have $Edu^2 + Gdv^2 + 2Fdudv = du^2 + dv^2$

QUESTION: Find the first fundamental form for the surface of revolution $\sigma(u,v) = [f(u)Cosv, f(u)Sinv, g(u)]$ assuming , $f(u) > 0\ \forall u$

SOLUTION: $\sigma(u,v) = [f(u)Cosv, f(u)Sinv, g(u)]$ where ${f'}^2 + {g'}^2 = 1 \therefore U \to (f,0,g)$ is unit speed curve.

$\Rightarrow \sigma_u = [f'(u)Cosv, f'(u)Sinv, g'(u)], \sigma_v = [-f(u)Sinv, f(u)Cosv, 0]$

Then $E = \sigma_u.\sigma_u = {f'}^2 + {g'}^2 = 1,, G = \sigma_v.\sigma_v = f^2(u)\ and\ F = \sigma_u.\sigma_v = 0$

For 1st fundamental form we have $Edu^2 + Gdv^2 + 2Fdudv = du^2 + f^2(u)dv^2$

Special Case: Take $u = \theta, v = \varphi, f(\theta) = Cos\theta, g(\theta) = Sin\theta$

Then we have $Edu^2 + Gdv^2 + 2Fdudv = d\theta^2 + Cos^2\theta d\varphi^2$

QUESTION: Find the first fundamental form for general cylinder $\sigma(u,v) = \vec{r}(u) + v\vec{a}$ where '$\vec{r}$' is unit speed curve and $\vec{a}$ is unit vector.

SOLUTION: $\sigma(u,v) = \vec{r}(u) + v\vec{a} \Rightarrow \sigma_u = \vec{r}'(u), \sigma_v = \vec{a}$

Then $E = \sigma_u.\sigma_u = \|\vec{r}'(u)\|^2 = 1,, G = \sigma_v.\sigma_v = \|\vec{a}\|^2 = 1\ and\ F = \sigma_u.\sigma_v = 0$

For 1st fundamental form we have $Edu^2 + Gdv^2 + 2Fdudv = du^2 + dv^2$

PRACTICE: Find the first fundamental form for the following surfaces;

i. $\sigma(u,v) = (SinhuSinhv, SinhuCoshv, Sinhu)$
ii. $\sigma(u,v) = (Coshu, Sinhu, v)$
iii. $\sigma(u,v) = (u, v, u^2 + v^2)$
iv. $\sigma(u,v) = (u - v, u - v, u^2 + v^2)$

QUESTION: on the surface of revolution $x = uCosv, y = uSinv, z = f(u)$ write down fundamental form of first order. Show that the parametric curves are orthogonal.

SOLUTION: $\vec{r} = [uCosv, uSinv, f(u)]$

$$\Rightarrow \vec{r}_1 = \frac{\partial \vec{r}}{\partial u} = [Cosv, Sinv, f'(u)] \Rightarrow \vec{r}_2 = \frac{\partial \vec{r}}{\partial v} = [-uSinv, uCosv, 0]$$

$$\Rightarrow E = {\vec{r}_1}^2 = Cos^2v + Sin^2v + {f'}^2 = 1 + {f'}^2 \Rightarrow G = {\vec{r}_2}^2 = u^2Sin^2v + u^2Cos^2v = u^2$$

$$\Rightarrow F = \vec{r}_1.\vec{r}_2 = [Cosv, Sinv, f'(u)].[-uSinv, uCosv, 0] = -uCosvSinv + uCosvSinv = 0$$

$\Rightarrow F = \vec{r}_1.\vec{r}_2 = 0$ show that both vectors are orthogonal.

Now for the fundamental form of fist order $ds^2 = Edu^2 + Gdv^2 + 2Fdudv$

$$\Rightarrow ds^2 = \left(1 + {f'}^2\right)du^2 + (u^2)dv^2 + 2(0)dudv \Rightarrow ds^2 = \left(1 + {f'}^2\right)du^2 + u^2dv^2$$

DIRECTION ON A SURFACE: Let $\vec{r} = \vec{r}(u,v)$ be a surface and let (du, dv) and $(\delta u, \delta v)$ denote the change in $\vec{r}$ in two directions then

$d\vec{r} = \vec{r}_1du + \vec{r}_2dv \ \ldots\ldots\ldots\ldots(i)$ and $\delta\vec{r} = \vec{r}_1\delta u + \vec{r}_2\delta v \ \ldots\ldots\ldots\ldots(ii)$

$$\Rightarrow d\vec{r}.\delta\vec{r} = \vec{r}_1.\vec{r}_1du\delta u + \vec{r}_1.\vec{r}_2du\delta v + \vec{r}_2.\vec{r}_1du\delta v + \vec{r}_2.\vec{r}_2dv\delta v$$

$$\Rightarrow d\vec{r}.\delta\vec{r} = Edu\delta u + F(du\delta v + du\delta v) + Gdv\delta v$$

Now write $|d\vec{r}| = ds$ also $|\delta\vec{r}| = \delta s$ and consider Ψ is the angle between two directions then $ds\delta s Cos\Psi = Edu\delta u + F(du\delta v + du\delta v) + Gdv\delta v$

The two directions are perpendicular if $Cos\Psi = 0$

We also note that $|d\vec{r} \times \delta\vec{r}| = ds\delta s Sin\Psi$
$\Longrightarrow |(\vec{r}_1 du + \vec{r}_2 dv) \times (\vec{r}_1 \delta u + \vec{r}_2 \delta v)| = ds\delta s Sin\Psi \;\; \Longrightarrow |\vec{r}_1 \times \vec{r}_2||du\delta v - dv\delta u| = ds\delta s Sin\Psi$

$\Longrightarrow H|du\delta v - dv\delta u| = ds\delta s Sin\Psi$

STANDARD UNIT NORMAL TO THE SURFACE: A surface 'S' defined by a mapping $\sigma: U \to R^3$ where $U \subseteq R^2$ containing a point $\vec{P}$ then standard unit normal to the surface is defined as $\vec{N}_\sigma = \vec{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$

OR The normal to the surface at any point is perpendicular to every tangent line $(\vec{r}_1 \; and \; \vec{r}_2)$ and its direction is along $|\vec{r}_1 \times \vec{r}_2|$ Thus $\Longrightarrow \vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} = \frac{\vec{r}_1 \times \vec{r}_2}{H}$

It is clear that $(\vec{r}_1 . \vec{n}) = 0$ also $(\vec{r}_2 . \vec{n}) = 0$ and we also get the following properties

1. $[\vec{n} \;, \vec{r}_1 \;, \vec{r}_2] = \vec{n}.(\vec{r}_1 \times \vec{r}_2) = \vec{n}.\vec{n}H = H$
2. $\vec{r}_1 \times \vec{n} = \vec{r}_1 \times \frac{\vec{r}_1 \times \vec{r}_2}{H} = \frac{1}{H}[\vec{r}_1 \times (\vec{r}_1 \times \vec{r}_2)] = \frac{1}{H}[(\vec{r}_1.\vec{r}_2)\vec{r}_1 - (\vec{r}_1.\vec{r}_1)\vec{r}_2] = \frac{1}{H}[F\vec{r}_1 - E\vec{r}_2]$
3. $\vec{r}_2 \times \vec{n} = \vec{r}_2 \times \frac{\vec{r}_1 \times \vec{r}_2}{H} = \frac{1}{H}[\vec{r}_2 \times (\vec{r}_1 \times \vec{r}_2)] = \frac{1}{H}[(\vec{r}_2.\vec{r}_2)\vec{r}_1 - (\vec{r}_2.\vec{r}_1)\vec{r}_2] = \frac{1}{H}[G\vec{r}_1 - F\vec{r}_2]$

SECOND FUNDAMENTAL FORM OF SURFACES OR FUNDAMENTAL FORM OF SECOND ORDER:

Let $\sigma: U \to R^3$ is a parameterization of a surface. Then the 2nd fundamental form is defined by the formula $\boldsymbol{Ldu^2 + Ndv^2 + 2Mdudv}$

Where $L = \vec{n}.\vec{r}_{11} = \vec{n}.\sigma_{uu} \quad M = \vec{n}.\vec{r}_{12} = \vec{n}.\sigma_{uv} \qquad N = \vec{n}.\vec{r}_{22} = \vec{n}.\sigma_{vv}$

The second order derivative or $\vec{r}$ with respect to 'u' and 'v' are denoted by

$$\vec{r}_{11} = \vec{r}_{uu} = \frac{\partial^2 \vec{r}}{\partial u^2} = \sigma_{uu} \qquad \vec{r}_{22} = \vec{r}_{vv} = \frac{\partial^2 \vec{r}}{\partial v^2} = \sigma_{vv} \qquad \vec{r}_{12} = \vec{r}_{uv} = \frac{\partial^2 \vec{r}}{\partial u \partial v} = \sigma_{uv}$$

Then "the fundamental magnitude of the second order is the resolved parts of the vectors $\vec{r}_{11}, \; \vec{r}_{22}, \; \vec{r}_{12}$ in the direction of normal to the surface" and they are denoted by

$L = \vec{n}.\vec{r}_{11} = \vec{n}.\sigma_{uu} \qquad M = \vec{n}.\vec{r}_{12} = \vec{n}.\sigma_{uv} \qquad N = \vec{n}.\vec{r}_{22} = \vec{n}.\sigma_{vv}$

Also keep in mind the relation $T^2 = LN - M^2$ which is not necessarily positive.

We can express scalar triple product by using L,M and N as follows;

1. $[\vec{r}_1 \;, \vec{r}_2, \; \vec{r}_{11}] = (\vec{r}_1 \times \vec{r}_2).\vec{r}_{11} = H\vec{n}.\vec{r}_{11} = HL$
2. $[\vec{r}_1 \;, \vec{r}_2, \; \vec{r}_{12}] = (\vec{r}_1 \times \vec{r}_2).\vec{r}_{12} = H\vec{n}.\vec{r}_{12} = HM$
3. $[\vec{r}_1 \;, \vec{r}_2, \; \vec{r}_{22}] = (\vec{r}_1 \times \vec{r}_2).\vec{r}_{22} = H\vec{n}.\vec{r}_{22} = HN$

THINGS TO REMEMBER:

Let $P(u, v)$ be the point of contact with parameter values 'u' and 'v' and $\vec{n}$ the unit normal. Then the position vector of a neighboring point $Q(u + du, v + dv)$ on the surface has the values $\vec{r} + (\vec{r}_1 du + \vec{r}_2 dv) + \frac{1}{2}(\vec{r}_{11} du^2 + \vec{r}_{22} dv^2 + 2\vec{r}_{12} dudv)$

Also the length of the perpendicular from 'Q' on the tangent plane at 'P' is the projection of the vector '$\overrightarrow{PQ}$' on the normal at 'P' and is therefore equal to
$ds^2 = \vec{n}.(\vec{r}_1 du + \vec{r}_2 dv) + \frac{1}{2}\vec{n}.(\vec{r}_{11} du^2 + \vec{r}_{22} dv^2 + 2\vec{r}_{12} dudv)$

As $\vec{n} \perp \vec{r}_1$ also $\vec{n} \perp \vec{r}_2$ thus the first expression vanishes and the net result so obtained is
$ds^2 = \frac{1}{2}\vec{n}.(\vec{r}_{11}du^2 + \vec{r}_{22}dv^2 + 2\vec{r}_{12}dudv) = \frac{1}{2}(\vec{n}.\vec{r}_{11}du^2 + \vec{n}.\vec{r}_{22}dv^2 + 2\vec{n}.\vec{r}_{12}dudv)$
$\boldsymbol{ds^2 = \frac{1}{2}(Ldu^2 + Ndv^2 + 2Mdudv)}$

Which is the length of perpendicular $\overrightarrow{\mathrm{PQ}}$ and is called **second order magnitude**. Or **Second fundamental form**. Where L,M,N are called the fundamental coefficients of Second order.

EXAMPLE: Consider the plane $\sigma(u,v) = \vec{a} + u\vec{p} + v\vec{q}$ where $\vec{a}$ is a point of plane and $\vec{p},\vec{q}$ are constant unit vectors parallel to the plane and perpendicular to each other then show
$$Ldu^2 + Ndv^2 + 2Mdudv = 0$$

SOLUTION: $\sigma(u,v) = \vec{a} + u\vec{p} + v\vec{q} \Longrightarrow \sigma_u = \vec{p} \Longrightarrow \sigma_{uu} = 0, \sigma_v = \vec{q} \Longrightarrow \sigma_{vv} = 0$ also $\sigma_{uv} = 0$

And $\vec{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{\vec{p}\times\vec{q}}{\|\vec{p}\times\vec{q}\|}$ then $L = \vec{n}.\sigma_{uu} = \vec{n}.0 = 0, M = \vec{n}.\sigma_{uv} = \vec{n}.0 = 0, N = \vec{n}.\sigma_{vv} = \vec{n}.0 = 0$

$\Longrightarrow Ldu^2 + Ndv^2 + 2Mdudv = 0$ after putting the values.

- Compute 2nd fundamental form of elliptical paraboloid $\sigma(u,v) = (u, v, u^2 + v^2)$

EXAMPLE: Compute 2nd fundamental form of a surface of revolution
$$\sigma(u,v) = [f(u)Cosv, f(u)Sinv, g(u)]$$

SOLUTION: $\sigma(u,v) = [f(u)Cosv, f(u)Sinv, g(u)]$ where we assume that assuming $, f(u) > 0\ \forall u$ and $f'^2 + g'^2 = 1 \therefore U \to (f, 0, g)$ is unit speed curve.

$\Longrightarrow \sigma_u = [f'(u)Cosv, f'(u)Sinv, g'(u)] \Longrightarrow \sigma_{uu} = [f''(u)Cosv, f''(u)Sinv, g''(u)]$
$\sigma_v = [-f(u)Sinv, f(u)Cosv, 0] \Longrightarrow \sigma_{vv} = [-f(u)Cosv, -f(u)Sinv, 0]$
also $\sigma_{uv} = [-f'(u)Sinv, f'(u)Cosv, 0]$

$\Longrightarrow \sigma_u \times \sigma_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ f'(u)Cosv & f'(u)Sinv & g'(u) \\ -f(u)Sinv & f(u)Cosv & 0 \end{vmatrix} = f[-g'(u)Cosv, -g'(u)Sinv, f']$ also $\|\sigma_u \times \sigma_v\| = f$

Now $\vec{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{f[-g'(u)Cosv, -g'(u)Sinv, f']}{f} = [-g'(u)Cosv, -g'(u)Sinv, f']$

then $L = \vec{n}.\sigma_{uu} = f'g'' - f''g'$, $\quad M = \vec{n}.\sigma_{uv} = \vec{n}.0 = 0, \quad N = \vec{n}.\sigma_{vv} = fg'$

$\Longrightarrow Ldu^2 + Ndv^2 + 2Mdudv = (f'g'' - f''g')du^2 + (fg')dv^2$ after putting the values.

QUESTION (PP): Calculate fundamental magnitude of first and second order for the surfaces $x = uCos\varphi, \quad y = uSin\varphi, \quad z = c\varphi$ where $u\ adn\ \varphi$ are parameters.

ANSWER: $\vec{r} = (x, \quad y, \quad z) = (uCos\varphi, \quad uSin\varphi, \quad c\varphi)$

$\Longrightarrow \vec{r}_1 = \frac{\partial\vec{r}}{\partial u} = [Cos\varphi, \quad Sin\varphi, \quad 0] \Longrightarrow \vec{r}_2 = \frac{\partial\vec{r}}{\partial\varphi} = [-uSin\varphi, \quad uCos\varphi, \quad c]$

Now the fundamental coefficients of first order are $\vec{r}_1^{\,2} = E, , \vec{r}_2^{\,2} = G\ and\ F = \vec{r}_1\vec{r}_2$

$\Longrightarrow E = \vec{r}_1^{\,2} = Cos^2\varphi + Sin^2\varphi = 1 \qquad \Longrightarrow G = \vec{r}_2^{\,2} = u^2Sin^2\varphi + u^2Cos^2\varphi + c^2 = u^2 + c^2$

$\Longrightarrow F = \vec{r}_1.\vec{r}_2 = [Cos\varphi, \ Sin\varphi, \ 0].[-uSin\varphi, \ uCos\varphi, \ c] = -uCos\varphi Sin\varphi + uCos\varphi Sin\varphi = 0$

$\Longrightarrow F = \vec{r}_1.\vec{r}_2 = 0$ show that both vectors are orthogonal.

Also $H^2 = EG - F^2 = 1(u^2 + c^2) - 0 = u^2 + c^2$

Also unit normal to the surface is $\vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} \Longrightarrow \vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{H}$

$$\Rightarrow \vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ Cos\varphi & Sin\varphi & 0 \\ -uSin\varphi & uCos\varphi & c \end{vmatrix}$$

$\Longrightarrow \vec{n} = \frac{[cSin\varphi, -cCos\varphi, \; u]}{H}$

Now the fundamental coefficients of second order are $\vec{r}_{11}, \; \vec{r}_{22}, \; \vec{r}_{12}$

For $\vec{r} = (x \,, \quad y \,, \quad z\,) = (uCos\varphi, \quad uSin\varphi, \quad c\varphi)$

$\Longrightarrow \vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial u^2} = (0, \quad 0, \quad 0) \qquad \Longrightarrow \vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial v^2} = (-uCos\varphi, \quad -uSin\varphi, \quad 0)$

$\Longrightarrow \vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial u \partial v} = (-Sin\varphi, \; Cos\varphi, \; 0)$

So the second order coefficients are $L = \vec{n}.\vec{r}_{11} = 0 \quad M = \vec{n}.\vec{r}_{12} = -\frac{c}{H} \quad N = \vec{n}.\vec{r}_{22} = 0$

And $T^2 = LN - M^2 = 0 - \left(-\frac{c}{H}\right)^2 = \frac{c^2}{H^2}$

QUESTION: Calculate fundamental magnitude of first and second order for the surfaces $x = a(u+v), \quad y = b(u+v), \quad z = uv$ where $u \; adn \; \varphi$ are parameters.

ANSWER: $\vec{r} = (x \,, \quad y \,, \quad z\,) = (a(u+v), \quad b(u+v), \quad uv)$

$\Longrightarrow \vec{r}_1 = \frac{\partial \vec{r}}{\partial u} = [a, \quad b, \quad v] \Longrightarrow \vec{r}_2 = \frac{\partial \vec{r}}{\partial v} = [a, \quad b, \quad u]$

Now the fundamental coefficients of first order are $\vec{r}_1^{\,2} = E,, \vec{r}_2^{\,2} = G \; and \; F = \vec{r}_1 \vec{r}_2$

$\Longrightarrow E = \vec{r}_1^{\,2} = a^2 + b^2 + v^2 \qquad \Longrightarrow G = \vec{r}_2^{\,2} = a^2 + b^2 + u^2$

$\Longrightarrow F = \vec{r}_1.\vec{r}_2 = [a, \quad b, \quad v].[a, \quad b, \quad u] = a^2 + b^2 + uv$

Also $H^2 = EG - F^2 = (a^2 + b^2 + v^2)(a^2 + b^2 + u^2) - (a^2 + b^2 + uv)^2$

$\Longrightarrow H^2 = EG - F^2 = a^2u^2 + b^2u^2 + a^2v^2 + b^2v^2 - 2uvb^2 - 2uva^2$

Also unit normal to the surface is $\vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{H} \Longrightarrow \vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}$

$$\Rightarrow \vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a & b & v \\ a & b & u \end{vmatrix}$$

$\Longrightarrow \vec{n} = \frac{b(u-v)\hat{\imath} + a(v-u)\hat{\jmath} + 0\hat{k}}{\sqrt{b^2(u-v)^2 + a^2(v-u)^2}}$

Now the fundamental coefficients of second order are $\vec{r}_{11}, \; \vec{r}_{22}, \; \vec{r}_{12}$

For $\vec{r} = (x \,, \quad y \,, \quad z\,) = (a(u+v), \quad b(u+v), \quad uv)$

$\Longrightarrow \vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial u^2} = (0, \quad 0, \quad 0) \Longrightarrow \vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial v^2} = (0, \quad 0, \quad 0) \quad \Longrightarrow \vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial u \partial v} = (0, \quad 0, \quad 1)$

So the second order coefficients are $L = \vec{n}.\vec{r}_{11} = 0 \quad M = \vec{n}.\vec{r}_{12} = 0 \quad N = \vec{n}.\vec{r}_{22} = 0$

And $T^2 = LN - M^2 = 0$

QUESTION: Taking x,y, as parameters, calculate the fundamental magnitudes and the unit normal to the surface $2z = ax^2 + 2hxy + by^2$

ANSWER: Given $\vec{r} = (x \;, \; y \;, \; z) = \left(x \;, \; y \;, \; \frac{ax^2 + 2hxy + by^2}{2}\right)$

$\Longrightarrow \vec{r}_1 = \frac{\partial \vec{r}}{\partial x} = \left(1 \;, \; 0 \;, \; \frac{2ax + 2hy}{2}\right) = (1 \;, \; 0 \;, \; ax + hy)$

$\Longrightarrow \vec{r}_2 = \frac{\partial \vec{r}}{\partial y} = \left(0 \;, \; 1 \;, \; \frac{2hx + 2by}{2}\right) = (0 \;, \; 1 \;, \; hx + by)$

Now $\Rightarrow E = \vec{r}_1^{\,2} = 1 + 0 + (ax + hy)^2 = 1 + (ax + hy)^2$

$\Rightarrow G = \vec{r}_2^{\,2} = 0 + 1 + (hx + by)^2 = 1 + (hx + by)^2$

$\Rightarrow F = \vec{r}_1 . \vec{r}_2 = 0 + 0 + (ax + hy)(hx + by)$

Also $H^2 = EG - F^2 = (ax + hy)^2 + (hx + by)^2 + 1$

Also unit normal to the surface is $\vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{H} \Rightarrow \vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}$

$\Rightarrow \vec{n} = \frac{[(-ax-hy),\ \ -hx-by,\ \ 1\,]}{\sqrt{(ax+hy)^2+(hx+by)^2+1}} = \frac{[(-ax-hy),\ \ -hx-by,\ \ 1\,]}{H}$

$$\boxed{\Rightarrow \vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a & b & v \\ a & b & u \end{vmatrix}}$$

Now the fundamental coefficients of second order are $\vec{r}_{11},\ \vec{r}_{22},\ \vec{r}_{12}$

For $\vec{r} = (x\ \ ,\ y\ ,\ z) = \left(x\ \ ,\ y\ ,\ \frac{ax^2+2hxy+by^2}{2}\right)$

$\Rightarrow \vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial x^2} = (0,\quad 0,\quad a)\ \Rightarrow \vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial y^2} = (0,\quad 0,\quad b)\quad \Rightarrow \vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial x \partial y} = (0,\quad 0,\quad h)$

So the second order coefficients are $L = \vec{n}.\vec{r}_{11} = \frac{[(-ax-hy),\ \ -hx-by,\ \ 1\,]}{H}.(0,\quad 0,\quad a) = \frac{a}{H}$

$M = \vec{n}.\vec{r}_{12} = \frac{[(-ax-hy),\ \ -hx-by,\ \ 1\,]}{H}.(0,\quad 0,\quad h) = \frac{h}{H}$

$N = \vec{n}.\vec{r}_{22} = \frac{[(-ax-hy),\ \ -hx-by,\ \ 1\,]}{H}.(0,\quad 0,\quad b) = \frac{b}{H}$

QUESTION: For the surfaces $x = uCos\varphi,\quad y = uSin\varphi,\quad z = f(u)$ where $u\ adn\ \varphi$ are parameters. Then show that $\boldsymbol{T^2 = \frac{u^3 f' f''}{H}}$

ANSWER: $\vec{r} = (x\ ,\quad y\ ,\quad z\,) = \left(uCos\varphi,\quad uSin\varphi,\quad f(u)\right)$

$\Rightarrow \vec{r}_1 = \frac{\partial \vec{r}}{\partial u} = \left(Cos\varphi,\quad Sin\varphi,\quad f'(u)\right) \Rightarrow \vec{r}_2 = \frac{\partial \vec{r}}{\partial \varphi} = [-uSin\varphi,\quad uCos\varphi,\quad 0]$

Now the fundamental coefficients of first order are $\vec{r}_1^{\,2} = E,, \vec{r}_2^{\,2} = G\ and\ F = \vec{r}_1 \vec{r}_2$

$\Rightarrow E = \vec{r}_1^{\,2} = Cos^2\varphi + Sin^2\varphi + [f'(u)]^2 = 1 + [f']^2\ \Rightarrow \boldsymbol{E = 1 + [f']^2}$

$\Rightarrow G = \vec{r}_2^{\,2} = u^2 Cos^2\varphi + u^2 Sin^2\varphi + 0 \Rightarrow \boldsymbol{G = u^2}$

$\Rightarrow F = \vec{r}_1 . \vec{r}_2 = [Cos\varphi,\quad Sin\varphi,\quad f'(u)].[-uSin\varphi,\quad uCos\varphi,\quad 0]$
$= -uSin\varphi Cos\varphi + uSin\varphi Cos\varphi + 0 \Rightarrow \boldsymbol{F = 0}$

Also $H^2 = EG - F^2 = (1 + [f']^2\,)(u^2\,) - 0 \Rightarrow H^2 = u^2(1 + [f']^2\,)$

Also unit normal to the surface is $\vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{H} \Rightarrow \vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}$

$\Rightarrow \boldsymbol{\vec{n} = \frac{u[Cos\varphi f',\quad -Sin\varphi f',\quad 1]}{H}}$

$$\boxed{\Rightarrow \vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ Cos\varphi & Sin\varphi & f'(u) \\ -uSin\varphi & uCos\varphi & 0 \end{vmatrix}}$$

Now the fundamental coefficients of second order are $\vec{r}_{11},\ \vec{r}_{22},\ \vec{r}_{12}$

For $\vec{r} = (x\ ,\quad y\ ,\quad z\,) = \left(uCos\varphi,\quad uSin\varphi,\quad f(u)\right)$

$\Rightarrow \vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial u^2} = (0,\quad 0,\quad f'')\ \Rightarrow \vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial \varphi^2} = [-Sin\varphi,\quad Cos\varphi,\quad 0]$

$\Rightarrow \vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial u \partial \varphi} = (-uCos\varphi,\quad -uSin\varphi,\quad 0)$

So the second order coefficients are $L = \vec{n}.\vec{r}_{11} = \frac{uf''}{H}\quad M = \vec{n}.\vec{r}_{12} = 0\quad N = \vec{n}.\vec{r}_{22} = \frac{u^2 f'}{H}$

And $T^2 = LN - M^2 = \frac{uf''}{H}.\frac{u^2 f'}{H} - 0 \Rightarrow \boldsymbol{T^2 = \frac{u^3 f' f''}{H}}$

QUESTION: For the surfaces $x = uCos\varphi$, $y = uSin\varphi$, $z = f(\varphi)$ where u and φ are parameters. Then show that $\boldsymbol{T^2 = \frac{f'}{H}}$

ANSWER: $\vec{r} = (x, \quad y, \quad z) = \big(uCos\varphi, \quad uSin\varphi, \quad f(\varphi)\big)$

$\Rightarrow \vec{r}_1 = \frac{\partial \vec{r}}{\partial u} = (Cos\varphi, \quad Sin\varphi, \ 0) \Rightarrow \vec{r}_2 = \frac{\partial \vec{r}}{\partial \varphi} = [-uSin\varphi, \quad uCos\varphi, \quad f'(\varphi)]$

Now the fundamental coefficients of first order are $\vec{r}_1^{\,2} = E, , \vec{r}_2^{\,2} = G \ and \ F = \vec{r}_1\vec{r}_2$

$\Rightarrow E = \vec{r}_1^{\,2} = Cos^2\varphi + Sin^2\varphi + 0 = 1 \ \Rightarrow \boldsymbol{E = 1}$

$\Rightarrow G = \vec{r}_2^{\,2} = u^2Cos^2\varphi + u^2Sin^2\varphi + [f'(\varphi)]^2 \Rightarrow \boldsymbol{G = u^2 + [f'(\varphi)]^2}$

$\Rightarrow F = \vec{r}_1.\vec{r}_2 = [Cos\varphi, \quad Sin\varphi, \quad 0].[-uSin\varphi, \quad uCos\varphi, \quad f'(\varphi)]$
$= -uSin\varphi Cos\varphi + uSin\varphi Cos\varphi + 0 \Rightarrow \boldsymbol{F = 0}$

Also $H^2 = EG - F^2 = 1(u^2 + [f'(\varphi)]^2) - 0 \Rightarrow H^2 = u^2 + [f'(\varphi)]^2$

Also unit normal to the surface is $\vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{H} \Rightarrow \vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}$

$\Rightarrow \vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ Cos\varphi & Sin\varphi & 0 \\ -uSin\varphi & uCos\varphi & f'(\varphi) \end{vmatrix}$

$\Rightarrow \boldsymbol{\vec{n} = \frac{[Sin\varphi f', \quad -Cos\varphi f', \quad u]}{H}}$

Now the fundamental coefficients of second order are $\vec{r}_{11}, \ \vec{r}_{22}, \ \vec{r}_{12}$

For $\vec{r} = (x, \quad y, \quad z) = \big(uCos\varphi, \quad uSin\varphi, \quad f(\varphi)\big)$

$\Rightarrow \vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial u^2} = (0, \quad 0, \quad 0) \ \Rightarrow \vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial \varphi^2} = (-uCos\varphi, \quad -uSin\varphi, \ f'')$
$\Rightarrow \vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial u \partial \varphi} = [-Sin\varphi, \quad Cos\varphi, \quad 0]$

So the second order coefficients are $L = \vec{n}.\vec{r}_{11} = 0 \quad M = \vec{n}.\vec{r}_{12} = -\frac{f'}{H} \quad N = \vec{n}.\vec{r}_{22} = \frac{uf''}{H}$

And $T^2 = LN - M^2 = 0.\frac{uf''}{H} - \left(-\frac{f'}{H}\right) \Rightarrow \boldsymbol{T^2 = \frac{f'}{H}}$

QUESTION: On the surface generated by binormal of twisted curve, the position vector of the current point may be expressed as $\vec{r} + u\vec{b}$ where $\vec{r}$ and $\vec{b}$ are functions of 's' take 'u' and's' as parameters and find fundamental magnitudes.

ANSWER: Let $\vec{R}$ be the position vector of any point on the surface. And $\vec{R} = \vec{r} + u\vec{b}$

$\Rightarrow \vec{R}_1 = \frac{\partial \vec{R}}{\partial u} = 0 + \vec{b} \Rightarrow \vec{R}_1 = \vec{b} \Rightarrow \vec{R}_2 = \frac{\partial \vec{R}}{\partial s} = \frac{d\vec{R}}{ds} + u\frac{d\vec{b}}{ds} = \vec{r}\,' + u\vec{b}' \Rightarrow \vec{R}_2 = \vec{t} - u\tau\vec{n}$
where $\vec{n}$ is perpendicular normal to the curve.

Now the fundamental coefficients of first order are $\vec{r}_1^{\,2} = E, , \vec{r}_2^{\,2} = G \ and \ F = \vec{r}_1\vec{r}_2$

$\Rightarrow E = \vec{R}_1^{\,2} = \vec{b}.\vec{b} = b^2 \ \Rightarrow \boldsymbol{E = b^2}$

$\Rightarrow G = \vec{R}_2^{\,2} = (\vec{t} - u\tau\vec{n})(\vec{t} - u\tau\vec{n}) = \vec{t}^2 + u^2\tau^2\vec{n}^2 \Rightarrow \boldsymbol{G = 1 + u^2\tau^2}$

$\Rightarrow F = \vec{R}_1.\vec{R}_2 = 0 - u\tau\vec{b}\vec{n} \Rightarrow \boldsymbol{F = 0}$

Also unit normal to the surface is $\vec{n} = \frac{\vec{R}_1 \times \vec{R}_2}{H} \Rightarrow \vec{n} = \frac{\vec{R}_1 \times \vec{R}_2}{|\vec{R}_1 \times \vec{R}_2|} \Rightarrow \boldsymbol{\vec{n} = \frac{\vec{n} + u\tau\vec{t}}{\sqrt{1 + u^2\tau^2}}}$

Now the fundamental coefficients of second order are $\vec{R}_{11}, \ \vec{R}_{22}, \ \vec{R}_{11}$

For $\vec{R} = \vec{r} + u\vec{b} \Rightarrow \vec{R}_{11} = \frac{\partial^2\vec{R}}{\partial u^2} = 0 \;\; \Rightarrow \vec{R}_{12} = \frac{\partial^2\vec{R}}{\partial u\partial s} = \vec{b}' = -\tau\vec{n}$

$\Rightarrow \boldsymbol{\vec{R}_{22}} = \frac{\partial^2\vec{R}}{\partial s^2} = \left(\vec{t} - u(\tau\vec{n}' + \tau'\vec{n})\right) = K\vec{n} - u\tau(\tau\vec{b} - K\vec{t}) - u\tau'\vec{n} = K\vec{n} - u\tau^2\vec{b} - u\tau K\vec{t} - u\tau'\vec{n}$

$\Rightarrow \vec{R}_{22} = (K - u\tau')\vec{n} - u\tau^2\vec{b} - u\tau K\vec{t}$

So the second order coefficients are $L = \vec{n}.\vec{R}_{11} = 0 \;\; M = \vec{n}.\vec{R}_{12} = -\frac{\tau}{H}$

$N = \vec{n}.\vec{R}_{22} = \frac{K+\tau^2u^2-u\tau\prime}{H}$ And $T^2 = LN - M^2 = 0.\frac{K+\tau^2u^2-u\tau\prime}{H} - \left(-\frac{\tau}{H}\right)^2 \Rightarrow \boldsymbol{T^2} = \boldsymbol{\frac{\tau^2}{H^2}}$

QUESTION: When the equation of the surface is given in Mango's form $z = f(x,y)$; x,y may be taken as parameters. Let 'P' , 'Q' be derivatives of 'z' of first order and let 'r','s','t' be those of second orders then show that $\boldsymbol{T^2} = \boldsymbol{\frac{rt-s^2}{H^2}}$ Also deduce that $\boldsymbol{T^2 = 0}$ for a developable surface.

ANSWER: Given that $z = f(x,y) \Rightarrow p = \frac{\partial z}{\partial x},\; q = \frac{\partial z}{\partial y},\; r = \frac{\partial^2 z}{\partial x^2},\; t = \frac{\partial^2 z}{\partial y^2},\; s = \frac{\partial^2 z}{\partial x\partial y}$

$\vec{r} = (x, \quad y, \quad z) \Rightarrow \vec{r}_1 = \frac{\partial\vec{r}}{\partial x} = \left(1, \quad 0, \; \frac{\partial z}{\partial x} = p\right) \Rightarrow \vec{r}_2 = \frac{\partial\vec{r}}{\partial y} = \left[0, \quad 1, \quad \frac{\partial z}{\partial y} = q\right]$

Now the fundamental coefficients of first order are $\vec{r}_1^{\,2} = E,, \vec{r}_2^{\,2} = G \; and \; F = \vec{r}_1\vec{r}_2$

$\Rightarrow \boldsymbol{E = \vec{r}_1^{\,2} = 1 + p^2 \;\; \Rightarrow G = 1 + q^2 \Rightarrow F = pq}$

Also $H^2 = EG - F^2 \Rightarrow \boldsymbol{H^2 = 1 + p^2 + q^2}$

Also unit normal to the surface is $\vec{n} = \frac{\vec{r}_1\times\vec{r}_2}{H} \Rightarrow \vec{n} = \frac{\vec{r}_1\times\vec{r}_2}{|\vec{r}_1\times\vec{r}_2|}$

$\Rightarrow \boldsymbol{\vec{n}} = \frac{[-\boldsymbol{p}, \quad -\boldsymbol{q}, \quad \boldsymbol{1}]}{\sqrt{\boldsymbol{p^2 + q^2 + 1}}}$

$$\boxed{\Rightarrow \vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & p \\ 0 & 1 & q \end{vmatrix}}$$

Now the fundamental coefficients of second order are $\vec{r}_{11},\; \vec{r}_{22},\; \vec{r}_{12}$

For $\vec{r} = (x, \quad y, \quad z)$

$\Rightarrow \vec{r}_{11} = \frac{\partial^2\vec{r}}{\partial x^2} = (0, \quad 0, \quad r) \Rightarrow \vec{r}_{22} = \frac{\partial^2\vec{r}}{\partial y^2} = (0, \quad 0, \quad t) \quad \Rightarrow \vec{r}_{12} = \frac{\partial^2\vec{r}}{\partial x\partial y} = (0, \quad 0, \quad s)$

So the second order coefficients are $L = \vec{n}.\vec{r}_{11} = \frac{r}{H} \;\; M = \vec{n}.\vec{r}_{12} = \frac{s}{H} \qquad N = \vec{n}.\vec{r}_{22} = \frac{t}{H}$

And $T^2 = LN - M^2 = \frac{r}{H}.\frac{t}{H} - \frac{s^2}{H^2} \Rightarrow \boldsymbol{T^2} = \boldsymbol{\frac{rt-s^2}{H^2}}$ **..........(i)**

Now for a developable surface

$\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2 = \frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} \Rightarrow s^2 = rt \Rightarrow rt - s^2 = 0 \Rightarrow (i) \Rightarrow \boldsymbol{T^2 = 0}$

QUESTION: Show that the curves $du^2 - (u^2 + c^2)d\varphi^2 = 0$ form an orthogonal system on the surfaces $x = uCos\varphi, \quad y = uSin\varphi, \quad z = c\varphi$ where u adn φ are parameters.

ANSWER: $\vec{r} = (x, \quad y, \quad z) = (uCos\varphi, \quad uSin\varphi, \quad c\varphi)$

$\Rightarrow \vec{r}_1 = \frac{\partial\vec{r}}{\partial u} = (Cos\varphi, \quad Sin\varphi, \; 0) \Rightarrow \vec{r}_2 = \frac{\partial\vec{r}}{\partial\varphi} = [-uSin\varphi, \quad uCos\varphi, \quad c]$

Now the fundamental coefficients of first order are $\vec{r}_1^{\,2} = E,, \vec{r}_2^{\,2} = G \; and \; F = \vec{r}_1\vec{r}_2$

$\Rightarrow E = \vec{r}_1^{\,2} = Cos^2\varphi + Sin^2\varphi + 0 = 1 \;\; \Rightarrow \boldsymbol{E = 1}$

$\Rightarrow G = \vec{r}_2^{\,2} = u^2Cos^2\varphi + u^2Sin^2\varphi + c^2 \Rightarrow \boldsymbol{G = u^2 + c^2}$

$\Rightarrow F = \vec{r}_1.\vec{r}_2 = [Cos\varphi, \quad Sin\varphi, \quad 0].[-uSin\varphi, \quad uCos\varphi, \quad c] = -uSin\varphi Cos\varphi + uSin\varphi Cos\varphi + 0$
$\Rightarrow \boldsymbol{F = 0}$

Also the directions are given by $du^2 - (u^2 + c^2)d\varphi^2 = 0 \Rightarrow du^2 = (u^2 + c^2)d\varphi^2$

$\Rightarrow \left(\frac{du}{d\varphi}\right)^2 = (u^2 + c^2) \Rightarrow \frac{du}{d\varphi} = \sqrt{(u^2 + c^2)} \ldots\ldots (i)\ also \Rightarrow \frac{\partial u}{\partial\varphi} = -\sqrt{(u^2 + c^2)} \ldots\ldots (ii)$

the two directions $\frac{du}{d\varphi}, \frac{\partial u}{\partial\varphi}$ on a surface are orthogonal if $E\frac{du}{d\varphi}.\frac{\partial u}{\partial\varphi} + F\left(\frac{du}{d\varphi} + \frac{\partial u}{\partial\varphi}\right) + G = 0$

now putting values in L.H.S.

$\Rightarrow E\frac{du}{d\varphi}.\frac{\partial u}{\partial\varphi} + F\left(\frac{du}{d\varphi} + \frac{\partial u}{\partial\varphi}\right) + G = -1.\sqrt{(u^2 + c^2)}\sqrt{(u^2 + c^2)} + (u^2 + c^2)$

$\Rightarrow E\frac{du}{d\varphi}.\frac{\partial u}{\partial\varphi} + F\left(\frac{du}{d\varphi} + \frac{\partial u}{\partial\varphi}\right) + G = -(u^2 + c^2) + (u^2 + c^2) = 0$

$\Rightarrow E\frac{du}{d\varphi}.\frac{\partial u}{\partial\varphi} + F\left(\frac{du}{d\varphi} + \frac{\partial u}{\partial\varphi}\right) + G = 0$ Thus two directions are orthogonal.

QUESTION (PP): Show that the differential equation of the orthogonal projection of the family of curves given by $P\delta u + Q\delta v = 0$ where 'P' and 'Q' are function of 'u' and 'v' is

$$(EQ - FP)du + (FQ - GP)dv = 0$$

ANSWER: The family of the curves is given by $P\delta u + Q\delta v = 0 \ldots\ldots\ldots\ldots (i)$

Diff.w.r.to 'v' $\Rightarrow P\frac{\delta u}{\delta v} + Q = 0 \Rightarrow \frac{\delta u}{\delta v} = -\frac{Q}{P}$

Now the condition for two curves to be orthogonal is $E\left(\frac{du}{dv}.\frac{\delta u}{\delta v}\right) + F\left(\frac{du}{dv} + \frac{\delta u}{\delta v}\right) + G = 0$

$\Rightarrow E\left(\frac{du}{dv}.\left(-\frac{Q}{P}\right)\right) + F\left(\frac{du}{dv} - \frac{Q}{P}\right) + G = 0 \qquad \because \frac{\delta u}{\delta v} = -\frac{Q}{P}$

$\Rightarrow -EQdu + FPdu - FQdu + GPdv = 0 \Rightarrow \boldsymbol{(EQ - FP)du + (FQ - GP)dv = 0}$ Required.

QUESTION: Find the tangent of the angle between two directions on the surface determined by the equation $Pdu^2 + Qdudv + Rdv^2 = 0$

(OR) if Ψ is the angle between two directions on the surface determined by $Pdu^2 + Qdudv + Rdv^2 = 0$ then show that $Tan\Psi = \frac{H\sqrt{Q^2-4PR}}{ER-EQ+GP}$

ANSWER: The given equation is $Pdu^2 + Qdudv + Rdv^2 = 0$

$\Rightarrow P\left(\frac{du^2}{dv^2}\right) + Q\left(\frac{du}{dv}\right) + R = 0 \qquad \therefore dividing\ by\ dv^2$

Let $\left(\frac{du}{dv}\right)$ and $\left(\frac{\delta u}{\delta v}\right)$ be the roots of the given above equation then

$sum\ of\ roots = \frac{du}{dv} + \frac{\delta u}{\delta v} = -\frac{Q}{P} \ldots\ldots\ldots (i) \quad Product\ of\ roots = \frac{du}{dv}.\frac{\delta u}{\delta v} = \frac{R}{P} \ldots\ldots\ldots (ii)$

$Difference\ of\ roots = \frac{du}{dv} - \frac{\delta u}{\delta v} = \frac{\sqrt{Q^2-4PR}}{P} \ldots\ldots\ldots (iii)$

Now we know that if Ψ is the angle between two directions on the surface then
$ds\delta sCos\Psi = Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v$ and $ds\delta sSin\Psi = H(du\delta v - dv\delta u)$

Hence $Tan\Psi = \frac{H(du\delta v - dv\delta u)}{Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v}$

$\Rightarrow Tan\Psi = \frac{H\left(\frac{du}{dv} - \frac{\delta u}{\delta v}\right)}{E\frac{du}{dv}.\frac{\delta u}{\delta v} + F\left(\frac{du}{dv} + \frac{\delta u}{\delta v}\right) + G} \qquad \therefore dividing\ by\ dv\delta v$

$\Rightarrow Tan\Psi = \frac{H\left(\frac{\sqrt{Q^2-4PR}}{P}\right)}{E\left(\frac{R}{P}\right) + F\left(-\frac{Q}{P}\right) + G} \quad \therefore using\ (i), (ii) and\ (iii) \quad \Rightarrow Tan\Psi = \frac{H\sqrt{Q^2-4PR}}{ER-EQ+GP}$

QUESTION: Prove that if θ is the angle between a direction on a surface and the curve $u = constant$ then $Cos\theta = \frac{1}{\sqrt{G}}\left(F\frac{du}{ds} + G\frac{dv}{ds}\right)$ and $Sin\theta = \frac{H}{\sqrt{G}}\left(\frac{du}{ds}\right)$

ANSWER: We know that if $d\vec{r}$ is the displacement corresponding to the increments du, dv $and\ \delta\vec{r}$ is the displacement corresponding to the increments $\delta u, \delta v$ then

$d\vec{r} = \vec{r}_1 du + \vec{r}_2 dv \ \ldots\ldots\ldots\ldots(i)$ and $\delta\vec{r} = \vec{r}_1\delta u + \vec{r}_2\delta v \ \ldots\ldots\ldots\ldots(ii)$

Then angle θ between two directions is

$ds\delta sCos\theta = Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v \ldots\ldots\ldots\ldots(iii)$

and $ds\delta sSin\theta = H(du\delta v - dv\delta u) \ldots\ldots\ldots\ldots(iv)$

now if the displacement is in the direction $u = constant$ then $\delta u = 0$

$(ii) \Rightarrow \delta\vec{r} = \vec{r}_2\delta v \Rightarrow |\delta\vec{r}| = |\vec{r}_2|\delta v \Rightarrow \delta s = \sqrt{G}\delta v$

$(iii) \Rightarrow ds\sqrt{G}\delta vCos\theta = 0 + F(du\delta v) + Gdv\delta v = \delta v(F\delta u + Gdv)$

$\Rightarrow ds\sqrt{G}Cos\theta = (Fdu + Gdv) \Rightarrow \boldsymbol{Cos\theta} = \frac{1}{\sqrt{G}}\left(\boldsymbol{F}\frac{\boldsymbol{du}}{\boldsymbol{ds}} + \boldsymbol{G}\frac{\boldsymbol{dv}}{\boldsymbol{ds}}\right)$

$(iv) \Rightarrow ds\sqrt{G}\delta vSin\theta = H(du\delta v - 0) \Rightarrow ds\sqrt{G}\delta vSin\theta = H(du\delta v) \Rightarrow ds\sqrt{G}Sin\theta = Hdu$

$\Rightarrow \boldsymbol{Sin\theta} = \frac{\boldsymbol{H}}{\sqrt{G}}\left(\frac{\boldsymbol{du}}{\boldsymbol{ds}}\right)$

QUESTION: If χ is the angle between a direction on a surface and the curve $u = constant$ then find $Cot\chi$

ANSWER: We know that if $d\vec{r}$ is the displacement corresponding to the increments du, dv $and\ \delta\vec{r}$ is the displacement corresponding to the increments $\delta u, \delta v$ then

$d\vec{r} = \vec{r}_1 du + \vec{r}_2 dv \ \ldots\ldots\ldots\ldots(i)$ and $\delta\vec{r} = \vec{r}_1\delta u + \vec{r}_2\delta v \ \ldots\ldots\ldots\ldots(ii)$

Then angle θ between two directions is

$ds\delta sCos\chi = Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v \ldots\ldots\ldots\ldots(iii)$

and $ds\delta sSin\chi = H(du\delta v - dv\delta u) \ldots\ldots\ldots\ldots(iv)$

now if the displacement is in the direction $u = constant$ then $\delta u = 0$

$(ii) \Rightarrow \delta\vec{r} = \vec{r}_2\delta v \Rightarrow |\delta\vec{r}| = |\vec{r}_2|\delta v \Rightarrow \delta s = \sqrt{G}\delta v$

$(iii) \Rightarrow ds\sqrt{G}\delta vCos\chi = 0 + F(du\delta v) + Gdv\delta v = \delta v(F\delta u + Gdv)$

$\Rightarrow ds\sqrt{G}Cos\chi = (Fdu + Gdv) \Rightarrow \boldsymbol{Cos\chi} = \frac{1}{\sqrt{G}}\left(\boldsymbol{F}\frac{\boldsymbol{du}}{\boldsymbol{ds}} + \boldsymbol{G}\frac{\boldsymbol{dv}}{\boldsymbol{ds}}\right)$

$(iv) \Rightarrow ds\sqrt{G}\delta vSin\chi = H(du\delta v - 0) \Rightarrow ds\sqrt{G}\delta vSin\chi = H(du\delta v) \Rightarrow ds\sqrt{G}Sin\chi = Hdu$

$\Rightarrow \boldsymbol{Sin\chi} = \frac{\boldsymbol{H}}{\sqrt{G}}\left(\frac{\boldsymbol{du}}{\boldsymbol{ds}}\right)$

Then $Cot\chi = \frac{Cos\chi}{Sin\chi} = \frac{\frac{1}{\sqrt{G}}\left(F\frac{du}{ds} + G\frac{dv}{ds}\right)}{\frac{H}{\sqrt{G}}\left(\frac{du}{ds}\right)} \Rightarrow \boldsymbol{Cot\chi} = \frac{\mathbf{1}}{\mathbf{H}}\left(\boldsymbol{F} + \boldsymbol{G}\frac{\boldsymbol{dv}}{\boldsymbol{du}}\right)$

QUESTION(PP): Prove that if θ is the angle between a direction on a surface and the curve $v = constant$ then $Cos\theta = \frac{1}{\sqrt{E}}\left(E\frac{du}{ds} + F\frac{dv}{ds}\right)$ and $Sin\theta = -\frac{H}{\sqrt{E}}\left(\frac{dv}{ds}\right)$

ANSWER: We know that if $d\vec{r}$ is the displacement corresponding to the increments du, dv $and\ \delta\vec{r}$ is the displacement corresponding to the increments $\delta u, \delta v$ then

$d\vec{r} = \vec{r}_1 du + \vec{r}_2 dv \ \ldots\ldots\ldots\ldots(i)$ and $\delta\vec{r} = \vec{r}_1\delta u + \vec{r}_2\delta v \ \ldots\ldots\ldots\ldots(ii)$

Then angle θ between two directions is

$ds\delta sCos\theta = Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v \ldots\ldots\ldots\ldots(iii)$

and $ds\delta sSin\theta = H(du\delta v - dv\delta u) \ldots\ldots\ldots\ldots(iv)$

now if the displacement is in the direction $v = constant$ then $\delta v = 0$

$(ii) \Rightarrow \delta\vec{r} = \vec{r}_1\delta u \Rightarrow |\delta\vec{r}| = |\vec{r}_1|\delta u \Rightarrow \delta s = \sqrt{E}\delta u$

$(iii) \Rightarrow ds\sqrt{E}\delta u Cos\theta = Edu\delta u + F(0 + dv\delta u) + 0 = \delta u(Edu + Fdv)$

$\Rightarrow ds\sqrt{E}Cos\theta = (Edu + Fdv) \Rightarrow \boldsymbol{Cos\theta} = \frac{1}{\sqrt{E}}\left(E\frac{du}{ds} + F\frac{dv}{ds}\right)$

$(iv) \Rightarrow ds\sqrt{E}\delta u Sin\theta = H(0 - dv\delta u) \Rightarrow ds\sqrt{E}\delta u Sin\theta = -H(dv\delta u)$

$\Rightarrow ds\sqrt{E}Sin\theta = -Hdv \quad \Rightarrow \boldsymbol{Sin\theta} = -\frac{H}{\sqrt{E}}\left(\frac{dv}{ds}\right)$

QUESTION(PP): Prove that if χ is the angle between a direction on a surface and the curve $v = constant$ then find $Cot\chi$

ANSWER: We know that if $d\vec{r}$ is the displacement corresponding to the increments du, dv $and\ \delta\vec{r}$ is the displacement corresponding to the increments $\delta u, \delta v$ then

$d\vec{r} = \vec{r}_1 du + \vec{r}_2 dv \ \ldots\ldots\ldots\ldots(i)$ and $\delta\vec{r} = \vec{r}_1\delta u + \vec{r}_2\delta v \ \ldots\ldots\ldots\ldots(ii)$

Then angle χ between two directions is

$ds\delta s Cos\chi = Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v \ldots\ldots\ldots\ldots(iii)$

and $\quad ds\delta s Sin\chi = H(du\delta v - dv\delta u) \ldots\ldots\ldots\ldots(iv)$

now if the displacement is in the direction $v = constant$ then $\delta v = 0$

$(ii) \Rightarrow \delta\vec{r} = \vec{r}_1\delta u \quad \Rightarrow |\delta\vec{r}| = |\vec{r}_1|\delta u \Rightarrow \delta s = \sqrt{E}\delta u$

$(iii) \Rightarrow ds\sqrt{E}\delta u Cos\chi = Edu\delta u + F(0 + dv\delta u) + 0 = \delta u(Edu + Fdv)$

$\Rightarrow ds\sqrt{E}Cos\chi = (Edu + Fdv) \Rightarrow \boldsymbol{Cos\chi} = \frac{1}{\sqrt{E}}\left(E\frac{du}{ds} + F\frac{dv}{ds}\right)$

$(iv) \Rightarrow ds\sqrt{E}\delta u Sin\chi = H(0 - dv\delta u) \Rightarrow ds\sqrt{E}\delta u Sin\boldsymbol{\chi} = -H(dv\delta u)$

$\Rightarrow ds\sqrt{E}Sin\chi = -Hdv \quad \Rightarrow \boldsymbol{Sin\chi} = -\frac{H}{\sqrt{E}}\left(\frac{dv}{ds}\right)$

Then $Cot\chi = \frac{Cos\chi}{Sin\chi} = \frac{\frac{1}{\sqrt{E}}\left(E\frac{du}{ds} + F\frac{dv}{ds}\right)}{-\frac{H}{\sqrt{E}}\left(\frac{dv}{ds}\right)} \Rightarrow \boldsymbol{Cot\chi} = -\frac{1}{\mathrm{H}}\left(E\frac{du}{dv} + F\right)$

DERIVATIVES OF $\vec{n}$ (By means of fundamental magnitudes)

We may express the derivatives of $\vec{n}$ in terms of $\vec{r}_1\ and\ \vec{r}_2$

we know that $\vec{n}.\vec{r}_1 = 0 \Rightarrow \vec{n}_1.\vec{r}_1 + \vec{n}.\vec{r}_{11} = 0 \qquad \therefore diff.w.r.to\ 'u'$

$\Rightarrow \vec{n}_1.\vec{r}_1 = -\vec{n}.\vec{r}_{11} = -L$

Similarly we can find $\vec{n}_1.\vec{r}_2 = -\vec{n}.\vec{r}_{12} = -M$,, $\vec{n}_2.\vec{r}_1 = -\vec{n}.\vec{r}_{21} = -M$,, $\vec{n}_2.\vec{r}_2 = -\vec{n}.\vec{r}_{22} = -N$

now we know that $\vec{n}.\vec{n} = 0 \Rightarrow \vec{n}_1.\vec{n} + \vec{n}.\vec{n}_1 = 0 \qquad \therefore diff.w.r.to\ 'u'$

$\Rightarrow \vec{n}.\vec{n}_1 = 0 \Rightarrow \vec{n}_1 \perp \vec{n}$ and therefore $\vec{n}_1$ is parallel to the plane determined by $\vec{r}_1\ and\ \vec{r}_2$

thus we can express $\vec{n}_1 = a\vec{r}_1 + b\,\vec{r}_2$ where 'a' and 'b' are to be determined as follows

forming the scalar product of each side with $\vec{r}_1\ and\ \vec{r}_2$ we get

$\Rightarrow \vec{n}_1.\vec{r}_1 = (a\vec{r}_1 + b\,\vec{r}_2).\vec{r}_1 = a\vec{r}_1^{\,2} + b\,\vec{r}_2.\vec{r}_1 \Rightarrow \boldsymbol{-L = aE + b\,F} \ldots\ldots\ldots\ldots(i)$

$\Rightarrow \vec{n}_1.\vec{r}_2 = (a\vec{r}_1 + b\,\vec{r}_2).\vec{r}_2 = a\vec{r}_1.\vec{r}_2 + b\,\vec{r}_2^{\,2} \Rightarrow \boldsymbol{-M = aF + b\,G} \ldots\ldots\ldots\ldots(ii)$

Multiplying eq (i) with F and (ii) with E and the subtracting we get

$-FL + EM = bF^2 - bEG \Rightarrow -(FL - EM) = -b(EG - F^2) \Rightarrow b = \frac{(FL-EM)}{(EG-F^2)} \Rightarrow \boldsymbol{b} = \frac{\boldsymbol{(FL-EM)}}{\boldsymbol{H^2}}$

Multiplying eq (i) with G and (ii) with F and the subtracting we get

$-FM + LG = aF^2 - aEG \Rightarrow -(FM - LG) = -a(EG - F^2) \Rightarrow a = \frac{(FM-LG)}{(EG-F^2)} \Rightarrow \boldsymbol{a} = \frac{\boldsymbol{(FM-LG)}}{\boldsymbol{H^2}}$

Then for $\vec{n}_1 = a\vec{r}_1 + b\,\vec{r}_2 \Rightarrow \frac{(FM-LG)}{H^2}\vec{r}_1 + \frac{(FL-EM)}{H^2}\,\vec{r}_2$

$\Rightarrow H^2\vec{n}_1 = (FM - LG)\vec{r}_1 + (FL - EM)\vec{r}_2 \ldots\ldots\ldots\ldots(iii)$

Similarly for $\vec{n}_2 = a\vec{r}_1 + b\,\vec{r}_2 \Longrightarrow \frac{(FM-LG)}{H^2}\vec{r}_1 + \frac{(FL-EM)}{H^2}\,\vec{r}_2$
$\Longrightarrow H^2\vec{n}_2 = (FN-GM)\vec{r}_1 + (FM-EN)\vec{r}_2 \ldots\ldots\ldots\ldots..(iv)$

Now form (iii) and (iv) we have

$\Longrightarrow T^2\vec{r}_1 = (FM-EN)\vec{n}_1 + (EM-FL)\vec{n}_2$ also $\Longrightarrow T^2\vec{r}_2 = (GM-FN)\vec{n}_1 + (FM-GL)\vec{n}_2$
Where $T^2 = LN - M^2$

POSSIBLE QUESTION: Express the derivative of $\vec{n}$ w.r.to 'u' as a linear combination of derivatives of $\vec{r}$

SOME USEFUL RESULTS

- **Prove that $H[\vec{n},\vec{n}_1,\vec{n}_2] = T^2$**

Proof: we know that $H^2\vec{n}_1 = (FM-LG)\vec{r}_1 + (FL-EM)\vec{r}_2 \ldots\ldots\ldots\ldots..(i)$
$\Longrightarrow H^2\vec{n}_2 = (FN-GM)\vec{r}_1 + (FM-EN)\vec{r}_2 \ldots\ldots\ldots\ldots..(ii)$

Then taking cross product of above both

$\Longrightarrow H^4(\vec{n}_1\times\vec{n}_2) = [(FM-LG)(FM-EN)-(FL-EM)(FN-GM)](\vec{r}_1\times\vec{r}_2)$

$\Longrightarrow H^4(\vec{n}_1\times\vec{n}_2) = H^2[LN-M^2](\vec{r}_1\times\vec{r}_2)$ After multiplication and simplifying

$\Longrightarrow H^4\vec{n}.(\vec{n}_1\times\vec{n}_2) = H^2T^2\vec{n}.(\vec{n}H)$ $\therefore \vec{n} = \frac{\vec{r}_1\times\vec{r}_2}{H}$

$\Longrightarrow H[\vec{n},\vec{n}_1,\vec{n}_2] = T^2$

- **Prove that $H[\vec{n},\vec{n}_1,\vec{r}_1] = EM - FL$**

Proof: we know that $H^2\vec{n}_1 = (FM-LG)\vec{r}_1 + (FL-EM)\vec{r}_2$

$\Longrightarrow H^2(\vec{n}_1\times\vec{r}_1) = 0 + (FL-EM)\vec{r}_2\times\vec{r}_1 = -(FL-EM)\vec{r}_1\times\vec{r}_2 = -(FL-EM)(\vec{n}H)$
$\Longrightarrow H^2\vec{n}.(\vec{n}_1\times\vec{r}_1) = \vec{n}.(EM-FL)(\vec{n}H)$ $\Longrightarrow \boldsymbol{H[\vec{n},\vec{n}_1,\vec{r}_1] = EM - FL}$

- **Prove that $H[\vec{n},\vec{n}_1,\vec{r}_2] = FM - GL$**

Proof: we know that $H^2\vec{n}_1 = (FM-LG)\vec{r}_1 + (FL-EM)\vec{r}_2$

$\Longrightarrow H^2(\vec{n}_1\times\vec{r}_2) = (FM-LG)\vec{r}_1\times\vec{r}_2 + 0 = (FM-LG)\vec{r}_1\times\vec{r}_2 = (FM-LG)(\vec{n}H)$
$\Longrightarrow H(\vec{n}_1\times\vec{r}_2) = (FM-LG)(\vec{n}) \Longrightarrow H\vec{n}.(\vec{n}_1\times\vec{r}_2) = \vec{n}.(FM-LG)(\vec{n})$
$\Longrightarrow \boldsymbol{H[\vec{n},\vec{n}_1,\vec{r}_2] = FM - GL}$

- **Prove that $H[\vec{n},\vec{n}_2,\vec{r}_1] = EN - FM$**

Proof: we know that $\Longrightarrow H^2\vec{n}_2 = (FN-GM)\vec{r}_1 + (FM-EN)\vec{r}_2$

$\Longrightarrow H^2(\vec{n}_2\times\vec{r}_1) = 0 + (FM-EN)\vec{r}_2\times\vec{r}_1 = -(FM-EN)\vec{r}_1\times\vec{r}_2 = (EN-FM)(\vec{n}H)$
$\Longrightarrow H^2\vec{n}.(\vec{n}_2\times\vec{r}_1) = \vec{n}.(EN-FM)(\vec{n}H) \Longrightarrow \boldsymbol{H[\vec{n},\vec{n}_2,\vec{r}_1] = EN - FM}$

- **Prove that $H[\vec{n},\vec{n}_2,\vec{r}_2] = FN - GM$**

Proof: we know that $\Longrightarrow H^2\vec{n}_2 = (FN-GM)\vec{r}_1 + (FM-EN)\vec{r}_2$

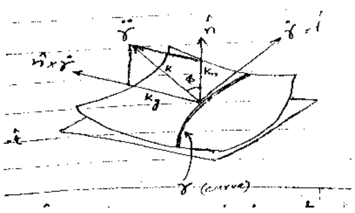
$\Longrightarrow H^2(\vec{n}_2\times\vec{r}_2) = (FN-GM)\vec{r}_1\times\vec{r}_2 + 0 = (FN-GM)(\vec{n}H)$
$\Longrightarrow H^2\vec{n}.(\vec{n}_2\times\vec{r}_2) = \vec{n}.(FN-GM)(\vec{n}H)$ $\Longrightarrow \boldsymbol{H[\vec{n},\vec{n}_2,\vec{r}_1] = FN - GM}$

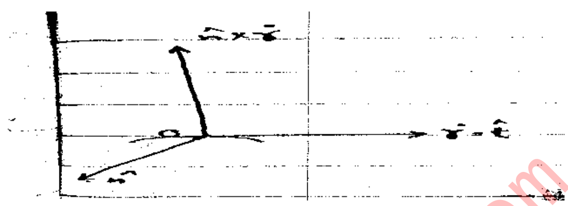
PRINCIPAL CURVATURES, PRINCIPAL SECTIONS, PRINCIPAL DIRECTIONS AND LINES OF CURVATURE

CURVATURE OF THE NORMAL SECTION: A normal section of a surface at a given point is the section of the surface by a plane containing the normal at the point. Such a section is plane curve, whose PRINCIPAL normal is parallel to the normal to the surface. We adopt the convection that the PRINCIPAL normal to the normal section is in the same direction as the direction of the normal to the surface.

NORMAL CURVATURE AND RADIUS OF NORMAL CURVATURE: Curvature of the normal section is called the normal curvature and its reciprocal is called the radius of the normal curvature. And it is denoted as $\Rightarrow K_n = \frac{L(du)^2+2Mu'v'+N(dv)^2}{Edu^2+2Fdudv+Gdv^2}$

OR if $\vec{r}$ is a unit speed curve in surface patch σ i.e. $\vec{r}(t) = [u(t), v(t)]$ and $\|\vec{r}'(t)\| = 1$





Then $\vec{r}' = \vec{t}$ is a unit tangent vector. Hence $\vec{r}' \perp \vec{n}$ $(unit\ normal\ vector\ of\ S)$ then by definition of cross product $\vec{n} \times \vec{r}'$ is also a unit vector perpendicular to $\vec{r}'$ and $\vec{n}$ unit vectors. Since $\vec{r}$ is a unit speed curve then $\vec{r}'' \perp \vec{r}'$ and hence the linear combination of $\vec{n}$ and $\vec{n} \times \vec{r}'$ is $\vec{r}'' = K_n\vec{n} + K_g(\vec{n} \times \vec{r}') \dots \dots \dots (i)$

Taking dot product of (i) with $\vec{n} \Rightarrow \vec{r}''.\vec{n} = K_n\vec{n}.\vec{n} + K_g(\vec{n} \times \vec{r}').\vec{n} = K_n(1) + K_g(0)$

$\Rightarrow \boldsymbol{K_n} = \boldsymbol{\vec{r}''.\vec{n}}$ is called **Normal Curvature**.

Taking dot product of (i) with $\vec{n} \times \vec{r}' \Rightarrow \vec{r}''.(\vec{n} \times \vec{r}') = K_n\vec{n}.(\vec{n} \times \vec{r}') + K_g(\vec{n} \times \vec{r}').(\vec{n} \times \vec{r}')$
$\Rightarrow \vec{r}''.(\vec{n} \times \vec{r}') = K_n(0) + K_g(1) \Rightarrow \boldsymbol{K_g} = \boldsymbol{\vec{r}''.(\vec{n} \times \vec{r}')}$ is called **Geodesic Curvature**.

REMARK:

since $K = \|\vec{r}''\| \Rightarrow K^2 = \|\vec{r}''\|^2 \Rightarrow K^2 = \vec{r}''.\vec{r}'' = [K_n\vec{n} + K_g(\vec{n} \times \vec{r}')].[K_n\vec{n} + K_g(\vec{n} \times \vec{r}')]$

$\Rightarrow K^2 = {K_n}^2\vec{n}.\vec{n} + K_nK_g(\vec{n} \times \vec{r}').\vec{n} + K_gK_n(\vec{n} \times \vec{r}').\vec{n} + {K_g}^2(\vec{n} \times \vec{r}').(\vec{n} \times \vec{r}')$

$\Rightarrow K^2 = {K_n}^2 + {K_g}^2 \Rightarrow K = \sqrt{{K_n}^2 + {K_g}^2}$ is actual curvature.

SYNCLASTIC AND ANTICLASTIC: Those portions of the surface on which the two principal curvatures have the same sign are said to be **Synclastic** e.g. the surface of an ellipsoid is Synclastic at all points, where those portions of the surface on which the two principal curvatures have opposite sign are said to be **Anticlastic** e.g. the surface of hyperbolic paraboloid is anticlastic at all points.

GUASS MAP:

for any regular parameterized surface $\delta: u \to R^3$, the Gauss map $g: \delta(u, v) \to S^2 = \xi v \epsilon R^3$ is defined as $g = \frac{\frac{\partial}{\partial u} \times \frac{\partial}{\partial v}}{\left\|\frac{\partial}{\partial u} \times \frac{\partial}{\partial v}\right\|}$

The Gauss map sends $\delta(u, v)$ of surface S to the point $N(u, v)$ of S^2

MEUNIER'S THEOREM: The curvature K_n of a normal section of a surface in any direction and the curvature $K$of any other section through the same tangent line are related as $\boldsymbol{K_n = KCos\theta \Rightarrow Cos\theta = \frac{K_n}{K}}$ where θ is the angle between the planes (Principal Normals) of the two sections.

PROOF (1ST METHOD): The angle between the two planes is the same as the angle between the PRINCIPAL normals of their sections. The unit PRINCIPAL normal of the normal section is clearly $\vec{n}$ and the unit PRINCIPAL normal of the other section, by Serret Frenet Formulae, is $\vec{t}' = \vec{r}'' = K\vec{n} \Rightarrow \vec{n} = \frac{\vec{r}''}{K}$ and hence θ is given by $Cos\theta = \vec{n}.\frac{\vec{r}''}{K}$ ……….. (i)

Now we know that $\vec{r}' = \frac{d\vec{r}}{ds} = \frac{\partial\vec{r}}{\partial u}.\frac{du}{ds} + \frac{\partial\vec{r}}{\partial v}.\frac{dv}{ds} \Rightarrow \vec{r}'' = \frac{d}{ds}\left(\frac{d\vec{r}}{ds}\right) = \frac{d}{ds}\left(\frac{\partial\vec{r}}{\partial u}.\frac{du}{ds} + \frac{\partial\vec{r}}{\partial v}.\frac{dv}{ds}\right)$

$\Rightarrow \vec{r}'' = \frac{d}{ds}\left(\frac{\partial\vec{r}}{\partial u}\right)\frac{du}{ds} + \frac{\partial\vec{r}}{\partial u}\frac{d^2u}{ds^2} + \frac{d}{ds}\left(\frac{\partial\vec{r}}{\partial v}\right)\frac{dv}{ds} + \frac{\partial\vec{r}}{\partial v}\frac{d^2v}{ds^2}$

$$\Rightarrow \vec{r}'' = \left(\frac{\partial^2\vec{r}}{\partial u^2}\frac{du}{ds} + \frac{\partial^2\vec{r}}{\partial u\partial v}\frac{dv}{ds}\right)\frac{du}{ds} + \frac{\partial\vec{r}}{\partial u}\frac{d^2u}{ds^2} + \left(\frac{\partial^2\vec{r}}{\partial v^2}\frac{dv}{ds} + \frac{\partial^2\vec{r}}{\partial u\partial v}\frac{du}{ds}\right)\frac{dv}{ds} + \frac{\partial\vec{r}}{\partial v}\frac{d^2v}{ds^2}$$

$$\Rightarrow \vec{r}'' = \vec{r}_{11}u'^2 + \vec{r}_{12}u'v' + \vec{n}.\vec{r}_1u'' + \vec{r}_{12}u'v' + \vec{r}_{22}v'^2 + \vec{n}.\vec{r}_2v''$$

$$\Rightarrow \vec{n}.\vec{r}'' = \vec{n}.\vec{r}_{11}u'^2 + \vec{n}.\vec{r}_{12}u'v' + \vec{n}.\vec{n}.\vec{r}_1u'' + \vec{n}.\vec{r}_{12}u'v' + \vec{n}.\vec{r}_{22}v'^2 + \vec{n}.\vec{n}.\vec{r}_2v''$$

$$\Rightarrow \vec{n}.\vec{r}'' = \vec{n}.\vec{r}_{11}u'^2 + 2\vec{n}.\vec{r}_{12}u'v' + 0 + \vec{n}.\vec{r}_{22}v'^2 + 0$$

$$\Rightarrow \vec{n}.\vec{r}'' = Lu'^2 + 2Mu'v' + Nv'^2 \text{ ……….. } (ii)$$

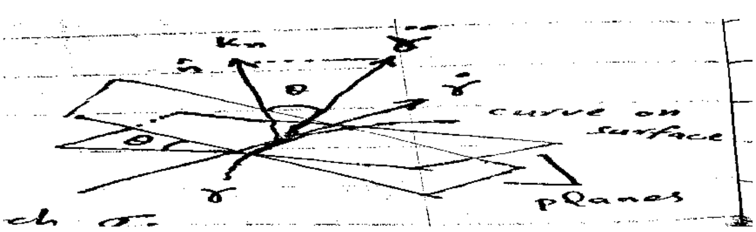
We note that u' and v' have the same value for both the curves at P, hence $\vec{n}.\vec{r}''$ have the same value for both the curves at this point.

Next we know consider the normal section using Serret Frenet Formulae
$\vec{t}' = \vec{r}'' = K_n\vec{n} \Rightarrow \vec{r}''.\vec{n} = K_n\vec{n}.\vec{n} \Rightarrow K_n = \vec{r}''.\vec{n}$ ……….. (iii)

$(i) \Rightarrow Cos\theta = \vec{n}.\frac{\vec{r}''}{K} \Rightarrow \boldsymbol{Cos\theta = \frac{K_n}{K} \Rightarrow K_n = KCos\theta}$ This is required.

PROOF (2nd METHOD): Suppose $\vec{r}$ is the unit speed curve on the surface patch σ then $\vec{r}''$ is perpendicular to $\vec{r}' = \hat{t}$ but $\vec{r}''$ makes angle θ with the principal unit normal $\vec{n}$ then

$\vec{r}''.\hat{n} = \|\vec{r}''\|\|\hat{n}\|Cos\theta$ ……….. (i)



Now since $K_n = \vec{r}''.\hat{n}$ and $K = \|\vec{r}''\|$ $also$ $\|\hat{n}\| = 1$ $(i) \Rightarrow \boldsymbol{K_n = KCos\theta \Rightarrow Cos\theta = \frac{K_n}{K}}$

CORROLLORY: Since

$K_n = \vec{r}''.\hat{n} \Rightarrow K_n = Lu'^2 + 2Mu'v' + Nv'^2 \Rightarrow K_n = \frac{L(du)^2+2Mu'v'+N(dv)^2}{(ds)^2}$

Then using first fundamental form we have $\Rightarrow \boldsymbol{K_n = \frac{L(du)^2+2Mdudv+N(dv)^2}{Edu^2+2Fdudv+Gdv^2}}$

QUESTION: If L,M,N vanish at all points then show that the surface is plane.

ANSWER: We know that $K_n = Lu'^2 + 2Mu'v' + Nv'^2 \implies K_n = 0 \quad \therefore L = M = N = 0$
$\implies$ All normal sections are straight line which is possible only if the surface is plane.

QUESTION: A real surface for which the equation $\frac{E}{L} = \frac{F}{M} = \frac{G}{N}$ holds is either spherical or plane.

Answer: Let $\frac{E}{L} = \frac{F}{M} = \frac{G}{N} = \propto$ where $\propto$ is any constant. Then $\implies E = L\propto, \quad F = M\propto, G = N\propto$

Now we know that $K_n = \frac{L(du)^2+2Mu'v'+N(dv)^2}{Edu^2+2Fdudv+Gdv^2} \implies K_n = \frac{L(du)^2+2Mu'v'+N(dv)^2}{L\propto du^2+2M\propto dudv+N\propto dv^2}$

$\implies K_n = \frac{L(du)^2+2Mu'v'+N(dv)^2}{\propto(L(du)^2+2Mu'v'+N(dv)^2)} \implies K_n = \frac{1}{\propto}$

Now if $\frac{1}{\propto} = 0$ then $K_n = 0$ at all points and the surface is plane. And if $\frac{1}{\propto} \neq 0\ (constant)$ then normal curvature at any point of the surface is constant. Which implies that surface is sphere.

QUESTION: Show that curvature K at any point 'P' of curve of intersection of two surfaces is given by $\boldsymbol{K^2Sin^2\theta = K_1^2 + K_2^2 - 2K_1K_2Cos\theta}$ where K_1, K_2 are two normal curvature of surfaces in the direction of curve at 'P' and θ is the angle between the normals at that point.

ANSWER: Let θ_1 be the angle between the principal normals of the curve of intersection and the normal section of the first surface at 'P'. Similarly θ_2 is the angle between the PRINCIPAL normals of the curve of intersection and the normal section of the second surface at 'P' then $\theta = \theta_1 + \theta_2$ applying Meunier's Theorem i.e. $\boldsymbol{K_n = KCos\theta}$
then $\quad K_1 = KCos\theta_1 \ldots\ldots(i) \quad and \quad K_2 = KCos\theta_2 \ldots\ldots(ii)$

$(i) \implies Cos\theta_1 = \frac{K_1}{K} \implies Sin\theta_1 = \sqrt{1 - \frac{K_1{}^2}{K^2}} = \sqrt{\frac{K^2-K_1{}^2}{K^2}}$

$(ii) \implies K_2 = KCos\theta_2 \implies K_2 = KCos(\theta - \theta_1) = K\{Cos\theta Cos\theta_1 + Sin\theta Sin\theta_1\}$

$\implies K_2 = K\left\{Cos\theta\left(\frac{K_1}{K}\right) + Sin\theta\left(\sqrt{\frac{K^2-K_1{}^2}{K^2}}\right)\right\} = K_1Cos\theta + \sqrt{K^2 - K_1{}^2}Sin\theta$

$\implies \sqrt{K^2 - K_1{}^2}Sin\theta = K_2 - K_1Cos\theta \implies \left(K^2 - K_1{}^2\right)Sin^2\theta = (K_2 - K_1Cos\theta)^2$

$\implies K^2Sin^2\theta - K_1{}^2Sin^2\theta = K_2{}^2 + K_1{}^2Cos^2\theta - 2K_1K_2Cos\theta$

$\implies K^2Sin^2\theta = K_2{}^2 + K_1{}^2Cos^2\theta + K_1{}^2Sin^2\theta - 2K_1K_2Cos\theta$

$\implies \boldsymbol{K^2Sin^2\theta = K_1^2 + K_2^2 - 2K_1K_2Cos\theta}$ which is required result.

ASYMPTOTIC CURVE: A curve $\vec{r}$ on a surface 'S' is called asymptotic if its normal curvature is everywhere zero i.e. $K_n = 0$

QUESTION: Prove that the asymptotic curve on the surface $\sigma(u,v) = (uCosv\,, uSinv, lnu)$ is given by $lnu = \pm(v + c)$ where 'c' is an arbitrary constant.

ANSWER: : $\sigma(u,v) = (uCosv\,, uSinv, lnu)$

$\implies \sigma_u = \left[Cosv, Sinv, \frac{1}{u}\right] \implies \sigma_{uu} = \left[0,0,-\frac{1}{u^2}\right]$

$\sigma_v = [-uSinv, uCosv, 0] \implies \sigma_{vv} = [-uCosv, -uSinv, 0]$ also $\sigma_{uv} = [-uSinv, uCosv, 0]$

$\implies \sigma_u \times \sigma_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ Cosv & Sinv & \frac{1}{u} \\ -uSinv & uCosv & 0 \end{vmatrix} = [-Cosv, -Sinv, u]$ also $\|\sigma_u \times \sigma_v\| = \sqrt{1+u^2}$

Now $\vec{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{[-Cosv,-Sinv,u]}{\sqrt{1+u^2}}$

then $L = \vec{n}.\sigma_{uu} = -\frac{1}{u\left(\sqrt{1+u^2}\right)}$, $\quad M = \vec{n}.\sigma_{uv} = 0, \quad N = \vec{n}.\sigma_{vv} = \frac{u}{\sqrt{1+u^2}}$

as we know that $K_n = Lu'^2 + 2Mu'v' + Nv'^2 = -\frac{1}{u\left(\sqrt{1+u^2}\right)}u'^2 + 0 + \frac{u}{\sqrt{1+u^2}}v'^2$

$-\frac{1}{u\left(\sqrt{1+u^2}\right)}u'^2 + \frac{u}{\sqrt{1+u^2}}v'^2 = 0$ $\qquad \therefore K = 0$ for asymptotic curve.

$\Rightarrow -\frac{1}{\sqrt{1+u^2}}\left[\frac{1}{u}u'^2 - uv'^2\right] = 0 \Rightarrow \frac{1}{u}u'^2 - uv'^2 = 0 \Rightarrow \frac{u'^2}{u^2} = v'^2 \Rightarrow \frac{u'}{u} = \pm v' \Rightarrow \int \frac{u'}{u}dt = \pm \int \frac{dv}{dt}dt$
$\Rightarrow \boldsymbol{lnu = \pm(v+c)}$

PRINCIPAL SECTION: the normal section of the surface having the maximum or minimum curvature is called principal section.

PRINCIPAL CURVATURE: the maximum or minimum values of normal curvature of principal sections are called principal curvatures. Its equation may be written as follows
$K_n{}^2H^2 - K_n(EN - 2FM + GL) + T^2 = 0$

OR let $\sigma(u,v)$ be a surface patch with 1st and 2nd fundamental forms respectively. Define symmetric 2×2 matrices $\mathcal{F}_I$ *and* $\mathcal{F}_{II}$ by

$\mathcal{F}_I = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$ *and* $\mathcal{F}_{II} = \begin{bmatrix} L & M \\ M & N \end{bmatrix}$ Then principal curvature of surface patches $\sigma(u,v)$ is the roots of the equation $det[\mathcal{F}_{II} - K\mathcal{F}_I] = 0$

HELICOIDS: A surface generated by a curve which is simultaneously rotated about a fixed axis and translated in the direction of the axis with a velocity proportional to the angular velocity of rotation. The plane sections through the axis are called Meridians.

QUESTION (PP): Calculate principal curvature of the Helicoid $\sigma(u,v) = (vCosu, vSinu, \lambda u)$

ANSWER: : $\sigma(u,v) = (vCosu, vSinu, \lambda u)$

$\Rightarrow \sigma_u = [-vSinu, vCosu, \lambda\,] \Rightarrow \sigma_{uu} = (-vCosu, -vSinu, 0)$
$\sigma_v = (Cosu, Sinu, 0) \Rightarrow \sigma_{vv} = [0,0,0]$ also $\sigma_{uv} = [-Sinu, Cosu, 0]$

Then $E = \sigma_u.\sigma_u = \|\sigma_u\|^2 = v^2 + \lambda^2, , G = \sigma_v.\sigma_v = \|\sigma_v\|^2 = 1$ *and* $F = \sigma_u.\sigma_v = 0$

$\Rightarrow \sigma_u \times \sigma_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -vSinu & vCosu & \lambda \\ Cosu & Sinu & 0 \end{vmatrix} = [-\lambda Sinu, \lambda Cosu, -v\,]$ also $\|\sigma_u \times \sigma_v\| = \sqrt{\lambda^2 + u^2}$

$\vec{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{[-\lambda Sinu, \lambda Cosu, -v\,]}{\sqrt{\lambda^2+u^2}}$ then $L = \vec{n}.\sigma_{uu} = 0$, $\quad M = \vec{n}.\sigma_{uv} = \frac{\lambda}{\sqrt{\lambda^2+u^2}}, \quad N = \vec{n}.\sigma_{vv} = 0$

Now for principal curvature $det[\mathcal{F}_{II} - K\mathcal{F}_I] = 0$ where $\mathcal{F}_I = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$ *and* $\mathcal{F}_{II} = \begin{bmatrix} L & M \\ M & N \end{bmatrix}$

Then $\mathcal{F}_{II} - K\mathcal{F}_I = \begin{bmatrix} L & M \\ M & N \end{bmatrix} - K\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} 0 & \frac{\lambda}{\sqrt{\lambda^2+u^2}} \\ \frac{\lambda}{\sqrt{\lambda^2+u^2}} & 0 \end{bmatrix} - K\begin{bmatrix} v^2+\lambda^2 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \mathcal{F}_{II} - K\mathcal{F}_I = \begin{bmatrix} 0 & \frac{\lambda}{\sqrt{\lambda^2+u^2}} \\ \frac{\lambda}{\sqrt{\lambda^2+u^2}} & 0 \end{bmatrix} - \begin{bmatrix} K(v^2+\lambda^2) & 0 \\ 0 & K \end{bmatrix} = \begin{bmatrix} K - (v^2+\lambda^2) & \frac{\lambda}{\sqrt{\lambda^2+u^2}} \\ \frac{\lambda}{\sqrt{\lambda^2+u^2}} & -K \end{bmatrix}$

$\Rightarrow det[\mathcal{F}_{II} - K\mathcal{F}_I] = \begin{vmatrix} K - (v^2+\lambda^2) & \frac{\lambda}{\sqrt{\lambda^2+u^2}} \\ \frac{\lambda}{\sqrt{\lambda^2+u^2}} & -K \end{vmatrix} = \frac{K^2(v^2+\lambda^2)^2 - \lambda^2}{v^2+\lambda^2}$

Then $det[\mathcal{F}_{II} - K\mathcal{F}_I] = 0 \Rightarrow \frac{K^2(v^2+\lambda^2)^2 - \lambda^2}{v^2+\lambda^2} = 0 \Rightarrow K^2(v^2+\lambda^2)^2 - \lambda^2 = 0 \Rightarrow \boldsymbol{K} = \pm\frac{\lambda}{v^2+\lambda^2}$

PRACTICE: Calculate principal curvature of the followings;

i. A unit sphere $\sigma(\theta,\varphi) = (Cos\theta Cos\varphi, Cos\theta Sin\varphi, Sin\theta)$
ii. A cylinder of radius '1' and axis is z – axis $\sigma(u,v) = (Cosv, Sinv, u)$
iii. Centroid $\sigma(u,v) = (CoshuCosv, CoshuSinv, u)$

PRINCIPAL DIRECTIONS: the tangent lines along the principal sections at a point are called principle directions.

LINE OF CURVATURE: A curve drawn on a surface is called a line of curvature iff the tangent line about any point of this curve gives one of two principal directions at that point. Its equation may be written as follows $\begin{vmatrix} (dv)^2 & -dudv & (du)^2 \\ L & M & N \\ E & F & G \end{vmatrix} = 0$

FIRST CURVATURE: The first curvature of the surface at any point may be defined as the sum of principal curvatures. We will denote it by 'J' and is given by
$J = K_a + K_b = \frac{1}{H^2}(EN - 2FM + GL)$

SECOND CURVATURE: The second curvature or specific curvature or Gauss curvature of the surface at any point may be defined as the product of principal curvatures. We will denote it by 'K' and is given by $K = K_a . K_b = \frac{T^2}{H^2}$

AMPLITUDE OF NORMAL CURVATURE: It may be defined as $A = \frac{1}{2}(K_b - K_a)$

MEAN NORMAL CURVATURE: It may be defined as $B = \frac{1}{2}(K_b + K_a)$

SURFACE OF CENTRES OR CENTRO - SURFACE: the locus of centre of curvature is a surface called Centro surface or surface of centre.

MINIMAL SURFACE: the surface is called minimal surface if first curvature is zero at all points on the surface.

NULL LINES: the null lines (or minimal curves) on a surface are defined as the curves of zero length.

QUESTION: Find the line of curvature and principal curvature for the surface (right helicoid) $\vec{r} = (uCos\varphi, \quad uSin\varphi, \quad c\varphi)$ and prove that it is a minimal surface.

SOLUTION: $\vec{r} = (uCos\varphi, \quad uSin\varphi, \quad c\varphi)$

$\Rightarrow \vec{r}_1 = (Cos\varphi, \quad Sin\varphi, \quad 0) \Rightarrow \vec{r}_2 = [-uSin\varphi, \quad uCos\varphi, \quad c]$

Now the fundamental coefficients of first order are $\vec{r}_1^{\,2} = E, , \vec{r}_2^{\,2} = G \text{ and } F = \vec{r}_1\vec{r}_2$

$\Rightarrow E = \vec{r}_1^{\,2} = Cos^2\varphi + Sin^2\varphi + 0 = 1 \Rightarrow \boldsymbol{E = 1}$

$\Rightarrow G = \vec{r}_2^{\,2} = u^2Cos^2\varphi + u^2Sin^2\varphi + c^2 \Rightarrow \boldsymbol{G = u^2 + c^2}$

$$\Rightarrow F = \vec{r}_1.\vec{r}_2 = [Cos\varphi, \quad Sin\varphi, \quad 0].[-uSin\varphi, \quad uCos\varphi, \quad c]$$
$$= -uSin\varphi Cos\varphi + uSin\varphi Cos\varphi + 0 \Rightarrow \boldsymbol{F = 0}$$

Now the fundamental coefficients of second order are $\vec{r}_{11}, \ \vec{r}_{22}, \ \vec{r}_{12}$

For $\vec{r} = (uCos\varphi, \quad uSin\varphi, \quad c\varphi) \Rightarrow \vec{r}_{11} = (0, \quad 0, \quad 0)$
$\Rightarrow \vec{r}_{22} = (-uCos\varphi, \quad -uSin\varphi, \quad 0) \Rightarrow \vec{r}_{12} = (-Sin\varphi, \quad Cos\varphi, \quad 0)$

So the second order coefficients are $L = \vec{n}.\vec{r}_{11} = 0$ $M = \vec{n}.\vec{r}_{12} = -\frac{c}{H}$ $N = \vec{n}.\vec{r}_{22} = 0$ also
$H = |\vec{r}_1 \times \vec{r}_2| = |(cSin\varphi, \quad cCos\varphi, \quad u)| = \sqrt{c^2 + u^2}$

Also unit normal to the surface is $\vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{H} \Longrightarrow \vec{n} = \frac{1}{H}(cSin\varphi, \quad cCos\varphi, \quad u)$
now equation of normal curvature is given as

$$\begin{vmatrix} (d\varphi)^2 & -dud\varphi & (du)^2 \\ L & M & N \\ E & F & G \end{vmatrix} = 0 \Longrightarrow \begin{vmatrix} (d\varphi)^2 & -dud\varphi & (du)^2 \\ 0 & -\frac{c}{H} & 0 \\ 1 & 0 & u^2 + c^2 \end{vmatrix} = 0 \Longrightarrow (d\varphi)^2 = \frac{(du)^2}{u^2+c^2}$$

$$\Longrightarrow d\varphi = \frac{du}{\sqrt{u^2+c^2}} \Longrightarrow \varphi = \pm Sinh\frac{u}{c} + A$$

Now equation of principal curvature is ${K_n}^2 H^2 - K_n(EN - 2FM + GL) + T^2 = 0$

$$\Longrightarrow {K_n}^2(u^2 + c^2) - K_n(0) - \frac{c^2}{u^2+c^2} = 0 \qquad \therefore T^2 = LN - M^2 = -\frac{c}{M}$$

$$\Longrightarrow K_n = \pm\frac{c}{u^2+c^2}$$

Also $J = K_a + K_b = \frac{1}{H^2}(EN - 2FM + GL) = 0$ thus given surface is minimal.

QUESTION: Find the principal curvature and principal direction on the surface of
$x = a(u + v), \ y = b(u - v), \ z = uv$

SOLUTION: $\vec{r} = (a(u + v), \ b(u - v), uv)$

$$\Longrightarrow E = {\vec{r}_1}^2 = a^2 + b^2 + v^2 \ \Longrightarrow G = {\vec{r}_2}^2 = a^2 + b^2 + u^2 \Longrightarrow F = \vec{r}_1.\vec{r}_2 = a^2 - b^2 + uv$$

So the second order coefficients are $L = \vec{n}.\vec{r}_{11} = 0$ $M = \vec{n}.\vec{r}_{12} = -\frac{2ab}{H}$ $N = \vec{n}.\vec{r}_{22} = 0$ also
$H = |\vec{r}_1 \times \vec{r}_2| = \sqrt{4a^2b^2 + a^2(u - v)^2 + b^2(u + v)^2}$ also $T^2 = LN - M^2 = -\frac{4a^2b^2}{H^2}$
now for principal directions

$$\begin{vmatrix} (dv)^2 & -dudv & (du)^2 \\ L & M & N \\ E & F & G \end{vmatrix} = 0 \Longrightarrow \begin{vmatrix} (dv)^2 & -dudv & (du)^2 \\ 0 & -\frac{2ab}{H} & 0 \\ a^2 + b^2 + v^2 & a^2 - b^2 + uv & a^2 + b^2 + u^2 \end{vmatrix} = 0$$

$$\Longrightarrow (dv)^2 = \frac{a^2+b^2+v^2}{a^2+b^2+u^2}(du)^2 \Longrightarrow dv = \sqrt{\frac{a^2+b^2+v^2}{a^2+b^2+u^2}}du$$

Now equation of principal curvature is ${K_n}^2 H^2 - K_n(EN - 2FM + GL) + T^2 = 0$

$$\Longrightarrow {K_n}^2 H^2 + 4ab\left(\frac{a^2-b^2+uv}{H}\right)K_n + T^2 = 0$$ Also $J = -4ab\left(\frac{a^2-b^2+uv}{H^3}\right)$ and $K_n = \frac{T^2}{H^2}$

UMBILIC POINT: when the normal curvature K_n is independent of the ratio $du: dv$ and has the same value for all direction through the point such a point is called umbilic on the surface.

THEOREM: Prove that the necessary and sufficient condition for the lines of curvature to be the parametric curves is that $F = 0, M = 0$

PROOF: Suppose that the lines of curvature are parametric curves i.e.
$u = cosntant; v = constant$

Now equation of normal curvature is given as $\begin{vmatrix} (dv)^2 & -dud\varphi & (du)^2 \\ L & M & N \\ E & F & G \end{vmatrix} = 0 \ \ldots\ldots\ldots (A)$

since the lines of curvature are parametric curves then the lines of curvature are mutually orthogonal. And being orthogonal curves $F = 0$

Now let $du = 0 \ i.e. u = cosntant \ and \ v = arbitrary$ also $F = 0$ then using all these values

$$(A) \Longrightarrow \begin{vmatrix} (dv)^2 & -dud\varphi & (0)^2 \\ L & M & N \\ E & 0 & G \end{vmatrix} = 0 \Longrightarrow (dv)^2(-MG) = 0 \ldots\ldots\ldots (B)$$

Also let $dv = 0$ $i.e. v = cosntant \ \ and \ u = arbitrary$ also $F = 0$ then using all these values

$$(A) \Rightarrow \begin{vmatrix} (0)^2 & -dud\varphi & (du)^2 \\ L & M & N \\ E & 0 & G \end{vmatrix} = 0 \Rightarrow (du)^2(EM) = 0 \dots\dots\dots (C)$$

as $H = \sqrt{EG - F^2} = \sqrt{EG - 0} \Rightarrow EG > 0 \Rightarrow EG \neq 0 \Rightarrow E \neq 0, G \neq 0$

so from (B) and (C) we conclude $M = 0$

CONVERSLY: Suppose $F = 0, M = 0$, to prove the lines of curvature to be the parametric curves we show that either $u = cosntant \ or \ v = constant$

Since equation of line of curvature is given as $\begin{vmatrix} (dv)^2 & -dudv & (du)^2 \\ L & M & N \\ E & F & G \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} (dv)^2 & -dudv & (du)^2 \\ L & 0 & N \\ E & 0 & G \end{vmatrix} = 0 \Rightarrow dudv(EN - GL) = 0 \Rightarrow dudv = 0 \ \ or \ (EN - GL) = 0$$

let $(EN - GL) = 0 \Rightarrow \frac{E}{N} = \frac{G}{L}$ which is condition of umbilie point and there is no umbilie point in this case so $dudv = 0 \ \Rightarrow either \ du = 0 \ or \ dv = 0$

$\Rightarrow$ **either $\boldsymbol{u = cosntant \ or \ v = constant}$ $\Rightarrow$ the lines of curvature to be the parametric curves**

THEOREM: Prove that the two principal directions at any point of a surface are orthogonal.

PROOF: let $u = \varphi_1(v) \dots\dots\dots..(i); u = \varphi_2(v) \dots\dots\dots..(ii)$ be two lines of curvature passing through point 'P' of the surface $S: \vec{r} = \vec{r}(u, v)$

$\Rightarrow \frac{d\varphi_1}{dv}, \frac{d\varphi_2}{dv}$ are the roots of the equation $\begin{vmatrix} (dv)^2 & -dudv & (du)^2 \\ L & M & N \\ E & F & G \end{vmatrix} = 0$

then the sum of the roots will be $\frac{d\varphi_1}{dv} + \frac{d\varphi_2}{dv} = -\frac{LG-NE}{LF-ME} \dots\dots\dots..(iii)$

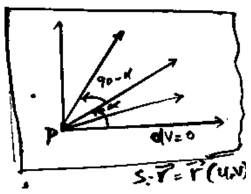
and the product of the roots will be $\frac{d\varphi_1}{dv} . \frac{d\varphi_2}{dv} = \frac{MG-NF}{LF-ME} \dots\dots\dots..(iv)$

Now the vectors $\vec{a} = \frac{\partial \vec{r}}{\partial u}\frac{d\varphi_1}{du} + \frac{\partial \vec{r}}{\partial v}.1 = \vec{r}_1\frac{d\varphi_1}{dv} + \vec{r}_2$ and $\vec{b} = \frac{\partial \vec{r}}{\partial u}\frac{d\varphi_2}{du} + \frac{\partial \vec{r}}{\partial v}.1 = \vec{r}_1\frac{d\varphi_2}{dv} + \vec{r}_2$ represent lines parallel to the tangent to the lines of curvature given by $(i) \ and \ (ii)$ at the point 'P' then consider $\vec{a}.\vec{b} = E\left(\frac{d\varphi_1}{dv}\frac{d\varphi_2}{dv}\right) + F\left(\frac{d\varphi_1}{dv} + \frac{d\varphi_2}{dv}\right) + G = E\left(\frac{MG-NF}{LF-ME}\right) + F\left(-\frac{LG-NE}{LF-ME}\right) + G$

$\Rightarrow \vec{a}.\vec{b} = 0 \Rightarrow$ principal directions at any point of a surface are orthogonal.

EULER'S THEOREM: If K_n is the normal curvature in a direction 'l'(i.e. $\frac{dv}{du}$) at any point 'P' of the surface $S: \vec{r} = \vec{r}(u,v)$and K_a and K_b are the principal curvatures at 'P' then $\boldsymbol{K_n = K_a Cos^2 \propto + K_b Sin^2 \propto}$ where $\propto$ is the angle which the direction 'l' makes with the $v = constant\ curve\ i.e. u = parametric\ curve.$

PROOF: let us assume that the parametric curves on the given surface 'S' are its lines of curvatures.



And as we know that the normal curvature in the direction at 'P' can be written as $K_n = \frac{L(du)^2+2Mdudv+N(dv)^2}{Edu^2+2Fdudv+Gdv^2} \dots\dots\dots(i)$ and using orthogonality of lines of curvature we have $F = 0 = M \Longrightarrow K_n = \frac{L(du)^2+N(dv)^2}{Edu^2+Gdv^2} \dots\dots\dots(ii)$ equivalently we can write $K_n = \frac{L(du)^2+N(dv)^2}{(ds)^2} \Longrightarrow K_n = L\left(\frac{du}{ds}\right)^2 + N\left(\frac{dv}{ds}\right)^2 \dots\dots\dots(iii)$

Now normal curvature along the principal direction for $u = constant\ curve$ becomes $K_a = \frac{L(du)^2}{Edu^2} = \frac{L}{E} \dots\dots\dots(iv)$ similarly $K_b = \frac{N(dv)^2}{Gdv^2} = \frac{N}{G} \dots\dots\dots(v)$
as $\propto$ is the angle which the direction 'l' makes with the $v = constant\ curve$ so
$Cos \propto = \frac{1}{\sqrt{E}}\left(E\frac{du}{ds} + F\frac{dv}{ds}\right) \Longrightarrow Cos \propto = \frac{1}{\sqrt{E}}\left(E\frac{du}{ds}\right)\ \ \therefore\ F = 0 \Longrightarrow Cos \propto = \sqrt{E}\frac{du}{ds}$
$\Longrightarrow \frac{du}{ds} = \frac{1}{\sqrt{E}} Cos \propto \cdots \dots\dots(vi)$

Now the angle between the direction 'l' and principle direction given by $u = constant$ is $(90-\propto)$ so $Cos(90-\propto) = \frac{1}{\sqrt{G}}\left(F\frac{du}{ds} + G\frac{dv}{ds}\right) \Longrightarrow Sin \propto = \frac{1}{\sqrt{G}}\left(G\frac{dv}{ds}\right)\ \ \therefore\ F = 0 \Longrightarrow Sin \propto = \sqrt{G}\frac{dv}{ds}$
$\Longrightarrow \frac{dv}{ds} = \frac{1}{\sqrt{G}} Sin \propto \cdots \dots\dots(vii)$

From equation $(iv) and (v)$ $\ L = K_a E$ and $N = K_b G$ and putting the values of $\frac{du}{ds}, \frac{dv}{ds}, L, N$ in
(iii) $\Longrightarrow K_n = K_a E\left(\frac{1}{\sqrt{E}} Cos \propto\right)^2 + K_b G\left(\frac{1}{\sqrt{G}} Sin \propto\right)^2 \Longrightarrow \boldsymbol{K_n = K_a Cos^2 \propto + K_b Sin^2 \propto}$

REMARK: if K_1 and K_2 are principal curvatures of a surface patch σ at a point 'P' then
(i) K_1 and K_2 are real members.
(ii) $K_1 = K_2 = K$ Then $\mathcal{F}_{II} = K\mathcal{F}_I$ and hence every tangent vector to σ at a point 'P' is principal curvature.
(ii) $K_1 \neq K_2$Then tow non – zero vectors $\vec{t}_1$ and $\vec{t}_2$ corresponding to K_1 and K_2 respectively are perpendicular.

PRINCIPAL VECTOR: If $T = \begin{bmatrix}\xi \\ \eta\end{bmatrix}$ satisfying equation $[\mathcal{F}_{II} - K\mathcal{F}_I]T = 0$ then the corresponding tangent vector $\vec{t} = \xi\sigma_u + \eta\sigma_v$ to the surface $\sigma(u,v)$ is called the principal vector corresponding to the principal curvature.

EULER'S THEOREM (PP) (another proof): Let $\vec{r}$ be a curve on a surface path σ and let K_1 *and* K_2 be the principal curvature with non – zero principal vectors $\vec{t}_1$ *and* $\vec{t}_2$ then

$\boldsymbol{K_n = K_1 Cos^2\theta + K_2 Sin^2\theta}$ where θ is the angle between $\vec{r}\,'$ *and* $\vec{t}_1$

PROOF: let us assume that $\vec{r}$ is a unit speed curve. Let $\vec{t}$ be a tangent vector of $\vec{r}$ also consider $\vec{t} = \xi\sigma_u + \eta\sigma_v$ then $T = \begin{bmatrix}\xi\\ \eta\end{bmatrix}$

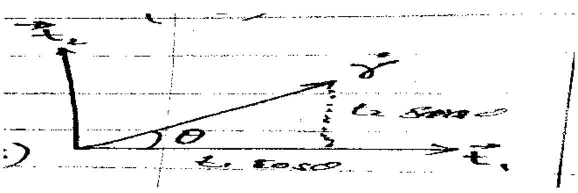
Case#1: let $K_1 = K_2 = K$ Then $\mathcal{F}_{II} = K\mathcal{F}_I$ and hence $K_n = T^t\mathcal{F}_{II}T = T^tK\mathcal{F}_IT = KT^t\mathcal{F}_IT$

$K_n = K\vec{t}.\vec{t} = K\|\vec{t}\| = K(1) = K(Cos^2\theta + Sin^2\theta) \Longrightarrow K_n = KCos^2\theta + KSin^2\theta$

result is true for $K_1 = K_2 = K$

Case#2: let $K_1 \neq K_2$Then tow non – zero vectors $\vec{t}_1$ and $\vec{t}_2$ corresponding to K_1 and K_2 respectively are perpendicular. We assume that $\vec{t}_1$ and $\vec{t}_2$ are unit vectors then

$\vec{t}_1 = \xi_1\sigma_u + \eta_1\sigma_v$ then $T_1 = \begin{bmatrix}\xi_1\\ \eta_1\end{bmatrix}$ and $\vec{t}_2 = \xi_2\sigma_u + \eta_2\sigma_v$ then $T_2 = \begin{bmatrix}\xi_2\\ \eta_2\end{bmatrix}$



Then $\vec{r}\,' = \vec{t} = \vec{t}_1Cos\theta + \vec{t}_2Sin\theta \Longrightarrow \vec{r}\,' = \xi\sigma_u + \eta\sigma_v = (\xi_1\sigma_u + \eta_1\sigma_v)Cos\theta + (\xi_2\sigma_u + \eta_2\sigma_v)Sin\theta$

$\Longrightarrow \xi = \xi_1Cos\theta + \xi_2Sin\theta$ and $\eta = \eta_1Cos\theta + \eta_2Sin\theta$

Then $\begin{bmatrix}\xi\\ \eta\end{bmatrix} = \begin{bmatrix}\xi_1\\ \eta_1\end{bmatrix}Cos\theta + \begin{bmatrix}\xi_2\\ \eta_2\end{bmatrix}Sin\theta \Longrightarrow T = T_1Cos\theta + T_2Sin\theta$

So the normal curvature of $\vec{r}$ is $K_n = T^t\mathcal{F}_{II}T = [T_1Cos\theta + T_2Sin\theta]^t\mathcal{F}_{II}[T_1Cos\theta + T_2Sin\theta]$

$\Longrightarrow K_n = Cos^2\theta(T_1)^t\mathcal{F}_{II}T_1 + Sin\theta Cos\theta[(T_1)^t\mathcal{F}_{II}T_2 + (T_2)^t\mathcal{F}_{II}T_1] + Sin^2\theta(T_2)^t\mathcal{F}_{II}T_2$

Since $[\mathcal{F}_{II} - K\mathcal{F}_I]T = 0 \Longrightarrow \mathcal{F}_{II}T = K\mathcal{F}_IT$

Then

$K_n = Cos^2\theta(T_1)^t(K_1\mathcal{F}_IT_1) + Sin\theta Cos\theta[K_2(T_1)^t\mathcal{F}_IT_2 + K_1(T_2)^t\mathcal{F}_IT_1] + Sin^2\theta(T_2)^t(K_2\mathcal{F}_IT_2)$

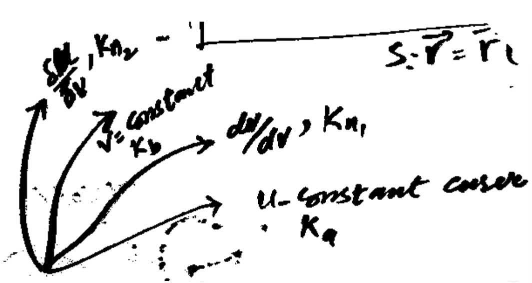
$K_n = K_1Cos^2\theta(T_1)^t\mathcal{F}_IT_1 + Sin\theta Cos\theta\left[K_2(\vec{t}_1.\vec{t}_2) + K_1(\vec{t}_2.\vec{t}_1)\right] + K_2Sin^2\theta(T_2)^t\mathcal{F}_IT_2$

$K_n = K_1Cos^2\theta(\vec{t}_1.\vec{t}_1) + Sin\theta Cos\theta[K_2(0) + K_1(0)] + K_2Sin^2\theta(\vec{t}_2.\vec{t}_2)$

$K_n = K_1Cos^2\theta(1) + K_2Sin^2\theta(1) \Longrightarrow \boldsymbol{K_n = K_1Cos^2\theta + K_2Sin^2\theta}$

COROLLARY: The sum of the normal curvatures in two directions at right angles is constant and equal to the sum of the principal curvature i.e. $K_{n_1} + K_{n_2} = K_a + K_b$

PROOF:



let the directions are $\frac{du}{dv}, \frac{\delta u}{\delta v}$ as shown in the figure then rotating the directions to coincide with $v = constant\ curve\ \ and\ \ u = constant\ curve$. respectively

then for $\frac{du}{dv}$ direction $\propto = 0$ and using the Euler's theorem i.e. $\boldsymbol{K_n = K_a Cos^2 \propto + K_b Sin^2 \propto}$
$\Rightarrow K_{n_1} = K_a Cos^2(0) + K_b Sin^2(0) \Rightarrow K_{n_1} = K_a \ldots\ldots\ldots\ldots(i)$

and for $\frac{\delta u}{\delta v}$ direction $\propto = 90°$ and using the Euler's theorem i.e. $\boldsymbol{K_n = K_a Cos^2 \propto + K_b Sin^2 \propto}$
$\Rightarrow K_{n_2} = K_a Cos^2(90°) + K_b Sin^2(90°) \Rightarrow K_{n_2} = K_b \ldots\ldots\ldots\ldots(ii)$

adding (i) and (ii) $\Rightarrow K_{n_1} + K_{n_2} = K_a + K_b$

QUESTION: If B is the mean normal curvature and A is the amplitude, deduce from Euler's theorem that $K_n = B - ACos2 \propto$

Solution: $R.H.S. = B - ACos2 \propto = \frac{1}{2}(K_b + K_a) - \frac{1}{2}(K_b - K_a)(2Cos^2 \propto -1)$
$= \frac{1}{2}(K_b + K_a) - \frac{1}{2}(K_b - K_a)Cos^2 + \frac{1}{2}(K_b - K_a) = K_b - \frac{1}{2}(K_b - K_a)Cos^2$
$= K_b(1 - Cos^2 \propto) + K_a Cos^2 = K_b Sin^2 \propto + K_a Cos^2 = K_n = L.H.S.$

QUESTION: If B is the mean normal curvature and A is the amplitude, deduce from Euler's theorem that $K_n - K_a = 2ASin^2 \propto$

Solution: $\boldsymbol{R.H.S. = 2ASin^2 \propto = 2\frac{1}{2}(K_b - K_a)Sin^2 \propto = K_b Sin^2 \propto - K_a(1 - Cos^2 \propto) = K_n - K_a}$

QUESTION: If B is the mean normal curvature and A is the amplitude, deduce from Euler's theorem that $K_b - K_n = 2ACos^2 \propto$

Solution: $R.H.S. = 2ACos^2 \propto = 2\frac{1}{2}(K_b - K_a)Cos^2 \propto = K_b(1 - Sin^2 \propto) - K_a Cos^2 \propto = K_b - K_n$

THEOREM: A surface is developable iff its specific curvature is zero at all points.

PROOF: Consider the surface $z = f(x, y)$ let $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $t = \frac{\partial^2 z}{\partial y^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$

$\vec{r} = (x, \quad y, \quad z) \Rightarrow \vec{r}_1 = \frac{\partial \vec{r}}{\partial x} = \left(1, \quad 0, \quad \frac{\partial z}{\partial x} = p\right) \Rightarrow \vec{r}_2 = \frac{\partial \vec{r}}{\partial y} = \left[0, \quad 1, \quad \frac{\partial z}{\partial y} = q\right]$

$\Rightarrow \boldsymbol{E = \vec{r}_1{}^2 = 1 + p^2} \Rightarrow \boldsymbol{G = 1 + q^2} \Rightarrow \boldsymbol{F = pq}$ also $H^2 = EG - F^2 \Rightarrow \boldsymbol{H^2 = 1 + p^2 + q^2}$

Also unit normal to the surface is $\vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{H} \Rightarrow \vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}$

$\Rightarrow \boldsymbol{\vec{n} = \frac{[-p, \quad -q, \quad 1]}{\sqrt{p^2 + q^2 + 1}}}$

$$\boxed{\Rightarrow \vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & p \\ 0 & 1 & q \end{vmatrix}}$$

$\Rightarrow \vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial x^2} = (0, \quad 0, \quad r) \Rightarrow \vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial y^2} = (0, \quad 0, \quad t) \Rightarrow \vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial x \partial y} = (0, \quad 0, \quad s)$

So the second order coefficients are $L = \vec{n}.\vec{r}_{11} = \frac{r}{H}$ $M = \vec{n}.\vec{r}_{12} = \frac{s}{H}$ $N = \vec{n}.\vec{r}_{22} = \frac{t}{H}$

And $T^2 = LN - M^2 = \frac{r}{H}.\frac{t}{H} - \frac{s^2}{H^2} \Rightarrow \boldsymbol{T^2 = \frac{rt - s^2}{H^2}}$

Now equation for curvature is $K^2H^2 - K(EN - 2FM + GL) + T^2 = 0$ and hence specific curvature will be $K = \frac{T^2}{H^2} = product\ of\ roots = \frac{rt - s^2}{H^2} = \frac{rt - s^2}{1 + p^2 + q^2}$

STEP-II: suppose that surface is developable then
$\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = \frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} \Rightarrow s^2 = rt \Rightarrow rt - s^2 = 0 \Rightarrow K = \frac{T^2}{H^2} = \frac{rt - s^2}{H^2} = \frac{0}{H^2} \Rightarrow K = 0$

CONVERSLY: suppose that surface curvature is identically zero then $K = 0$
$\Rightarrow rt - s^2 = 0 \Rightarrow s^2 = rt \Rightarrow \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = \frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} \Rightarrow$ the surface is developable.

QUESTION: Find the equation for the principal curvatures and the differential equation of the lines of curvature, of the surface $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

SOLUTION: given $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \Rightarrow \vec{r} = \left(x\,, \quad y\,, \quad \frac{x^2}{2a^2} + \frac{y^2}{2b^2}\right)$

$\Rightarrow \vec{r}_1 = \frac{\partial \vec{r}}{\partial x} = \left(1, \quad 0, \ \frac{x}{a}\right) \Rightarrow \vec{r}_2 = \frac{\partial \vec{r}}{\partial y} = \left[0, \quad 1, \quad \frac{y}{b}\right]$

$\Rightarrow E = \vec{r}_1^{\,2} = 1 + \frac{x^2}{a^2} \Rightarrow G = 1 + \frac{y^2}{b^2} \Rightarrow F = \frac{xy}{ab}$ also $H = |\vec{r}_1 \times \vec{r}_2| = \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1}$

Also unit normal to the surface is $\vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{H} \Rightarrow \vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}$

$\Rightarrow \vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & p \\ 0 & 1 & q \end{vmatrix}$

$\Rightarrow \vec{n} = \frac{\left[-\frac{x}{a}, \quad -\frac{y}{b}, \quad 1\right]}{\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1}}$

$\Rightarrow \vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial x^2} = \left(0, \quad 0, \quad \frac{1}{a}\right) \Rightarrow \vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial y^2} = \left(0, \quad 0, \quad \frac{1}{b}\right) \Rightarrow \vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial x \partial y} = (0, \quad 0, \quad 0)$

So the second order coefficients are $L = \vec{n}.\vec{r}_{11} = \frac{1}{aH}$ $M = \vec{n}.\vec{r}_{12} = 0$ $N = \vec{n}.\vec{r}_{22} = \frac{1}{bH}$

And $T^2 = LN - M^2 = \frac{1}{abH^2}$

Now differential equation for principal curvature is $K^2H^2 - K(EN - 2FM + GL) + T^2 = 0$

$\Rightarrow K^2H^2 - K\left(\frac{a^2+x^2}{bH} - \frac{b^2+y^2}{aH}\right) + T^2 = 0$and hence specific curvature will be

$K = \frac{T^2}{H^2} = product\ of\ roots = \frac{1}{abH^4}$ Also $J = \left(\frac{a^2+x^2}{b} - \frac{b^2+y^2}{a}\right)\frac{1}{H^3}$

Now for the differential equation of the lines of curvature we have

$$\begin{vmatrix} (dy)^2 & -dxdy & (dx)^2 \\ L & M & N \\ E & F & G \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} (dy)^2 & -dxdy & (dx)^2 \\ \frac{1}{aH} & 0 & \frac{1}{bH} \\ 1 + \frac{x^2}{a^2} & \frac{xy}{ab} & 1 + \frac{y^2}{b^2} \end{vmatrix} = 0$$

$$\Rightarrow -\frac{xy}{ab^2H}(dy)^2 + \left[\left(1 + \frac{y^2}{b^2}\right)\frac{1}{aH} - \left(1 + \frac{x^2}{a^2}\right)\frac{1}{bH}\right]dxdy + \frac{xy}{a^2bH}(dx)^2 = 0$$

$$\Rightarrow -\frac{xy}{ab^2H}\left(\frac{dy}{dx}\right)^2 + \left[\left(\frac{b^2+y^2}{b^2}\right)\frac{1}{aH} - \left(\frac{a^2+x^2}{a^2}\right)\frac{1}{bH}\right]\frac{dy}{dx} + \frac{xy}{a^2bH} = 0$$

$$\Rightarrow -\frac{xy}{ab^2H}\left(\frac{dy}{dx}\right)^2 + \frac{1}{abH}\left[\left(\frac{b^2+y^2}{b}\right) - \left(\frac{a^2+x^2}{a}\right)\right]\frac{dy}{dx} + \frac{xy}{a^2bH} = 0$$

$$\Rightarrow -\frac{1}{b}\left(\frac{dy}{dx}\right)^2 + \frac{1}{xy}\left[\left(\frac{b^2+y^2}{b}\right) - \left(\frac{a^2+x^2}{a}\right)\right]\frac{dy}{dx} + \frac{1}{a} = 0$$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 - \frac{b}{xy}\left[\left(\frac{b^2+y^2}{b}\right) - \left(\frac{a^2+x^2}{a}\right)\right]\frac{dy}{dx} - \frac{b}{a} = 0$ is the required answer.

PRACTICE: Find the equation for the principal curvatures and the differential equation of the lines of curvature, of the surfaces

i. $3z = ax^3 + by^3$
ii. $z = cTan^{-1}\frac{y}{x}$

QUESTION: At the point of intersection of the paraboloid $xy = cz$ with the hyperboloid $x^2 + y^2 - z^2 + c^2 = 0$, the principal radii of the paraboloid are $\frac{z^2(1\pm\sqrt{2})}{c}$

SOLUTION: given paraboloid $xy = cz \Rightarrow \vec{r} = \left(x, \quad y, \quad \frac{xy}{c}\right)$

$\Rightarrow \vec{r}_1 = \frac{\partial \vec{r}}{\partial x} = \left(1, \quad 0, \quad \frac{y}{c}\right) \Rightarrow \vec{r}_2 = \frac{\partial \vec{r}}{\partial y} = \left[0, \quad 1, \quad \frac{x}{c}\right]$

$\Rightarrow E = \vec{r}_1{}^2 = 1 + \frac{y^2}{c^2} \Rightarrow G = 1 + \frac{x^2}{c^2} \Rightarrow F = \frac{xy}{c^2}$ also $H = |\vec{r}_1 \times \vec{r}_2| = \sqrt{\frac{[-y, \quad -x, \quad c]}{c}} = \frac{y^2}{c^2} + \frac{x^2}{c^2} + \frac{1}{c}$

Also unit normal to the surface is $\vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{H} \Rightarrow \vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}$

$$\boxed{\Rightarrow \vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & \frac{y}{c} \\ 0 & 1 & \frac{x}{c} \end{vmatrix}}$$

$\Rightarrow \vec{n} = \frac{[-y, \quad -x, \quad c]}{cH}$

$\Rightarrow \vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial x^2} = (0, \quad 0, \quad 0) \Rightarrow \vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial y^2} = (0, \quad 0, \quad 0) \Rightarrow \vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial x \partial y} = \left(0, \quad 0, \quad \frac{1}{c}\right)$

So the second order coefficients are $L = \vec{n}.\vec{r}_{11} = 0 \quad M = \vec{n}.\vec{r}_{12} = \frac{1}{cH} \quad N = \vec{n}.\vec{r}_{22} = 0$

And $T^2 = LN - M^2 = \frac{-1}{c^2H^2}$ and We note that at a point of intersection with hyperboloid $x^2 + y^2 - z^2 + c^2 = 0$ is $H^2 = \frac{z^2}{c^2}$ and $T^2 = \frac{-1}{z^2}$

Now differential equation for principal radii is $T^2\rho^2 - (EN - 2FM + GL)\rho + H^2 = 0$
$\Rightarrow -\frac{\rho^2}{z^2} - \left(0 - 2\frac{z}{c}\frac{1}{cH} + 0\right) + H^2 = 0 \Rightarrow \rho^2c^2 - 2cz^2H\rho - z^4 = 0$
$\Rightarrow \rho = \frac{2cz^2 \pm \sqrt{4c^2z^2 + 4c^2z^4}}{2c^2} = \frac{z^2 \pm z^2\sqrt{2}}{c} = \frac{z^2(1\pm\sqrt{2})}{c}$

SURFACE OF REVOLUTION: A surface generated by the rotation of the plane curve about an axis in its plane is called a **surface of revolution**.

If z – axis is taken as the axis of revolution and 'u' denotes the distance of a point from z – axis then surface may be expressed as $x = uCos\varphi, \quad y = uSin\varphi, \quad z = f(u)$

We may also use $\sigma(u, v) = (uCosv, \quad uSinv, \quad f(u))$

QUESTION (PP;2016): show that if a surface of revolution is a minimal surface then

$$u\frac{d^2f}{du^2} + \frac{df}{du}\left[1 + \left(\frac{df}{du}\right)^2\right] = 0 \Rightarrow uf''(u) + f'(u)\left[1 + \left(f'(u)\right)^2\right] = 0$$

SOLUTION: let the surface of revolution is $\vec{r} = (uCos\varphi, \quad uSin\varphi, \quad f(u))$

$\Rightarrow \vec{r}_1 = (Cos\varphi, \quad Sin\varphi, \quad f_1) \Rightarrow \vec{r}_2 = [-uSin\varphi, \quad uCos\varphi, \quad f_2]$

$\Rightarrow E = \vec{r}_1{}^2 = 1 + f_1{}^2 \Rightarrow G = \vec{r}_2{}^2 = u^2 \Rightarrow F = \vec{r}_1.\vec{r}_2 = 0$

For $\vec{r} = (uCos\varphi, \quad uSin\varphi, \quad f(u)) \Rightarrow \vec{r}_{11} = (0, \quad 0, \quad f_{11})$
$\Rightarrow \vec{r}_{22} = (-uCos\varphi, \quad -uSin\varphi, \quad 0) \Rightarrow \vec{r}_{12} = (-Sin\varphi, \quad Cos\varphi, \quad 0)$

also
$H = |\vec{r}_1 \times \vec{r}_2| = |(-uf_1Cos\varphi, \quad -uf_1Sin\varphi, \quad u)| = \sqrt{(-uf_1Cos\varphi)^2 + (-uf_1Sin\varphi)^2 + u^2}$
$\Rightarrow H = |\vec{r}_1 \times \vec{r}_2| = \sqrt{u^2 + u^2f_1{}^2} = \sqrt{(1 + f_1{}^2)u^2}$

Also unit normal to the surface is $\vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{H} \Rightarrow \vec{n} = \frac{1}{H}(-uf_1Cos\varphi, \quad -uf_1Sin\varphi, \quad u)$

So the second order coefficients are $L = \vec{n}.\vec{r}_{11} = \frac{uf_{11}}{H} \quad M = \vec{n}.\vec{r}_{12} = 0 \quad N = \vec{n}.\vec{r}_{22} = \frac{u^2f_1}{H}$

Now equation of principal curvature is $K^2H^2 - K(EN - 2FM + GL) + T^2 = 0$

$\Rightarrow u^4\left(1+{f_1}^2\right)^2K^2 - K\left[\frac{(1+{f_1}^2)u^2f_1}{H} + \frac{u^3f_{11}}{H}\right] + \frac{u^3f_{11}f_1}{H^2} = 0$

$\Rightarrow u^4\left(1+{f_1}^2\right)^2K^2 - K\left[f_1\sqrt{(1+{f_1}^2)u^2} + \frac{u^3f_{11}}{\sqrt{(1+{f_1}^2)u^2}}\right] + \frac{u^3f_{11}f_1}{(1+{f_1}^2)u^2} = 0$

$\Rightarrow u^6\left(1+{f_1}^2\right)^4K^2 - K\left[{f_1}^3\sqrt{(1+{f_1}^2)u^2} + u^3f_{11}\sqrt{(1+{f_1}^2)u^2}\right] + u^3f_{11}f_1 = 0$

$\Rightarrow u^6\left(1+{f_1}^2\right)^4K^2 - K\sqrt{(1+{f_1}^2)u^2}\left[f_1(1+{f_1}^2)u^2 + u^3f_{11}\right] + u^3f_{11}f_1 = 0$

Then $J = K_a + K_b = \frac{\sqrt{(1+{f_1}^2)u^2}\left[f_1(1+{f_1}^2)u^2+u^3f_{11}\right]}{u^6\left(1+{f_1}^2\right)^4}$

Now given surface is minimal then curvature will be zero then $\left[f_1\left(1+{f_1}^2\right)u^2 + u^3f_{11}\right] = 0$

$\Rightarrow f_1\left(1+{f_1}^2\right) + uf_{11} = 0 \Rightarrow u\frac{d^2f}{du^2} + \frac{df}{du}\left[1 + \left(\frac{df}{du}\right)^2\right] = 0$

EXAMPLE: On the surface formed by the revolution of a parabola about its directrix, one principal curvature is double the other.

SOLUTION: consider a parabola in yz – plane with its directrix along z – axis then surface of revolution will be $\vec{r} = \left(yCos\varphi,\quad ySin\varphi,\quad f(y)\right)$ where $f(y) = z = 2\sqrt{a(y-a)}$

$$\Rightarrow \vec{r}_1 = \left(Cos\varphi,\quad Sin\varphi,\quad f_1 = \frac{\partial f}{\partial y} = \frac{a}{\sqrt{(y-a)}}\right) \Rightarrow \vec{r}_2 = \left[-uSin\varphi,\quad uCos\varphi,\quad f_2 = \frac{\partial f}{\partial \varphi} = 0\right]$$

$$\Rightarrow E = {\vec{r}_1}^2 = 1 + \frac{a}{(y-a)} = \frac{y}{(y-a)} \Rightarrow G = {\vec{r}_2}^2 = y^2 \Rightarrow F = \vec{r}_1.\vec{r}_2 = 0$$

For $\vec{r} = \left(uCos\varphi,\quad uSin\varphi,\quad f(y)\right) \Rightarrow \vec{r}_{11} = \left(0,\quad 0,\quad f_{11} = \frac{\partial^2 f}{\partial y^2} = \frac{-\sqrt{a}}{2(y-a)^{3/2}}\right)$

$\Rightarrow \vec{r}_{22} = (-yCos\varphi,\quad -ySin\varphi,\quad 0) \Rightarrow \vec{r}_{12} = (-Sin\varphi,\quad Cos\varphi,\quad 0)$

also

$H = |\vec{r}_1 \times \vec{r}_2| = |(-yf_1Cos\varphi,\quad -yf_1Sin\varphi,\quad y)| = \sqrt{(-yf_1Cos\varphi)^2 + (-yf_1Sin\varphi)^2 + y^2}$

$\Rightarrow H = |\vec{r}_1 \times \vec{r}_2| == \sqrt{\frac{ay^2}{(y-a)} + y^2} = \frac{y^3}{(y-a)}$

Also unit normal to the surface is $\vec{n} = \frac{\vec{r}_1\times\vec{r}_2}{H} \Rightarrow \vec{n} = \frac{1}{H}(-yf_1Cos\varphi,\quad -yf_1Sin\varphi,\quad y)$

$L = \vec{n}.\vec{r}_{11} = \frac{-\sqrt{a}y}{2(y-a)^{3/2}} \qquad M = \vec{n}.\vec{r}_{12} = 0 \qquad N = \vec{n}.\vec{r}_{22} = \frac{-\sqrt{a}y^2}{H\sqrt{(y-a)}}$

Then from the Euler's theorem $K_a = \frac{L}{E} = \frac{-\sqrt{a}}{2H\sqrt{(y-a)}}$ and $K_b = \frac{N}{G} = \frac{\sqrt{a}}{H\sqrt{(y-a)}}$

Thus one principal curvature is double the other .i.e $K_b = 2K_a$

EXAMPLE: Find the equation for the principal radii, the lines of curvature and the first and second curvature of the surface $x = uCos\theta, \quad y = uSin\theta, \quad z = f(\theta)$

SOLUTION: consider a parabola in yz – plane with its directrix along z – axis then surface of revolution will be $\vec{r} = \left(uCos\theta, \quad uSin\theta, \quad f(\theta)\right)$

$$\Rightarrow \vec{r}_1 = \left(Cos\theta, \quad Sin\theta, \ f_1 = \frac{\partial f}{\partial u} = 0\right) \Rightarrow \vec{r}_2 = \left[-uSin\theta, \quad uCos\theta, \quad f_2 = \frac{\partial f}{\partial \theta}\right]$$

$$\Rightarrow E = \vec{r}_1^{\,2} = 1 \Rightarrow G = \vec{r}_2^{\,2} = u^2 + f_2^{\,2} \Rightarrow F = \vec{r}_1.\vec{r}_2 = 0$$

For $\vec{r} = \left(uCos\theta, \quad uSin\theta, \quad f(\theta)\right) \Rightarrow \vec{r}_{11} = (0, \quad 0, \quad 0)$

$$\Rightarrow \vec{r}_{22} = \left(-uCos\theta, \quad -uSin\theta, \quad f_{22}\right) \Rightarrow \vec{r}_{12} = (-Sin\theta, \quad Cos\theta, \quad 0)$$

also $H = |\vec{r}_1 \times \vec{r}_2| = \sqrt{f_2^{\,2} + u^2}$

Also unit normal to the surface is $\vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{H} \Rightarrow \vec{n} = \frac{1}{H}(f_2Sin\theta, \quad -f_2Cos\theta, \ u)$

$L = \vec{n}.\vec{r}_{11} = 0 \quad M = \vec{n}.\vec{r}_{12} = -\frac{f_2}{H} \quad N = \vec{n}.\vec{r}_{22} = \frac{uf_{22}}{H}$ also $T^2 = LN - M^2 = -\frac{f_2^{\,2}}{H^2}$

Now equation of principal curvature is $K^2H^2 - K(EN - 2FM + GL) + T^2 = 0$

$$\Rightarrow \frac{H^2}{\rho^2} - \frac{1}{\rho}\left[\frac{uf_{22}}{H}\right] - \frac{f_2^{\,2}}{H^2} = 0 \Rightarrow \frac{f_2^{\,2}+u^2}{\rho^2} - \frac{1}{\rho}\left[\frac{uf_{22}}{\sqrt{f_2^{\,2}+u^2}}\right] - \frac{f_2^{\,2}}{H^2} = 0$$

$$\Rightarrow (f_2^{\,2} + u^2) - \rho\left[\frac{uf_{22}}{\sqrt{f_2^{\,2}+u^2}}\right] - \frac{f_2^{\,2}}{f_2^{\,2}+u^2}\rho^2 = 0 \Rightarrow f_2^{\,2}\rho^2 + u^2 - uf_{22}\sqrt{f_2^{\,2} + u^2}\rho - \left(f_2^{\,2} + u^2\right)^2 = 0$$

Then $J = \frac{uf_{22}}{H^3} = \frac{uf_{22}}{(f_2^{\,2}+u^2)^{3/2}}$ And $K = -\frac{f_2^{\,2}}{(f_2^{\,2}+u^2)^2}$

Then lines of curvature are given by

$$\begin{vmatrix} (d\theta)^2 & -dud\theta & (du)^2 \\ L & M & N \\ E & F & G \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} (d\theta)^2 & -dud\theta & (du)^2 \\ 0 & -\frac{f_2}{H} & \frac{uf_{22}}{H} \\ 1 & 0 & u^2 + f_2^{\,2} \end{vmatrix} = 0$$

$\Rightarrow -f_2\left(f_2^{\,2} + u^2\right)(d\theta)^2 + f_2dud\theta + f_2(du)^2 = 0$ is the equation of lines for curvature.

QUESTION: Show that the line of curvature of the paraboloid $xy = az$ lie on the surface $Sinh^{-1}\frac{x}{a} + Sinh^{-1}\frac{y}{a} = constant$

SOLUTION: given paraboloid $xy = cz \Rightarrow \vec{r} = \left(x, \quad y, \quad \frac{xy}{a}\right)$

$$\Rightarrow \vec{r}_1 = \frac{\partial \vec{r}}{\partial x} = \left(1, \quad 0, \ \frac{y}{a}\right) \Rightarrow \vec{r}_2 = \frac{\partial \vec{r}}{\partial y} = \left[0, \quad 1, \quad \frac{x}{a}\right]$$

$$\Rightarrow E = \vec{r}_1^{\,2} = 1 + \frac{y^2}{a^2} \Rightarrow G = 1 + \frac{x^2}{a^2} \Rightarrow F = \frac{xy}{a^2}, \ H = |\vec{r}_1 \times \vec{r}_2| = \sqrt{\frac{[-y, \quad -x, \ a]}{a}} = \frac{y^2}{a^2} + \frac{x^2}{a^2} + \frac{1}{a}$$

Also unit normal to the surface is $\vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{H} \Rightarrow \vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}$

$$\Rightarrow \vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & \frac{y}{a} \\ 0 & 1 & \frac{x}{a} \end{vmatrix}$$

$$\Rightarrow \vec{n} = \frac{[-y, \quad -x, \ a]}{aH}$$

$$\Rightarrow \vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial x^2} = (0, \quad 0, \quad 0) \Rightarrow \vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial y^2} = (0, \quad 0, \quad 0) \quad \Rightarrow \vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial x \partial y} = \left(0, \quad 0, \quad \frac{1}{a}\right)$$

So the second order coefficients are $L = \vec{n}.\vec{r}_{11} = 0$ $M = \vec{n}.\vec{r}_{12} = \frac{1}{aH}$ $N = \vec{n}.\vec{r}_{22} = 0$

And $T^2 = LN - M^2 = \frac{-1}{a^2H^2}$

Then lines of curvature are given by

$$\begin{vmatrix} (dy)^2 & -dydx & (dx)^2 \\ L & M & N \\ E & F & G \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} (dy)^2 & -dydx & (dx)^2 \\ 0 & \frac{1}{aH} & 0 \\ 1+\frac{y^2}{a^2} & \frac{xy}{a^2} & 1+\frac{x^2}{a^2} \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{aH}\frac{a^2+x^2}{a^2}(dy)^2 + \frac{1}{aH}\frac{a^2+y^2}{a^2}(dx)^2 = 0 \Rightarrow \frac{(dy)^2}{\sqrt{a^2+x^2}} + \frac{(dx)^2}{\sqrt{a^2+y^2}} = 0$$
$$\Rightarrow Sinh^{-1}\frac{x}{a} + Sinh^{-1}\frac{y}{a} = constant$$

QUESTION: Calculate fundamental magnitude of first and second order for the surfaces $x = uCos\varphi, \quad y = uSin\varphi, \quad z = f(u) + cv$ where u adn φ are parameters.

ANSWER: $\vec{r} = (x, \quad y, \quad z) = (uCos\varphi, \quad uSin\varphi, \quad f(u) + cv)$

$$\Rightarrow \vec{r}_1 = \frac{\partial \vec{r}}{\partial u} = [Cos\varphi, \quad Sin\varphi, \quad f_1] \Rightarrow \vec{r}_2 = \frac{\partial \vec{r}}{\partial v} = [-uSin\varphi, \quad uCos\varphi, \quad c]$$

Now the fundamental coefficients of first order are $\vec{r}_1^{\,2} = E, , \vec{r}_2^{\,2} = G$ and $F = \vec{r}_1\vec{r}_2$

$$\Rightarrow E = \vec{r}_1^{\,2} = 1 + f_1^{\,2} \qquad \Rightarrow G = \vec{r}_2^{\,2} = u^2 + c^2 \Rightarrow F = \vec{r}_1.\vec{r}_2 = f_1c$$

Also $H^2 = EG - F^2 = 1(u^2 + c^2) - 0 = (cSinv - f_1uCosv)i - (cCosv + f_1uSinv)j$

Also unit normal to the surface is $\vec{n} = \frac{\vec{r}_1\times\vec{r}_2}{|\vec{r}_1\times\vec{r}_2|} \Rightarrow \vec{n} = \frac{\vec{r}_1\times\vec{r}_2}{H}$

$$\Rightarrow \vec{r}_1\times\vec{r}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ Cos\varphi & Sin\varphi & f_1 \\ -uSin\varphi & uCos\varphi & c \end{vmatrix}$$

$$\Rightarrow \vec{n} = \frac{[cSinv - f_1uCosv, - cCosv - f_1uSinv, \ u]}{H}$$

Now the fundamental coefficients of second order are $\vec{r}_{11}, \ \vec{r}_{22}, \ \vec{r}_{12}$

For $\vec{r} = (x, \quad y, \quad z) = (uCos\varphi, \quad uSin\varphi, \quad f(u) + cv)$

$$\Rightarrow \vec{r}_{11} = \frac{\partial^2\vec{r}}{\partial u^2} = (0, \quad 0, \quad f_{11}) \qquad \Rightarrow \vec{r}_{22} = \frac{\partial^2\vec{r}}{\partial v^2} = (-uCos\varphi, \quad -uSin\varphi, \quad 0)$$
$$\Rightarrow \vec{r}_{12} = \frac{\partial^2\vec{r}}{\partial u\partial v} = (-Sin\varphi, \ Cos\varphi, \ 0)$$

So the second order coefficients are $L = \vec{n}.\vec{r}_{11} = \frac{uf_{11}}{H}$

$$M = \vec{n}.\vec{r}_{12} = \frac{[cSinv - f_1uCosv, - cCosv - f_1uSinv, \ u]}{H}.(-Sin\varphi, \ Cos\varphi, \ 0) = -\frac{c}{H}$$
$$N = \vec{n}.\vec{r}_{22} = \frac{[cSinv - f_1uCosv, - cCosv - f_1uSinv, \ u]}{H}.(-uCos\varphi, \quad -uSin\varphi, \quad 0) = \frac{u^2f_1}{H}$$

And $T^2 = LN - M^2 = \frac{uf_{11}}{H}\frac{u^2f_1}{H} - \left(-\frac{c}{H}\right)^2 = \frac{u^3f_1f_{11}}{H^2} - \frac{c}{H}$

Now equation of principal curvature is $K^2H^2 - K(EN - 2FM + GL) + T^2 = 0$
and the curvature are given by

$$J = K_a + K_b = \frac{1}{H^2}(EN - 2FM + GL) = \frac{u^2f_1^{\,3} + (u^2+2c^2)f_1 + uf_{11}(u^2+c^2)}{H^3}$$

And $K = K_a.K_b = \frac{T^2}{H^2} = \frac{u^3f_1f_{11} - c^2}{H^4}$

QUESTION: Find the principal curvature and the line of curvature on the surface generated by the tangent to the twisted curve.

SOLUTION: the position vector of the current point on the surface is $\vec{R} = \vec{r} + u\vec{t}$

$$\Rightarrow \vec{R}_1 = \frac{\partial \vec{R}}{\partial u} = \vec{t}' \Rightarrow \vec{R}_2 = \frac{\partial \vec{R}}{\partial s} = \vec{r}' + u\vec{t}' = \vec{t} + uK\vec{n}$$

$$\Rightarrow E = \vec{R}_1.\vec{R}_1 = 1 \Rightarrow G = \vec{R}_2.\vec{R}_2 = 1 + u^2K^2 \Rightarrow F = \vec{R}_1.\vec{R}_2 = 0 \;,\; H = \left|\vec{R}_1 \times \vec{R}_2\right| = uK$$

Also unit normal to the surface is $\vec{n} = \frac{\vec{R}_1 \times \vec{R}_2}{H} \Rightarrow \vec{n} = \frac{uK\vec{b}}{uK} = \vec{b}$

$$\Rightarrow \vec{R}_{11} = \frac{\partial^2 \vec{R}}{\partial u^2} = 0 \quad \Rightarrow \vec{R}_{22} = \frac{\partial^2 \vec{R}}{\partial s^2} = \vec{t}' + uK\vec{n}' + uK'\vec{n} = -uK^2\vec{t} + (K + uK')\vec{n} + uK\tau\vec{b}$$
$$\Rightarrow \vec{R}_{12} = \frac{\partial^2 \vec{R}}{\partial u \partial s} = \vec{t}' = K\vec{n}$$

$L = \vec{n}.\vec{R}_{11} = 0 \quad M = \vec{n}.\vec{R}_{12} = 0 \qquad N = \vec{n}.\vec{R}_{22} = uK\tau$ And $T^2 = LN - M^2 = 0$

Then lines of curvature are given by

$$\begin{vmatrix} (du)^2 & -duds & (ds)^2 \\ L & M & N \\ E & F & G \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} (du)^2 & -duds & (ds)^2 \\ 0 & 0 & uK\tau \\ 1 & 0 & 1 + u^2K^2 \end{vmatrix} = 0$$

$$\Rightarrow uK\tau duds + uK\tau(ds)^2 = 0 \Rightarrow ds(du + ds) = 0 \Rightarrow ds = 0 \; or \; (du + ds) = 0$$
$$\Rightarrow s = constant \; or \quad s + u = constant$$

Now equation of principal curvature is $\tilde{K}^2H^2 - \tilde{K}(EN - 2FM + GL) + T^2 = 0$
$\Rightarrow \tilde{K}^2u^2K^2 - \tilde{K}(uK\tau - 0 - 0) + 0 = 0 \Rightarrow \tilde{K}\left(uK\tilde{K} - \tau\right) = 0 \quad \Rightarrow K_a = 0, K_b = \frac{\tau}{uK}$ required.

ISOMETRY: If S_1 and S_2 are surfaces, a smooth mapping $f: S_1 \to S_2$ is called locally isometry if length of the curve on S_1 and length of the curve on S_2 are same.

Every local isometry is a local diffeomorphism, called isometry.

THEOREM: A diffeomorphism $f: S_1 \to S_2$ is an isometry if for any surface patch $\sigma_1$of S_1 and $\sigma_2$of S_2 have the same fundamental form.

PROOF: For surfaces S_1 and S_2 let $\sigma_1 = \sigma_1(u, v)$ and $\sigma_2 = \sigma_2(u, v)$ respectively. And suppose S_1 and S_2 have the same fundamental form.i.e.

$E_1du^2 + G_1dv^2 + 2F_1dudv = E_2du^2 + G_2dv^2 + 2F_2dudv \quad \dots\dots\dots.(i)$
Then we have to prove for isometry $l(\vec{r}_1) = l(\vec{r}_2)$

For this let $\vec{r}_1(t) = \sigma_1\big(u(t), v(t)\big)$ and $\vec{r}_2(t) = \sigma_2\big(u(t), v(t)\big)$ are two curves on surfaces S_1 and S_2 respectively. Then

$\|\vec{r}_1{}'(t)\| = \sqrt{E_1du^2 + G_1dv^2 + 2F_1dudv}$ and $\|\vec{r}_2{}'(t)\| = \sqrt{E_2du^2 + G_2dv^2 + 2F_2dudv}$

Now
$l(\vec{r}_1) = \int\|\vec{r}_1{}'(t)\|dt = \int\sqrt{E_1du^2 + G_1dv^2 + 2F_1dudv}dt = \int\sqrt{E_2du^2 + G_2dv^2 + 2F_2dudv}dt$
$l(\vec{r}_1) = \int\|\vec{r}_1{}'(t)\|dt = \int\|\vec{r}_2{}'(t)\|dt = l(\vec{r}_2) \Rightarrow l(\vec{r}_1) = l(\vec{r}_2) \Rightarrow S_1$ and S_2 are isometric.

CONVERSLY: Suppose that S_1 and S_2 are isometric. i.e. $l(\vec{r}_1) = l(\vec{r}_2)$ then we have to prove that both have same fundamental form.

Given $l(\vec{r}_1) = l(\vec{r}_2) \Longrightarrow \int \|\vec{r}_1{}'(t)\| dt = \int \|\vec{r}_2{}'(t)\| dt \Longrightarrow \|\vec{r}_1{}'(t)\| = \|\vec{r}_2{}'(t)\|$
$\Longrightarrow \int \sqrt{E_1 du^2 + G_1 dv^2 + 2F_1 dudv} = \int \sqrt{E_2 du^2 + G_2 dv^2 + 2F_2 dudv}$
$\Longrightarrow E_1 du^2 + G_1 dv^2 + 2F_1 dudv = E_2 du^2 + G_2 dv^2 + 2F_2 dudv$

$\Longrightarrow$ both surfaces have same fundamental form.

- **PRACTICE:** Show that following surfaces have isometry;

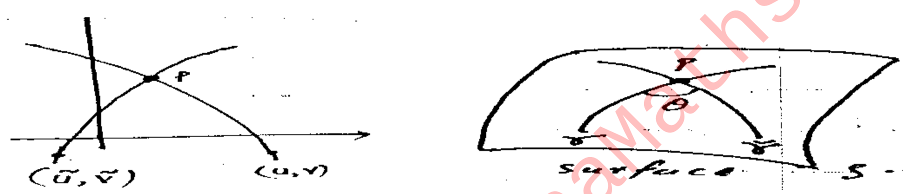
$\sigma(u, v) = (CoshuCosv, CoshuSinv, u)$ and $\bar{\sigma}(u, v) = (uCosv, uSinv, v)$

Hint: find only the 1st fundamental form of both and if both are of same fundamental forms then both are isometric.

CONFORMAL MAPPING OF SURFACES: Suppose that two curves $\vec{r}$ and $\bar{r}$ on a surface 'S' intersect at point 'P' then the angle θ between curves $\vec{r}$ and $\bar{r}$ at 'P' is equal to the angle between their tangent, is define as $Cos\theta = \frac{\vec{r}'.\ \bar{r}'}{\|\vec{r}'\|.\ \|\bar{r}'\|}$ … … (i)

OR If S_1 and S_2 are surfaces, a diffeomorphism $f: S_1 \to S_2$ is said to be conformal if the angle of intersection, between the intersecting curves $\vec{r}_1$ and $\bar{r}_1$ on S_1 and intersecting curves $\vec{r}_2$ and $\bar{r}_2$ on S_2 are same. (in short $f: S_1 \to S_2$ is conformal iff it preserves the angles)

ANGLE IN TERMS OF 1ST FUNDAMENTAL FORM OF SURFACES:



Suppose $\vec{r}(t) = \sigma\big(u(t), v(t)\big)$ and $\bar{r}(t) = \sigma\big(\bar{u}(t), \bar{v}(t)\big)$ are two smooth curves on surface 'S' then suppose that for some parametric value we have $\sigma\big(u(t_0), v(t_0)\big) = P = \sigma\big(\bar{u}(t_0), \bar{v}(t_0)\big)$ then by using chain rule we have $\vec{r}' = \sigma_u u' + \sigma_v v'$ and $\bar{r}' = \sigma_u \bar{u}' + \sigma_v \bar{v}'$

Then $\vec{r}'.\ \bar{r}' = (\sigma_u u' + \sigma_v v').(\sigma_u c + \sigma_v \bar{v}') = E(u'\bar{u}') + F(u'\bar{v}' + v'\bar{u}') + G(v'\bar{v}')$

Now just replacing $\vec{r}$ and $\bar{r}$ we get

Then $\vec{r}'.\ \vec{r}' = E(u')^2 + 2F(u'v') + G(v')^2 = \|\vec{r}'\|^2$

Then $\bar{r}'.\ \bar{r}' = E(\bar{u}')^2 + 2F(\bar{u}'\bar{v}') + G(\bar{v}')^2 = \|\bar{r}'\|^2$

$(i) \Longrightarrow Cos\theta = \frac{\vec{r}'.\ \bar{r}'}{\|\vec{r}'\|.\ \|\bar{r}'\|} == \frac{E(u'\bar{u}') + F(u'\bar{v}' + v'\bar{u}') + G(v'\bar{v}')}{\sqrt{E(u')^2 + 2F(u'v') + G(v')^2}.\sqrt{E(\bar{u}')^2 + 2F(\bar{u}'\bar{v}') + G(\bar{v}')^2}}$

$\Longrightarrow \theta = Cos^{-1}\left(\frac{E(u'\bar{u}') + F(u'\bar{v}' + v'\bar{u}') + G(v'\bar{v}')}{\sqrt{E(u')^2 + 2F(u'v') + G(v')^2}.\sqrt{E(\bar{u}')^2 + 2F(\bar{u}'\bar{v}') + G(\bar{v}')^2}}\right)$

EXAMPLE: The parametric curves on surface patch $\sigma(u, v)$ can be parameterized by $\vec{r}(t) = \sigma(u_0, t)$ and $\bar{r}(t) = \sigma(t, v_0)$ then find the angle between them.

PROOF: Given $\vec{r}(t) = \sigma(u_0, t)$ and $\bar{r}(t) = \sigma(t, v_0)$
and $u(t) = u_0, v(t) = t$ also $\bar{u}(t) = t, \bar{v}(t) = v_0$ where u_0 *and* v_0 are constants and also $u'(t) = 0, v'(t) = 1$ and $\bar{u}'(t) = 1, \bar{v}'(t) = 0$

Now using $\Longrightarrow \theta = Cos^{-1}\left(\frac{E(u'\bar{u}') + F(u'\bar{v}' + v'\bar{u}') + G(v'\bar{v}')}{\sqrt{E(u')^2 + 2F(u'v') + G(v')^2}.\sqrt{E(\bar{u}')^2 + 2F(\bar{u}'\bar{v}') + G(\bar{v}')^2}}\right) = Cos^{-1}\left(\frac{F}{\sqrt{GE}}\right)$

THEOREM: A diffeomorphism $f: S_1 \to S_2$ is said to be conformal iff for any surface patch σ_1 on S_1 and $\bar{\sigma}_2: fo\sigma_1$ on S_2 both have 1st fundamental form, are proportional.

PROOF: Suppose σ_1 and $\bar{\sigma}_2: fo\sigma_1$ are two surface patches of surfaces S_1 and S_2 respectively. Now we suppose that their 1st fundamental forms are proportional. i.e.

$(E_1du^2 + G_1dv^2 + 2F_1dudv) = \lambda(E_2du^2 + G_2dv^2 + 2F_2dudv\ )$ where $\lambda(u,v)$ is smooth function. Then we have $E_1 = \lambda E_2\ , G_1 = \lambda G_1\ , F_1 = \lambda F_2$ ……… … (i)

Now we have to prove for conformal $\theta_1 = \theta_2$

Let θ_1 be the angle of intersection of the curves $\vec{r}_1$ and $\bar{r}_1$ on S_1 and θ_2 be the angle of intersection of the curves $\vec{r}_2$ and $\bar{r}_2$ on S_2

Since $\Longrightarrow Cos\theta_1 = \frac{E_1(u'\bar{u}'\ )+F_1(u'\bar{v}'+v'\bar{u}')+G_1(v'\bar{v}')}{\sqrt{E_1(u')^2+2F_1(u'v')+G_1(v')^2}.\sqrt{E_1(\bar{u}')^2+2F_1(\bar{u}'\bar{v}'\ )+G_1(\bar{v}'\ )^2}}$

Then by using (i) $\Longrightarrow Cos\theta_1 = \frac{\lambda[E_2(u'\bar{u}'\ )+F_2(u'\bar{v}'+v'\bar{u}')+G_2(v'\bar{v}')]}{\lambda\left[\sqrt{E_2(u')^2+2F_2(u'v')+G_2(v')^2}.\sqrt{E_2(\bar{u}')^2+2F_2(\bar{u}'\bar{v}'\ )+G_2(\bar{v}'\ )^2}\right]}$

$\Longrightarrow Cos\theta_1 = Cos\theta_2 \Longrightarrow \theta_1 = \theta_2 \Longrightarrow f$ is conformal.

CONVERSLY: Suppose that f is conformal. i.e. $\theta_1 = \theta_2 \Longrightarrow Cos\theta_1 = Cos\theta_2$

$$\Longrightarrow \frac{E_1(u'\bar{u}'\ ) + F_1(u'\bar{v}' + v'\bar{u}') + G_1(v'\bar{v}')}{\sqrt{E_1(u')^2 + 2F_1(u'v') + G_1(v')^2}.\sqrt{E_1(\bar{u}')^2 + 2F_1(\bar{u}'\bar{v}'\ ) + G_1(\bar{v}'\ )^2}}$$
$$= \frac{E_2(u'\bar{u}'\ ) + F_2(u'\bar{v}' + v'\bar{u}') + G_2(v'\bar{v}')}{\sqrt{E_2(u')^2 + 2F_2(u'v') + G_2(v')^2}.\sqrt{E_2(\bar{u}')^2 + 2F_2(\bar{u}'\bar{v}'\ ) + G_2(\bar{v}'\ )^2}} \dots\dots (i)$$

Suppose for required result $\vec{r}(t) = \sigma_1(a+t,b)$ and $\bar{r}(t) = \sigma_1(a + tCos\varphi,\ b + tSin\varphi)$ where a, φ are constants, and

$u(t) = a + t \Longrightarrow u' = 1\ , v(t) = b \Longrightarrow v' = 0\ , \bar{u}(t) = a + tCos\varphi \Longrightarrow \bar{u}' = Cos\varphi\ ,$
$\bar{v}(t) = b + tSin\varphi \Longrightarrow \bar{v}' = Sin\varphi$

$(i) \Longrightarrow \frac{E_1Cos\varphi+F_1Sin\varphi}{\sqrt{E_1}.\sqrt{E_1Cos^2\varphi+2F_1Sin\varphi Cos\varphi+G_1Sin^2\varphi}} = \frac{E_2Cos\varphi+F_2Sin\varphi}{\sqrt{E_2}.\sqrt{E_2Cos^2\varphi+2F_2Sin\varphi Cos\varphi+G_2Sin^2\varphi}}$

$(i) \Longrightarrow \frac{(E_1Cos\varphi+F_1Sin\varphi)^2}{E_1.(E_1Cos^2\varphi+2F_1Sin\varphi Cos\varphi+G_1Sin^2\varphi)} = \frac{(E_2Cos\varphi+F_2Sin\varphi)^2}{E_2.(E_2Cos^2\varphi+2F_2Sin\varphi Cos\varphi+G_2Sin^2\varphi)}$ … … (ii)

$(E_1Cos\varphi + F_1Sin\varphi)^2 = {E_1}^2Cos^2\varphi + {F_1}^2Sin^2\varphi + 2E_1F_1Sin\varphi Cos\varphi$

$(E_1Cos\varphi + F_1Sin\varphi)^2 = {E_1}^2Cos^2\varphi + {F_1}^2Sin^2\varphi + 2E_1F_1Sin\varphi Cos\varphi + E_1G_1Sin^2\varphi - E_1G_1Sin^2\varphi$

$(E_1Cos\varphi + F_1Sin\varphi)^2 = E_1(E_1Cos^2\varphi + 2F_1Sin\varphi Cos\varphi - G_1Sin^2\varphi) - (E_1G_1 - {F_1}^2)Sin^2\varphi$

$$(ii) \Longrightarrow \frac{E_1(E_1Cos^2\varphi + 2F_1Sin\varphi Cos\varphi - G_1Sin^2\varphi) - (E_1G_1 - {F_1}^2)Sin^2\varphi}{E_1.(E_1Cos^2\varphi + 2F_1Sin\varphi Cos\varphi + G_1Sin^2\varphi)}$$
$$= \frac{E_2(E_2Cos^2\varphi + 2F_2Sin\varphi Cos\varphi - G_2Sin^2\varphi) - (E_2G_2 - {F_2}^2)Sin^2\varphi}{E_2.(E_2Cos^2\varphi + 2F_2Sin\varphi Cos\varphi + G_2Sin^2\varphi)}$$

$\Longrightarrow 1 - \frac{(E_1G_1-{F_1}^2)Sin^2\varphi}{E_1.(E_1Cos^2\varphi+2F_1Sin\varphi Cos\varphi+G_1Sin^2\varphi)} = 1 - \frac{(E_2G_2-{F_2}^2)Sin^2\varphi}{E_2.(E_2Cos^2\varphi+2F_2Sin\varphi Cos\varphi+G_2Sin^2\varphi)}$

$\Longrightarrow \frac{(E_1G_1-{F_1}^2)Sin^2\varphi}{E_1.(E_1Cos^2\varphi+2F_1Sin\varphi Cos\varphi+G_1Sin^2\varphi)} = \frac{(E_2G_2-{F_2}^2)Sin^2\varphi}{E_2.(E_2Cos^2\varphi+2F_2Sin\varphi Cos\varphi+G_2Sin^2\varphi)}$

$\Longrightarrow (E_2Cos^2\varphi + 2F_2Sin\varphi Cos\varphi + G_2Sin^2\varphi) = \frac{E_1(E_2G_2-{F_2}^2)}{E_2.(E_1G_1-{F_1}^2)}(E_1Cos^2\varphi + 2F_1Sin\varphi Cos\varphi + G_1Sin^2\varphi)$

Put $\lambda = \frac{E_1(E_2G_2 - F_2{}^2)}{E_2.(E_1G_1 - F_1{}^2)}$ then

$\Longrightarrow (E_2Cos^2\varphi + 2F_2Sin\varphi Cos\varphi + G_2Sin^2\varphi) = \lambda(E_1Cos^2\varphi + 2F_1Sin\varphi Cos\varphi + G_1Sin^2\varphi)$

$\Longrightarrow (E_2 - \lambda E_1)Cos^2\varphi + 2(F_2 - \lambda F_1)Sin\varphi Cos\varphi + (G_2 - \lambda G_1)Sin^2\varphi = 0$

This condition holds when $(E_2 - \lambda E_1) = 0 \Longrightarrow E_2 = \lambda E_1$, $(F_2 - \lambda F_1) = 0 \Longrightarrow F_2 = \lambda F_1$ and $(G_2 - \lambda G_1) = 0 \Longrightarrow G_2 = \lambda G_1$

Hence prove d the 1st fundamental forms are proportional.

THEOREM: Prove that the first fundamental forms of σ_1 *and* σ_2 are proportional
OR prove that $f: \sigma_1 \to \sigma_2$ is conformal.

PROOF: **For 1st fundamental form of σ_2**

Since $\sigma_2(u,v) = (u,v,0) \Longrightarrow (\sigma_2)_u = (1,0,0)$ *and* $(\sigma_2)_v = (0,1,0)$

$E_2 = (\sigma_2)_u.(\sigma_2)_u = \|(\sigma_2)_u\|^2 = 1$,$G_2 = (\sigma_2)_v.(\sigma_2)_v = \|(\sigma_2)_v\|^2 = 1$, $F_2 = (\sigma_2)_u.(\sigma_2)_v = 0$

Then $E_2du^2 + G_2dv^2 + 2F_2dudv = du^2 + dv^2$ is required.

For 1st fundamental form of σ_1

Since $\sigma_1(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, 1 - \frac{2u}{u^2+v^2+1}\right)$

$\Longrightarrow (\sigma_1)_u = \left[\frac{(u^2+v^2+1)2 - 2u(2u)}{(u^2+v^2+1)^2}, \frac{2v(2u)}{(u^2+v^2+1)^2}, \frac{2(2u)}{(u^2+v^2+1)^2}\right]$

$\Longrightarrow (\sigma_1)_u = \left[\frac{2u^2+2v^2+2-4u^2}{(u^2+v^2+1)^2}, \frac{4uv}{(u^2+v^2+1)^2}, \frac{4u}{(u^2+v^2+1)^2}\right]$

and $(\sigma_1)_v = \left[\frac{2u(2v)}{(u^2+v^2+1)^2}, \frac{(u^2+v^2+1)2 - 2v(2v)}{(u^2+v^2+1)^2}, \frac{4v}{(u^2+v^2+1)^2}\right]$

$\Longrightarrow (\sigma_1)_v = \left[\frac{4uv}{(u^2+v^2+1)^2}, \frac{2u^2+2v^2+2-4v^2}{(u^2+v^2+1)^2}, \frac{4v}{(u^2+v^2+1)^2}\right]$

Now $E_1 = (\sigma_1)_u.(\sigma_1)_u = \|(\sigma_1)_u\|^2 = \frac{4}{(u^2+v^2+1)^2}$,$G_1 = (\sigma_1)_v.(\sigma_1)_v = \|(\sigma_1)_v\|^2 = \frac{4}{(u^2+v^2+1)^2}$, $F_1 = (\sigma_1)_u.(\sigma_1)_v = 0$

Thus $E_1du^2 + G_1dv^2 + 2F_1dudv = \frac{4}{(u^2+v^2+1)^2}[du^2 + dv^2]$

Put $\lambda = \frac{4}{(u^2+v^2+1)^2} = \lambda(u,v)$ then

Then $(E_1du^2 + G_1dv^2 + 2F_1dudv) = \lambda(E_2du^2 + G_2dv^2 + 2F_2dudv\ )$ hence proved.

The 1st fundamental forms are proportional. $\Longrightarrow f: \sigma_1 \to \sigma_2$ is conformal.

QUESTION: Show that every isometry is conformal mapping. Give an example of a conformal mapping which is not an isometry.

SOLUTION: If $f: S_1 \to S_2$ is an isometry between two surfaces S_1 *and* S_2 then their fundamental forms are equal and hence proportional by $\lambda = 1$ $(1st\ FF\ of\ S_1 = 1st\ FF\ of S_2\ )$ so every isometry is conformal.

The Stereographic projection π is an example of conformal mapping but not an isometry because $\lambda \neq 1$

QUESTION: Show that Mercator's parameterization the surface (sphere) $\sigma(u,v) = (SechuCosv, SechuSinv, Tanhu)$ is conformal.

SOLUTION: Given (say) $\sigma_1(u,v) = (SechuCosv, SechuSinv, Tanhu)$

$\Longrightarrow (\sigma_1)_u = (SechuTanhuCosv, SechuTanhuSinv, Sech^2u)$
$and\ (\sigma_1)_v = (-SechuSinv, SechuCosv\ , 0)$

Now $E_1 = (\sigma_1)_u.(\sigma_1)_u = \|(\sigma_1)_u\|^2 = Sech^2u$,$G_1 = (\sigma_1)_v.(\sigma_1)_v = \|(\sigma_1)_v\|^2 = Sech^2u$,
$F_1 = (\sigma_1)_u.(\sigma_1)_v = 0$

$E_1du^2 + G_1dv^2 + 2F_1dudv = Sech^2u[du^2 + dv^2] = Sech^2u[E_2du^2 + G_2dv^2 + 2F_2dudv]$

$\Longrightarrow 1st\ FF\ of\ S_1 = 1st\ FF\ of S_2$

- **QUESTION:** Show that Enneper's surface (sphere) $\sigma(u,v) = \left(u - \frac{u^3}{3} + uv^2\ , v - \frac{v^3}{3} + vu^2\ , u^2 - v^2\right)$ is conformaly parameterized.

QUESTION: let $f(x)$ be a smooth function and $\sigma(u,v) = \big(uCosv, uSinv, f(u)\big)$ be a surface obtained by rotating the curve $z = f(x)$ in xz – plane around z – axis, then find all functions f for which σ is conformal.

SOLUTION: Given (say) $\sigma_1(u,v) = \big(uCosv, uSinv, f(u)\big)$

$\Longrightarrow (\sigma_1)_u = \big(Cosv, Sinv, f'(u)\big)\ and\ (\sigma_1)_v = (-uSinv, uCosv, 0)$

Now $E_1 = (\sigma_1)_u.(\sigma_1)_u = \|(\sigma_1)_u\|^2 = 1 + [f'(u)]^2$,$G_1 = (\sigma_1)_v.(\sigma_1)_v = \|(\sigma_1)_v\|^2 = u^2$,
$F_1 = (\sigma_1)_u.(\sigma_1)_v = 0$

$E_1du^2 + G_1dv^2 + 2F_1dudv = [1 + [f'(u)]^2]du^2 + u^2dv^2 \ldots\ldots\ldots (i)$ and hence mapping is conformal to xz – plane $\sigma_2(u,v) = (u, 0, v)$ iff whose 1st fundamental form is
$E_2du^2 + G_2dv^2 + 2F_2dudv = du^2 + dv^2$

Now equation (i) holds when $1 + [f'(u)]^2 = u^2 \Longrightarrow [f'(u)]^2 = u^2 - 1$
$\Longrightarrow [f'(u)] = \pm\sqrt{u^2 - 1} \Longrightarrow f(u) = \pm \int \sqrt{u^2 - 1}\, du \Longrightarrow f(u) = \pm \left[\frac{u\sqrt{u^2-1}}{2} - \frac{1}{2}Cosh^{-1}u\right] + c$

QUESTION: let σ be a ruled surface $\sigma(u,v) = \vec{r}(u) + v\vec{\delta}(u)$ where $\vec{r}$ is a unit speed curve in R^3 and $\vec{\delta}$ is a unit vector for all 'u' then prove that σ is conformal.

SOLUTION: Given that $\vec{r}$ is a unit speed curve i.e. $\|\vec{r}'(u)\| = 1$ and $\vec{\delta}$ is a unit vector i.e. $\|\vec{\delta}(u)\| = 1$ and let $\sigma_1(u,v) = \vec{r}(u) + v\vec{\delta}(u) \Longrightarrow (\sigma_1)_u = \vec{r}'(u) + v\vec{\delta}'(u)\ and\ (\sigma_1)_v = \vec{\delta}(u)$

Now $E_1 = \big(\vec{r}'(u) + v\vec{\delta}'(u)\big).\big(\vec{r}'(u) + v\vec{\delta}'(u)\big) = \|\vec{r}'(u)\|^2 + 2v\big(\vec{r}'.\vec{\delta}'\big) + v^2\big\|\vec{\delta}'(u)\big\|^2$
$\Longrightarrow E_1 = 1 + 2v\big(\vec{r}'.\vec{\delta}'\big) + v^2\big(\vec{\delta}'.\vec{\delta}'\big)$,

$G_1 = (\sigma_1)_v.(\sigma_1)_v = \big(\vec{\delta}.\vec{\delta}\big) = \big\|\vec{\delta}(u)\big\|^2 = 1$,

$F_1 = \big(\vec{r}'(u) + v\vec{\delta}'(u)\big).\vec{\delta}(u) = \vec{r}'.\vec{\delta} + v\vec{\delta}'.\vec{\delta} = \big(\vec{r}'.\vec{\delta}\big) + 0 = \big(\vec{r}'.\vec{\delta}\big)\ \because \vec{\delta} \perp \vec{\delta}'\ as\ \vec{t} \perp \vec{n}$

Hence fundamental form is

$$E_1du^2 + G_1dv^2 + 2F_1dudv = \big[1 + 2v\big(\vec{r}'.\vec{\delta}'\big) + v^2\big(\vec{\delta}'.\vec{\delta}'\big)\big]du^2 + dv^2 + 2\big(\vec{r}'.\vec{\delta}\big)dudv \ldots\ldots (i)$$

And hence mapping is conformal to $\sigma_2(u,v) = (u, 0, v)$ iff whose 1st fundamental form is
$E_2du^2 + G_2dv^2 + 2F_2dudv = du^2 + dv^2$

Now equation (i) holds when $\left[1+2v(\vec{r}'.\vec{\delta}')+v^2(\vec{\delta}'.\vec{\delta}')\right]=1\ and\ 2(\vec{r}'.\vec{\delta})=0$
$\Rightarrow 2v(\vec{r}'.\vec{\delta}')+v^2(\vec{\delta}'.\vec{\delta}')=0\ and\ (\vec{r}'.\vec{\delta})=0$

$\Rightarrow (2v\vec{r}'+v^2\vec{\delta}').\vec{\delta}'=0 \Rightarrow (2v\vec{r}'+v^2\vec{\delta}')\neq 0\ then\ \vec{\delta}'=0$
$\Rightarrow \vec{\delta}(u)=constant\ i.e. independent\ of\ u$

$also\ (\vec{r}'.\vec{\delta})=0 \Rightarrow integrating\ w.r.to'u'\ we\ get\ \vec{r}.\vec{\delta}=cosntant$

$\Rightarrow \sigma$ is conformal iff $\vec{\delta}=cosntant$ and $\vec{r}$ is contained in a plane i.e. $\vec{r}.\vec{\delta}=d(say)$

QUESTION: Show that the surface patch $\sigma(u,v)=[f(u,v),g(u,v),0]$ where $f\ and\ g$ are smooth functions on uv – plane is conformal if and only if
$f_u=g_v\ and\ f_v=-g_u\ (CR\ eq's)$ or $f_u=-g_v\ and\ f_v=g_u(anti\ CR\ eq's)$

SOLUTION: Given $\sigma(u,v)=[f(u,v),g(u,v),0] \Rightarrow \sigma_u=(f_u,g_u,0)\ \ and\ \sigma_v=(f_v,g_v,0)$

Now $E_1=\sigma_u.\sigma_u=\|\sigma_u\|^2={f_u}^2+{g_u}^2$, $G_1=\sigma_v.\sigma_v=\|\sigma_v\|^2={f_v}^2+{g_v}^2$,
$F_1=\sigma_u.\sigma_v=f_uf_v+g_ug_v$

Hence fundamental form is

$$E_1du^2+G_1dv^2+2F_1dudv=\left[{f_u}^2+{g_u}^2\right]du^2+\left[{f_v}^2+{g_v}^2\right]dv^2+(f_uf_v+g_ug_v)dudv$$

Now σ is conformal to the uv – plane iff ${f_u}^2+{g_u}^2=1\ \ldots\ldots(i)$, ${f_v}^2+{g_v}^2=1\ \ldots\ldots(ii)$,
$f_uf_v+g_ug_v=1\ \ldots\ldots(iii)$

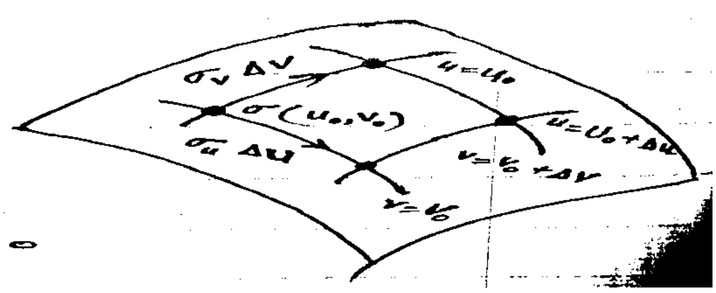
Now if $z=f_u+ig_u$ and $w=f_v+ig_v$ then σ is conformal iff $z\bar{z}=w\bar{w}\ \ \ldots\ldots(iv)$

Now we have $z\bar{w}+\bar{z}w=0 \Rightarrow z^2w\bar{w}+z\bar{z}w^2=0$ i.e. $\times\ ing\ with\ zw \Rightarrow z^2z\bar{z}+z\bar{z}w^2=0$
$\Rightarrow (z^2+w^2)|z|^2=0 \Rightarrow |z|^2\neq 0\ ,(z^2+w^2)=0 \Rightarrow z=\pm iw$

If $z=+iw \Rightarrow f_u+ig_u=i(f_v+ig_v) \Rightarrow f_u+ig_u=if_v-g_v \Rightarrow f_u=-g_v\ and\ f_v=g_u$

If $z=-iw \Rightarrow f_u+ig_u=-i(f_v+ig_v) \Rightarrow f_u+ig_u=-if_v+g_v \Rightarrow f_u=g_v\ and\ f_v=-g_u$

SURFACE AREA: Suppose that $\sigma: U\rightarrow R^3$ is a surface patch on a surface S. The image of σ is covered by the two families of parameterized curves obtained setting
$u=constant\ and\ v=constant$ respectively. Fixing $(u_0,v_0)\epsilon U$



Let $\sigma=\sigma(u,v) \Rightarrow d\sigma=\sigma_u\Delta u+\sigma_v\Delta v$

If $\Rightarrow v=constant\ then\ \Delta v=0 \Rightarrow d\sigma=\vec{a}=\sigma_u\Delta u$

and $\Rightarrow u=constant\ then\ \Delta u=0 \Rightarrow d\sigma=\vec{b}=\sigma_v\Delta v$

now as $area\ of\ parallelogram=\|\vec{a}\times\vec{b}\|=\|\sigma_u\Delta u\ \times\sigma_v\Delta v\|=\|\sigma_u\ \times\sigma_v\|\Delta u\Delta v$

then surface area is defined as “ The $A_\sigma(R)$ of part $\sigma(R)$ of surface patch $\sigma: U\rightarrow R^3$ corresponding to a region $R\subseteq U$ is $A_\sigma(R)=\iint\|\sigma_u\ \times\sigma_v\|dudv$

PREPOSITION: Show that $\|\sigma_u \times \sigma_v\| = \sqrt{EG - F^2}$

SOLUTION: Suppose $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are vectors in R^3 then we use the result

$(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{b}.\vec{c})(\vec{a}.\vec{d})$

Now $\|\sigma_u \times \sigma_v\|^2 = (\sigma_u \times \sigma_v).(\sigma_u \times \sigma_v) = (\sigma_u . \sigma_u).(\sigma_v . \sigma_v) - (\sigma_u . \sigma_v).(\sigma_u . \sigma_v)$
$\|\sigma_u \times \sigma_v\|^2 = EG - (\sigma_u . \sigma_v)^2 = EG - F^2 \Longrightarrow \|\sigma_u \times \sigma_v\| = \sqrt{EG - F^2}$

REMARK:

- We can also write surface are definition in the form $A_\sigma(R) = \iint \sqrt{EG - F^2} dudv$
- For a regular surface $EG - F^2 > 0$ (always) everywhere. Since for regular surface $\sigma_u \times \sigma_v$ is never zero.

PREPOSITION (PP): Show that the area of surface patch is unchanged by reparameterization.

SOLUTION: let $\sigma: U \to R^3$ be a surface patch and $\bar{\sigma}: \bar{U} \to R^3$ be a reparameterization of σ with reparameterization map $\varphi: \bar{U} \to U$ thus if $\varphi(\bar{u}, \bar{v}) = (u, v)$ we have $\bar{\sigma}(\bar{u}, \bar{v}) = \sigma(u, v) \ldots.(i)$ let $\bar{R} \subseteq \bar{U}$ be a region and let $R = \varphi(\bar{R}) \subseteq U$ then we have to show $A_\sigma(R) = A_{\bar{\sigma}}(\bar{R})$

Let $A_{\bar{\sigma}}(\bar{R}) = \iint \|\bar{\sigma}_{\bar{u}} \times \bar{\sigma}_{\bar{v}}\| d\bar{u}d\bar{v} \ldots\ldots(ii)$

Since $\bar{\sigma}_{\bar{u}} = \sigma_u \frac{\partial u}{\partial \bar{u}} + \sigma_v \frac{\partial v}{\partial \bar{u}}$ and $\bar{\sigma}_{\bar{v}} = \sigma_u \frac{\partial u}{\partial \bar{v}} + \sigma_v \frac{\partial v}{\partial \bar{v}}$ then

$\bar{\sigma}_{\bar{u}} \times \bar{\sigma}_{\bar{v}} = \left(\sigma_u \frac{\partial u}{\partial \bar{u}} + \sigma_v \frac{\partial v}{\partial \bar{u}}\right) \times \left(\sigma_u \frac{\partial u}{\partial \bar{v}} + \sigma_v \frac{\partial v}{\partial \bar{v}}\right)$

$\bar{\sigma}_{\bar{u}} \times \bar{\sigma}_{\bar{v}} = (\sigma_u \times \sigma_u) \frac{\partial u}{\partial \bar{u}} \frac{\partial u}{\partial \bar{v}} + (\sigma_u \times \sigma_v) \frac{\partial u}{\partial \bar{u}} \frac{\partial v}{\partial \bar{v}} + (\sigma_v \times \sigma_u) \frac{\partial v}{\partial \bar{u}} \frac{\partial u}{\partial \bar{v}} + (\sigma_v \times \sigma_v) \frac{\partial v}{\partial \bar{u}} \frac{\partial v}{\partial \bar{v}}$

$\bar{\sigma}_{\bar{u}} \times \bar{\sigma}_{\bar{v}} = 0 + (\sigma_u \times \sigma_v) \frac{\partial u}{\partial \bar{u}} \frac{\partial v}{\partial \bar{v}} + (\sigma_v \times \sigma_u) \frac{\partial v}{\partial \bar{u}} \frac{\partial u}{\partial \bar{v}} + 0 = \left[\frac{\partial u}{\partial \bar{u}} \frac{\partial v}{\partial \bar{v}} - \frac{\partial v}{\partial \bar{u}} \frac{\partial u}{\partial \bar{v}}\right] (\sigma_u \times \sigma_v)$

$\bar{\sigma}_{\bar{u}} \times \bar{\sigma}_{\bar{v}} = \begin{vmatrix} \frac{\partial u}{\partial \bar{u}} & \frac{\partial u}{\partial \bar{v}} \\ \frac{\partial v}{\partial \bar{u}} & \frac{\partial v}{\partial \bar{v}} \end{vmatrix} (\sigma_u \times \sigma_v) = \frac{\partial(u,v)}{\partial(\bar{u},\bar{v})} (\sigma_u \times \sigma_v) = J(\bar{\varphi})(\sigma_u \times \sigma_v)$

Replacing u *by* $\bar{u}$ *and* v *by* $\bar{v}$ $\bar{\sigma}_{\bar{u}} \times \bar{\sigma}_{\bar{v}} = \frac{\partial(u,v)}{\partial(u,v)} (\sigma_u \times \sigma_v) = (\sigma_u \times \sigma_v) \Longrightarrow \bar{\sigma}_{\bar{u}} \times \bar{\sigma}_{\bar{v}} = (\sigma_u \times \sigma_v)$
$(ii) \Longrightarrow A_{\bar{\sigma}}(\bar{R}) = \iint \|\bar{\sigma}_{\bar{u}} \times \bar{\sigma}_{\bar{v}}\| d\bar{u}d\bar{v} = \iint \|\sigma_u \times \sigma_v\| dvdv = A_\sigma(R) \Longrightarrow \boldsymbol{A_\sigma(R) = A_{\bar{\sigma}}(\bar{R})}$

EXAMPLE: Consider the torous $\sigma(u, v) = [(b + aSinv)Cosu,\ (b + aSinv)Sinu, aCosv]$ where $0 \le u, v \le 2\pi$ then find its surface area.

SOLUTION: Given $\sigma(u, v) = [(b + aSinv)Cosu,\ (b + aSinv)Sinu, aCosv]$
$\Longrightarrow \sigma_u = [-(b + aSinv)Sinu,\ (b + aSinv)Cosu, 0]$
$and\ \sigma_v = [aCosvCosu,\ aCosvSinu, -aSinv]$

Now $E = \sigma_u.\sigma_u = \|\sigma_u\|^2 = (b + aSinv)^2$ $, = \sigma_v.\sigma_v = \|\sigma_v\|^2 = a^2$, $F = \sigma_u.\sigma_v = 0$

For surface are we have $A_\sigma(R) = \iint \sqrt{EG - F^2} dudv = \int_0^{2\pi} \int_0^{2\pi} \sqrt{a^2(b + aSinv)^2 - 0} dudv$
$A_\sigma(R) = \int_0^{2\pi} du \int_0^{2\pi} a(b + aSinv) dv = |u|_0^{2\pi}.|abv - a^2 Cosv|_0^{2\pi} = 4\pi^2 ab$

GEODESICS: A curve $\vec{r}$ on a surface 'S' is called a geodesic if $\vec{r}''$ is zero or perpendicular to the tangent plane of surface at point $\vec{r}(t)$ i.e. parallel to its unit normal for all values of parameter t

PREPOSITION: Any geodesic has constant speed. i.e. $\|\vec{r}'\| = c \Longrightarrow \frac{d}{dt}\|\vec{r}'\| = 0$

PROOF: Let $\vec{r}(t)$ be a geodesic on surface then we have

$\frac{d}{dt}\|\vec{r}'\|^2 = \frac{d}{dt}(\vec{r}'.\vec{r}') = 2\vec{r}''.\vec{r}' \ldots\ldots(i)$ since $\vec{r}$ is a geodesic so $\vec{r}''$ is perpendicular to the tangent plane therefore $\vec{r}'' \perp \vec{r}'$ (tangent vector at 'P') $\Longrightarrow \vec{r}''.\vec{r}' = 0$

$(i) \Longrightarrow \frac{d}{dt}\|\vec{r}'\|^2 = 0 \Longrightarrow 2\|\vec{r}'\|\frac{d}{dt}\|\vec{r}'\| = 0 \Longrightarrow 2\|\vec{r}'\| \neq 0, \frac{d}{dt}\|\vec{r}'\| = 0$

PREPOSITION (PP;2018):

A curve on a surface is geodesic iff its geodesic curvature is zero everywhere.

PROOF: It is sufficient to consider a unit speed curve $\vec{r}$ contained in a surface patchσ. Let $\vec{n}$ be the standard unit normal of σ so that $K_g = \vec{r}''.(\vec{n} \times \vec{r}') \ldots\ldots(i)$

Since $\vec{r}$ is geodesic then $\vec{r}'' \parallel \vec{n}$ and obviously $\vec{r}'' \perp (\vec{n} \times \vec{r}') \Longrightarrow \vec{r}''.(\vec{n} \times \vec{r}') = 0 \Longrightarrow K_g = 0$

CONVERSLY: Suppose that $K_g = 0 \Longrightarrow \vec{r}''.(\vec{n} \times \vec{r}') = 0 \Longrightarrow \vec{r}'' \perp (\vec{n} \times \vec{r}') \Longrightarrow \vec{r}'' \perp \vec{r}' \; or \; \vec{r}'' \parallel \vec{n}$ $\Longrightarrow \vec{r}$ is geodesic.

PREPOSITION: Any straight line on a surface is geodesic.

PROOF: For a straight line we have $\vec{r}(t) = \vec{a} + \vec{b}t \Longrightarrow \vec{r}'(t) = \vec{b} \Longrightarrow \vec{r}''(t) = 0 \Longrightarrow \vec{r}$ is geodesic.

Question: If K and τ are the curvature and torsion of the geodesic then prove that $\tau = (K - K_a)(K_b - K)$

PROOF: Since $\tau = (K_b - K_a)Sin\psi Cos\psi$ Also $K = K_a Cos^2\psi + K_b Sin^2\psi$

where ψ is the angle between two directions geodesic tangent and line of curvature,

Now $(K - K_a) = K_a Cos^2\psi + K_b Sin^2\psi - K_a = K_a(Cos^2\psi - 1) + K_b Sin^2\psi$

$(K - K_a) = -K_a Sin^2\psi + K_b Sin^2\psi = Sin^2\psi(K_b - K_a)$

Also $(K_b - K) = K_b - K_a Cos^2\psi - K_b Sin^2\psi = K_b(1 - Sin^2\psi) - K_a Cos^2\psi$

$(K_b - K) = K_b Cos^2\psi - K_a Cos^2\psi = Cos^2\psi(K_b - K_a)$

Therefore $(K - K_a)(K_b - K) = [Sin^2\psi(K_b - K_a)][Cos^2\psi(K_b - K_a)]$

$(K - K_a)(K_b - K) = Sin^2\psi Cos^2\psi(K_b - K_a)^2$

Since $\tau = (K_b - K_a)Sin\psi Cos\psi$

Therefore $(K - K_a)(K_b - K) = \tau^2$

Implies $\tau \neq (K - K_a)(K_b - K)$

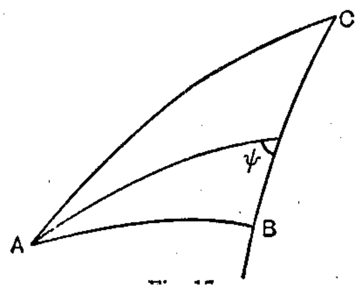
Our correction is wrong and it is dimensionally incorrect.

And the correct answer is $(K - K_a)(K_b - K) = \tau^2$

GEODESIC TRIANGLE: A curvilinear triangle ABC bounded by a geodesics.

GAUSS'S THEOREM: The whole second curvature of a geodesic triangle is equal to the excess of the sum of the triangle over two right angles.

PROOF: Let us choose geodesic polar coordinates with the vertex A as pole. Then the specific curvature is $K = -\frac{1}{D}\frac{\partial^2 D}{\partial u^2}$ and the area of an element of the surface is Ddudv. Consequently the whole second curvature of the geodesic triangle is $\Omega = \iint KdS = -\iint \frac{\partial^2 D}{\partial u^2} dudv$



Integrate first w.r.to 'u' from the pole A to the side BC.

Then since at the pole D_1 is equal to unity we find on integration

$\Omega = \int (1 - D_1)dv \Rightarrow \Omega = \int 1dv + \int (-D_1)dv \Rightarrow \Omega = \int dv + \int d\Psi$ Where $(-D_1)dv = d\Psi$

Now $\int_B^C dv = m\angle A$ and $\int d\Psi = \mathrm{C} - (\pi - B)$ then

$\Omega = \int dv + \int d\Psi \;\; = A + B + C - \pi$ This is required whole second curvature of triangle.

GAUSS'S EQUATIONS

$$EK = (\Gamma_{uu}^{v})_v - (\Gamma_{uv}^{v})_u + \Gamma_{uu}^{u}\Gamma_{uv}^{v} + \Gamma_{uu}^{v}\Gamma_{vv}^{v} - (\Gamma_{uv}^{v})^2$$

$$FK = (\Gamma_{uv}^{u})_u - (\Gamma_{uu}^{u})_v + \Gamma_{uv}^{v}\Gamma_{uv}^{u} - \Gamma_{uu}^{v}\Gamma_{vv}^{u}$$

$$FK = (\Gamma_{uv}^{v})_u - (\Gamma_{vv}^{v})_u + \Gamma_{uv}^{u}\Gamma_{uv}^{v} - \Gamma_{vv}^{u}\Gamma_{uu}^{v}$$

$$GK = (\Gamma_{vv}^{u})_u - (\Gamma_{uv}^{u})_v + \Gamma_{vv}^{u}\Gamma_{uu}^{u} + \Gamma_{vv}^{v}\Gamma_{uv}^{u} - (\Gamma_{uv}^{u})^2 - \Gamma_{uv}^{v}\Gamma_{uv}^{u}$$

CODAZZI'S EQUATIONS

$$L_v - M_u = L\Gamma_{uv}^{u} + M(\Gamma_{uv}^{u} - \Gamma_{uu}^{u}) - N\Gamma_{uu}^{v}$$

$$M_v - N_u = L\Gamma_{vv}^{u} + M(\Gamma_{vv}^{v} - \Gamma_{uv}^{u}) - N\Gamma_{uv}^{v}$$

حرفِ آخر (25-05-2020)

خوش رہیں خوشیاں بانٹیں اور جہاں تک ہو سکے دوسروں کے لیے آسانیاں پیدا کریں۔

اللہ تعالیٰ آپ کو زندگی کے ہر موڑ پر کامیابیوں اور خوشیوں سے نوازے۔(اٰمین)

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