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METRIC SPACE★★★

⇒ **Metric Space :-** Let X be a non-empty set. Let $d: X \times X \to \mathbb{R}$ be a non-negative real valued function which satisfies the following axioms

(i) $d(x_1, x_2) = 0 \iff x_1 = x_2 \quad \forall\ x_1, x_2 \in X$

(ii) $d(x_1, x_2) = d(x_2, x_1) \quad \forall\ x_1, x_2 \in X$ [Symmetric Property]

(iii) $d(x_1, x_2) \leq d(x_1, x_3) + d(x_3, x_2) \quad \forall\ x_1, x_2, x_3 \in X$ (Triangle inequality)

Then d is called metric on X and pair (X, d) is called metric space.

★★★

Question:- Let $\mathbb{R}$ be a set of real numbers. Define $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $d(x_1, x_2) = |x_1 - x_2| \quad \forall\ x_1, x_2 \in \mathbb{R}$. Show that "$d$" is metric on $\mathbb{R}$.

Solution:-

(i) As $|x_1 - x_2| \geq 0$

$\Rightarrow d(x_1, x_2) \geq 0 \quad \forall\ x_1, x_2 \in \mathbb{R}$

(ii) Let $d(x_1, x_2) = 0$

$\iff |x_1 - x_2| = 0$

$\iff x_1 - x_2 = 0$

$\iff x_1 = x_2 \quad \forall\ x_1, x_2 \in \mathbb{R}$

(iii) $d(x_1, x_2) = |x_1 - x_2|$

$= |-1(x_2 - x_1)|$

$= |-1|\,|x_2 - x_1| \quad \because |ab| = |a|\,|b|$

$= 1\,|x_2 - x_1|$

$= |x_2 - x_1|$

$d(x_1, x_2) = d(x_2, x_1) \quad \forall\ x_1, x_2 \in \mathbb{R}$

(iv) $d(x_1, x_3) = |x_1 - x_3|$

$= |(x_1 - x_2) + (x_2 - x_3)|$

$\leq |x_1 - x_2| + |x_2 - x_3|$

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$d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3), \forall x_1, x_2, x_3 \in \mathbb{R}$

$\Rightarrow$ d is metric on R

$\Rightarrow$ $(\mathbb{R}, d)$ is metric space

★★★

Question:- Let $X =]0,1[\subseteq \mathbb{R}$. Show that $d: X \times X \to \mathbb{R}$ define $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$ is metric on $\mathbb{R}$

Solution:- (i) $\left|\frac{1}{x} - \frac{1}{y}\right| \geq 0$

$\Rightarrow d(x,y) \geq 0 \quad \forall x, y \in X$

(ii) Let $d(x,y) = 0$

$\Leftrightarrow \left|\frac{1}{x} - \frac{1}{y}\right| = 0$

$\Leftrightarrow \frac{1}{x} - \frac{1}{y} = 0$

$\Leftrightarrow \frac{1}{x} = \frac{1}{y}$

$\Leftrightarrow x = y \quad \forall x, y \in X$

صرف zero کا mod ہی zero کے برابر ہوتا ہے اور جب دو غرز کے Reciprocal برابر ہوں تو وہ دو غرز بھی برابر ہوتے ہیں

(iii) $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$

$= \left|\frac{1}{y} - \frac{1}{x}\right|$

mod کے اندر والی ویلیوز کو آپس میں change کر سکتے ہیں

$d(x,y) = d(y,x) \quad \forall x, y \in X$

(iv)

To Prove $d(x,y) + d(y,z) \geq d(x,z) \quad \forall x, y, z \in X$

Now $d(x,z) = \left|\frac{1}{x} - \frac{1}{z}\right|$

$= \left|\left(\frac{1}{x} - \frac{1}{y}\right) + \left(\frac{1}{y} - \frac{1}{z}\right)\right|$

$\leq \left|\frac{1}{x} - \frac{1}{y}\right| + \left|\frac{1}{y} - \frac{1}{z}\right|$

$\Rightarrow d(x,z) \leq d(x,y) + d(y,z) \quad \forall x, y, z \in X$

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$\Rightarrow$ d is metric on X
$\Rightarrow$ (X, d) is metric space.

* * *

Note:- $d(x,y) = |x-y|$ is called a usual metric on $\mathbb{R}$.

* * *

Question:- Let X be the set of all continuous real valued functions defined on $[0,1]$. for $f, g \in X$ defined $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$. Prove that (X, d) is a metric space.

Solution:- (i)

As $|f(x) - g(x)| \geq 0$

$\Rightarrow \int_0^1 |f(x) - g(x)|\, dx \geq 0$

$\Rightarrow d(f,g) \geq 0 \quad \forall\ f, g \in X$

(ii) Let $d(f,g) = 0$

$\Leftrightarrow \int_0^1 |f(x) - g(x)| = 0$

$\Leftrightarrow |f(x) - g(x)| = 0$

$\Leftrightarrow f(x) - g(x) = 0$

$\Leftrightarrow f(x) = g(x) \quad \forall\ x \in]0,1[$

$\Leftrightarrow f = g \quad \forall\ f, g \in X$

(iii) $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$

$= \int_0^1 |g(x) - f(x)|\, dx$

$d(f,g) = d(g,f) \quad \forall\ f, g \in X$

(iv) $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$

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$$d(f,g) = \int_0^1 |f(x) - h(x) + h(x) - g(x)|\, dx$$

$$= \int_0^1 |(f(x) - h(x)) + (h(x) - g(x))|\, dx$$

$$\leq \int_0^1 |f(x) - h(x)| + |h(x) - g(x)|\, dx$$

$$= \int_0^1 |f(x) - h(x)|\, dx + \int_0^1 |h(x) - g(x)|\, dx$$

$$= d(f,h) + d(h,g)$$

$$\Rightarrow d(f,g) \leq d(f,h) + d(h,g) \quad \forall\ f,g,h \in X$$

$\Rightarrow$ d is metric on X

$\Rightarrow (X,d)$ is metric space.

★★★

Question:- Let $X = \mathbb{R}$ and $d'(x_1,x_2) = |x_1| + |x_2|$ $\forall$ $x_1, x_2 \in \mathbb{R}$ then d' is not a metric space Prove?

Solution:- (i) As $|x_1| \geq 0$ & $|x_2| \geq 0$

$\Rightarrow |x_1| + |x_2| \geq 0 \quad \forall\ x_1, x_2 \in \mathbb{R}$

$\Rightarrow d' \geq 0$

(ii) Let $d'(x_1,x_2) = 0$

$\Rightarrow |x_1| + |x_2| = 0$

$\Rightarrow |x_1| = 0$ & $|x_2| = 0$

$\Rightarrow x_1 = 0$ & $x_2 = 0$

$\Rightarrow x_1 = x_2 \quad \forall\ x_1, x_2 \in \mathbb{R}$

Conversely Let $x_1 = x_2$

$\Rightarrow |x_1| = |x_2|$

$\Rightarrow |x_1| + |x_2| = 0$

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$\Rightarrow \quad d'(x_1, x_2) \neq 0$

$\Rightarrow$ d' is not a metric space.

★★★

Question:- Let $\mathbb{R}^n$ be the set of all ordered n-tuples that is $\mathbb{R}^n = \{x = (x_1, x_2, x_3, \cdots, x_n) : x_i \in \mathbb{R}\}$ define $d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$ Show that d is metric on $\mathbb{R}^n$.

Solution:- (i) As $\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \geq 0$

$\Rightarrow d(x,y) \geq 0 \quad \forall x, y \in \mathbb{R}^n$

(ii) Let $d(x,y) = 0$

$\Leftrightarrow \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = 0$

$\Leftrightarrow \sum_{i=1}^{n} (x_i - y_i)^2 = 0$

$\Leftrightarrow (x_i - y_i)^2 = 0$

$\Leftrightarrow x_i - y_i = 0$

$\Leftrightarrow x_i = y_i \quad \forall i = 1, 2, 3, \cdots, n$

$\Leftrightarrow (x_1, x_2, x_3, \cdots, x_n) = (y_1, y_2, y_3, \cdots, y_n)$

$\Leftrightarrow x = y \quad \forall x, y \in \mathbb{R}^n$

(iii) As $d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

$= \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2}$

$d(x,y) = d(y,x) \quad \forall x, y \in \mathbb{R}^n$

(iv) Let

$x = (x_1, x_2, x_3, \cdots, x_n)$, $y = (y_1, y_2, y_3, \cdots, y_n)$

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and $z = (z_1, z_2, z_3, \cdots, z_n)$, where $x, y, z \in \mathbb{R}^n$

Now $$d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

$$= \sqrt{\sum_{i=1}^{n} [(x_i - z_i) + (z_i - y_i)]^2}$$

$$= \left[\sum_{i=1}^{n} [(x_i - z_i) + (z_i - y_i)]^2\right]^{1/2}$$

$$\leq \left[\sum_{i=1}^{n} (x_i - z_i)^2\right]^{1/2} + \left[\sum_{i=1}^{n} (z_i - y_i)^2\right]^{1/2}$$ by Minkowski's inequality

$$= d(x,z) + d(z,y)$$

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y) \quad \forall\ x, y, z \in \mathbb{R}^n$

$\Rightarrow d$ is metric on $\mathbb{R}^n$

$\Rightarrow (\mathbb{R}^n, d)$ is a metric space.

Note:- d is called usual metric on $\mathbb{R}^n$

* * *

Question:- Show that $d_0(x,y) = \sup_{i=1}^{n} |x_i - y_i|$ is metric on $\mathbb{R}^n$.

Solution:- (i) As $\sup_{i=1}^{n} |x_i - y_i| \geq 0$

$\Rightarrow d_0(x,y) \geq 0$

(ii) Let $d_0(x,y) = 0 \quad \forall\ x, y \in \mathbb{R}^n$

$\Leftrightarrow \sup_{i=1}^{n} |x_i - y_i| = 0$

$\Leftrightarrow |x_i - y_i| = 0$

$\Leftrightarrow x_i - y_i = 0$

$\Leftrightarrow x_i = y_i$

$\Leftrightarrow (x_1, x_2, x_3, \cdots, x_n) = (y_1, y_2, y_3, \cdots, y_n)$

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$\Leftrightarrow \quad x = y \qquad \forall \; x, y \in \mathbb{R}^n$

(iii) $d_0(x,y) = \sup_{i=1}^{n} |x_i - y_i|$

$= \sup_{i=1}^{n} |y_i - x_i|$

$d_0(x,y) = d_0(y,x) \qquad \forall \; x, y \in \mathbb{R}^n$

(iv) Let $x = (x_1, x_2, x_3, \cdots, x_n)$, $y = (y_1, y_2, y_3, \cdots y_n)$ & $z = (z_1, z_2, z_3, \cdots, z_n)$ where $x, y, z \in \mathbb{R}^n$

Now $d_0(x,y) = \sup_{i=1}^{n} |x_i - y_i|$

$= \sup_{i=1}^{n} |x_i - z_i + z_i - y_i|$

$= \sup_{i=1}^{n} |(x_i - z_i) + (z_i - y_i)|$

$\leq \sup_{i=1}^{n} |x_i - z_i| + \sup_{i=1}^{n} |z_i - y_i|$

$= d_0(x,z) + d_0(z,y)$

$\Rightarrow d_0(x,y) \leq d_0(x,z) + d_0(z,y) \qquad \forall \; x, y, z \in \mathbb{R}^n$

$\Rightarrow d$ is metric on $\mathbb{R}^n$

* * *

Question:- Let (X,d) be the metric space & $d' : X \times X \to \mathbb{R}$ be defined by

$$d'(x_1, x_2) = \frac{d(x_1, x_2)}{1 + d(x_1, x_2)} \qquad \forall \; x_1, x_2 \in X$$

Prove that d' is metric on X.

Solution:-

(i) As $d(x_1, x_2) \geq 0$ $\because d$ is metric

$\Rightarrow \dfrac{d(x_1, x_2)}{1 + d(x_1, x_2)} \geq 0$

$\Rightarrow d'(x_1, x_2) \geq 0$

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(ii) Let $d'(x_1, x_2) = 0$

$\Leftrightarrow \dfrac{d(x_1, x_2)}{1 + (x_1, x_2)} = 0$

$\Leftrightarrow d(x_1, x_2) = 0$

$\Leftrightarrow x_1 = x_2$ $\because d$ is metric

(iii) $d'(x_1, x_2) = \dfrac{d(x_1, x_2)}{1 + d(x_1, x_2)}$

$= \dfrac{d(x_2, x_1)}{1 + d(x_2, x_1)}$ $\because d$ is metric

$\Rightarrow d'(x_1, x_2) = d'(x_2, x_1) \quad \forall\ x_1, x_2 \in X$

(iv) Let $x_1, x_2, x_3 \in X$

Then $d'(x_1, x_3) = \dfrac{d(x_1, x_3)}{1 + d(x_1, x_3)}$

As $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$ $\because d$ is metric

$\Rightarrow \dfrac{d(x_1, x_3)}{1 + d(x_1, x_3)} \leq \dfrac{d(x_1, x_2) + d(x_2, x_3)}{1 + d(x_1, x_2) + d(x_2, x_3)}$ $\because$ If $a \leq b \Rightarrow \frac{a}{1+a} \leq \frac{b}{1+b}$

$\Rightarrow d'(x_1, x_3) \leq \dfrac{d(x_1, x_2)}{1 + d(x_1, x_2) + d(x_2, x_3)} + \dfrac{d(x_2, x_3)}{1 + d(x_1, x_2) + d(x_2, x_3)}$

$\leq \dfrac{d(x_1, x_2)}{1 + d(x_1, x_2)} + \dfrac{d(x_2, x_3)}{1 + d(x_2, x_3)}$

$\Rightarrow d'(x_1, x_3) \leq d'(x_1, x_2) + d'(x_2, x_3) \quad \forall\ x_1, x_2, x_3 \in X$

$\because d$ is metric on X

$\Rightarrow (d, X)$ is metric space.

★ ★ ★

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Question:- Show that every non-empty set can be given a metric & Hence can be converted into metric space.

Solution:- Let X be a non-empty set

Define $d: X \times X \longrightarrow \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Now we show that d is metric on X

(i) $d(x,y) = 0$ or $d(x,y) = 1$

$\Rightarrow d(x,y) \geq 0 \quad \forall\ x, y \in X$

(ii) let $d(x,y) = 0$

$\Longleftrightarrow x = y$ by definition of d

$\forall\ x, y \in X$

(iii) Case 1 If $x = y$

$\Rightarrow d(x,y) = 0$ As $x = y \Rightarrow y = x \Rightarrow d(y,x) = 0$

$\Rightarrow d(x,y) = d(y,x)$

Case 2 If $x \neq y$

$\Rightarrow d(x,y) = 1$

As $x \neq y \Rightarrow y \neq x$

$\Rightarrow d(y,x) = 1$

$\Rightarrow d(x,y) = d(y,x) \quad \forall\ x, y \in X$

$\therefore$ in either case $d(x,y) = d(y,x)$

(iv) Case 1 let $x, y, z \in X$

If $x = y = z$

Then $d(x,y) = 0$, $d(y,z) = 0$, $d(x,z) = 0$

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$\Rightarrow d(x,y) = d(x,z) + d(z,y)$

Case 2 when $x \neq y \neq z$

Then $d(x,y)=1$, $d(x,z)=1$, $d(z,y)=1$

$\Rightarrow d(x,y) = 1 < 1+1$

$\Rightarrow d(x,y) < d(x,z)+d(z,y)$

The other case can be verified similarly

Therefore in either case

$$d(x,y) \leq d(x,z)+d(z,y)$$

$\Rightarrow$ d is metric on X

★★★

★ A sequence is said to be bounded if set of the points of the sequence is bounded.

★★★

Q:- Let l^∞ be the space of all bounded sequences define d on l^∞ by $d(x,y) = \sup_{i=1}^{n} |x_n - y_n|$ then d is metric on l^∞, where

$$x = \{x_n\},\ y = \{y_n\} \in l^\infty$$

Solution:-

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⇒ **Open Ball or Open Sphere:-** Let (X,d) be a metric space. For any point $x \in X$ and real number $r > 0$. The subset $B(x,r) = \{y \mid y \in X \wedge d(x,y) < r\}$ is called open ball or open sphere with centre at x and radius r. It is also denoted by $S_r(x)$ or $B_r(x)$

⇒ **Closed Ball or Closed Sphere:-** Let (X,d) be a metric space. For any point $x \in X$ and real number $r > 0$ the subset $\overline{B}(x,r) = \{y \mid y \in X \wedge d(x,y) \leq r\}$ is called closed ball or closed sphere.

* * *

Question:- Find open ball in $\mathbb{R}$.

Solution:- Let $X = \mathbb{R}$ and metric on $\mathbb{R}$ is defined as $d(x,y) = |x-y|$

Now $B(x,r) = \{y \mid y \in \mathbb{R} \wedge d(y,x) < r\}$

$= \{y \mid y \in \mathbb{R} \wedge |y-x| < r\}$

$= \{y \mid y \in \mathbb{R} \wedge -r < y-x < r\}$

$= \{y \mid y \in \mathbb{R} \wedge x-r < y < x+r\}$

$=]x-r, x+r[$

Therefore in $\mathbb{R}$ open balls are open intervals.

* * *

Question:- Find open ball for discreate metric space.

Solution:- The discreate metric space is defined as

$$d(x_1,x_2) = \begin{cases} 0 & \text{if } x_1 = x_2 \\ 1 & \text{if } x_1 \neq x_2 \end{cases} \quad \forall\, x_1, x_2 \in X$$

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The open ball is defined as

$$B(x,r) = \{y \mid y \in X \wedge d(x,y) < r\}$$

Case 1:- When $r = 1$

$$B(x,r) = \{y \mid y \in X \wedge d(y,x) < 1\}$$
$$= \{y \mid y \in X \wedge d(y,x) = 0\}$$
$$= \{y \mid y \in X \wedge y = x\}$$
$$= \{x\}$$

Case 2:- When $r < 1$

let $r = 1 - h$, $0 < h < 1$

$$B(x,r) = \{y \mid y \in X \wedge d(x,y) < 1 - h\}$$
$$= \{y \mid y \in X \wedge d(y,x) = 0\}$$
$$= \{y \mid y \in X \wedge y = x\}$$
$$= \{x\}$$

Case 3:- When $r > 1$

let $r = 1 + h$, $h > 0$

$$B(x,r) = \{y \mid y \in X \wedge d(y,x) < 1 + h\}$$
$$= \{y \mid y \in X \wedge d(y,x) = 0 \text{ , } d(y,x) = 1\}$$
$$= \{y \mid y \in X \wedge y = x \text{ , } y \neq x\}$$
$$= X$$

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⇒ **Distance Between Sets :-** Let A and B be the subsets of a metric space (X,d). The distance between A and B is defined as

$$d(A,B) = \inf_{\substack{a\in A \\ b\in B}} d(a,b)$$

If $A=\{x\}$ is a singleton subset of X then $d(A,B)$ is denoted by $d(x,B)$ and is called the distance of the point x from the set B. In this case

$$d(x,B) = \inf_{b\in B} d(x,b)$$

* * *

Example :- $X=\mathbb{R}$, $d(x,y)=|x-y|$ $\forall\, x,y\in\mathbb{R}$
$A=\{1,2,3\}$, $B=\{4,5\}$ find distance b/w A & B.

Solution :- $A=\{1,2,3\}$, $B=\{4,5\}$

$d(1,4)=|1-4|=3$, $d(1,5)=|1-5|=4$

$d(2,4)=|2-4|=2$, $d(2,5)=|2-5|=3$

$d(3,4)=|3-4|=1$, $d(3,5)=|3-5|=2$

$$d(A,B) = \inf_{\substack{a\in A \\ b\in B}} d(a,b)$$

$$= \inf\{3,4,2,3,1,2\}$$

$$= 1$$

Distance b/w A & B is 1

* * *

⇒ **Diameter of A Set :-** For any non-empty subset A of a metric space (X,d) the diameter of A is denoted by $\delta(A)$ and is defined as $d(A) = \delta(A) = \sup_{x,y\in A} d(x,y)$

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Example:- Find the diameter of $A=\{1,2,3\}$ in metric space $X=\mathbb{R}$

Solution:- $X=\mathbb{R}$, $A=\{1,2,3\}$

Usual metric on $\mathbb{R}$ is defined as

$$d(x,y) = |x-y|$$

Now $d(1,1) = |1-1| = 0$, $d(1,2) = |1-2| = 1$

$d(1,3) = |1-3| = 2$, $d(2,2) = |2-2| = 0$

$d(2,3) = |2-3| = 1$, $d(3,3) = |3-3| = 0$

$$d(A) = \delta(A) = \sup_{x,y \in A} d(x,y)$$

$$= \sup\{0,1,2,0,1,0\}$$

$$= 2$$

Diameter of set A is 2

Theorem:- Let (X,d) be a metric space & A be a subset of X. Then for any $x,y \in X$

$$|d(x,A) - d(y,A)| \leq d(x,y)$$

Proof:- for any $z \in A$

$$d(x,z) \leq d(x,y) + d(y,z) \quad \text{---------} \quad (1)$$

Now $$d(x,A) = \inf_{z \in A} d(x,z)$$

$$\leq \inf_{z \in A} \{d(x,y) + d(y,z)\} \quad \text{using (1)}$$

$$= \inf_{z \in A} d(x,y) + \inf_{z \in A} d(y,z)$$

$$= d(x,y) + \inf_{z \in A} d(y,z)$$

$$d(x,A) \leq d(x,y) + d(y,A)$$

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$$d(x,A) - d(y,A) \leq d(x,y) \quad \text{---------- (2)}$$

Interchanging x with y

$$d(y,A) - d(x,A) \leq d(y,x)$$

$$-\{d(x,A) - d(y,A)\} \leq d(x,y) \quad \text{---------- (3)}$$

from eqn (2) & (3)

$$|d(x,A) - d(y,A)| \leq d(x,y)$$ proved

* * *

Theorem:- Show that diameter of a closed ball $\bar{B}(a,r)$ is less or equal to $2r$

i.e $\delta(\bar{B}(a,r)) \leq 2r$

Proof:- Let $(x,y) \in \bar{B}(a,r)$

As $x \in \bar{B}(a,r)$ & $d(x,a) \leq r$ ---------- (1)

& $y \in \bar{B}(a,r)$ & $d(y,a) \leq r$ ---------- (2)

Now $d(x,y) \leq d(x,a) + d(a,y)$

$d(x,y) \leq d(x,a) + d(y,a)$

$\leq r + r$ using (1) & (2)

$d(x,y) \leq 2r$

Taking supremum on both sides

$$\Rightarrow \sup_{x,y \in \bar{B}(a,r)} d(x,y) \leq \sup_{x,y \in \bar{B}(a,r)} 2r$$

$\delta(\bar{B}(a,r)) \leq 2r$ Proved.

* * *

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⇒ **Definition:-** Let (X,d) be a metric space and $A \subseteq X$. Then A is said to be bounded if $\delta(A)$ is finite.

Theorem: Let (X,d) be a metric space and A, B be the two bounded subsets of X then prove that $A \cup B$ is bounded.

Proof: Let $x, y \in A \cup B$

Then $x, y \in A$ or $x, y \in B$ or $x \in A$ & $y \in B$

Case 1:-

If $(x,y) \in A$

Then $d(x,y) \leq \delta(A)$

$\sup_{x,y \in A \cup B} d(x,y) \leq \delta(A)$

$\delta(A \cup B) \leq \delta(A)$

As A is bounded so $\delta(A)$ is finite

$\Rightarrow$ R.H.S is finite. So left hand side is finite

So $(A \cup B)$ is finite.

Case 2:- If $x, y \in B$

Then $d(x,y) \leq \delta(B)$

$\sup_{x,y \in A \cup B} d(x,y) \leq \delta(B)$

$\delta(A \cup B) \leq \delta(B)$

As B is bounded so $\delta(B)$ is finite

So R.H.S is finite $\Rightarrow$ L.H.S is finite

$\Rightarrow \delta(A \cup B)$ is finite.

case 3:- If $x \in A$ & $y \in B$

Then $d(x,y) \leq d(x,a) + d(a,y)$ where $a \in A$

$d(x,y) \leq d(x,a) + d(a,b) + d(b,y)$ where $b \in B$

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$$\sup_{x,y\in A\cup B} d(x,y) \leq \sup_{x\in A} d(x,a) + d(a,b) + \sup_{y\in B} d(b,y)$$

$$\delta(A\cup B) \leq \delta(A) + d(a,b) + \delta(B)$$

$\Rightarrow$ R.H.S is finite ($\because$ A & B are finite)

$\Rightarrow$ L.H.S is finite

$\Rightarrow$ $A\cup B$ is bounded.

$\Rightarrow$ **Open Set:-** Let (X,d) be a metric space and $A \subseteq X$ then A is called an open set if for each $x \in A$ there exist an open ball $B(x,r)$ s.t $x \in B(x,r) \subseteq A$

Theorem- Show that every subset in a discrete metric space is an open set.

Proof Let (X,d) be a discrete metric space then

$$d(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x\neq y \end{cases}$$

Let $A \subseteq X$. To prove A is open

If $A = \phi$ Then A is open

If $A \neq \phi$ Then to prove A is open

Let $x \in A$ and $r = 1/2$

$$B(x,\tfrac{1}{2}) = \{y \in X \mid d(x,y) < 1/2\}$$

$$= \{y \in X \mid d(x,y) = 0\}$$

$$= \{y \in X \mid y = x\}$$

$$B(x,\tfrac{1}{2}) = \{x\} \subseteq A$$

$\Rightarrow B(x,\frac{1}{2}) \subseteq A \Rightarrow x \in B(x,\frac{1}{2}) \subseteq A$

$\therefore$ A is open.

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Theorem:- Every open ball in a metric space (X,d) is an open set.

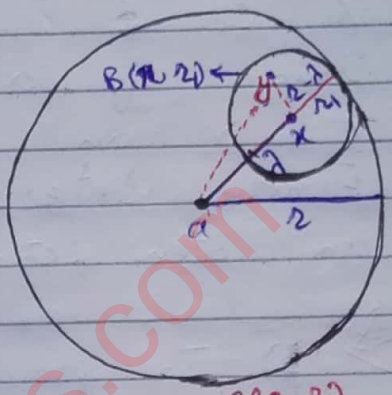
Proof Let $B(a,r)$ be an open ball in a metric space (X,d) with centre "a" & radius "r"

Let $x \in B(a,r)$

$\Rightarrow d(x,a) < r$

let $d(x,a) = \lambda$ ——— ①

& $r_1 = r - \lambda$ ——— ②



~~Now~~

~~$d(x,a) \leq d(y,x) + d(x,a)$~~

~~$d(y,a)$~~ Consider $B(x,r_1)$

We claim that $B(x,r_1) \subseteq B(x,r)$

Let $y \in B(x,r_1)$

$d(y,x) < r_1$ ——— ③

Now $d(y,a) \leq d(y,x) + d(x,a)$

$d(y,a) < r_1 + \lambda$ by ① & ②

$\Rightarrow d(y,a) < r - \lambda + \lambda$ by ②

$\Rightarrow d(y,a) < r \Rightarrow y \in B(a,r)$

As $y \in B(x,r_1)$ & then $y \in B(a,r)$

$\Rightarrow B(x,r_1) \subseteq B(a,r)$

$\therefore B(a,r)$ is an open set.

★★★

Example:- Let $X = \mathbb{R}$ and $d(x,y) = |x-y|$. Let $x \in X$ and $r > 0$. Find an open set in $\mathbb{R}$.

Solution:- $B(x,r) = \{y \in X \mid d(x,y) < r\}$

$= \{y \in X \mid |y-x| < r\}$

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$B(x,2) =]x-2, x+2[$

Since we know that every open ball is an open set therefore $]x-2, x+2[$ is an open set.

***

Theorem:- Let (X,d) be a metric space then

(i) X & ϕ are open set

(ii) Union of any number of open sets in X is an open set.

(iii) Finite Intersection of open sets in X is an open set.

Proof 1) To prove X is an open set

Let $x \in X$ & $r > 0$

Then $B(x,r) = \{y \in X \mid d(y,x) < r\} \subseteq X$

$\Rightarrow X$ is an open set.

To prove ϕ is an open set. We have to prove for every $x \in \phi$ there exist an open ball $B(x,r)$ s.t $x \in B(x,r) \subseteq \phi$

But ϕ has no element so condition is trivially satisfied. Hence ϕ is open.

2) To prove union of any number of open set in X is an open set.

Let $\{U_\alpha : \alpha \in I\}$ be a family of open sets in X. To Prove $\bigcup_{\alpha \in I} U_\alpha$ is open set

Let $x \in \bigcup_{\alpha \in I} U_\alpha$

$\Rightarrow x \in U_\alpha$ for some $\alpha \in I$

As $x \in U_\alpha$ and U_α is an open set then $\exists$ an open ball $B(x,r)$ s.t

$x \in B(x,r) \subseteq U_\alpha$

As $U_\alpha \subseteq \bigcup_{\alpha \in I} U_\alpha$

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$\Rightarrow x \in B(x,r) \subseteq U_\alpha \subseteq \bigcup_{\alpha\in I} U_\alpha$

$\Rightarrow x \in B(x,r) \subseteq \bigcup_{\alpha\in I} U_\alpha$

$\Rightarrow \bigcup_{\alpha\in I} U_\alpha$ is open set.

(3) To prove finite intersection of open sets in X is an open set in X.

Let $\{U_i : i = 1,2,3,\cdots n\}$ be a finite family of open sets in X.

To Prove $\bigcap_{i=1}^{n} U_i$ is an open set.

Let $x \in \bigcap_{i=1}^{n} U_i \Rightarrow x \in U_i \ \forall i = 1,2,3,\cdots,n$

As $x \in U_i$ and U_i is an open set. Then there exist an open ball $B(x,r_i)$ s.t

$x \in B(x,r_i) \subseteq U_i$, $i = 1,2,3,\cdots,n$

Put $r = \min\{r_1, r_2, r_3, \cdots, r_n\}$

We claim that

$B(x,r) \subseteq B(x,r_i) \ \forall i$

Let $y \in B(x,r)$

$d(y,x) < r \leq r_i \quad \forall i$

$d(y,x) < r_i$

$y \in B(x,r_i)$

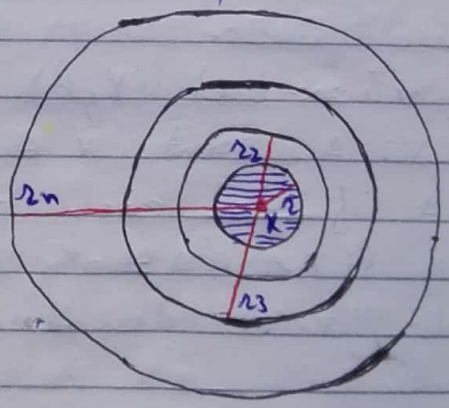
$B(x,r) \subseteq B(x,r_i) \quad \forall i$

As $B(x,r) \subseteq B(x,r_i) \subseteq U_i \quad \forall i$

$\Rightarrow B(x,r) \subseteq U_i \quad \forall i$

$\Rightarrow B(x,r) \subseteq \bigcap_{i=1}^{n} U_i$

$\Rightarrow \bigcap_{i=1}^{n} U_i$ is an open set.



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*Definition:- Let (X,d) be a metric space and $A \subseteq X$. A point $x \in A$ is called an interior point of A if there exist an open ball $B(x,r)$ s.t $x \in B(x,r) \subseteq A$.

Set of all interior points of A is called interior of A and it is denoted by $Int(A)$ or A°.

*Definition:- Let (X,d) be a metric space and $A \subseteq X$. a point $x \in X$ is called exterior point of A if $\exists$ an open ball $B(x,r)$ s.t $B(x,r) \cap A = \phi$ or $B(x,r) \subseteq X - A$.

The set of all exterior points of A is called exterior of A and it is denoted by $Ext(A)$.

Result:- If $A \cap B = \phi$ then $A \subseteq B'$ and $B \subseteq A'$

Note:- If x is an exterior point of A then
$B(x,r) \cap A = \phi \Rightarrow B(x,r) \subseteq A'$
$\Rightarrow x$ is an interior point of $A' = X - A$
$\Rightarrow Ext(A) = Int(A')$

*Definition:- Let (X,d) be a metric space and $A \subseteq X$. A point $x \in X$ is called boundry point of A if for each open ball $B(x,r)$
$B(x,r) \cap A \neq \phi$ & $B(x,r) \cap A' \neq \phi$
And it is denoted by $b(A)$. [Boundry or Frantier Point]

***

Q:- Let $X = \mathbb{R}$ and $A = [0,1]$. Find interior, exterior and boundry of A.

Solution:- Let $x = 0 \in A$, $r > 0$

$B(x,r) =]x-r, x+r[$

$B(0,r) =]0-r, 0+r[$

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$B(0,r) =]-r, r[\not\subseteq A$

$\Rightarrow$ 0 is not an interior point of A

Similarly 1 is not an interior point of A.

$\Rightarrow Int(A) =]0,1[$

Now

$A' = \mathbb{R} - A \Rightarrow A' = \mathbb{R} - [0,1]$

$=]-\infty, 0[\cup]1, \infty[$

$Int(A') =]-\infty, 0[\cup]1, \infty[$

$Ext(A) = Int(A')$

$Ext(A) =]-\infty, 0[\cup]1, \infty[$

Now $B(0,r) \cap A \neq \phi$

& $B(0,r) \cap A' \neq \phi$

$\Rightarrow$ 0 is an boundry point of A

Similary 1 is an boundry point of A

$\Rightarrow b(A) = \{0,1\}$

* * *

Theorem:- Let (X,d) be a metric space and $A \subseteq X$. Then A is open set if and only if A is union of open balls.

Proof) Suppose A is an open set

To prove A is the union of open balls

Let $x \in A$. As A is open set then $\exists$ an open ball $B(x, r_x)$ s.t

$x \in B(x, r_x) \subseteq A$

$\Rightarrow \{x\} \subseteq B(x, r_x) \subseteq A$

$\Rightarrow \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} (x, r_x) \subseteq \bigcup_{x \in A} A$

$\Rightarrow A \subseteq \bigcup_{x \in A} (x, r_x) \subseteq A$

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$$\Rightarrow \quad A = \bigcup_{x \in A} B(x, r_x)$$

Conversely, Let A is the union of open balls.
To prove A is an open set
As every open ball is an open set and union of open sets is open. Therefore "A" is the unions of open balls is open
$\Rightarrow$ A is open set.

* * *

***Limit Point:-** Let (X,d) be a metric space and $A \subseteq X$. A point $x \in X$ is called limit point of A if each open ball with centre x contains at least one point of A different from x.
i.e $B(x,r) \cap A \setminus \{x\} \neq \phi$

***Derived Set:-** The set of all limit points of A is called derived set of A. and it is denoted by $D(A)$ or A^d or A'.

***Closed Set:-** Let (X,d) be a metric space and $A \subseteq X$ then A is closed if $D(A) \subseteq A$ or A is closed if it contains all of its limit points.

* * *

Example:- Let $X = \mathbb{R}$, $A \subseteq X$ and $A = [0,1[$. Find limit point of A and show that A is not closed.

Solution: $B(x,2) =]x-2, x+2[$

$B(0,2) =]0-2, 0+2[$

$=]-2, 2[$

Now $A \setminus \{0\} =]0,1[$

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$B(0,2) \cap A\setminus\{0\} =]-2,2[\cap]0,1[$

$=]0,2[\neq \phi$

$\Rightarrow$ 0 is the limit point of A

Similarly 1 is the limit point of A

$\Rightarrow D(A) = [0,1] \not\subseteq A$

$\Rightarrow$ A is not closed.

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Example:- Show that N = The set of natural numbers is closed in $\mathbb{R}$.

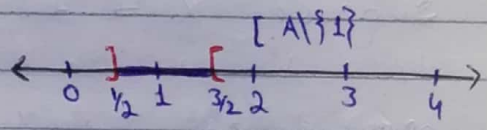
Solution:- $X = \mathbb{R}$, $A = N = \{1,2,3,\dots\}$

Let $x = 1$, $r = 1/2$

$B(x,r) =]x-r, x+r[$

$B(1, \frac{1}{2}) =]1-\frac{1}{2}, 1+\frac{1}{2}[$

$=]\frac{1}{2}, \frac{3}{2}[$



$A\setminus\{1\} = \{2,3,4,\dots\}$

$B(1,\frac{1}{2}) \cap A\setminus\{1\} = \phi$

$\Rightarrow$ 1 is not the limit point of A

Similarly 2,3,4,.... are not limit points of A

$\Rightarrow D(A) = \phi \subseteq N \Rightarrow D(A) \subseteq N$

$\Rightarrow$ N is closed.

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Theorem:- Let (X,d) be a metric space and $A \subseteq X$ then A is closed iff A' is open set.

Proof Suppose A is closed

To prove A' is open set

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Let $x \in A' \Rightarrow x \notin A$

As A is closed Then $D(A) \subseteq A$

$\Rightarrow x \notin D(A)$

$\Rightarrow$ x is not the limit point of A.

So there exist an open ball $B(x,r)$ s.t

$B(x,r) \cap A \setminus \{x\} = \phi$

As $x \notin A \Rightarrow A \setminus \{x\} = A$

$\Rightarrow B(x,r) \cap A = \phi$

$\Rightarrow B(x,r) \subseteq A'$ $\because A \cap B = \phi$ Then $A \subseteq B'$ & $B \subseteq A'$

As $x \in B(x,r) \subseteq A'$

$\Rightarrow A'$ is an open set.

Conversely suppose A' is an open set

To prove A is closed set.

We have to prove $D(A) \subseteq A$

Let $x \in D(A)$ To prove $x \in A$

Suppose $x \notin A \Rightarrow x \in A'$

As A' is an open set so there exist an open ball $B(x,r)$ s.t $x \in B(x,r) \subseteq A'$

As $A \cap A' = \phi$

Therefore $A \cap B(x,r) = \phi$

As $x \notin A \Rightarrow A \setminus \{x\} = A$

$\Rightarrow B(x,r) \cap A \setminus \{x\} = \phi$

$\Rightarrow x$ is not the limit point of A

$\Rightarrow x \notin D(A)$

Which is a contradiction. So our supposition is wrong. Hence $x \in A$

$\Rightarrow D(A) \subseteq A$

$\Rightarrow$ A is closed set.

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Theorem:- Let (X,d) be a metric space then
(i) ϕ & X are closed sets
(ii) Union of finite number of closed sets is closed
(iii) Intersection of any number of closed sets is closed.

Proof (i) (a) As X is open
$\Rightarrow X'$ is closed $\Rightarrow \phi$ is closed $\because X' = \phi$

(b) As ϕ is open
$\Rightarrow \phi'$ is closed $\Rightarrow X$ is closed $\because \phi' = X$

(ii) Let $\{U_i : i = 1,2,3,\ldots,n\}$ be a finite family of closed sets

To prove $\bigcup_{i=1}^{n} U_i$ is closed set

$$\left(\bigcup_{i=1}^{n} U_i\right)' = \bigcap_{i=1}^{n} U_i' \quad \text{——— } \circledast$$

As U_i is closed $\Rightarrow (U_i)'$ is open set
As finite intersection of open sets is open

$\Rightarrow \bigcap_{i=1}^{n} U_i'$ is open

$\Rightarrow \left(\bigcup_{i=1}^{n} U_i\right)'$ is open by $\circledast$

$\Rightarrow \bigcup_{i=1}^{n} U_i$ is closed

$\Rightarrow$ Union of finite number of closed sets is closed

(iii) Let $\{U_\alpha : \alpha \in I\}$ be a family of closed sets in X. To prove $\bigcap_{\alpha \in I} U_\alpha$ is closed

Now By Demargon's Law

$$\left(\bigcap_{\alpha \in I} U_\alpha\right)' = \bigcup_{\alpha \in I} U_\alpha'$$

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As U_α is an closed set

$\Rightarrow U'_\alpha$ is an open set

As union of any number of open sets is open

$\Rightarrow \bigcup_{\alpha \in I} U'_\alpha$ is an open set

$\Rightarrow \left(\bigcap_{\alpha \in I} U_\alpha\right)'$ is an open set

$\Rightarrow \bigcap_{\alpha \in I} U_\alpha$ is a closed set.

***

Q:- Show that in a metric space (X, d) every singlton set is closed. Also show that every finite subset of X is closed.

Solution:- Let (X, d) be a metric space & $x_0 \in X$. To prove $\{x_0\}$ is closed set. We have to prove $\{x_0\}'$ is an open set.

Let $x \in \{x_0\}'$

$\Rightarrow x \notin \{x_0\} \Rightarrow x \neq x_0$

$\therefore d(x, x_0) > 0$

Let $r = d(x, x_0)$

We claim that $B(x, r) \subseteq \{x_0\}'$

Let $y \in B(x, r)$

$d(y, x) < r \Rightarrow d(y, x) < d(x, x_0)$

$\Rightarrow y \neq x_0$

$\Rightarrow y \notin \{x_0\} \Rightarrow y \in \{x_0\}'$

$\Rightarrow B(x, r) \subseteq \{x_0\}'$

$\Rightarrow \{x_0\}'$ is an open set.

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$\Rightarrow \{x_0\}$ is an closed set.

Hence in a metric space every singleton set is closed set.

Let $A = \{x_1, x_2, x_3, \cdots, x_n\}$ be a finite set in (X, d)

As $A = \{x_1, x_2, x_3, \cdots, x_n\}$

$= \{x_1\} \cup \{x_2\} \cup \{x_3\} \cup \cdots \cup \{x_n\}$

$\Rightarrow A = \bigcup_{i=1}^{n} \{x_i\}$

As each x_i is closed & finite union of closed sets is close

$\Rightarrow \bigcup_{i=1}^{n} \{x_i\}$ is closed set.

$\Rightarrow$ A is closed.

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Note:- Infinite intersection of open sets may not be open set.

Example:- Let $X = \mathbb{R}$, $U_n = \left]-\frac{1}{n}, \frac{1}{n}\right[$ are open sets in $\mathbb{R}$.

Then $\bigcap_{n=1}^{n} U_n = \{0\}$ It is not an open set.

Hence infinite intersection of open sets in metric space (X, d) may not be open.

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Theorem:- Prove that every closed ball in a metric space (X, d) is a closed set.

Proof:- Let $\bar{B}(a, r)$ be a closed ball in a metric space (X, d) with centre "a" and radius r.

To prove $\bar{B}(a, r)$ is closed set.

For this we have to prove $\{\bar{B}(a, r)\}'$ is open set.

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Let $x \in \{\bar{B}(a,r)\}' \Rightarrow x \notin \bar{B}(a,r)$

$\Rightarrow d(x,a) > r$

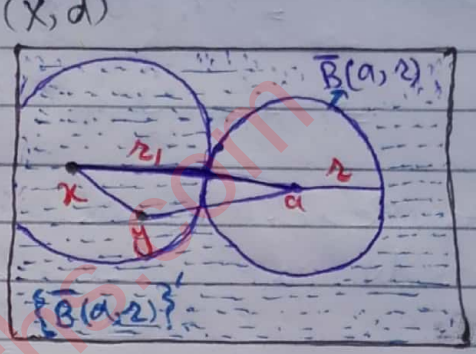
$d(x,a) - r > 0$

Let $r_1 = d(x,a) - r$

We claim that $B(x,r_1) \subseteq \{\bar{B}(a,r)\}'$

Let $y \in B(x,r_1)$

$\Rightarrow d(y,x) < r_1$



By Triangle Inequality

$d(x,a) \leq d(x,y) + d(y,a)$

$d(x,a) < r_1 + d(y,a)$

or $d(x,a) - r_1 < d(y,a)$

or $d(y,a) > d(x,a) - r_1$

$\Rightarrow d(y,a) > d(x,a) - d(x,a) + r$

$d(y,a) > r \Rightarrow y \notin \bar{B}(a,r)$

$\Rightarrow y \in \{\bar{B}(a,r)\}'$

So $B(x,r_1) \subseteq \{\bar{B}(a,r)\}'$

$\Rightarrow \{\bar{B}(a,r)\}'$ is open set

$\Rightarrow \bar{B}(a,r)$ is closed set.

★ ★ ★

Closure:- Let (X,d) be a metric space and $A \subseteq X$. Then closure of A is the union of A & the set of limit points of A. And it is denoted by $\bar{A}$

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Theorem:- Let (X,d) be a metric space and A, B subsets of X. Then (i) $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$
(ii) $\overline{A \cup B} = \bar{A} \cup \bar{B}$ (iii) $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$
(iv) $\bar{\phi} = \phi$ (v) $\bar{X} = X$

Proof (i) Given $A \subseteq B$. To prove $\bar{A} \subseteq \bar{B}$

Let $x \in \bar{A} = A \cup D(A)$

$\Rightarrow x \in A$ or $x \in D(A)$

If $x \in A$ and $A \subseteq B \Rightarrow x \in B$

As $B \subseteq \bar{B} = B \cup D(B) \Rightarrow x \in \bar{B}$

$\Rightarrow \bar{A} \subseteq \bar{B}$ proved in this case.

Now if $x \notin A \Rightarrow x \in D(A)$

$\Rightarrow x$ is the limit point of A. Then for each ball $B(x,2)$ s.t

$B(x,2) \cap A \setminus \{x\} \neq \phi$

As $A \subseteq B \Rightarrow A \setminus \{x\} \subseteq B \setminus \{x\}$

$\therefore B(x,2) \cap B \setminus \{x\} \neq \phi \Rightarrow x$ is limit point of B

$\Rightarrow x \in D(B)$

$\Rightarrow x \in B \cup D(B) \Rightarrow x \in \bar{B}$

$\Rightarrow \bar{A} \subseteq \bar{B}$

$\Rightarrow$ In either case $\bar{A} \subseteq \bar{B}$

(ii) As $A \subseteq A \cup B$ & $B \subseteq A \cup B$

$\Rightarrow \bar{A} \subseteq \overline{A \cup B}$ & $\bar{B} \subseteq \overline{A \cup B}$

$\Rightarrow \bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$ ——(1)

Let $x \in \overline{A \cup B}$

$\Rightarrow x \in A \cup B$ or $x \in D(A \cup B)$

$\Rightarrow x \in A \cup B$ or x is limit point of $A \cup B$

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If $x \in A \cup B$ Then $x \in A$ or $x \in B$

$\Rightarrow x \in \bar{A}$ or $x \in \bar{B}$

$\Rightarrow x \in \bar{A} \cup \bar{B}$

Now If x is the limit point of $A \cup B$

Then for each ball $B(x, r)$

$B(x, r) \cap A \cup B \setminus \{x\} \neq \phi$

$\Rightarrow B(x, r) \cap A \setminus \{x\} \neq \phi$

or $B(x, r) \cap B \setminus \{x\} \neq \phi$

$\Rightarrow x$ is limit point of A or x is limit point of B.

$\Rightarrow x \in D(A)$ or $x \in D(B)$

$\Rightarrow x \in \bar{A}$ or $x \in \bar{B}$

$\Rightarrow x \in \bar{A} \cup \bar{B}$

So in either case $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$ — ②

from ① & ②

$$\overline{A \cup B} = \bar{A} \cup \bar{B}$$

(iii) To prove $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

As $A \cap B \subseteq A \Rightarrow \overline{A \cap B} \subseteq \bar{A}$

Also $A \cap B \subseteq B \Rightarrow \overline{A \cap B} \subseteq \bar{B}$

As $\overline{A \cap B} \subseteq \bar{A}$ & $\overline{A \cap B} \subseteq \bar{B}$

$\Rightarrow \overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

(iv), (v)

As ϕ is closed then $\bar{\phi} = \phi$ By Result

As X is closed then $\bar{X} = X$ — " —

Result :- A is closed if & only if $\bar{A} = A$

★ ★ ★

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Remark:- Converse of (iii) is not true in general.

Example:- Let $X = \mathbb{R}$, $A =]1,2[$, $B =]2,3[$

$\bar{A} = [1,2]$, $\bar{B} = [2,3]$

$\bar{A} \cap \bar{B} = \{2\}$ ——— ①

$A \cap B = \phi$, $\overline{A \cap B} = \bar{\phi} = \phi$

$\overline{A \cap B} = \phi$ — ②

$\Rightarrow \bar{A} \cap \bar{B} \neq \overline{A \cap B}$ by ① & ②

——— * * * ———

Theorem:- Let (X,d) be a metric space and $A \subseteq X$ then (i) A° is the union of all open sets which are contained in A.

(ii) A° is an open set.

(iii) A° is largest open set contained in A

(iv) $A^\circ = A$ iff A is an open set.

Proof (i) Let $\gamma = \{U_\alpha : U_\alpha$ is an open set in X and $U_\alpha \subseteq A\}$

To prove $A^\circ = \bigcup_{\alpha \in I} U_\alpha = \cup\gamma$

Let $x \in \cup\gamma \Rightarrow x \in \bigcup_{\alpha \in I} U_\alpha$

$\Rightarrow x \in U_\alpha$ for some $\alpha \in I$

As U_α is an open set then there exist an open ball $B(x,r)$ s.t

$x \in B(x,r) \subseteq U_\alpha$

As $U_\alpha \subseteq A \Rightarrow x \in B(x,r) \subseteq U_\alpha \subseteq A$

$\Rightarrow x \in B(x,r) \subseteq A$ Transitive property

$\Rightarrow x$ is an interior point of A

$\Rightarrow x \in A^\circ \Rightarrow \cup\gamma \subseteq A^\circ$ ——— ①

Conversely $x \in A^\circ$

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$\Rightarrow$ x is an interior point of A then there exist an open ball $B(x,r)$ s.t

$$x \in B(x,r) \subseteq A$$

As $B(x,r)$ is an open set & $B(x,r) \subseteq A$

$\Rightarrow B(x,r) \in \gamma$

As $x \in B(x,r) \in \gamma \Rightarrow x \in \cup\gamma$

$\Rightarrow A^\circ \subseteq \cup\gamma$ ——— (2)

from (1) & (2) we conclude that

$$A^\circ = \cup\gamma$$

$$\Rightarrow A^\circ = \bigcup_{\alpha\in I} U_\alpha \text{ where } U_\alpha \subseteq A.$$

(ii) Let $\gamma = \{U_\alpha : U_\alpha$ is an open set in X & $U_\alpha \subseteq A\}$

Then $A^\circ = \cup\gamma$

$$= \bigcup_{\alpha\in I} U_\alpha \text{ where } U_\alpha \subseteq A$$

As each U_α is an open set and union of open sets is open

Therefore $\cup\gamma$ is open

$\Rightarrow A^\circ$ is open

(iii) Let $\gamma = \{U_\alpha : U_\alpha$ is an open set in X and $U_\alpha \subseteq A\}$

Then $A^\circ = \bigcup_{\alpha\in I} U_\alpha = \cup\gamma$

Let W is the largest open set contains in A.

To prove $A^\circ = W$

As W is an open set & $W \subseteq A$

$\Rightarrow W \in \gamma \Rightarrow W \subseteq \cup\gamma$

$W \subseteq A^\circ$ ——— (3) $\because \cup\gamma = A^\circ$

As A° is an open set and $A^\circ \subseteq A$. But W is the largest open set contains in A.

$\therefore A^\circ \subseteq W$ ——— (4)

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from ① & ② $A^\circ = W$

Hence A° is the largest ~~point~~ open set contained in A.

(iv) To prove $A^\circ = A$ if & only if A is an open set.

Let $A^\circ = A$

To prove A is an open set

Let $x \in A = A^\circ \Rightarrow x \in A^\circ$

$\Rightarrow$ x is an interior point of A

Then there exist an open ball $B(x,r)$ s.t

$x \in B(x,r) \subseteq A \Rightarrow$ A is an open set

Conversely given A is an open set

To prove $A^\circ = A$

Given $A^\circ \subseteq A$ ——— ①

Let $x \in A$

As A is an open set then $\exists$ an open ball $B(x,r)$ s.t

$x \in B(x,r) \subseteq A \Rightarrow$ x is an interior point of A

$\Rightarrow x \in A^\circ \quad \because A \subseteq A^\circ$ ——— ②

from ① & ② $A^\circ = A$.

* * *

Theorem:- Let (X,d) be a metric space & $A, B \subseteq X$ then (i) If $A \subseteq B$ then $A^\circ \subseteq B^\circ$

(ii) $(A \cap B)^\circ = A^\circ \cap B^\circ$

(iii) $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$

Proof (i) Given $A \subseteq B$. To prove $A^\circ \subseteq B^\circ$

Let $x \in A^\circ \Rightarrow$ x is an interior point of A

$\Rightarrow$ There exist an open ball $B(x,r)$ s.t

$x \in B(x,r) \subseteq A$

As $A \subseteq B \Rightarrow x \in B(x,r) \subseteq A \subseteq B$

$\Rightarrow x \in B(x,r) \subseteq B$

$\Rightarrow$ x is an interior point of B

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(ii) To prove $(A \cap B)^\circ = A^\circ \cap B^\circ$

As $(A \cap B) \subseteq A \Rightarrow (A \cap B)^\circ \subseteq A^\circ$ using ①

& $(A \cap B) \subseteq B \Rightarrow (A \cap B)^\circ \subseteq B^\circ$

Now as $(A \cap B)^\circ \subseteq A^\circ$ & $(A \cap B)^\circ \subseteq B^\circ$

$\Rightarrow (A \cap B)^\circ \subseteq A^\circ \cap B^\circ$ —— ①

Now to prove $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$

Let $x \in A^\circ \cap B^\circ$

$\Rightarrow x \in A^\circ$ & $x \in B^\circ$

As $x \in A^\circ \Rightarrow x$ is an interior point of A

So $\exists$ an open ball $B(x, r)$ such that

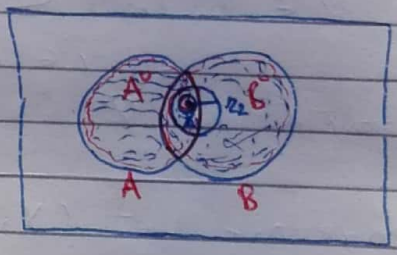
$x \in B(x, r) \subseteq A$

As $x \in B^\circ \Rightarrow x$ is an interior point of B

So there exist an open ball $B(x, r)$

s.t $x \in B(x, r_2) \subseteq B$

Put $r = \min(r_1, r_2)$



We claim that $B(x, r) \subseteq B(x, r_i)$ $i = 1, 2$

Let $y \in B(x, r)$

$\Rightarrow d(y, x) < r \leq r_i \Rightarrow d(y, x) < r_i$

$\Rightarrow y \in B(x, r_i)$

Therefore $B(x, r) \subseteq B(x, r_i)$

Therefore $B(x, r) \subseteq B(x, r_1) \subseteq A$

Also $B(x, r) \subseteq B(x, r_2) \subseteq B$

$\Rightarrow x \in B(x, r) \subseteq A \cap B$

$\Rightarrow x$ is an interior point of $A \cap B$

$\Rightarrow x \in (A \cap B)^\circ$

$\Rightarrow A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$ —— ②

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From eqn ① & ②

$$A^\circ \cap B^\circ = (A \cap B)^\circ$$

(iii) To prove $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$

As $A \subseteq A \cup B \Rightarrow A^\circ \subseteq (A \cup B)^\circ$

Also $B \subseteq A \cup B \Rightarrow B^\circ \subseteq (A \cup B)^\circ$

$\therefore A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$

★★★

Note:- $(A \cup B)^\circ \not\subseteq A^\circ \cup B^\circ$ in general.

Example:-

Let $X = \mathbb{R}$, $d(x,y) = |x-y|$

Let $A = [1,2]$, $B = [2,3]$

Then $A^\circ =]1,2[$ & $B^\circ =]2,3[$

$A^\circ \cup B^\circ =]1,2[\cup]2,3[$

$=]1,3[- \{2\}$ ——— ①

$A \cup B = [1,2] \cup [2,3] = [1,3]$

$(A \cup B)^\circ =]1,3[$ ——— ②

from eqn ① & ② $(A \cup B)^\circ \not\subseteq A^\circ \cup B^\circ$

★★★

Theorem:- Let (X, d) be a metric space and $A \subseteq X$ then

(i) $\bar{A}$ is the intersection of all closed superset of A

(ii) $\bar{A}$ is the closed subset of X

(iii) $\bar{A} = A$ iff A is closed

(iv) $\bar{A}$ is the smallest closed superset of A.

Proof:- (i) Let $\gamma = \{F \mid F \text{ is closed in } X, A \subseteq F\}$ be a family of all closed supersets of A.

To prove $\bar{A} = \cap\gamma$

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Let $x \in \bar{A} = A \cup D(A)$ Then $x \in A$ or $x \in D(A)$

If $x \in A$ & $A \subseteq F$ for every closed superset F of A

$\Rightarrow x \in F \Rightarrow x \in \cap \gamma$

If $x \notin A$ then $x \in D(A) \Rightarrow x$ is the limit point of $A \Rightarrow$ for each open ball $B(x,r)$

$B(x,r) \cap A \setminus \{x\} \neq \phi$

As $A \subseteq F \Rightarrow A \setminus \{x\} \subseteq F \setminus \{x\}$

$\Rightarrow B(x,r) \cap F \setminus \{x\} \neq \phi$

$\Rightarrow x$ is the limit point of $F \Rightarrow x \in D(F)$

As F is closed so $D(F) \subseteq F$

$\Rightarrow x \in F$ for every closed superset F of A

$\Rightarrow x \in \cap \gamma$

$\therefore \bar{A} \subseteq \cap \gamma$

Conversely let $x \in \cap \gamma$

To prove $x \in \bar{A}$

Suppose $x \notin \bar{A}$

$\Rightarrow x \notin A$ & $x \notin D(A)$

As $x \notin D(A)$ then x is not the limit point of A.

$\Rightarrow \exists$ an open ball $B(x,r)$ s.t

$B(x,r) \cap A \setminus \{x\} = \phi$

As $x \notin A \therefore A \setminus \{x\} = A$

$\therefore B(x,r) \cap A = \phi \Rightarrow A \subseteq \{B(x,r)\}'$

As $B(x,r)$ is an open set

$\Rightarrow \{B(x,r)\}'$ is closed set

Also $A \subseteq \{B(x,r)\}'$

$\Rightarrow \{B(x,r)\}'$ is the closed superset of A

Therefore $\{B(x,r)\}' \in \gamma$

As $x \in B(x,r)$

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$\Rightarrow x \notin \{B(x,\epsilon)\}' \quad \Rightarrow x \notin \cap\gamma$

which is a contradiction.

So our supposition is wrong

$\therefore x \in \bar{A} \Rightarrow \cap\gamma \subseteq \bar{A}$ —— ②

from eqn ① & ② $\bar{A} = \cap\gamma$

(ii) Let $\gamma = \{F \mid F \text{ is closed in } X, A \subseteq F\}$

Then $\bar{A} = \cap\gamma$ (By part 1)

As intersection of closed sets is closed

$\Rightarrow \cap\gamma$ is closed set

$\Rightarrow \bar{A}$ is closed

(iii) To prove $\bar{A} = A$ iff A is closed

Given $\bar{A} = A$ to prove A is closed

As $\bar{A} = A \cup D(A)$

$\Rightarrow D(A) \subseteq \bar{A}$

Given $\bar{A} = A \Rightarrow D(A) \subseteq A$

$\Rightarrow A$ is closed.

Conversely Given A is closed

To prove $\bar{A} = A$

As $A \subseteq \bar{A}$ —— ①

As A is closed $\Rightarrow D(A) \subseteq A$

$\Rightarrow A \cup D(A) \subseteq A \cup A \Rightarrow A \cup D(A) \subseteq A$

$\Rightarrow \bar{A} \subseteq A$ —— ②

from ① & ② $\bar{A} = A$

(iv) Let $\gamma = \{F \mid F \text{ is closed in } X, A \subseteq F\}$ be a family of all closed supersets of A

Then $\bar{A} = \cap\gamma$

Let W be the closed superset of A

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To prove $\bar{A} \subseteq W$

As W is closed and $A \subseteq W$

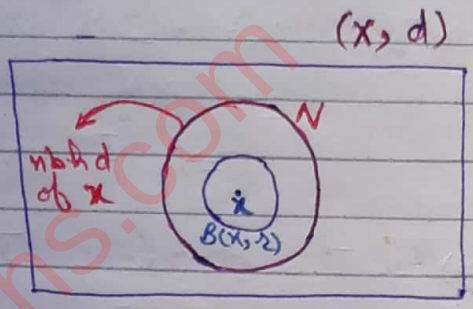
then $W \in \gamma$

$\Rightarrow \cap\gamma \subseteq W$ $[A \cap B \subseteq A]$

$\Rightarrow \bar{A} \subseteq W$ But W is arbitrary

Hence $\bar{A}$ is the smallest closed superset of A.

* **Neighbourhood:-** Let (X,d) be a metric space, $N \subseteq X$ then N is called nbhd of x if there exist an open ball $B(x,r)$ s.t $x \in B(x,r) \subseteq N$



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Theorem:- Let (X,d) be a metric space & U be subset of X then U is an open set if & only if U is the nbhd of each of its point.

Proof: Given U is an open set.

To prove U is the nbhd of each of its point.

Let $x \in U$. As U is an open set then $\exists$ an open ball $B(x,r)$ st

$x \in B(x,r) \subseteq U$

So U is the nbhd of x

As x is arbitrary point. So U is the nbhd of each of its points.

Conversely Given U is the nbhd of each of its point.

To prove U is an open set.

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Let $x \in U$ and U is the nbhd of x. Then $\exists$ an open ball $B(x,r)$ s.t

$$x \in B(x,r) \subseteq U$$

So U is an open set.

* * *

Theorem:- Let (X,d) be a metric space and $A \subseteq X$. Let x be a limit point of A & $x \notin A$. Then every open ball with centre at x contains infinite number of points of A.

Proof Given x is a limit point of A

To prove each open ball $B(x,r)$ contains infinite number of points of A.

Suppose $B(x,r)$ contains finite number of points of A.

i.e $B(x,r) \cap A = \{x_1, x_2, x_3, \cdots, x_n\}$

As $x \notin A \Rightarrow x \notin B(x,r) \cap A$

$\Rightarrow x \neq x_i$ for $i = 1,2,3,\cdots,n$

$d(x, x_i) > 0$

Put $r_i = d(x, x_i)$, $i = 1,2,3,\cdots n$

Put $r^* = \min(r_1, r_2, \cdots r_n)$

We claim that $B(x, r^*)$ does not contains any point of A.

Now $B(x,r^*) = \{y \mid y \in X,\ d(y,x) < r^*\}$

Let $y \in B(x,r^*) \Rightarrow d(y,x) < r^* \leq r_i$

$\Rightarrow d(y,x) < r_i \Rightarrow d(y,x) < d(x,x_i)$

$\Rightarrow y \neq x_i$

$\Rightarrow B(x,r^*)$ does not contains any points of A.

$\Rightarrow B(x,r^*) \cap A = \phi$

As $x \notin A \Rightarrow A \setminus \{x\} = A$

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$\Rightarrow B(x, r^*) \cap A \setminus \{x\} = \phi$

$\Rightarrow x$ is not limit point of A which is a contradiction.

So our supposition is wrong

$\Rightarrow$ Each open ball with centre at x contains infinite numbers of points of A.

★ **Definition:-** Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0$

or $\lim_{n\to\infty} x_n = x$

or for $\epsilon > 0$ there exist a +ve integer N s.t $d(x_n, x) < \epsilon \quad \forall\ n \geq N$

or $x_n \in B(x, \epsilon) \quad \forall\ n \geq N$

i.e each ball with centre x contains all the points of the sequence except a finite number of points.

Example:- Let $X = \mathbb{R}$, $d(x, y) = |x - y|$

Sequence $1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots$

Let $\epsilon = 1/10$, $1, \frac{1}{2}, \frac{1}{3}, \cdots \longrightarrow 0$

$d(1, 0) = |1 - 0| = 1 \not< \epsilon$

$d(\frac{1}{2}, 0) = |\frac{1}{2} - 0| = 1/2 \not< \epsilon$

$\vdots$

$d(\frac{1}{9}, 0) = |\frac{1}{9} - 0| = 1/9 \not< \epsilon$, $d(\frac{1}{10}, 0) = |\frac{1}{10} - 0| = \frac{1}{10} \not< \epsilon$

$d(\frac{1}{11}, 0) = |\frac{1}{11} - 0| = \frac{1}{11} < \epsilon$ & so on $< \epsilon$

$\Rightarrow N = 11$ s.t

$d(x_n, x) < \epsilon \quad \forall\ n \geq 11$

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⇒ **Bounded Sequence:-** A sequence $\{x_n\}$ in metric (X,d) is said to be bounded if there exist some +ve real number λ (However large) s.t $d(x_n,x) \leq \lambda$, $\forall$ n

Here $x \in X$ is some fixed element.

***

Theorem:- Let (X,d) be a metric space

(i) Every convergent sequence is bounded & limit of the sequence is unique.

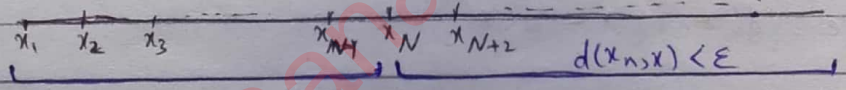
(ii) If $x_n \to x$ & $y_n \to y$ then $d(x_n,y_n) \to d(x,y)$.

Proof:-

(i) Let $\{x_n\}$ be a convergent sequence & $x_n \to x$. To prove $\{x_n\}$ is bounded.

As $\{x_n\}$ is convergent then for $\varepsilon > 0$ there exist a +ve integer N s.t

$$d(x_n,x) < \varepsilon \quad \forall n \geq N$$



Put $a = \max\{d(x,x_1), d(x,x_2), \cdots, d(x_{N-1},x)\}$

$\therefore d(x_n,x) \leq a$ for $n = 1,2,\cdots,N-1$

$d(x_n,x) \leq a < a+\varepsilon$

$\Rightarrow d(x_n,x) < a+\varepsilon \quad \forall \quad n=1,2,3,\cdots,N-1$

$\Rightarrow$ Sequence is bounded.

Uniqueness:- To prove limit is unique

Suppose $x_n \to x$ & $x_n \to y$

To prove $x = y$

$0 \leq d(x,y) \leq d(x,x_n) + d(x_n,y)$, $x_n \in X$

when $n \to \infty$ $x_n \to x \Rightarrow d(x_n,x) \to 0$

& $x_n \to y \Rightarrow d(x_n,y) \to 0$

$\Rightarrow 0 \leq d(x,y) \leq 0+0 \Rightarrow 0 \leq d(x,y) \leq 0$

$\Rightarrow d(x,y) = 0 \Rightarrow x = y \Rightarrow$ limit is unique.

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(ii) Given $x_n \to x$ & $y_n \to y$

To prove $d(x_n, y_n) \to d(x, y)$

$d(x_n, y_n) \leq d(x_n, x) + d(x, y_n)$

$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n)$

$d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y, y_n)$ ——— ①

Inter change x_n with x & y_n with y

$d(x, y) - d(x_n, y_n) \leq d(x, x_n) + d(y_n, y)$

$\Rightarrow -\{d(x_n, y_n) - d(x, y)\} \leq d(x_n, x) + d(y_n, y)$ ——— ②

from ① & ②

$|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y)$

As $x_n \to x \Rightarrow d(x_n, x) = 0$

& $y_n \to y \Rightarrow d(y_n, y) = 0$

Therefore $|d(x_n, y_n) - d(x, y)| \leq 0$

$\Rightarrow d(x_n, y_n) \to d(x, y)$

★★★

⇒ **Cauchy Sequence:-** Let (X, d) be a metric space & $\{x_n\}$ be a sequence in X then $\{x_n\}$ is said to be cauchy sequence if for $\varepsilon > 0$ there exist a +ve integer N s.t

$d(x_n, x_m) < \varepsilon \quad \forall\ m, n \geq N$

★★★

Theorem:- Prove that every convergent sequence is cauchy & also discuss its inverse.

Proof Let $\{x_n\}$ be a convergent sequence in X & let $x_n \to x$

To prove $\{x_n\}$ is a cauchy sequence.

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As $x_n \to x$ then for $\varepsilon > 0$ there exist a +ve integer N s.t

$$d(x_n, x) < \varepsilon/2 \quad \forall\ n \geq N$$

Let $m, n \geq N$

$\Rightarrow d(x_m, x) < \varepsilon/2$ ——— ①

& $d(x_n, x) < \varepsilon/2$ ——— ②

Now $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m)$

$< \varepsilon/2 + \varepsilon/2$ using ① & ②

$\Rightarrow d(x_n, x_m) < \varepsilon \quad \forall\ m, n \geq N$

$\Rightarrow \{x_n\}$ is cauchy sequence.

So every convergent is cauchy sequence.

* Converse of the above theorem is not true in general. i.e a cauchy sequence may or may not be convergent.

Let $X =]0,1[$ & sequence $= \{\frac{1}{n}\}$ then $\{\frac{1}{n}\}$ is a cauchy sequence in $]0,1[$

but $\frac{1}{n} \to 0 \notin X$ as $n \to \infty$

So every cauchy sequence may not be convergent.

⇒

★★★

*Definition:- Let (X,d) be a metric space & $\{x_n\}$ be a sequence in X. then $\{x_n\}$ converges to a point x (write as $x_n \to x$) if for every $\varepsilon > 0$ there exist a +ve integer "n_0" such that $d(x_n, x) < \varepsilon$ whenever $n \geq n_0$ or in other words

$$\lim_{n \to \infty} d(x_n, x) = 0$$

Then we write $\lim_{n \to \infty} x_n = x$

Then x is called the limit point of $\{x_n\}$

⇒

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Theorem:- If a convergent sequence in a metric space has infinitely many distinct points then its limit is a limit point of the set of the points of the sequence.

Proof) Let (X,d) be a metric space & $\{x_n\}$ be a convergent sequence in X s.t $x_n \to x$. Let A be the set of points of the sequence $\{x_n\}$.

To prove x is a limit point of A. Let us consider any open ball $B(x,r)$. As $x_n \to x$ so $B(x,r)$ contains all the points of the sequence except a finite number of points.

So $B(x,r)$ contains infinite many points of A.

i.e $B(x,r) \cap A \setminus \{x\} \neq \phi$

$\Rightarrow x \in D(A) \Rightarrow x$ is limit point of A.

Theorem:- Let (X,d) be a metric space & $M \subseteq X$. Then (1) $x \in \bar{M}$ iff there exist a sequence $\{x_n\}$ in M s.t $x_n \to x$

(2) M is closed if & only if $\{x_n\}$ is a sequence in M s.t $x_n \to x$ then $x \in M$.

Proof) 1) Let $x \in \bar{M} \Rightarrow x \in M \cup D(M)$

$\Rightarrow x \in M$ or $x \in D(M)$

If $x \in M$ then we can make a constant sequence $(x, x, x, \ldots) \to x$

If $x \notin M \Rightarrow x \in D(M)$

$\Rightarrow x$ is the limit point of M

So for each +ve integer $n = 1, 2, 3, \ldots$ $B(x, \frac{1}{n})$ contains $x_n \in M$ s.t $x_n \neq x$

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$\{x_n\}$ is a sequence in M

As $x_n \in B(x, \frac{1}{n}) \Rightarrow d(x_n, x) < \frac{1}{n}$

when $n \to \infty$ $d(x_n, x) \to 0$

$\Rightarrow x_n \to x$

Conversely given that $\{x_n\}$ is a sequence in M s.t. $x_n \to x$

If $x \in M$ & $M \subseteq \bar{M} \Rightarrow x \in \bar{M}$

If $x \notin M$

To prove $x \in D(M)$ or x is a limit point of M. Let $B(x, r)$ be an open ball with centre at x & radius r.

As $x_n \to x$ then $B(x, r)$ contains all the points of sequence $\{x_n\}$ except a finite number of points.

As $\{x_n\}$ is a sequence in M

Then $B(x, r)$ contains the point of M different from x

i.e $B(x, r) \cap M \setminus \{x\} \neq \phi$

$\Rightarrow$ x is a limit point of M

$\Rightarrow x \in D(M) \Rightarrow x \in M \cup D(M)$

$\Rightarrow x \in \bar{M}$

(2) Suppose $\{x_n\}$ is a sequence in M s.t $x_n \to x$ then $x \in M$

To prove M is closed.

As $M \subseteq \bar{M}$ —— ①

Let $x \in \bar{M}$ then by part ① there exist a sequence $\{x_n\}$ in M s.t $x_n \to x$

Then by given condition $x \in M$

$\Rightarrow \bar{M} \subseteq M$ —— ②

from ① & ② $M = \bar{M}$

$\Rightarrow$ M is closed.

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Conversely given M is closed.

To prove if there exist a sequence $\{x_n\}$ in M s.t $x_n \to x$ then $x \in M$.

Let $\{x_n\}$ be a sequence in M s.t $x_n \to x$

As $\{x_n\}$ is a sequence in M s.t $x_n \to x$ then by part ① $x \in \overline{M}$

$\Rightarrow x \in \overline{M} = M$ ∵Given M is closed

$\Rightarrow x \in M$.

Theorem **Complete Metric Space:-** A metric space (X, d) is said to be complete if every cauchy sequence in X converges to a point x. $[x \in X]$

Theorem:- A subspace Y of a complete metric space X is complete iff Y is closed in X.

Proof Given Y is complete.

To prove Y is closed in X.

Since $Y \subseteq \overline{Y}$ — ①

Let $x \in \overline{Y}$ then there exist a sequence $\{x_n\}$ in Y s.t $x_n \to x$

As every convergent sequence is cauchy

∴ $\{x_n\}$ is cauchy in Y also Y is complete $\Rightarrow x_n \to x \in Y$

$\Rightarrow \overline{Y} \subseteq Y$ — ②

From ① & ② $Y = \overline{Y} \Rightarrow Y$ is closed.

Conversely given Y is closed.

To prove Y is complete.

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Let $\{x_n\}$ be cauchy sequence in Y as $Y \subseteq X$
$\Rightarrow \{x_n\}$ is cauchy sequence in X.
Given X is complete
$\therefore x_n \to x \in X$ (Definition of comple
As $\{x_n\}$ is sequence in Y and $x_n \to x$
$\Rightarrow x \in \bar{Y} = Y$ {By converse of previous theorem}
$\Rightarrow x \in Y \quad \Rightarrow Y$ is complete.

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⇒ **Decreasing Sequence:-** A sequence $\{A_n\}$ of subsets of X is said to be decreasing sequence if $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \dots$

⇒ **Nested Sequence:-** Let (X,d) be a metric space & $\{A_n\}$ be a sequence of non-empty subsets of X then this sequence is called Nested sequence if

(i) $A_n \supseteq A_{n+1}$, $\forall\ n \geq 1$

(ii) $\delta(A_n) \to 0$, as $n \to \infty$

★ ★ ★

Cantor's Intersection theorem:- OR Nested Interval Property:- Let (X,d) be a complete metric space & $\{F_n\}$ be a decreasing sequence of non-empty closed subsets of X s.t $\delta(F_n) \to 0$ as $n \to \infty$ then $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point

OR

Let (X,d) be a complete metric space & $\{F_n\}$ be a nested sequence of closed sets then $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

Proof: Let we prove that $\bigcap_{n=1}^{\infty} F_n$ is non-empty

As F_n is non-empty $\forall\ n$

Let $x_n \in F_n$ & $x_m \in F_m$

Suppose $m \geq n$ then $F_m \subseteq F_n$

As $x_m \in F_m$ & $F_m \subseteq F_n$

$\Rightarrow x_m \subseteq F_n$

Then x_n & $x_m \subseteq F_n$

$d(x_n, x_m) \leq \delta(F_n) \to 0$ as $n \to \infty$

$\Rightarrow \{x_n\}$ is cauchy sequence & as X is complete metric space so $x_n \to x \in X$

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Here arises two cases.

Case 1:- If $\{x_n\}$ contains finite number of distinct point then x is that point which repeat infinitly many time. then there exist +ve integer n_0 s.t

$$x \in F_n \quad \forall\ n \geq n_0$$

But as $\{F_n\}$ is decreasing $\Rightarrow x \in F_n, \forall n$

$$\Rightarrow x \in \cap F_n \quad \text{so } \cap F_n \neq \phi$$

Case 2:- If $\{x_n\}$ contains infinite number of distinct point then x is the limit point of the set of the points of the sequence

For any integer $n \geq n_0$ the elements $x_n, x_{n+1}, \ldots$ are all in F_n.

Since x is a limit point of the set of these points of F_n,

It is a limit point of F_n also.

Since F_n is closed so $x \in F_n, \forall n$

$$\text{So } x \in \bigcap_{n=1}^{\infty} F_n$$

$$\Rightarrow \bigcap_{n=1}^{\infty} F_n \neq \phi$$

Now we prove that $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

Let $x, y \in \bigcap_{n=1}^{\infty} F_n$

$$\Rightarrow x, y \in F_n \quad \forall n$$

$$d(x,y) \leq \delta(F_n)$$

$$d(x,y) \leq \delta(F_n) \text{ as } n \to \infty \Rightarrow d(x,y) \leq 0$$

$$\Rightarrow d(x,y) = 0 \Rightarrow x = y$$

$\therefore$ Intersection contains exactly one point

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BASIC TOPOLOGY ★★★

⇒ **Topology:-** Let X be a non-empty set & τ be a collection of subsets of X. then τ is called Topology if

(i) ϕ & X belogs to τ

(ii) The intersection of any two sets in τ belongs to τ

(iii) The union of any number of sets in τ belongs to τ

The member of τ are then called τ-open sets or simply open sets. & X together with τ i.e (X, τ) is called a topological space.

The set X is called its ground set & the element of X are called its point.

Example:- Let $X = \{1, 2, 3\}$

$\tau = \{\phi, X\}$

Then τ is a Topology on X

(i) $\phi, X \in \tau$

(ii) let $u_1 = \phi$ & $u_2 = X \in \tau$

Now $u_1 \cap u_2 = \phi \cap X = \phi \in \tau$

(iii) $u_1 \cup u_2 = \phi \cup X$

$= X \in \tau$

$\Rightarrow \tau$ is topology on X

(X, τ) is topological space.

Note:- If $X \neq \phi$ then $\tau = \{\phi, X\}$ is called indiscreate Topology

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Example:- Let $X = \{a, b, c\}$
& $\tau = \{\phi, X, \{a\}, \{b\}\}$

Then τ is not a Topology because if
$u_1 = \{a\}$ & $u_2 = \{b\}$
then $u_1 \cup u_2 = \{a\} \cup \{b\} = \{a, b\} \notin \tau$

Example:- Let $X \neq \phi$ & $\tau = P(X)$ = Power set of X
$P(X) = \{A : A \subseteq X\}$

Then τ is a Topology on X.
(i) $\phi \subseteq X \Rightarrow \phi \in \tau$
$X \subseteq X \Rightarrow X \in \tau$

(ii) If $u_1, u_2 \in \tau$
Then $u_1 \subseteq X$ & $u_2 \subseteq X$
$u_1 \cap u_2 \subseteq X \Rightarrow u_1 \cap u_2 \in \tau$

(iii) Let $\{u_\alpha : \alpha \in I\}$ be collection of τ
since $u_\alpha \subseteq X$ for each $\alpha \in I$
$\Rightarrow \bigcup_{\alpha \in I} u_\alpha \subseteq X \Rightarrow \bigcup_{\alpha \in I} u_\alpha \in \tau$

$\Rightarrow \tau$ is Topology on X. Thus $P(X)$ is Topology on X.

Note:- It is called the discreate Topology.

***Coarser & Finer Topologies:-** It is some time convenient to make a comparison of Topologies on a given non-empty set X by using the inclusion relation of sets

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★ If τ_1 & τ_2 are two Topologies on X & $\tau_1 \subseteq \tau_2$ Then τ_1 is said to be coarser or (weaker) then τ_2. Alternatively τ_2 is said to be finer (or stronger) than τ_1.

★ Two Topologies on X are called incomparable if neither is coarser than the other & neither is the finer than the other.

e.g $X = \{1, 2\}$, $\tau_1 = \{\phi, X, \{1\}\}$, $\tau_2 = \{\phi, X, \{2\}\}$
Then neither $\tau_1 \subseteq \tau_2$ nor $\tau_2 \subseteq \tau_1$

Note:- The indiscrete Topology contained in any Topology on X is therefore the coarsest (weakest) Topology on X.

At the other extreme is the discrete Topology which containe any Topology on X & is therefore finest (strongest) Topology on X.

Examples:- Let X be a non empty set & τ be the collection of ϕ & all those subsets of X whose complements are finite. Show that τ is a topology. (Co-finite Topology)

Solution:- Here $\tau = \phi \cup \{A : A \subseteq X, A' \text{ is finite}\}$

(i) $\phi \in \tau$

Now as $X \subseteq X$, $X' = X - X = \phi$ is finite

$\Rightarrow X \in \tau$

$\Rightarrow \phi, X \in \tau$

(ii) Let $A, B \in \tau$ then A' & B' are finite

Now $(A \cap B)' = A' \cup B'$ (By Demorga's Law)

As R.H.S is finite being union of finite sets

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$\Rightarrow$ L.H.S is finite $\Rightarrow$ $(A \cap B)'$ is finite

$\Rightarrow A \cap B \in \tau$

(iii) Let $\{A_\alpha : \alpha \in I\}$ be a subfamily of τ

$\Rightarrow A_\alpha'$ are finite, $\forall \alpha \in I$

Now $\left(\bigcup_{\alpha \in I} A_\alpha\right)' = \bigcap_{\alpha \in I} A_\alpha'$ (By Demorgan's Law)

As A_α' are finite $\Rightarrow \bigcap_{\alpha \in I} A_\alpha'$ is finite

$\Rightarrow \left(\bigcup_{\alpha \in I} A_\alpha\right)'$ is finite $\Rightarrow \bigcup_{\alpha \in I} A_\alpha \in \tau$

$\Rightarrow \tau$ is topology on X

Theorem:- Let τ_1 & τ_2 be two Topologies on a non-empty set X. Then $\tau_1 \cap \tau_2$ is also a Topology on X.

Proof:

(i) As $\phi, X \in \tau_1$ $\because \tau_1$ is Topology

& $\phi, X \in \tau_2$ $\because \tau_2$ " "

$\Rightarrow \phi, X \in \tau_1 \cap \tau_2$

(ii) Let $U_1, U_2 \in \tau_1 \cap \tau_2$

$\Rightarrow U_1, U_2 \in \tau_1$ & $U_1, U_2 \in \tau_2$

$\Rightarrow U_1 \cap U_2 \in \tau_1$ & $U_1 \cap U_2 \in \tau_2$ $\because$ Both τ_1 & τ_2 are Topologies

$\Rightarrow U_1 \cap U_2 \in \tau_1 \cap \tau_2$

(iii) Let $\{U_\alpha : \alpha \in I\}$ be any subcollection of $\tau_1 \cap \tau_2$

As $U_\alpha \in \tau_1 \cap \tau_2$, $\forall \alpha \in I$

Then $U_\alpha \in \tau_1$ & $U_\alpha \in \tau_2$ $\forall \alpha \in I$

$\Rightarrow \bigcup_{\alpha \in I} U_\alpha \in \tau_1$ & $\bigcup_{\alpha \in I} U_\alpha \in \tau_2$ $\because \tau_1$ & τ_2 are Topologies

$\Rightarrow \bigcup_{\alpha \in I} U_\alpha \in \tau_1 \cap \tau_2 \Rightarrow \tau_1 \cap \tau_2$ is Topology on X.

Example:- Let $X = \{1, 2, 3\}$

$\tau_1 = \{\phi, X, \{1\}\}$, $\tau_2 = \{\phi, X, \{2\}\}$

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$\tau_1 \cup \tau_2 = \{\phi, X, \ldots\}$

is not Topology because Let $U_1 = \{1\}$, $U_2 = \{2\} \in \tau_1 \cup \tau_2$

But $U_1 \cup U_2 = \{1\} \cup \{2\} = \{1,2\} \notin \tau_1 \cup \tau_2$

$\Rightarrow \tau_1 \cup \tau_2$ is not a topology.

Open Set:- Let (X, τ) be a topological space and $A \subseteq X$. Then A is called an open set if $A \in \tau$. e.g $X \neq \phi$, $\tau = \{\phi, X\}$ then ϕ & X are open sets.

Theorem:- Let (X, τ) be a Topological space then

(i) X and ϕ are open sets

(ii) Finite intersection of open sets is open

(iii) Union of any number of open sets in X is open.

Proof (i) As X & $\phi \in \tau$ $\because \tau$ is Topology on X

$\Rightarrow$ X and ϕ are open sets

(ii) Let $\{A_i : i = 1, 2, 3, \ldots, n\}$ be a finite family of open sets in X.

As $A_i \in \tau$, $\forall i$ ($\because A_i$ are open sets)

As τ is Topology on X

$\Rightarrow \bigcap_{i=1}^{n} A_i \in \tau \Rightarrow \bigcap_{i=1}^{n} A_i$ is open set.

(iii) Let $\{A_\alpha : \alpha \in I\}$ be a family of open sets in X.

To prove $\bigcup_{\alpha \in I} A_\alpha$ is open in X

As $\{A_\alpha : \alpha \in I\} \in \tau$ ($\because A_\alpha$ are open sets)

And τ is a Topology

$\Rightarrow \bigcup_{\alpha \in I} A_\alpha \in \tau \Rightarrow \bigcup_{\alpha \in I} A_\alpha$ is open set.

Theorem:- Show that every metric space is a Topological space.

Proof Let (X, d) be a metric space and

$\tau = \{U / U \subseteq X$ and U is an open set$\}$

(i) As ϕ is an open set in X $\Rightarrow \phi \in \tau$

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& as X is also an open set in $X \Rightarrow X \in \tau$

$\Rightarrow \phi$ and $X \in \tau$

(ii) Let U_1 and $U_2 \in \tau$ then U_1 & U_2 are open sets in X

$\Rightarrow U_1 \cap U_2$ is an open set in X { ∵ finite intersection of open sets is open in Metric space

$\Rightarrow U_1 \cap U_2 \in \tau$

(iii) Let $\{U_\alpha : \alpha \in I\}$ is a sub family of τ

To prove $\bigcup_{\alpha \in I} U_\alpha \in \tau$

As $U_\alpha \in \tau \Rightarrow U_\alpha$ are open sets

$\Rightarrow \bigcup_{\alpha \in I} U_\alpha$ is an open set { ∵ Union of any number of open sets is open in metric space

$\Rightarrow \bigcup_{\alpha \in I} U_\alpha \in \tau$

$\Rightarrow \tau$ is topology on X

Hence every metric space is a Topological space.

Note:- The Topology determined on Metric space is called Metric Topology.

⇒ **Neighbourhood:-** Let (X, τ) be a Topological space and let x be a point of X. A subset N of X is called a nbhd of x iff there exist an open set U such that $x \in U$ and $U \subseteq N$ (or for short $x \in U \subseteq N$)

Example Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{b\}, \{b, c\}\}$ find the nbhd of points of X.

Solution Let $x = a \in X \subseteq X$

$\Rightarrow X$ is nbhd of a

If $x_1 = b$ then $\{b\}$, $\{b, c\}$, $\{a, b\}$ and X are

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nbhds of b. Similarly the point c has two nbhds $\{b, c\}$ and X.

Theorem:- Let (X, τ) is a Topological space then a subset A of X is open iff A is a nbhd of each of its points.

Proof Given A is open. To prove A is nbhd of each of its points.

Let $x \in A$. As $A \subseteq A$ and A is open

$\Rightarrow x \in A \subseteq A \Rightarrow$ A is nbhd of x

Since x is arbitrary point of A

$\Rightarrow$ A is nbhd of each of its point.

Conversely Let A is nbhd of each of its point. To prove A is open

Let $x \in A$. Since A is nbhd of x then there exist an open ~~ball~~ set U_x s.t

$$x \in U_x \subseteq A \Rightarrow \{x\} \subseteq U_x \subseteq A$$

$$\Rightarrow \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} U_x \subseteq A \Rightarrow A \subseteq \bigcup_{x \in A} U_x \subseteq A$$

$$\Rightarrow A = \bigcup_{x \in A} U_x$$

Since U_x is open and union of open sets is open therefore A is open.

⇒ Closed Set:- Let (X, τ) be a topological space and $A \subseteq X$ then A is said to be closed if its A' is an open set i.e $A' \in \tau$

Example:- Let $X = \{a, b, c, d\}$ and Let $\tau = \{\phi, X, \{a\}, \{b\}\}$ then the closed set in (X, τ) are $X, \phi, \{b, c, d\}, \{c, d\}$ which are the complements of $\phi, X, \{a\}, \{a, b\}$ (open sets)

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Theorem:- Let (X, τ) be a topological space then

(1) X & ϕ are closed

(2) finite union of closed set is closed

(3) Intersection of any number of closed sets is closed.

Proof:

(i) As $X' = \phi \in \tau$

$\Rightarrow X'$ is open $\Rightarrow X$ is closed

As $\phi' = X \in \tau$

$\Rightarrow \phi'$ is open $\Rightarrow \phi$ is closed

(ii) Let $\{F_i : i = 1, 2, 3, \cdots n\}$ be a finite family of closed sets

To prove $\bigcup_{i=1}^{n} F_i$ is closed.

i.e $\left(\bigcup_{i=1}^{n} F_i\right)'$ is open

Now $\left(\bigcup_{i=1}^{n} F_i\right)' = \bigcap_{i=1}^{n} F_i'$ By Demorgan's law

As F_i is closed $\Rightarrow F_i'$ are open sets

As intersection of finite number of open sets in an open set

$\therefore \bigcap_{i=1}^{n} F_i'$ is an open set

$\Rightarrow \left(\bigcup_{i=1}^{n} F_i\right)'$ is open $\Rightarrow \bigcup_{i=1}^{n} F_i$ is closed

(iii) Let $\{F_\alpha : \alpha \in I\}$ be a family of closed sets

To prove $\bigcap_{\alpha \in I} F_\alpha$ is closed

Now $\left(\bigcap_{\alpha \in I} F_\alpha\right)' = \bigcup_{\alpha \in I} F_\alpha'$

As F_α are closed sets $\Rightarrow F_\alpha'$ are open sets

As union of any number of open sets is an open set

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$\therefore \bigcup_{\alpha \in I} F\alpha'$ is an open set

$\Rightarrow \left(\bigcap_{\alpha \in I} F\alpha\right)'$ is an open set $\Rightarrow \bigcap_{\alpha \in I}$ is closed

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Theorem:- Show that if τ_1 & τ_2 are two Topologies on X then Topology $\tau_1 \cap \tau_2$ is coarser (weaker) than τ_1 as well as τ_2 and finer (Stronger) than any Topology which is coarser (weaker) then both.

Proof) Let τ_1 & τ_2 be two topologies.

To prove $\tau_1 \cap \tau_2$ is weaker than τ_1 & τ_2

As $\tau_1 \cap \tau_2 \subseteq \tau_1$ & $\tau_1 \cap \tau_2 \subseteq \tau_2$

$\Rightarrow \tau_1 \cap \tau_2$ is coarser (weaker) than τ_1 & τ_2

Now to prove $\tau_1 \cap \tau_2$ is finer (stronger) than any topology which is contained in both τ_1 & τ_2

Let $\tau^* \subseteq \tau_1$ & $\tau^* \subseteq \tau_2$

As $\tau^* \subseteq \tau_1$ & $\tau^* \subseteq \tau_2$

$\tau^* \subseteq \tau_1 \cap \tau_2 \Rightarrow \tau_1 \cap \tau_2$ is finer than τ^*

Since τ^* is an arbitrary Topology so $\tau_1 \cap \tau_2$ is stronger than every topology which is contained in both τ_1 & τ_2

⇒Closure of a Set:- Let (X, τ) be a topological space & $A \subseteq X$. then closure of A $(\bar{A})$ is the intersection of all closed supersets of A

Example:- Let $X = \{1, 2, 3, 4\}$

$\tau = \{\phi, X, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$

Let $A = \{4\}$

closed sets:- $X, \phi, \{2, 3, 4\}, \{3, 4\}, \{4\}$

closed super sets of A are $X, \{2, 3, 4\}, \{3, 4\}, \{4\}$

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⇒ Closure of a subset depends upon the topology of X.

$\bar{A} = X \cap \{2,3,4\} \cap \{3,4\} \cap \{4\}$

$\bar{A} = \{4\}$

Example:- Let $X = \{1,2,3,4\}$

$\tau = \{\phi, X, \{1,2\}\}$ Let $A = \{4\}$

closed sets:- $X, \phi, \{3,4\}$

closed supersets of A are $X, \{3,4\}$

$\bar{A} = X \cap \{3,4\}$

$\Rightarrow \bar{A} = \{3,4\}$

Results:- Let (X, τ) be a Topological space and $A \subseteq X$ (i) A is closed iff $\bar{A} = A$

(ii) If $A \subseteq B$ then $\bar{A} \subseteq \bar{B}$

Proof:

(i) Suppose A is closed

To prove $\bar{A} = A$, As $A \subseteq \bar{A}$ ——①

As $A \subseteq A$ and A is closed

∴ A is closed superset of A. As $\bar{A}$ is the intersection of all closed supersets of A

∴ $\bar{A} \subseteq A$ ——②

from ① & ② $\bar{A} \subseteq A$ & $A \subseteq \bar{A}$

$\Rightarrow \bar{A} = A$

Conversely suppose $\bar{A} = A$

As $\bar{A}$ = intersection of all closed supersets of A and intersection of closed sets is closed

Therefore $\bar{A}$ is closed

As $A = \bar{A} \Rightarrow$ A is closed

(ii) Given $A \subseteq B$

To prove $\bar{A} \subseteq \bar{B}$

As $B \subseteq \bar{B}$ (Always True)

& $A \subseteq B$ ∴ $A \subseteq B \subseteq \bar{B}$

$\Rightarrow A \subseteq \bar{B}$

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As $\bar{B}$ is closed & $A \subseteq \bar{B}$
$\therefore \bar{B}$ is closed superset of A
But $\bar{A}$ = intersection of all closed supersets of A. Therefore $\bar{A} \subseteq \bar{B}$

Theorem:- Let (X, τ) be a Topological space and $A, B \subseteq X$ Then

(i) $\bar{\phi} = \phi$ (ii) $\bar{X} = X$ (iii) $\bar{\bar{A}} = \bar{A}$

(iv) $\overline{A \cup B} = \bar{A} \cup \bar{B}$ (v) $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

Proof 1) As ϕ is closed $\Rightarrow \bar{\phi} = \phi$ (By Previous theorem)

2) As X is closed $\Rightarrow \bar{X} = X$

3) To prove $\bar{\bar{A}} = \bar{A}$

Let $\bar{A} = C$ As $\bar{A}$ is closed $\Rightarrow C$ is closed

As C is closed $\Rightarrow \bar{C} = C$

$\Rightarrow \bar{\bar{A}} = \bar{A} \quad \because \bar{A} = C$

4) To prove $\overline{A \cup B} = \bar{A} \cup \bar{B}$

As $A \subseteq A \cup B \Rightarrow \bar{A} \subseteq \overline{A \cup B}$

As $B \subseteq A \cup B \Rightarrow \bar{B} \subseteq \overline{A \cup B}$

As $\bar{A} \subseteq \overline{A \cup B}$ & $\bar{B} \subseteq \overline{A \cup B}$

$\Rightarrow \bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$ ——— ①

As $A \subseteq \bar{A}$ & $B \subseteq \bar{B} \Rightarrow A \cup B \subseteq \bar{A} \cup \bar{B}$

As $\bar{A}$ & $\bar{B}$ are closed sets

$\therefore \bar{A} \cup \bar{B}$ is closed $\because$ union of closed sets is closed

Also $A \cup B \subseteq \bar{A} \cup \bar{B}$

$\Rightarrow \bar{A} \cup \bar{B}$ is closed super set of $A \cup B$

As $\overline{A \cup B}$ = intersection of all closed supersets of $A \cup B$

$\Rightarrow \overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$ ——— ②

from ① & ②

$\overline{A \cup B} = \bar{A} \cup \bar{B}$

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(v) To prove $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

As $A \cap B \subseteq A \Rightarrow \overline{A \cap B} \subseteq \bar{A}$

As $A \cap B \subseteq B \Rightarrow \overline{A \cap B} \subseteq \bar{B}$

As $\overline{A \cap B} \subseteq \bar{A}$ & $\overline{A \cap B} \subseteq \bar{B}$

$\Rightarrow \overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

Coverse of (v) is not true in general

Example :- 1) Let $X = \mathbb{R}$ & $A =]1,2[$, $B =]2,3[$

$\bar{A} = [1,2]$ & $\bar{B} = [2,3]$

$\bar{A} \cap \bar{B} = \{2\}$ —— ① $A \cap B = \phi$

$\overline{A \cap B} = \bar{\phi} = \phi$ —— ②

from ① & ②

$\bar{A} \cap \bar{B} \not\subseteq \overline{A \cap B}$

2) Let $X = \{a, b, c\}$ & $\tau = \{X, \phi, \{a\}, \{b\}, \{a,b\}\}$

Let $A = \{a\}$ & $B = \{b, c\}$

$\bar{A}$ = Intersection of closed super sets of A

$\bar{A} = \{a, c\}$ Also $\bar{B} = \{b, c\}$

Hence $\bar{A} \cap \bar{B} = \{c\}$

but $A \cap B = \phi \Rightarrow \overline{A \cap B} = \bar{\phi} = \phi$

So $\bar{A} \cap \bar{B} \neq \overline{A \cap B}$

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⇒ **Neighbourhood:-** Let (X,τ) be a topological space and $x \in X$. then a subset N of X is said to be nbhd of x if there exist some open set U in X such that $x \in U \subseteq N$

Example:- Let $X = \{1,2,3,4\}$, $\tau = \{\phi, X, \{1\}, \{4\}, \{2,3\}, \{1,2,3\}, \{1,4\}, \{2,3,4\}\}$. $N = \{1,2,4\}$

Now As $1 \in \{1\} \subseteq N$

$\Rightarrow N$ is neighbourhood of 1

N is not nbhd of 2.

As 3 does not belong to N. $\Rightarrow N$ is not neighbourhood of 3.

As $4 \in \{1,4\} \subseteq N$

$\therefore N$ is the nbhd of 4.

So N is the nbhd of 1 and 4.

***Remarks:-** If N is an open set. Then as for each $x \in N \Rightarrow x \in N \subseteq N$. So it means N is the nbhd of each of its element. It is called open neighbourhood.

⇒ **Interior Point:-** Let (X,τ) be a topological space and $A \subseteq X$. Then an element $x \in A$ is said to be an interior point of A if $\exists$ some open set U in X such that $x \in U \subseteq A$. In other words, x is an interior point of A if & only if A is neighbourhood of x

OR x is an interior point of A if $\exists$ an open set U containing x such that $U \cap A' = \phi$

⇒ **Interior of A Set:-** Let (X,τ) be a topological space and $A \subseteq X$ then interior of A is the union of all open sets which are contained in A. In other words if $\{G_i : i \in I\}$ be a family of open sets contained in A then $A^\circ = \bigcup_i G_i$

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Furthermore A° is the largest open set contained in A. i.e if G is an open subset of A then $G \subset A^\circ \subset A$

★ $A^\circ \subset A$.

Theorem – Let (X, τ) be a topological space and $A, B \subset X$ then show that

1) A° is an open set 2) $A^\circ = A$ iff A is open
3) $X^\circ = X$ & $\phi^\circ = \phi$ 4) If $A \subseteq B$ then $A^\circ \subseteq B^\circ$
5) $(A \cap B)^\circ = A^\circ \cap B^\circ$ 6) $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$
7) $(A^\circ)^\circ = A^\circ$

Proof 1) As A° = Union of all open sets contained in A. As union of any number of open sets is an open set

$\Rightarrow A^\circ$ is an open set.

2) Suppose $A = A^\circ$

To prove A is an open set

As A° is an open set and $A = A^\circ$

so A is an open set

Conversely suppose A is an open set

To prove $A = A^\circ$

As $A^\circ \subseteq A$ —— ①

As A° = Union of all open sets contained in A

Also A is an open set & $A \subseteq A$

$\Rightarrow$ A is an open set containd in A

But A° = union of all open sets contained in A.

$\Rightarrow A \subseteq A^\circ$ —— ②

from ① & ② $A = A^\circ$

3) As ϕ & X are open sets

$\Rightarrow \phi^\circ = \phi$ & $X^\circ = X$ (by ②)

4) ~~If $A \subseteq B$~~ As $A^\circ \subseteq A$ & $A \subseteq B$

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$\Rightarrow A^\circ \subseteq A \subseteq B \quad \Rightarrow A^\circ \subseteq B$

$\Rightarrow A^\circ$ is an open set contained in B

But B° = union of all open sets contained in B

$\Rightarrow A^\circ \subseteq B^\circ$

Alternate:- To prove $A^\circ \subseteq B^\circ$

Let $x \in A^\circ \Rightarrow x$ is an interior point of A

so $\exists$ an open set U s.t $x \in U \subseteq A$

But $A \subseteq B \Rightarrow x \in U \subseteq A \subseteq B$

$\Rightarrow x \in U \subseteq B \Rightarrow x$ is an interior pt of B

$\Rightarrow x \in B^\circ \Rightarrow A^\circ \subseteq B^\circ$

5) As $A \cap B \subseteq A \Rightarrow (A \cap B)^\circ \subseteq A^\circ$

As $A \cap B \subseteq B \Rightarrow (A \cap B)^\circ \subseteq B^\circ$

As $(A \cap B)^\circ \subseteq A^\circ$ & $(A \cap B)^\circ \subseteq B^\circ$

$\Rightarrow (A \cap B)^\circ \subseteq A^\circ \cap B^\circ$ ——— ①

Conversely $A^\circ \subseteq A$ & $B^\circ \subseteq B$

$\Rightarrow A^\circ \cap B^\circ \subseteq A \cap B$.

As A° & B° are open sets then

$A^\circ \cap B^\circ$ is an open set

$\Rightarrow A^\circ \cap B^\circ$ is an open set contained in $A \cap B$

But $(A \cap B)^\circ$ = union of all open sets contained in $A \cap B$

$\Rightarrow A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$ ——— ②

From ① & ②

$(A \cap B)^\circ = A^\circ \cap B^\circ$

6) To prove $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$

As $A \subseteq A \cup B \Rightarrow A^\circ \subseteq (A \cup B)^\circ$

Also $B \subseteq A \cup B \Rightarrow B^\circ \subseteq (A \cup B)^\circ$

As $A^\circ \subseteq (A \cup B)^\circ$ & $B^\circ \subseteq (A \cup B)^\circ$

$\Rightarrow A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$

Example:- Let $X = \{a, b, c, d\}$

$\tau = \{\phi, X, \{a, b\}, \{c, d\}\}$

Let $A = \{a\}$, $B = \{b, d\}$

clearly $A^\circ = \phi$ & $B^\circ = \phi$

But $A \cup B = X$

So $(A \cup B)^\circ = X^\circ = X$

But $A^\circ \cup B^\circ = \phi \cup \phi = \phi$

So $A^\circ \cup B^\circ \neq (A \cup B)^\circ$

7) Let $A^\circ = C$. As A° is open set

$\Rightarrow C$ is an open set

As C is an open set $\Rightarrow C^\circ = C$

$\Rightarrow (A^\circ)^\circ = A^\circ \quad (\because A^\circ = C$

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⇒ **Exterior point:-** Let (X, τ) be a topological space and $A \subseteq X$ then $x \in X$ is said to be an exterior point of A if x is an interior point of A'. i.e x is said to be exterior point of A if there exist some open set U such that $x \in U \subseteq A'$

OR x is exterior point of A if there exist open set U containing x s.t $U \cap A = \phi$

⇒ **Boundry Point:-** Let (X, τ) be a topological space and $A \subseteq X$ then $x \in X$ is said to be boundry point or frontier point of A if x is neither the interior point of A nor the exterior point of A'. In other words $x \in X$ is said to be boundry point of $A \subseteq X$ if for every open set U containing x

$$U \cap A \neq \phi \quad \text{and} \quad U \cap A' \neq \phi$$

* **Remark:-** A° or $Int(A)$ denotes the set of all interior points of A, $Ext(A)$ denotes the set of all exterior points of A, $b(A)$ or $Fr(A)$ denotes the set of all boundry points of A.

Example:- Let $X = \{1, 2, 3, 4\}$

$\tau = \{\phi, X, \{1,2\}, \{2,3\}, \{2\}, \{1,2,3\}\}$ and $A = \{2, 3, 4\}$ then find A', $Ext(A)$, $b(A)$

Solution If $A = \{2, 3, 4\}$ then $A' = \{1\}$

$1 \in b(A)$, $2 \in Int(A)$

$3 \in Int(A)$, $4 \in b(A)$

$Int(A) = \{2, 3\}$, $Ext(A) = \phi$

$b(A) = \{1, 4\}$

Note that:-

(i) $Int(A) \cup Ext(A) \cup b(A) = X$

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(ii) $Int(A) \cap Ext(A) = \phi$

(iii) $Int(A) \cap b(A) = \phi$

(iv) $b(A) \cap Ext(A) = \phi$

Theorem:- Let (X, τ) be a topological space and $A \subseteq X$, then $\bar{A} = \{x / x \in X$ and each open nbhd of x intersect $A\} = \{x / x \in X$ and for each open nbhd u of x, $u \cap A \neq \phi\}$

Proof: Let $B = \{x / x \in X \wedge$ for each open nbhd u of x $u \cap A \neq \phi\}$

To prove $\bar{A} = B$

Let $x \in \bar{A}$ To prove $x \in B$

Suppose $x \notin B \Rightarrow \exists$ an open nbhd u of x s.t $u \cap A = \phi$

$\Rightarrow A \subseteq u'$

As u is open $\Rightarrow u'$ is closed superset of A.

As $\bar{A}$ = Intersection of all closed supersets of A

$\Rightarrow \bar{A} \subseteq u'$

Since $x \in u \Rightarrow x \notin u' \Rightarrow x \notin \bar{A}$

which is a contradiction

So our supposition is wrong $\therefore x \in B$

$\Rightarrow \bar{A} \subseteq B$ ——— (1)

Conversely Let $x \in B$. To prove $x \in \bar{A}$

Suppose $x \notin \bar{A} \Rightarrow \exists$ a closed superset F of A s.t $x \notin F \Rightarrow x \in F'$

As F is closed $\Rightarrow F'$ is an open set.

Moreover F' is nbhd of x because open set is nbhd of each of its points

As $F \cap F' = \phi$ & $A \subseteq F \Rightarrow A \cap F' = \phi$

$\Rightarrow x \notin B$

which is a contradiction

So our suppositian is wrong $\therefore x \in \bar{A}$

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$\Rightarrow B \subseteq \bar{A}$ ——— ②

From ① & ②

$\bar{A} = B$

⇒ Limit Point of a Set:- Let (X, τ) be a topological space and $A \subseteq X$. Then a point $x \in X$ is called limit point of A if for each open set u s.t $x \in u$

$u \cap A \setminus \{x\} \neq \phi$

The set of all limit points of A is called derived set of A and it is denoted by $D(A)$, A^d, A'

Example:- $X = \{1, 2, 3, 4\}$

$\tau = \{\phi, X, \{1\}, \{2,3\}, \{1,2,3\}\}$

Let $A = \{2,3,4\}$ Take $x = 4$

Take $u = X \in \tau$ s.t $x \in u$

$A \setminus \{4\} = \{2,3\}$ $\quad u \cap A \setminus \{4\} = X \cap \{2,3\}$

$= \{2,3\} \neq \phi$

$\Rightarrow 4$ is a limit point

Take $x = 1$. Take $u = X$ s.t $1 \in u$

$A \setminus \{1\} = \{2,3,4\}$, $u \cap A \setminus \{x\} = X \cap \{2,3,4\}$

$= \{2,3,4\} \neq \phi$

Now Take $u = \{1\}$ s.t $1 \in u$

$A \setminus \{x\} = \{2,3,4\}$, $u \cap A \setminus \{1\} = \{1\} \cap \{2,3,4\} = \phi$

$\Rightarrow x$ is not limit point of A.

Remark:- An element $x \in X$ can never be limit point of any subset A of X if $\{x\} \in \tau$ because $\{x\} \cap A \setminus \{x\} = \phi$

Theorem:- Let (X, τ) be a topological space and $A \subseteq X$. Then

1) $\bar{A} = A \cup D(A)$
2) A is closed iff $D(A) \subseteq A$.

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1)

Proof) Let $x \in \bar{A}$. To prove $x \in A \cup D(A)$

If $x \in A$ then $x \in A \cup D(A)$

If $x \notin A$ then we have to prove $x \in D(A)$

Suppose $x \notin D(A)$

$\Rightarrow$ x is not a limit point of set A

then $\exists$ atleast one open set U s.t $x \in U$ and $U \cap A \setminus \{x\} = \phi$

As $x \notin A \Rightarrow A \setminus \{x\} = A$

$\therefore U \cap A = \phi \Rightarrow A \subseteq U'$

As $x \in U \Rightarrow x \notin U'$

As U is an open set $\Rightarrow U'$ is closed

Also $A \subseteq U'$

$\Rightarrow U'$ is closed superset of A

As $\bar{A}$ = intersection of all closed supsets of A

$\therefore \bar{A} \subseteq U'$

As $\bar{A} \subseteq U'$ and $x \notin U' \Rightarrow x \notin \bar{A}$

So our supposition is wrong

$\therefore x \in D(A) \Rightarrow x \in A \cup D(A)$

$\Rightarrow \bar{A}$ is subset of $A \cup D(A)$ ——— (1)

Conversely

Let $x \in A \cup D(A)$

$\Rightarrow x \in A$ or $x \in D(A)$

If $x \in A \Rightarrow x \in \bar{A}$ $(\because A \subseteq \bar{A})$

If $x \notin A$ then $x \in D(A)$

To prove $x \in \bar{A}$

Suppose $x \notin \bar{A} \Rightarrow \exists$ a closed superset of A say F s.t $x \notin F \Rightarrow x \in F'$

As F is closed $\Rightarrow F'$ is open

$\Rightarrow F'$ is open set containing x

As $F \cap F' = \phi$ and $A \subseteq F$

$\Rightarrow A \cap F' = \phi$

Since $x \notin A \Rightarrow A \setminus \{x\} = A$

$\Rightarrow A \setminus \{x\} \cap F' = \phi$ or $F' \cap A \setminus \{x\} = \phi$

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$\Rightarrow$ x is not the limit point of A

$\Rightarrow x \notin D(A)$

Which is a contradiction. So our supposition is wrong

Hence $x \in \bar{A}$

$\therefore A \cup D(A) \subseteq \bar{A}$ ——— ②

from ① & ② $\bar{A} = A \cup D(A)$

2) To prove A is closed iff $D(A) \subseteq A$

Let A is closed

To prove $D(A) \subseteq A$

Let $x \in D(A) \Rightarrow x \in A \cup D(A) = \bar{A}$

$\Rightarrow x \in \bar{A} = A$ ($\because A$ is closed iff $A = \bar{A}$)

$\Rightarrow x \in A \Rightarrow D(A) \subseteq A$

Conversely suppose $D(A) \subseteq A$

To prove A is closed

For this we have to show that $\bar{A} = A$

As $A \subseteq \bar{A}$ ——— ①

As $D(A) \subseteq A$ & $A \subseteq A$

$\Rightarrow$ ~~$A \subseteq D(A)$~~ $A \cup D(A) \subseteq A$

$\Rightarrow \bar{A} \subseteq A$ ——— ②

from ① & ② $\bar{A} = A$

$\Rightarrow A$ is closed.

Theorem:- Let (X, τ) be a topological space then any closed set A of X is the disjoint union of A° & $b(A)$

Proof) We have to show That

(i) $A = A^\circ \cup b(A)$

(ii) $A^\circ \cap b(A) = \phi$

(i) Let $x \in A$

If $x \in A^\circ$ then $x \in A^\circ \cup b(A)$

So nothing is left to prove

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If $x \notin A^\circ$. To prove $x \in b(A)$
Suppose $x \notin b(A)$
then $x \in Int(A) \cup Ext(A)$
but $x \notin A^\circ$ so $x \in Ext(A)$
$\Rightarrow$ $\exists$ an open set V s.t
$x \in V \subseteq A' \Rightarrow x \in A' \Rightarrow x \notin A$
which is a contradiction.
So our supposition is wrong
$\therefore x \in b(A)$
$\Rightarrow x \in A^\circ \cup b(A)$ so $A \subseteq A^\circ \cup b(A)$ ——— ①

Converse
Let $x \in A^\circ \cup b(A)$
$\Rightarrow x \in A^\circ$ or $x \in b(A)$
If $x \in A^\circ$ & $A^\circ \subseteq A \Rightarrow x \in A$
Then nothing is left to prove
If $x \notin A^\circ \Rightarrow x \in b(A)$
To prove $x \in A$
Suppose $x \notin A \Rightarrow x \in A'$
As A is closed $\Rightarrow A'$ is an open set
Also $A \cap A' = \phi$
$\therefore x$ is not a boundry point of A
$\Rightarrow x \notin b(A)$
which is a contradiction
So our supposition is wrong $\therefore x \in A$
$\therefore A^\circ \cup b(A) \subseteq A$ ——— ②
from ① & ② $A = A^\circ \cup b(A)$

(ii) To prove $A^\circ \cap b(A) = \phi$
Let $A^\circ \cap b(A) \neq \phi$
Let $x \in A^\circ \cap b(A)$
$\Rightarrow x \in A^\circ$ and $x \in b(A)$
If $x \in A^\circ$ then there exist an open set U containing x s.t
$x \in U \subseteq A$

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$\Rightarrow U \cap A' = \phi$ $\because A \cap B = \phi$ iff $A \subseteq B'$ & $B \subseteq A'$

$\Rightarrow x \notin b(A)$ ($\because$ if $x \in b(A)$, $U \cap A' \neq \phi$, $U \cap A \neq \phi$ which is a contradiction. because $x \in b(A)$ So our suppocition is wrong

Hence $A^\circ \cap b(A) = \phi$

Definition:- Let (X, τ) be a topological space and $A \subseteq X$. then a point $x \in A$ is said to be a isolated point of A if x is not the limit point of A. i.e there exist an open set U containing x s.t $U \cap A \setminus \{x\} = \phi$

The set of all isolated points is denoted by A^*

i.e $A^* = \{x : x \in A$ and x is not a limit point$\}$

Theorem:- Let (X, τ) be a topological space then any closed subset A of X is disjoint union of the set of isolated points of A and the set of limit points of A.

Proof: We have to prove

(i) $A = D(A) \cup A^*$

(ii) $D(A) \cap A^* = \phi$

(i) Let $x \in A$. To prove $x \in D(A) \cup A^*$

If $x \in A^*$ then $x \in D(A) \cup A^*$ then nothing is left to prove

If $x \notin A^*$ To prove $x \in D(A)$

Suppose $x \notin D(A)$

$\Rightarrow$ x is not a limit point of A.

$\Rightarrow$ x is isolated point of A

$\Rightarrow x \in A^*$

A contradiction because $x \notin A^*$

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So our supposition is wrong

Hence $x \in D(A) \Rightarrow x \in D(A) \cup A^*$

$\Rightarrow A \subseteq D(A) \cup A^*$ ——— ①

Conversely let $x \in D(A) \cup A^*$

$\Rightarrow x \in D(A)$ or $x \in A^*$

If $x \in A^*$ and $A^* \subseteq A \Rightarrow x \in A$

If $x \notin A^*$ then $x \in D(A)$

To prove $x \in A$

As A is closed $\Rightarrow D(A) \subseteq A$

$\Rightarrow x \in A$

$\Rightarrow D(A) \cup A^* \subseteq A$ ——— ②

from ① & ②

$A = D(A) \cup A^*$

(ii) $D(A) \cap A^* = \phi$ obvious.

⇒ **Dense Set:-** Let (X, τ) be a topological space and $A \subseteq X$ then A is called dence in X if $\bar{A} = X$.

Example:- Let $X = \{1,2,3,4,5\}$, $\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$

Let $A = \{1,2\}$ Show that A is dence in X.

Solution

closed sets are $X, \phi, \{2,3,4,5\}, \{1,3,4,5\}, \{3,4,5\}$

closed super set of A is X only

therefore $\bar{A} = X$

$\Rightarrow$ A is dence in X.

Example:- Let $X = \{1,2,3,4,5\}$, $\tau = \{\phi, X, \{1,2\}, \{1\}, \{2\}\}$

& $B = \{2,3\}$ B is Dence or not.

Solution

closed sets are $X, \phi, \{2,3,4,5\}$

$\{1,3,4,5\}, \{3,4,5\}$

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closed supersets of B are $X, \{2,3,4,5\}$

$\overline{B} = X \cup \{2,3,4,5\}$

$\overline{B} = \{2,3,4,5\} \neq X$

$\Rightarrow B$ is not dence in X

Example:- $X = \mathbb{R}$, $\tau = P(X) = 2^X$

Then $\overline{\mathbb{R}} = \mathbb{R}$ So $\mathbb{R}$ is dence in $\mathbb{R}$

Because for any proper subset of $\mathbb{R}$ i.e $A \subseteq \mathbb{R}$ but $A \neq \mathbb{R}$ Then A is closed then $\overline{A} = A \Rightarrow \overline{A} \neq \mathbb{R}$ $(\because A \neq \mathbb{R})$

$\Rightarrow$ A is not dence set in $\mathbb{R}$

So $\mathbb{R}$ is only dence set in $\mathbb{R}$

Note:- In discrete topology every subset is open as well as closed.

⇒ **Denumerable Set:-** a set A is said to be Denumerable if it has one-to-one correspondence with set of natural numbers or there exist a function $f: A \to N$ which is bijective.

⇒ **Countable Set:-** A set A is said to be countable if it is either finite or denumerable.

Example:- $A = \{1,2,3\}$, $N = \{1,2,3,\ldots\}$

{* Infinite countable set is called denumerable}

⇒ **Separable Space:-** Let (X, τ) be a topological space then (X, τ) is said to be separable if it has a countable dence set.

Example:- Let $X = \{1,2,3\}$

$\tau = \{\phi, X, \{1\}\}$ then $\overline{X} = X$ (always True)

$\Rightarrow$ X is dence in (X, τ)

As X is finite so X is countable

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$\Rightarrow (X, \tau)$ has a countable dence set
$\Rightarrow (X, \tau)$ is separable.

Example 2:- $X = \{1, 2, 3, 4, 5\}$
$\tau = \{\phi, X, \{1\}, \{2\}, \{1, 2\}\}$ Take $A = \{1, 2\}$
Then $\bar{A} = X$ (already proved
$\Rightarrow$ A is dence in X
Also A is countable
Hence (X, τ) is separable.

$\Rightarrow$ **Base:-** Let (X, τ) be a topological space and β is subfamily of τ then β is called base for τ (or for X). If each $u \in \tau$ can be written as the unions of members of β

Example:- Let $X = \{a, b\}$
$\tau = \{\phi, X, \{a\}, \{b\}\}$ $\beta = \{\phi, \{a\}, \{b\}\}$
(i) $\beta \subseteq \tau$. Let $\phi \in \beta \Rightarrow \phi \cup \phi = \phi$
Let $X \in \tau \Rightarrow X = \{a\} \cup \{b\}$
$\{a\} \in \tau$ then $\{a\} = \{a\} \cup \{a\}$
$\{b\} \in \tau$ then $\{b\} = \{b\} \cup \{b\}$
note that each $u \in \tau$ can be written as the union of members of β

Example:- Let $X = \{1, 2, 3\}$
$\tau = \{\phi, X, \{1\}, \{2, 3\}\}$ $\beta = \{\phi, \{1\}, \{2, 3\}\}$
(i) $\beta \subseteq \tau$
(ii) $X = \{1\} \cup \{2, 3\}$
$\{1\} \in \tau$, $\{1\} = \{1\} \cup \{1\}$, $\{1\} \in \beta$
$\{2, 3\} \in \tau$, $\{2, 3\} = \{2, 3\} \cup \{2, 3\}$, $\{2, 3\} \in \beta$
Then β is base for τ

Example:- $X = \{1, 2, 3\}$, $\tau = \{\phi, X, \{1\}, \{2, 3\}\}$
$\beta_1 = \{\{1\}\} \subseteq \tau$ but $X, \{2, 3\}$

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can not be expressed as union of elements of β_1 $\Rightarrow$ β_1 is not base.

2)

$\beta_2 = \{\{1\}, \{2\}, \{3\}\}$

β_2 is not a base for τ $\because \beta_2 \notin \tau$

Although each $U \in \tau$ can be expressed as union of members of β

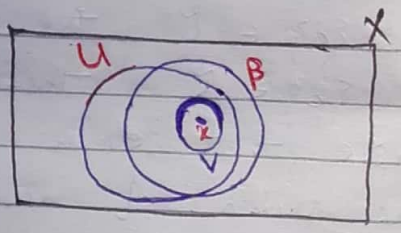
Remark:- It is not necessary that $\phi \in \beta$. Because to express $\phi \in \tau$ as a union of members of β we have an empty subfamily 'S' of β s.t $US = \phi$

Note:- $\phi \subseteq \beta$

$S = \{\phi : \phi \subseteq \beta\} \subseteq \beta$ is empty subfamily.

Theorem: Let (X, τ) be a topological space & β be a subfamily of τ, Then β is a base for X iff for each $U \in \tau$, $x \in U$ $\exists$ a $V \in \beta$ s.t $x \in V \subseteq U$

Proof: Suppose β is base for X.



To prove for each $U \in \tau$ s.t $x \in U$ $\exists$ a $V \in \beta$ s.t $x \in V \subseteq U$

Let $U \in \tau$ s.t $x \in U$

As $U \in \tau$ & β is base for τ then $\exists$ a subfamily $\{B_\alpha : \alpha \in I\}$ of β s.t $U = \bigcup_{\alpha \in I} B_\alpha$, $B_\alpha \in \beta$

As $x \in U = \bigcup_{\alpha \in I} B_\alpha \Rightarrow x \in \bigcup_{\alpha \in I} B_\alpha$

$\Rightarrow x \in B_\alpha$ for som $\alpha \in I$

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But $B_\alpha \subseteq \bigcup_{\alpha \in I} B_\alpha = U \Rightarrow B_\alpha \subseteq U$

$\Rightarrow x \in B_\alpha \subseteq U$

Let $B_\alpha = V \in \beta \Rightarrow x \in V \subseteq U$

Conversely

Suppose that for each $U \in \tau$ and $x \in U$ there exist $V \in \beta$ s.t $x \in V \subseteq U$

To prove β is a base for τ

Let $U \in \tau$ & $x \in U$

Then by given condition there exist $V_x \in \beta$ s.t $x \in V_x \subseteq U$

$\Rightarrow \{x\} \subseteq V_x \subseteq U$

$\Rightarrow \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} V_x \subseteq U$

$\Rightarrow U \subseteq \bigcup_{x \in U} V_x \subseteq U$

$\Rightarrow U = \bigcup_{x \in U} V_x$

$\Rightarrow \beta$ is base for X.

Theorem :- Let X be a non-empty set. A family β of subsets of X is a base for some topology on X iff

(i) $X = \bigcup_{\alpha \in I} B_\alpha$ where $B_\alpha \in \beta$

(ii) For B_1 & $B_2 \in \beta$ and $x \in B_1 \cap B_2$ then there exist $B_3 \in \beta$ s.t $x \in B_3 \subseteq B_1 \cap B_2$

Proof Given β is base for some topology τ on X.

To prove (i) & (ii) conditions are hold

(i) As $X \in \tau$ & β is base for τ Then $\exists$ a subfamily $\{B_\alpha : \alpha \in I\}$ of β s.t

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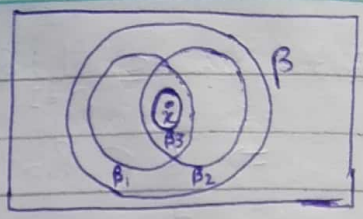
(ii)

$$X = \bigcup_{\alpha \in I} B_\alpha$$



(ii)

Let B_1 & $B_2 \in \beta \subseteq \tau$

$\Rightarrow B_1, B_2 \in \tau$

$\Rightarrow B_1 \cap B_2 \in \tau$ ($\because \tau$ is a topology

Let $x \in B_1 \cap B_2$

As $B_1 \cap B_2 \in \tau$ & $x \in B_1 \cap B_2$

& β is a base for τ

Then by a well know theorem $\exists$

$V \in \beta$ s.t $x \in V \subseteq B_1 \cap B_2$

Take $V = B_3 \in \beta$

$\Rightarrow x \in B_3 \subseteq B_1 \cap B_2$

Conversely

Suppose condition (i) & (ii) holds

To prove β is a base for some topology τ on X.

Let τ be the collection of all possible unions of members of β

We claim that τ is a topology on X & β is base for this topology

(i) As $X = \bigcup_{\alpha \in I} B_\alpha$, $B_\alpha \in \beta$

$\Rightarrow X \in \tau$ $\because$ every element of τ is the union of elements of β

Let γ be the empty collection of elements of β $\Rightarrow \phi = \cup \gamma$

$\Rightarrow \phi \in \tau$

(ii)

Let U_1 & $U_2 \in \tau$ Then there exist sub-families $\{B_\alpha : \alpha \in I\}$ & $\{B_{\alpha'} : \alpha' \in I'\}$ of β

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s.t $U_1 = \bigcup_{\alpha\in I} B_\alpha$ & $U_2 = \bigcup_{\alpha'\in I'} B_{\alpha'}$

Now $U_1 \cap U_2 = \left(\bigcup_{\alpha\in I} B_\alpha\right) \cap \left(\bigcup_{\alpha'\in I'} B_{\alpha'}\right)$

$U_1 \cap U_2 = \bigcup (B_\alpha \cap B_{\alpha'})$ — ① distributive law

— (*1)

Let $x \in B_\alpha \cap B_{\alpha'}$ where $B_\alpha, B_{\alpha'} \in \beta$

Then by give condition (ii) there exist $B_x \in \beta$ s.t

$x \in B_x \subseteq B_\alpha \cap B_{\alpha'}$

$\Rightarrow \{x\} \subseteq B_x \subseteq B_\alpha \cap B_{\alpha'}$

$\bigcup_{x\in B_\alpha\cap B_{\alpha'}} \{x\} \subseteq \bigcup_{x\in B_\alpha\cap B_{\alpha'}} B_x \subseteq B_\alpha \cap B_{\alpha'}$

$\Rightarrow B_\alpha \cap B_{\alpha'} \subseteq \bigcup_{x\in B_\alpha\cap B_{\alpha'}} B_x \subseteq B_\alpha \cap B_{\alpha'}$

$\Rightarrow B_\alpha \cap B_{\alpha'} = \bigcup_{x\in B_\alpha\cap B_{\alpha'}} B_x$ — (*2)

Put in ①

$U_1 \cap U_2 = \bigcup (\bigcup B_x)$

$\Rightarrow U_1 \cap U_2 \in \tau$

(iii) Let $\{U_\alpha : \alpha \in I\}$ be a subfamily of τ

As $U_\alpha \in \tau$ so $\exists$ a subcollection $\{B_{\alpha'} : \alpha' \in I'\}$ of β s.t

$U_\alpha = \bigcup_{\alpha'\in I'} B_{\alpha'}$

$\Rightarrow \bigcup_{\alpha\in I} U_\alpha = \bigcup_{\alpha\in I} \left(\bigcup_{\alpha'\in I'} B_{\alpha'}\right)$ — (*3)

$\Rightarrow \bigcup_{\alpha\in I} U_\alpha \in \tau$

$\Rightarrow \tau$ is topology on X & β is base for this topology

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Explanation:- (*1) $U B_x$ is the union of members of B. so $U(UB_x)$ is also the union of members of B.

(*2) $\underset{\alpha \in I'}{U} B\alpha'$ is the union of members of B

$\Rightarrow \underset{\alpha \in I}{U}(U B\alpha')$ is the union of members of B.

⇒ **Second Countable Space:-** A topological space (X, τ) is said to be 2nd-countable space if it has a countable base.

Example:- $X = \{1, 2\}$

$\tau = \{\phi, X, \{1\}, \{2\}\}$

$B = \{\{1\}, \{2\}\}$ is a base for X.

As B is finite so is countable.

Hence X has a countable base.

So X is 2nd countable.

⇒ Next theorem is on page 51.

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⇒ **Open Cover:-** Let (X, τ) be a topological space. A system $\{U_\alpha : \alpha \in I\}$ of open sets is said to be an open cover for X if $X = \bigcup_{\alpha \in I} U_\alpha$

Example:- $X = \{1, 2\}$, $\tau = \{\phi, X, \{1\}, \{2\}\}$

→ $U_1 = \{1\}$; $U_2 = \{2\}$

Then $X = \bigcup_{i=1}^{n} U_i$

So $\{U_1, U_2\}$ is an open cover for X

→ $U_1 = \{1\}$, $U_2 = \{2\}$, $U_3 = \phi$

Then $\{U_1, U_2, U_3\}$ is also an open cover for X

→ $U_1 = \{1\}$, $U_2 = \{X\}$

Then $\{U_1, U_2\}$ is an open cover for X

→ If $U_1 = X$ then $\{U_1\}$ is also an open cover for X

⇒ **Lindelof Space:-** A topological space (X, τ) is said to be lindelof space if every open cover has a countable subcover

Example:- $X = \{1, 2, 3\}$ $\tau = \{\phi, X, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}\}$

Then $\gamma_1 = \{\{2\}, \{3\}, \{1\}, \{2,3\}\}$ is an open cover for X

Let $\gamma_2 = \{\{1\}, \{2\}, \{3\}\}$

∴ γ_2 is a subcover and also countable

⇒ X is lindelof space.

⇒ **Lindelof Theorem:-** Let X be a 2nd countable space. If a non empty set G is represented as a union of open sets $\{G_i\}$, Then G can be represented as countable union of G_i's

OR Let X be a 2nd countable space. If a non empty set G is represented as a union of open sets then G can be represented as a countable union of open sets.

Proof: Let $\{G_i\}$ be collection of open sets in

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X s.t $G = \bigcup G_i$

Since X is 2nd countable space, So X has a countable base.

Let $\beta = \{B_n\}$ be a countable base for X.

Now let $x \in G \Rightarrow x \in \bigcup G_i$ $(\because G = \bigcup G_i)$

$\Rightarrow x \in G_i$ for some i

Now as $x \in G_i$, G_i is an open set and $\beta = \{B_n\}$ is a countable base for X then there exist $B_{nx} \in \beta$ s.t

$$x \in B_{nx} \subseteq G_i$$

$$\Rightarrow \{x\} \subseteq B_{nx} \subseteq G_i$$

$$\Rightarrow \bigcup_{x \in G} \{x\} \subseteq \bigcup_{x \in G} B_{nx} \subseteq G_i$$

$$\Rightarrow G \subseteq \bigcup_{x \in G} B_{nx} \subseteq G_i \subseteq G$$

$$\Rightarrow G \subseteq \bigcup_{x \in G} B_{nx} \subseteq G$$

$$\Rightarrow G = \bigcup_{x \in G} B_{nx}$$

Now as $\beta = \{B_n\}$ is countable so $\{B_{nx} : B_{nx} \in \beta \text{ and } \bigcup_{x \in G} B_{nx} = G\}$ is countable.

Thus for each basic open set in B_{nx} we can choose a set G_i which contains it (i.e $B_{nx} \subseteq G_i$) Then the class $\{G_i\}$ which arise in this way is countable and its union is G.

$B_{1x} \in G_1, B_{2x} \in G_2, \ldots, B_{nx} \in G_n, \ldots$

$B_{1x} \in \beta, B_{2x} \in \beta, \ldots, B_{nx} \in \beta \Rightarrow \{B_{1x}, B_{2x}, \ldots B_{nx}\} \in \beta$

$\Rightarrow$ As $\{B_n\}$ is countable so $\{B_{nx}\}$ is also countable

$G = \bigcup_{x \in G} B_{nx}$, For B_{1x} $\exists G_1 \in \{G_i\}$ s.t $B_{1x} \subseteq G_1$

" B_{2x} $\exists G_2$ " " " $B_{2x} \subseteq G_2$

" B_{nx} $\exists G_n$ " " " $B_{nx} \subseteq G_n$

$\Rightarrow G = \bigcup B_{nx} \subseteq \bigcup_{i \in N} G_i \subseteq \bigcup_{i \in I} G_i = G \Rightarrow G \subseteq \bigcup_{i \in N} \subseteq G \Rightarrow G = \bigcup_{i \in N} G_i$

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⇒ Relative Topology or Subspace Topology:-

Let (X,τ) be a topological space and Y be a non empty subset of X. Then collection $\tau_Y = \{U \cap Y : U \in \tau\}$ is a topology on Y and is called relative topology

Note:- If τ_Y is a relative topology on Y then (Y,τ_Y) is subspace of (X,τ)

Example:- $X = \{1,2,3\}$

$\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$ Let $Y = \{2,3\}$

$\tau_Y = \{\phi \cap Y, X \cap Y, \{1\} \cap Y, \{2\} \cap Y, \{1,2\} \cap Y\}$

$= \{\phi, Y, \{2\}\}$

Then τ_Y is a topology on Y

Note:- $\tau_1 = \{\phi, Y\}$ is topology on Y (Indiscrete)

$\tau_2 = P(Y)$ is also " " " (Discrete)

$\tau_3 = \{\phi, Y, \{3\}\}$ is another topology on Y

But τ_1, τ_2, τ_3 are not relative topologies.

Theorem:- Every subspace of a 2nd countable space is 2nd - countable.

Proof: Let X be a 2nd countable space and Y be a subspace of X.

To prove Y is 2nd countable

As X is 2nd countable, so X has a countable base.

Let $\beta = \{B_n : n \in \mathbb{N}\}$ be a countable base for X. Put $\beta_Y = \{B_n \cap Y : B_n \in \beta\}$

We claim that β_Y is a countable base for Y. As β is countable so β_Y is countable

Let U be an open set in Y and $x \in U$

As U is an open set in Y so there exist

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an open set V in X s.t $U = V \cap Y$

$x \in U = V \cap Y$

$\Rightarrow x \in V$ & $x \in Y$

As $V \in \tau$, $x \in V$

and β is base for X then $\exists$ an element $B_n \in \beta$ s.t $x \in B_n \subseteq V$

As $B_n \subseteq V \Rightarrow B_n \cap Y \subseteq V \cap Y = U$

$\Rightarrow x \in B_n \cap Y \subseteq U$

where $B_n \cap Y \in B_y$

$\Rightarrow B_y$ is a base for Y.

⇒ **First Countable Space:-** A topological space (X, τ) is said to be first countable space if it has countable local base at each of its point where local base is defined as...

* **Local base (Neighbourhood base) or Base at a point:-** Let (X, τ) be a topological space and $x \in X$ then a collection B_x (containing x) is called local base at x if for each $U \in \tau$ s.t $x \in U$ there exist $V \in B_x$ s.t

$x \in V \subseteq U$

Example:- Let $X = \{1, 2, 3, 4\}$

$\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}, \{3,4\}\}$

Take $x = 1 \in X$

Then $B_1 = \{\{1\}, \{1,2\}\}$ is a local base at x

Let $U = X$, $1 \in U$, $V = \{1\} \in B_1$

$\Rightarrow 1 \in V \subseteq U$

Let $U = \{1\} \in \tau$

$1 \in U$, $V = \{1\} \in B_1$

Then $1 \in V \subseteq U$

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$U = \{1,2\} \in \tau$, $1 \in U$, $V = \{1,2\} \subseteq B_1$

So $1 \in V \subseteq U$

So B_1 is a local base at $x=1$

Note:- 1) B_1 is a local base for 1 but is not a base for whole space

2) local base is defined point to point.

Theorem:- Prove that every 2nd-countable space is 1st countable.

Proof:- Let X be a 2nd countable space.

To prove X is 1st countable space.

For this. Let $x \in X$ be an arbitrary point of X we have to prove that X has a countable local base for this x.

Since X is 2nd countable space so X has a countable base.

Let $\beta = \{B_n\}$ be a countable base for X.

$\beta_x = \{B_{nx} : x \in B_{nx}, B_{nx} \in \beta\}$

Then $\beta_x \subseteq \beta \Rightarrow \beta_x$ is countable ($\because \beta$ is countable

Further $\beta_x \subseteq \tau$ ($\because \beta_x \subseteq \beta \subseteq \tau$)

Now, let $U \in \tau$ such that $x \in U$. Then as β is base so there exist $V \in \beta$ s.t $x \in V \subseteq U$. Then by the construction of β_x, $V \in \beta_x$ so we have $V \in \beta_x$ s.t

$x \in V \subseteq U$

Hence β_x is local base at x.

$\Rightarrow$ X is first countable space.

Remark:- Converse of the above theorem is not true in general. i.e a first countable space need not to be a 2nd countable space. i.e if X is an uncountable

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$\beta_y = \{U \cap Y : U \in \beta_x\} \subseteq \tau_y = \{U \cap Y : U \in \tau\}$

set & τ_D is discrete topology on X then (X, τ_D) is first countable space because for every $x \in X$ we have countable local base.

$\beta_x = \{\{x\}\}$ for $x \in X$ But (X, τ_D) is not 2nd countable because the smallest base for X that can be considered is $B = \{\{x\} : x \in X\}$ which is not countable.

Theorem:- Every subspace of a first countable space is first countable.

Proof: Let X be a first countable space and Y be a subspace of X

To prove Y is first countable.

Let $x \in Y \subseteq X \Rightarrow x \in X$

As X is first countable so X has countable local base at x say β_x

Put $\beta_y = \{U \cap Y : U \in \beta_x\}$

Clearly $\beta_y \subseteq \tau_y$ Furthermore as β_x is countable so β_y is countable

We claim that β_y is local base at $x \in Y$

Let V be an open set in Y s.t $x \in V$.

As V is an open set in Y so there exit an open set V_1 in X s.t $V = V_1 \cap Y$

As $x \in V = V_1 \cap Y$

$\Rightarrow x \in V_1$ and $x \in Y$. As $V_1 \in \tau$, $x \in V_1$ and β_x is a local base at x so there exist $U \in \beta_x$ s.t $x \in U \subseteq V_1$

As $U \subseteq V_1$

$U \cap Y \subseteq V_1 \cap Y = V$

$\Rightarrow x \in U \cap Y \subseteq V$, where $U \cap Y \in \beta_y$

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Hence subspace of a first countable space is first countable.

Theorem: Every closed subspace of a lindelof space is lindelof.

Proof: Let X be a lindelof space & Y be a closed subspace of X. To prove Y is lindelof. Let $\{U_\alpha : \alpha \in I\}$ be an open cover for Y i.e $Y = \bigcup_{\alpha\in I} U_\alpha$

As U_α is an open set in Y then $\exists$ an open set V_α in X s.t $U_\alpha = V_\alpha \cap Y$

As Y is closed in $X \Rightarrow Y'$ is open in X

Now $U_\alpha = V_\alpha \cap Y \Rightarrow U_\alpha \subseteq V_\alpha$

$$\Rightarrow \bigcup_{\alpha\in I} U_\alpha \subseteq \bigcup_{\alpha\in I} V_\alpha$$

$$\Rightarrow Y \subseteq \bigcup_{\alpha\in I} V_\alpha$$

As $X = Y \cup Y' \subseteq (\bigcup_{\alpha\in I} V_\alpha) \cup Y' \subseteq X$

$$\Rightarrow X = \bigcup_{\alpha\in I} V_\alpha \cup Y'$$

$\Rightarrow \{Y', V_\alpha : \alpha \in I\}$ is an open cover for X. Since X is lindelof so this open cover for X has a countable subcover. Let $\{Y', \bigcup_n V_n\}$ be a countable subcover for X

$$\Rightarrow X = (\bigcup_n V_n) \cup Y'$$

$$\Rightarrow Y \subseteq X = (\bigcup_n V_n) \cup Y'$$

$$\Rightarrow Y \subseteq (\bigcup_n V_n) \cup Y' \Rightarrow Y \subseteq \bigcup_n V_n \quad (\because Y \not\subseteq Y')$$

$$\Rightarrow Y = (\bigcup_n V_n) \cap Y \Rightarrow Y = \bigcup_n (V_n \cap Y) \quad \text{(associative}$$

$$\Rightarrow Y = \bigcup_n U_n$$

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$\Rightarrow \{U U_n\}$ is a countable subcover for Y
Hence Y is lindelof.

Theorem:- Let X is an uncountable set with cofinite topology then X is neither first countable nor 2nd countable.

Proof:- Suppose X is first countable. Then for each $x \in X$, there exist a countable local base for $x \in X$.
Let $\beta = \{B_n\}$ be a countable local base at $x \in X$. Since $\beta \subseteq \tau_c$ (By def of local base)
So for each $B_n \in \beta \Rightarrow B_n \in \tau_c$ so for all n B_n' is finite
$\Rightarrow \bigcup_n B_n'$ is countable

Put $G = \bigcup_n B_n' \Rightarrow G$ is countable
As G is countable & X is uncountable
so $G' = X \setminus G$ is uncountable
Then $\exists\ y \in G'$ s.t $x \neq y$
As $y \in G'$ so $y \notin G \Rightarrow y \notin \bigcup_n B_n'$
$\Rightarrow y \in (\bigcup_n B_n')' \Rightarrow y \in \bigcap_n B_n \Rightarrow y \in B_n\ \forall n$ *
Put $U = X - \{y\}$ then $U' = \{y\} \Rightarrow U \in \tau_c$

Hence, U is an open set and $x \in U$ & β is local base at x. then there exist $B_n \in \beta$ s.t $x \in B_n \subseteq U$
Now as $y \notin U$ and $B_n \subseteq U \Rightarrow y \notin B_n$ which is a contradiction
$\therefore y \in B_n\ \forall n$
So our suppositin is wrong. Hence

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X is not first countable.

Now suppose X is 2nd countable, Then by a well known theorem X is first countable which is contradiction because X is not 1st countable (Proved above) So our supposition is wrong Hence X is not 2nd countable.

Theorem:- Every separable metric space is 2nd countable.

Proof

Let (X,d) be a separable metric space To prove X is 2nd countable (i.e its base exist which is countable)

As X is separable so X has a countable dense set in X. Let "A" be countable dense set in X i.e $\bar{A} = X$

Let $\mathcal{B} = \{B(x,q) / x \in A, q \in \mathbb{Q}\}$

clearly $\mathcal{B} \subseteq \tau$

Let G be open set in X and $y \in G$

As G is open in X and X is metric space so there exist an open ball $B(y,q)$ s.t

$$y \in B(y,q) \subseteq G$$

Now consider cocentric open ball $B(y, q/3)$

As A is dense in X i.e $\bar{A} = X$

Also $y \in G \subseteq X \Rightarrow y \in X = \bar{A}$

$\Rightarrow y \in \bar{A}$

$\Rightarrow$ each open nbhd of y must intersect A.

As $B(y, q/3)$ is an open set (∵ Every open ball is open set

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and every open set is open nbhd of each of its point

So $B(y, 2/3)$ is open nbhd of 'y'

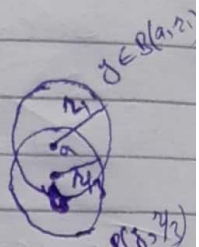
So $B(y, 2/3)$ intersect A

So $B(y, 2/3) \cap A \neq \phi$

Let $a \in B(y, 2/3) \cap A$

$\Rightarrow a \in B(y, 2/3)$ and $a \in A$

$\Rightarrow d(a, y) < 2/3$



Let z_1 be a rational number s.t

$2/3 < z_1 < 2\,2/3$

We claim that $B(a, z_1) \subseteq B(y, 2)$

Let $z \in B(a, z_1) \Rightarrow d(z, a) < z_1$

Now $d(z, y) \leq d(z, a) + d(a, y)$

$< z_1 + 2/3$

$< \frac{2z}{3} + \frac{2}{3}$ $\quad \because z_1 < \frac{2z}{3}$

$\Rightarrow d(z, y) < z \Rightarrow z \in B(y, z)$

$\Rightarrow B(a, z_1) \subseteq B(y, z)$

As $y \in B(a, z_1) \subseteq B(y, z) \subseteq G$

$\Rightarrow y \in B(a, z_1) \subseteq G$

Note that $B(a, z_1) \in \gamma$ $\quad (\because a \in A, z_1 \in \mathbb{Q}$

By Theorem on page 44 B
$\because G \in \tau$ & $B(a, z_1) \in \beta = \gamma$ → Base

So γ is base for X (By Previous theorem)

As A and $\mathbb{Q}$ are both countable so γ is countable. So γ is a countable base

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for X

Therefore X is 2nd countable.

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CHAPTER # 2

SEPARATION AXIOMS.

⇒ **Axioms:-** Axiom is a Mathematical statement without proof but consider to be true.

Note:- This chapter is known as seperation axioms because in this chapter we study the space point to point and seperate, Then by using open sets that's why known as separation axiom.

⇒ **T_0-Space:-** A topological space (X,τ) is said to be T_0-space if for each $x,y \in X$ s.t $x \neq y$ either $\exists$ an open set u s.t $x \in u$, $y \notin u$ or $\exists$ an open set v s.t $y \in v$, $x \notin v$.

Example 1:- $X = \{1,2\}$, $\tau = \{\phi, X, \{1\}, \{2\}\}$

Let $x = 1$, $y = 2 \in X$ and $x \neq y$

Let $u = \{1\} \in \tau$

$x \in u$, $y \notin u$

$\Rightarrow$ X is T_0-space.

Note:- This criteria must be hold for each pair of point of X

Example 2:- Let $X = \{1,2,3\}$

$\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$

Then $x = 1, y = 2 \in X$ s.t $x \neq y$

$u = \{1\} \in \tau$ s.t

$x \in u$, $y \notin u$.

Now Let $x = 1, y = 3 \in X$ s.t $x \neq y$

$u = \{1\} \in \tau$ s.t

$x \in u$, $y \notin u$

Now consider $x = 2, y = 3$ & $x \neq y$

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Take $V = \{1, 2\} \in \tau$ s.t
$x \in V$ & $y \notin V$
$\Rightarrow X$ is T_0-space.

Example 3:- (Counter Example)
$X = \{1, 2, 3\}$ $\tau = \{\phi, X, \{1\}\}$
Take $x = 2$, $y = 3 \in X$ s.t $x \neq y$
Then their does not exist any open set $U \in \tau$
s.t $x \in U$, $y \notin U$ or
there does not exist any open set V
s.t $y \in V$ s.t $x \notin V$
$\Rightarrow X$ is not T_0-space.

Note:- ایسی space جو T_0 نہ ہو اس میں کوئی ایسا open set
exist نہیں کرتا جو دونوں میں سے ایک point کو contain کرے۔

Theorem:- Every Subspace of a T_0-space is T_0-space.

Proof:- Let X be a T_0-space and Y be a subspace of X.
To prove Y is T_0-space.
Let $x, y \in Y$ s.t $x \neq y$
As $x, y \in Y \subseteq X$
$\Rightarrow x, y \in X$
As $x, y \in X$, $x \neq y$ and X is T_0 space,
Then their exist an open set U s.t
$x \in U$, $y \notin U$ or their exist an open set V s.t $y \in V$, $x \notin V$
Put $U_1 = U \cap Y$, $V_1 = V \cap Y$

Then U_1 & V_1 are open sets in Y
As $x \in U$, $x \in Y$
$\Rightarrow x \in U \cap Y = U_1$

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$\Rightarrow x \in U_1$

As $y \notin U$, $y \in Y \Rightarrow y \notin U \cap Y = U_1$

$\Rightarrow x \in U_1$, $y \notin U_1$

Similarly we can prove that

$y \in V$, $x \notin V$

$\Rightarrow$ Y is T_0-space

Theorem:- (V.Imp) A topological space (X, τ) is T_0-space iff $a, b \in X$ s.t $a \neq b$ then

$$\overline{\{a\}} \neq \overline{\{b\}}$$

Proof:-

Suppose X is T_0-space

To prove $a, b \in X$ s.t $a \neq b$ then $\overline{\{a\}} \neq \overline{\{b\}}$

As $a, b \in X$, $a \neq b$ and X is T_0-space, so there exist an open set U s.t

$a \in U$, $b \notin U$ (say)

As $a \in U$ & $b \notin U \Rightarrow U \cap \{b\} = \phi$

$\Rightarrow a \notin \overline{\{b\}}$

but $a \in \overline{\{a\}} \Rightarrow \overline{\{a\}} \neq \overline{\{b\}}$

$\because \bar{A} = \{a : \text{each open nbh } U \text{ of } a, U \cap A \neq \phi\}$
$a \in A \subseteq \bar{A} \Rightarrow a \in \bar{A}$

Conversely suppose for $a, b \in X$ s.t $a \neq b$ then $\overline{\{a\}} \neq \overline{\{b\}}$

To prove X is a T_0-space

Suppose X is not a T_0-space

Then every open set (say) U containing a also contains b

$\Rightarrow U \cap \{b\} \neq \phi$ (Previous theorem)

$\Rightarrow a \in \overline{\{b\}}$ ⊛ $\Rightarrow \{a\} \subseteq \overline{\{b\}}$

$\Rightarrow \overline{\{a\}} \subseteq \overline{\{b\}}$ ($\because$ If $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$)

$\Rightarrow \overline{\{a\}} \subseteq \{b\}$ ——— (1) ($\because \bar{\bar{A}} = \bar{A}$)

by $a \in U$ & U is open ⊛
$\Rightarrow$ U is open nbhd of a
& $U \cap \{b\} \neq \phi \Rightarrow a \in \overline{\{b\}}$

Similarly we can prove

$\overline{\{b\}} \subseteq \overline{\{a\}}$ ——— (2)

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from ① & ② $\overline{\{a\}} = \overline{\{b\}}$
which is a contradiction. So our supposition is wrong
Hence X is T_0 space.

⇒ **T_1-Space:-** A topological space (X, τ) is said to be T_1-space if for every $x, y \in X$ s.t $x \neq y$ there exist two open sets U, V s.t $x \in U$, $y \notin U$ & $y \in V$, $x \notin V$

Note:- open sets U & V are not necessarily disjoint.

Example:- $X = \{1, 2\}$
$\tau = \{\phi, X, \{1\}, \{2\}\}$
Consider $x = 1$, $y = 2 \in X$ s.t $x \neq y$
$U = \{1\}$, $V = \{2\} \in \tau$
Note that $x \in U$, $y \notin U$ &
$y \in V$, $x \notin V$
⇒ X is T_1-space.

Theorem:- Prove that every T_1-space is T_0-space.

Proof:
Let X be a T_1-space.
To prove X is T_0 space
Let $x, y \in X$ and $x \neq y$
As X is T_1 space then there exist two open sets U & V s.t $x \in U$, $y \notin U$ &
$y \in V$, $x \notin V$
In particular ∃ an open set U s.t
$x \in U$ & $y \notin U$
⇒ X is T_0-space.

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Theorem:- Every subspace of a T_1-space is T_1.

Proof: Let X be a T_1 space & Y be a subspace of X

To prove Y is T_1-space

Let $x, y \in Y$ s.t $x \neq y$

As $x, y \in Y \subseteq X$

$\Rightarrow x, y \in X$

As $x, y \in X$ & $x \neq y$ and X is T_1-space

So there exist two open sets U, V s.t

$x \in U$, $y \notin U$

& $y \in V$, $x \notin V$

Put $U_1 = U \cap Y$, $V_1 = V \cap Y$

Then U_1 & V_1 are open sets in Y

Since $x \in U$, $x \in Y$

$\Rightarrow x \in U \cap Y = U_1$

As $y \notin U$, $y \in Y \Rightarrow y \notin U \cap Y = U_1$

$\Rightarrow x \in U_1$, $y \notin U_1$

Similarly $y \in V_1$, $x \notin V_1$

$\Rightarrow Y$ is T_1-space.

Hence every subspace of T_1-space is T_1.

Theorem:- A topological space (X, τ) is T_1-space iff each singleton subset of X is closed.

Proof:

Suppose X is T_1-space

To prove each singleton subset of X is closed

Let $\{x\}$ be a singleton subset of X

For this we will prove $\{x\}'$ is an open set.

Let $y \in \{x\}'$

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$\Rightarrow y \notin \{x\} \quad \Rightarrow y \neq x$

As $x, y \in X$, $x \neq y$ and X is a T_1-space so there exist two open sets U & V s.t $x \in U_x$, $y \notin U_x$ &
$y \in V_y$, $x \notin V_y$

Now as $V_y \subseteq X$ and $x \notin V_y$.

$\Rightarrow V_y \subseteq \{x\}'$

As $y \in V_y$ & $V_y \subseteq \{x\}'$

$\Rightarrow y \in V_y \subseteq \{x\}'$

$\Rightarrow \{x\}'$ is open [illegible] $\Rightarrow \{x\}$ is closed

extra

$y \in V_y \subseteq \{x\}'$

$\Rightarrow \{y\} \subseteq V_y \subseteq \{x\}'$

$\Rightarrow \bigcup_{y \in \{x\}'} \{y\} \subseteq \bigcup_{y \in \{x\}'} V_y \subseteq \{x\}'$

$\Rightarrow \{x\}' \subseteq \bigcup_{y \in \{x\}'} V_y \subseteq \{x\}'$

$\Rightarrow \{x\}' = \bigcup_{y \in \{x\}'} V_y$

As $V_y \in \tau$ and τ is a topology then $\bigcup_{y \in \{x\}'} V_y \in \tau$

$\Rightarrow \{x\}' \in \tau \Rightarrow \{x\}'$ is an open set
$\Rightarrow \{x\}$ is closed

Hence every singleton subset of X is closed.

Conversely suppose that every singleton subset of X is closed.

To prove X is T_1-space

Let $x, y \in X$ s.t $x \neq y$

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Put $U = \{y\}'$ & $V = \{x\}'$

As each singleton subset of X is closed

$\Rightarrow \{y\}$ & $\{x\}$ are closed sets

$\Rightarrow \{y\}'$ and $\{x\}'$ are open sets

i.e U & V are open sets

As $x \notin \{y\}$ $(\because x \neq y$

$\Rightarrow x \in \{y\}' = U$

As $y \in \{y\} \Rightarrow y \notin \{y\}' = U$

$\Rightarrow x \in U$ & $y \notin U$

Similarly we can prove $y \in V$, $x \notin V$

$\Rightarrow X$ is T_1-space.

Rough :- $A \subseteq B \Rightarrow B' \subseteq A'$ (Always True)

$x \notin V_y \Rightarrow x \in V_y'$

$\Rightarrow \{x\} \subseteq V_y' \Rightarrow (V_y')' \subseteq \{x\}'$

$\Rightarrow V_y \subseteq \{x\}'$

Theorem :- Every finite T_1-space is discrete.

Proof) Let X be a finite T_1-space

To prove X is discrete

Let $A \subseteq X$

It is enough to prove that

A is closed

As X is finite

$\Rightarrow A$ is finite

Let $A = \{x_1, x_2, \dots, x_n\}$

$= \{x_1\} \cup \{x_2\} \cup \dots \cup \{x_n\}$

$$A = \bigcup_{i=1}^{n} \{x_i\} \quad \text{——— (1)}$$

Given X is T_1-space

X= Discrete اگر X کا ہر open subset ہے اور closed ہے۔ مثال کے طور پر $X = \{1, 2\}$ then $P(X)$ $\{\phi, X, \{1\}, \{2\}$

کیونکہ $X = \{1, 2, 3\}$

$\{1, 2\}$ & $\{3\}$ are subs

As $A = \{3\} \Rightarrow A' = \{1, 2\}$

ہر subset کسی نہ کسی closed subset کا complimnt ہے اور ہر closed کا compli- open ہے لہٰذا A بھی open ہوگا

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$\Rightarrow \{x_i\}$ are closed sets

As we know that union of finite number of closed sets is closed

$\Rightarrow \bigcup_{i=1}^{n} \{x_i\}$ is closed set

$\Rightarrow A$ is closed set

$\Rightarrow X$ is discrete set

مارے closed subsets ہیں لہٰذا
ان کے open complement ہیں
A کسی closed کا complement
ہے لہٰذا open ہے

Theorem:- A topological space (X, τ) is T_1-space iff each subset of X is the intersection of its open supersets.

Proof: Let X be T_1-space.

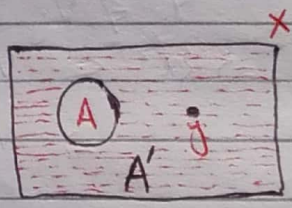
To prove each subset of X is the intersection of its open supersets

$X = T_1$ if $A \subseteq X$
Then $A = \bigcap_{\alpha \in I} U_\alpha$
$A \subseteq U_\alpha$, $U_\alpha \in \tau$

Let $A \subseteq X$

Let $y \in X$ s.t $y \notin A$

$\Rightarrow y \in A'$



As X is T_1-space. so each singleton subset of X is closed

$\Rightarrow \{y\}$ is closed $\Rightarrow \{y\}'$ is an open set

As $A \subseteq X$ & $y \notin A \Rightarrow y \in A' \Rightarrow \{y\} \subseteq A'$

$\Rightarrow A \subseteq \{y\}' \quad \forall\, y \in A'$

we claim that

$$A = \bigcap_{y \in A'} \{y\}'$$

As $A \subseteq \{y\}' \quad \forall\, y \in A'$

$$\Rightarrow A \subseteq \bigcap_{y \in A'} \{y\}' \quad \text{——— (1)}$$

Let $z \in \bigcap_{y \in A'} \{y\}'$

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$\Rightarrow z \in \{y\}' \quad \forall\ y \in A'$

$\Rightarrow z \notin \{y\} \quad \Rightarrow z \neq y$

$\Rightarrow z \notin A' \quad (\because y \in A') \quad \Rightarrow z \in A$

$\Rightarrow \bigcap_{y \in A'} \{y\}' \subseteq A \quad \text{——— ②}$

from ① & ② $A = \bigcap_{y \in A'} \{y\}'$

Hence A can be written as the intersection of its open supersets.

Conversely ⇒ Correction at page 63

Given that each subset of X can be written as the intersection of its open supersets. To prove X is T_1-space

Let $x, y \in X$ s.t $x \neq y$

Suppose X is not T_1-space

Then each open set in X containing 'x' also containing 'y' & conversely

Now by given condition $\{x\}$ is the intersection of its open supersets

$\Rightarrow \{x\}$ is an open set. Since intersection of open sets is open.

Then $y \in \{x\}$ ∵ each open set containing x also containing y

$\Rightarrow y = x$

which is a contradiction.

So our supposition is wrong

Hence X is T_1-space.

Note:- T_0-space need not to be T_1-space.

$X = \{1, 2\}$, $\tau = \{\phi, X, \{1\}\}$

Then X is T_0 but not T_1

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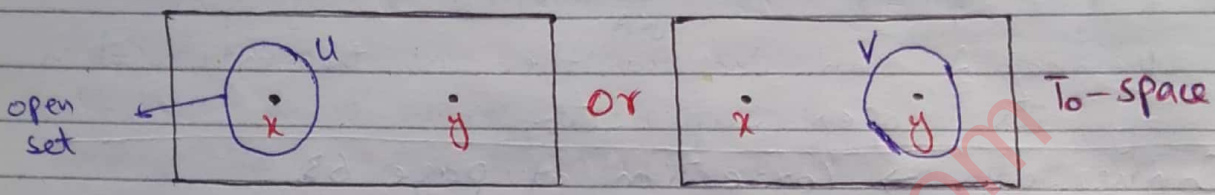
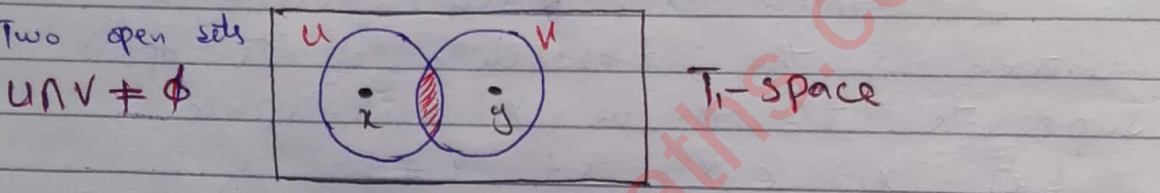
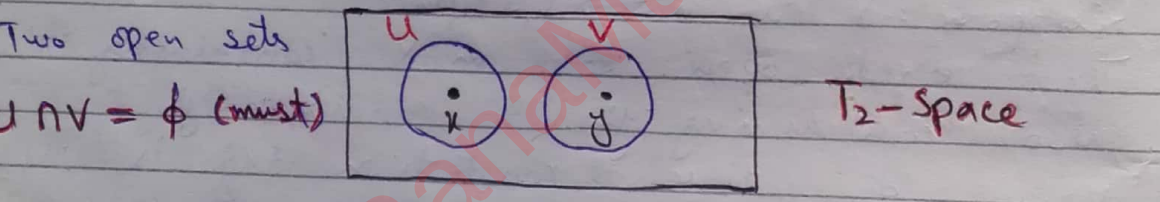
⇒ T_2-Space:-(Housdorff):- A topological space (X, τ) is said to be T_2-space if for each $x, y \in X$ s.t $x \neq y$ then $\exists$ two open sets U & V s.t $x \in U$, $y \in V$ & $U \cap V = \phi$







Theorem:- Every T_2-space is T_1-space.

Proof:-

Let X be T_2-space

To prove X is T_1-space.

Let $x, y \in X$ s.t $x \neq y$

As X is T_2-space then their exist two open sets U & V s.t

$x \in U$, $y \in V$ and $U \cap V = \phi$

As $y \in V$ & $U \cap V = \phi \Rightarrow y \notin U$

Also $x \in U$ & $U \cap V = \phi \Rightarrow x \notin V$

$\Rightarrow x \in U$, $y \notin U$

and $y \in V$, $x \notin V$

$\Rightarrow X$ is T_1-space

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Hence every T_2-space is T_1-space.

Counter Example:-

Let $X = \mathbb{N}$, τ = Cofinit topology

Let $x, y \in X$ s.t $x \neq y$

Let $U = X - \{y\}$, $V = X - \{x\}$

$\Rightarrow U' = \{y\}$ & $V' = \{x\}$

which are finite

$\Rightarrow$ U & V are open sets

Also note that $x \in U$ and $y \notin U$

and $y \in V$ & $x \notin V$

$\therefore X$ is T_1-space

To prove X is not T_2-space

Suppose X is T_2-space

Then $U \cap V = \phi \Rightarrow (U \cap V)' = \phi'$

$\Rightarrow (U \cap V)' = X = \mathbb{N}$

$\Rightarrow U' \cup V' = X = \mathbb{N}$

Since $U' = \{y\}$ & $V' = \{x\}$ are finite

$\Rightarrow$ L.H.S = finite

$\Rightarrow \mathbb{N}$ is finite

which is a contradiction. So our supposition is wrong

$\Rightarrow$ X is not T_2-space.

Theorem: Every Subspace of T_2-space is T_2-space.

Proof Let X be a T_2-space and Y be a subspace of X.

To prove Y is T_2-space.

Let $x, y \in Y$ s.t $x \neq y$

As $y \subseteq X \Rightarrow x, y \in X$ s.t $x \neq y$

As X is T_2-space

Then $\exists$ two open sets U & V s.t

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$x \in U$ & $y \in V$ and $U \cap V = \phi$

Put $U_1 = U \cap Y$ & $V_1 = V \cap Y$

Then U_1 & V_1 are open sets in Y

As $x \in U$ & $x \in Y$

$\Rightarrow x \in U \cap Y = U_1 \Rightarrow x \in U_1$

& As $y \in V$ and $y \in Y$

$\Rightarrow y \in V \cap Y = V_1 \Rightarrow y \in V_1$

$\Rightarrow x \in U_1$ & $y \in V_1$

Furthermore

$$U_1 \cap V_1 = (U \cap Y) \cap (V \cap Y)$$
$$= (U \cap V) \cap Y$$
$$= \phi \cap Y$$
$$U_1 \cap V_1 = \phi$$

$\Rightarrow Y$ is T_2 - space

Hence every subspace of T_2 - space is T_2 - space.

Theorem: Show that every T_2 - space is T_0 - space.

Proof: Since every T_2 - space is T_1 - space and every T_1 - space is T_0 - space.

So every T_2 - space is T_0 - space.

Theorem: Every metric space is T_2 - space.

Proof:

Let (X, d) be a metric space.

To prove X is T_2 - space

Let $x, y \in X$ s.t $x \neq y$

As $x, y \in X$ and $x \neq y$ and X is a metric space

$\therefore d(x, y) > 0$

Let $d(x, y) = r$

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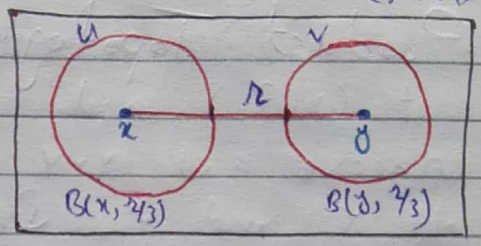
(X,d)



Let $U = B(x, 2/3)$

$V = B(y, 2/3)$

clearly $x \in U$, $y \in V$

and U & V are open sets b/c in a metric space open balls are open sets.

We claim that

$U \cap V = \phi$

Suppose $U \cap V \neq \phi$

Let $z \in U \cap V \Rightarrow z \in U$ & $z \in V$

$\Rightarrow z \in B(x, 2/3)$ & $z \in B(y, 2/3)$

$\Rightarrow d(z,x) < 2/3$ and $d(z,y) < 2/3$

As $2 = d(x,y) \leq d(x,z) + d(z,y)$

$d(x,y) < 2/3 + 2/3$

$= 2\,2/3$

$\Rightarrow 2 < 2\,2/3$

which is a contradiction

So our supposition is wrong

Hence $U \cap V = \phi$

$\Rightarrow X$ is T_2 - space

So every metric space is T_2 - space.

Q: Prove that $\mathbb{R}^n$, ℓ^∞ or any metric space is T_2-space and T_1-space and T_0-space.

Solution

It is enough to prove that every metric space is T_2-space (Previous Theorem)

And Then T_2 is T_1 & T_1 is T_0

$\Rightarrow \mathbb{R}^n$, ℓ^∞ and all metric spaces are T_2, T_1, T_0 - spaces

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⇒ Product Topology or Tychonoff Top...

Let (X, τ_1) and (Y, τ_2) be two topological spaces and $X \times Y$ is the cartesian product of X and Y. Define a subset $U \times V$ of $X \times Y$ to be an open set in $X \times Y$ iff $U \in \tau_1$ & $V \in \tau_2$ then the class of all subsets $U \times V$ of $X \times Y$ called product topology on $X \times Y$.

Example:- Let $X = \{1, 2, 3\}$, $\tau_1 = \{\phi, X, \{1\}, \{2, 3\}\}$

$Y = \{a, b, c, d\}$, $\tau_2 = \{\phi, Y, \{a, b\}\}$

$X \times Y = \{(x, y) : x \in X \text{ \& } y \in Y\}$

$= \{(1,a), (1,b), (1,c), (1,d), (2,a), (2,b), (2,c), (2,d), (3,a), (3,b), (3,c), (3,d)\}$

τ = Product Topology

$= \{U \times V : U \in \tau_1 \text{ \& } V \in \tau_2\}$

$= \{\phi \times \phi, \phi \times Y, \phi \times \{a, b\}, X \times \phi, X \times Y, X \times \{a, b\}, \{1\} \times \phi, \{1\} \times Y, \{1\} \times \{a, b\}, \{2, 3\} \times \phi, \{2, 3\} \times Y, \{2, 3\} \times \{a, b\}\}$

Theorem:- The following statements about a topological space are equivallent.

(i) X is a T_2 – space

(ii) The diagonal $D = \{(x, x) : x \in X\}$ is closed in $X \times X$

Proof (i) ⇒ (ii)

Given X is a T_2-space

To prove D is closed in $X \times X$

$X = \{1, 2, 3\}$

$\tau = \{\phi, X, \{1\}, \{2\}, \{1, 2\}\}$

$X \times X = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

$D = \{(1,1), (2,2), (3,3)\}$

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For this we will prove D' is an open set in $X \times X$

Let $(x,y) \in D'$

$\Rightarrow (x,y) \notin D$

$\Rightarrow x \neq y$

ایسے order pair جنکے Co-ordinates
same ہوں diagonal کہلاتے ہیں
T_2 کا Diagonal ہمیشہ close ہوتا
ہے اور اس کا converse بھی
True ہے

As $x,y \in X$, $x \neq y$ and X is T_2-space

Then there exist two open sets U_x & V_y

s.t $x \in U_x$, $y \in V_y$

and $U_x \cap V_y = \phi$

Now let $(u,v) \in U_x \times V_y$

$\Rightarrow u \in U_x$ and $v \in V_y$

$\Rightarrow u \neq v$ $(\because U_x \cap V_y = \phi)$

$\Rightarrow (u,v) \notin D \Rightarrow (u,v) \in D'$

$\Rightarrow U_x \times V_y \subseteq D'$

As $(x,y) \in U_x \times V_y \subseteq D'$

$\Rightarrow \{(x,y)\} \subseteq U_x \times V_y \subseteq D'\}$

$$\Rightarrow \bigcup_{(x,y)\in D'} \{(x,y)\} \subseteq \bigcup_{(x,y)\in D'} (U_x \times V_y) \subseteq D'$$

$$\Rightarrow D' \subseteq \bigcup_{(x,y)\in D'} (U_x \times V_y) \subseteq D'$$

$$\Rightarrow D' = \bigcup_{(x,y)\in D'} (U_x \times V_y)$$

$\Rightarrow D'$ is an open set in $X \times X$

$\Rightarrow D$ is closed in $X \times X$

Conversely suppose that D is closed in $X \times X$

To prove X is T_2-space

Let $x,y \in X$ s.t $x \neq y$

$\Rightarrow (x,y) \notin D \Rightarrow (x,y) \in D'$

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As D is closed $\Rightarrow D'$ is open in $X \times X$

As D' is an open set in $X \times X$ and $(x,y) \in D'$ then there exist and open set $U_x \times V_y$ in $X \times X$ s.t

$$(x,y) \in U_x \times V_y \subseteq D'$$

$$\Rightarrow (x,y) \in U_x \times V_y \ \&$$

$$\Rightarrow x \in U_x \ \& \ y \in V_y$$

Next we have to show that $U_x \cap V_y = \phi$

Suppose $U_x \cap V_y \neq \phi$

Let $z \in U_x \cap V_y$

$$\Rightarrow z \in U_x \ \& \ z \in V_y$$

$$\Rightarrow (z,z) \in U_x \times V_y \subseteq D'$$

$$\Rightarrow (z,z) \in D'$$

$\Rightarrow (z,z) \notin D$ which is a contradiction

So our supposition is wrong

Hence $U_x \times V_y = \phi$

$\Rightarrow X$ is a T_2-space.

Theorem: A topological space X is T_2-space iff for any two distinct points $a, b \in X$ there are closed sets C_1 & C_2 s.t

$a \in C_1$, $b \notin C_1$

$b \in C_2$, $a \notin C_2$ and $C_1 \cup C_2 = X$.

Proof: Let X be T_2-space.

As $a, b \in X$, $a \neq b$ and X is T_2-space then there exist two open sets U & V s.t $a \in U$, $b \in V$

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and $U \cap V = \phi$

Put $C_1 = V'$ & $C_2 = U'$

Then C_1 and C_2 are closed sets

As $a \in U \Rightarrow a \notin U' = C_2 \Rightarrow a \notin C_2$

Also $b \in V \Rightarrow b \notin V' = C_1 \Rightarrow b \notin C_1$

As $a \in U$, $U \cap V = \phi \Rightarrow a \notin V$

$\Rightarrow a \in V' = C_1 \Rightarrow a \in C_1$

Also $b \notin U \Rightarrow b \in U' = C_2$

$\Rightarrow b \in C_2$

As $U \cap V = \phi$

$\Rightarrow [U \cap V]' = \phi' \Rightarrow U' \cup V' = X$ ($\because \phi' = X$

$\Rightarrow C_2 \cup C_1 = X \Rightarrow C_1 \cup C_2 = X$

$\Rightarrow a \in C_1, b \notin C_1$ and $b \in C_2, a \notin C_2$

and $C_1 \cup C_2 = X$

Conversely

Suppose for $a, b \in X$ s.t $a \neq b$

there exist two closed sets C_1 and C_2 s.t

$a \in C_1, b \notin C_1$ and $b \in C_2, a \notin C_2$

and $C_1 \cup C_2 = X$

To prove X is T_2 - space

Let $a, b \in X$ s.t $a \neq b$

put $U = C_2'$ & $V = C_1'$

$\Rightarrow U$ and V are open sets in X.

As $b \notin C_1 \Rightarrow b \in C_1' = V \Rightarrow b \in V$

Also $a \notin C_2 \Rightarrow a \in C_2' = U \Rightarrow a \in U$

As $C_1 \cup C_2 = X$

$\Rightarrow [C_1 \cup C_2]' = X' \Rightarrow C_1' \cap C_2' = \phi$

$\Rightarrow V \cap U = \phi \Rightarrow U \cap V = \phi$

$\Rightarrow X$ is T_2 - space

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⇒ **Definition:-** Let (X, τ) be a topological space. A sequence $\{x_n\}$ in X is said to be converge to a point $x \in X$ if every open set U containing 'x' there is a natural number n_0 s.t

$$x_n \in U \quad \forall \; n \geq n_0$$

Note:- In Mathematics Uniqueness is always proved by Contradiction.

Theorem: Let X be T_2- space then any sequence in X can converge to at the most one point. **OR** In a T_2- space limit of the sequence is unique.

Proof: Suppose $\{x_n\}$ is a sequence in X and $x_n \to x \in X$

Also $x_n \to y \in X$ and $x \neq y$

As $x, y \in X$, $x \neq y$ and X is a T_2- space Then there exist two open sets U & V s.t

$x \in U$ and $y \in V$ and $U \cap V = \phi$

As $x_n \to x$ and $x \in U$ then there exist a natural number n_0 (n_1) s.t

$$x_n \in U \quad \forall \; n \geq n_1$$

Also $x_n \to y$ and $y \in V$ then there exist a natural number n_0 (n_2) s.t

$$x_n \in V \quad \forall \; n \geq n_2$$

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Put $n_0 = \max(n_1, n_2)$

Then $x_n \in U$ and $x_n \in V \quad \forall\, n \geq n_0$

$\Rightarrow x_n \in U \cap V \quad \forall\, n \geq n_0$

$\Rightarrow U \cap V \neq \phi$

which is a contradiction. So our Supposition is wrong. ($\because U \cap V = \phi$)

Hence limit point of a sequence in T_2 - space is unique.

Q:- What is the special property (Main Property) of T_2 - space.

Ans:-

Main property of T_2-space is that limit point of convergent sequence always unique.

Correction:- Conversely given that each subset of X can be written as the intersection of its open supersets. Let $x, y \in X$ s.t $x \neq y$

Then $\{x\}$ is the intersection of its open supersets and $\{y\}$ is also the intersection of its open supersets.

So there is an open set "U" which contans 'x' but not 'y' [for otherwise x & y both will be in the intersection of open super sets of $\{x\}$ and $\{y\}$] and similarly there is an superset V which contains y but not x

Hence X is T_1 - space.

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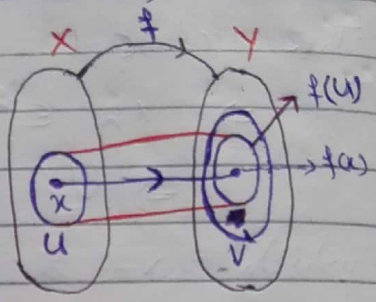
⇒ **Continuous Function:-** Let (X, τ_1) and (Y, τ_2) be two top-spaces. A function $f: X \to Y$ is said to be continuous at $x \in X$ if for each open set V containing $f(x)$ $\exists$ an open set U containing x s.t $f(U) \subseteq V$

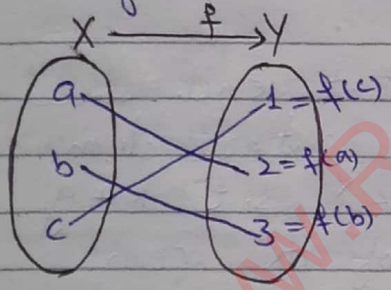


f is said to be continuous on X if it is continuous for each point of X.

Example:- $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a, b\}, \{c\}\}$

$Y = \{1, 2, 3\}$, $\tau_2 = \{\phi, Y, \{1\}, \{2, 3\}\}$

and $f: X \to Y$ is defined by following diagram.



Let $x = a$, $f(a) = 2$

Take $V = \{2, 3\}$, $f(a) = 2$

Then take $U = \{a, b\}$ then $a \in U$ s.t $f(U) = \{2, 3\} \subseteq V$

Now Take $V = Y$, $f(a) = 2 \Rightarrow 2 \in V$

then take $U = X$ s.t $a \in U$ and $f(U) = Y$

$\Rightarrow f(a) \in f(U) \subseteq V$

$\Rightarrow f$ is continuous on "a"

Theorem:- Let (X, τ_1) & (Y, τ_2) be the two top-spaces. A function $f: X \to Y$ is said to be continuous if and only if inverse image of each open set is open.

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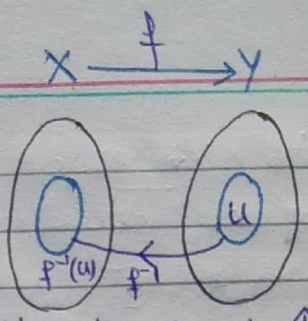

i.e if U is open in τ_2 then $f^{-1}(U) \in \tau_1$ is open

Proof: Suppose $f: X \to Y$ is continuous.

To prove inverse image of each open set in open in X

Let G be an open set in Y

To prove $f^{-1}(G)$ is open in X.

If $G = \phi$ then $f^{-1}(G) = \phi$ & Hence open So nothing is left to prove.

If $G \neq \phi$ then $f^{-1}(G) \neq \phi$

Let $x \in f^{-1}(G) \Rightarrow f(x) \in G$

As $f(x) \in G$ and f is continuous at x So there exist an open set U_x in X s.t $x \in U_x$ and $f(U_x) \subseteq G \Rightarrow U_x \subseteq f^{-1}(G)$

As $x \in U_x \subseteq f^{-1}(G) \Rightarrow \{x\} \subseteq U_x \subseteq f^{-1}(G)$

$$\bigcup_{x \in f^{-1}(G)} \{x\} \subseteq \bigcup_{x \in f^{-1}(G)} U_x \subseteq f^{-1}(G)$$

$$\Rightarrow f^{-1}(G) \subseteq \bigcup_{x \in f^{-1}(G)} U_x \subseteq f^{-1}(G)$$

$$\Rightarrow f^{-1}(G) = \bigcup_{x \in f^{-1}(G)} U_x$$

As U_x is open in X and union of any number of open sets is open so

$\bigcup_{x \in f^{-1}(G)} U_x$ is open. $\Rightarrow f^{-1}(G)$ is open

therefore inverse image of each open set is open.

Conversely suppose that inverse image of each open set is open.

To prove f is continuous on X.

Let $x \in X$. Let V be an open set in Y s.t $f(x) \in V \Rightarrow x \in f^{-1}(V)$

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As inverse image of each open set is open Therefore $f^{-1}(V)$ is an open set in X.

Put $f^{-1}(V) = U$ then $x \in U$

As $f(f^{-1}(V)) \subseteq V \Rightarrow f(U) \subseteq V$

$\therefore f$ is continuous.

Theorem: Let (X, τ_1) and (Y, τ_2) be the two topological spaces then $f: X \to Y$ is continuous iff inverse image of closed set is closed in X.

Proof: Given f is continuous.

To prove inverse image of each closed set is closed.

Let "C" be a closed set in Y

$\Rightarrow C'$ is an open set in Y

$\Rightarrow f^{-1}(C')$ is an open set in X (By Previous)

Now $f^{-1}(C') = f^{-1}(Y - C)$

$= f^{-1}(Y) - f^{-1}(C)$

$= X - f^{-1}(C) = \{f^{-1}(C)\}'$

As $f^{-1}(C')$ is an open set in X

$\Rightarrow \{f^{-1}(C)\}'$ is an open set in X

$\Rightarrow f^{-1}(C)$ is closed in X

Conversely given inverse image of each closed set is closed.

To prove f is continuous.

Let G be an open set in Y

$\Rightarrow G'$ is closed in Y

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As inverse image of each closed set is closed

$\Rightarrow f^{-1}(G')$ is closed in X

$f^{-1}(G') = f^{-1}(Y - G) = f^{-1}(Y) - f^{-1}(G)$

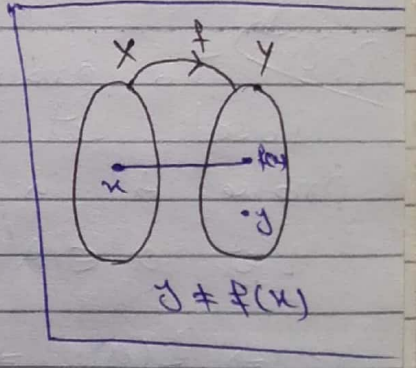
$= X - f^{-1}(G) = \{f^{-1}(G)\}'$

As $f^{-1}(G')$ is closed

$\Rightarrow \{f^{-1}(G)\}'$ is closed $\Rightarrow f^{-1}(G)$ is open

So by previous theorem f is continuous.

Theorem: Let X be a topological space and Y be a T_2-space if $f: X \to Y$ is continuous then the graph $G = \{(x, y) \mid y = f(x)\}$ is closed in $X \times Y$



Proof: To prove "G" is closed we have to prove G' is open set in $X \times Y$

Let $(x, y) \in G' \Rightarrow (x, y) \notin G$

$\Rightarrow y \neq f(x)$

As $y, f(x) \in Y$ and $y \neq f(x)$ and Y is T_2-space so there exist two open sets V and V_1 in Y s.t $y \in V$, $f(x) \in V_1$ and $V \cap V_1 = \phi$

As V_1 is open in Y and f is continuous therefore $f^{-1}(V_1)$ is open in X.

As $f(x) \in V_1 \Rightarrow x \in f^{-1}(V_1) = U$

$\Rightarrow x \in U, y \in V$

$\Rightarrow (x, y) \in U \times V \subseteq G'$

But $U \times V$ is open

$U \times V \subseteq G'$

$(x, y) \in U \times V \Rightarrow x \in U, y \in V$

As $V \cap V_1 = \phi \Rightarrow f^{-1}(V \cap V_1) = f^{-1}(\phi)$

$f^{-1}(V) \cap f^{-1}(V_1) = \phi$

As $x \in U = f^{-1}(V_1)$

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in $X \times Y$
$\Rightarrow G'$ is open in $X \times Y$
$\Rightarrow G$ is closed in $X \times Y$

$\Rightarrow x \in f^{-1}(V_1)$
$\Rightarrow x \in f^{-1}(V_1) \Rightarrow f(x) \in V_1$
$\Rightarrow f(x) \notin V$ and $y \in V \Rightarrow f(x) \neq y$
$\Rightarrow (x,y) \notin G \Rightarrow (x,y) \in G'$
$\Rightarrow (U \times V) \subseteq G'$

Note:- $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$
$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
$f^{-1}(\phi) = \phi$

Theorem:- Let (X, τ) be a topological space and Y be a T_2-space if functions $f, g : X \to Y$ are continuous then $A = \{x : x \in X \text{ and } f(x) = g(x)\}$ is closed in X.

Proof: To prove A is a closed set we have to prove A' is an open set.

Let $a \in A' \Rightarrow a \notin A$

$\Rightarrow f(a) \neq g(a)$

As $f(a), g(a) \in Y$, $f(a) \neq g(a)$ and Y is T_2-space then $\exists$ two open sets U and V s.t

$f(a) \in U$, $g(a) \in V$ and $U \cap V = \phi$

$\Rightarrow a \in f^{-1}(U) = U_1$

$\Rightarrow a \in g^{-1}(V) = V_1$

$\Rightarrow a \in f^{-1}(U) \cap g^{-1}(V)$

i.e $a \in U_1 \cap V_1$

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As f and g are continuous functions so U_1 and V_1 are open sets in X
Also note that
$$a \in U_1 \cap V_1 \subseteq A'$$
$\Rightarrow A'$ is an open set in X because $U_1 \cap V_1$ is open in X
$\Rightarrow$ A is closed in X.

Theorem:- A first countable space is T_2-space iff every convergent sequence has a unique limit point.

Proof:- Suppose X is 1st countable space which is also T_2.

To prove every convergent sequence has a unique Limit point.

As we know that in a T_2-space every convergent sequence has a unique limit point so every sequence in given space has unique limit point.

Conversely suppose in 1st countable space X every convergent sequence has a unique limit point.

To prove X is T_2-space.

Let $a, b \in X$ s.t $a \neq b$

Suppose X is not a T_2-space. As X is not a T_2-space, so every open set which containg 'a' also contain 'b'

Let $\{U_n\}$ and $\{V_n\}$ be countable nested local bases at 'a' and 'b' respectively. Then
$$U_n \cap V_n \neq \emptyset \quad \forall n$$
$$\Rightarrow a_n \in U_n \cap V_n \quad , n = 1, 2, \cdots, n$$

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Then $\{a_n\}$ converge to both 'a' & 'b'
$\Rightarrow \{a_n\}$ has not unique limit point which is a contradiction.
So our supposition is wrong.
Hence X is T_2-space.

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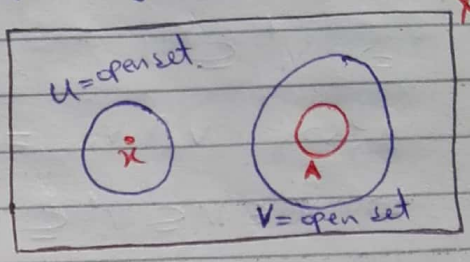
⇒ **Regular Space:-** A topological space (X, τ) is said to be regular if for every $x \in X$ and for any closed subset A of X s.t $x \notin A$ there exist two open sets U and V s.t $x \in U$, $A \subseteq V$ and $U \cap V = \phi$



⇒ **T_3- Space:-** A regular T_1- space is called T_3- space
[For T_3- space check conditions of T_1 and regularity]

Theorem:- The following statements about a topological space are equivallent.
(i) X is regular
(ii) for any open set U s.t $x \in U$ there is an open set V s.t $x \in \bar{V} \subseteq U$
(iii) each element of X has a local base consisting of closed sets.

Proof: (i) → (ii)
Given X is regular.
To prove condition (ii) holds
Let U be an open set in X s.t $x \in U$
As $x \in U \Rightarrow x \notin U'$
As U is an open set so U' is closed set and $x \notin U'$
As X is regular space so ∃ two open sets V and V_1 s.t
$x \in V$ and $U' \subseteq V_1$
& $V \cap V_1 = \phi$

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As $U' \subseteq V_1 \Rightarrow V_1' \subseteq (U')'$ [$\because$ if $A \subseteq B \Rightarrow B' \subseteq A'$]

$\Rightarrow V_1' \subseteq U$

Also $V \cap V_1 = \phi \Rightarrow V \subseteq V_1'$

$\Rightarrow x \in V \subseteq V_1' \subseteq U$ ——— ①

As V_1 is open $\Rightarrow V_1'$ is closed and $V \subseteq V_1'$

$\Rightarrow V_1'$ is closed superset of V

But $\bar{V}$ = Intersection of all closed superset of V

$\Rightarrow \bar{V} \subseteq V_1'$

$\Rightarrow V \subseteq \bar{V} \subseteq V_1'$ ——— ② [$\because$ for any set $A \Rightarrow A \subseteq \bar{A}$]

from ① & ②

$x \in V \subseteq \bar{V} \subseteq V_1' \subseteq U$

$\Rightarrow x \in \bar{V} \subseteq U$

(ii) $\Rightarrow$ (iii) Given for any open set U in X s.t $x \in U$ there exist an open set V containg x s.t $x \in \bar{V} \subseteq U$

To prove condition (iii) holds

Let U be an open set s.t $x \in U$

Let $\beta = \{\bar{V} : x \in \bar{V} \subseteq U\}$ Then by given condition there exist an open set V s.t $x \in \bar{V} \subseteq U$, $\bar{V} \in \beta$

Then β is a local base at "x" whose elements are closed sets because closure is always a closed set.

(iii) $\Rightarrow$ (i) Given that each point of X

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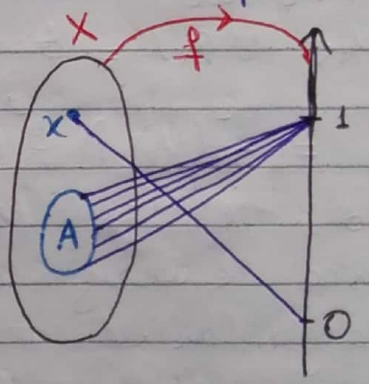
has a local base which contains closed sets. To prove X is regular.
Let $x \in X$ and A be closed set in X s.t $x \notin A$
As $x \notin A \Rightarrow x \in A'$
As A is closed $\Rightarrow$ A' is open
As A' is open and $x \in A'$ then by condition (iii) there is a closed set B in local base s.t $x \in B \subseteq A'$
$\Rightarrow B \subseteq A' \Rightarrow (A')' \subseteq B' \Rightarrow A \subseteq B'$
choose $U = B$, $V = B'$
then $x \in U$, $A \subseteq V$ and $U \cap V = B \cap B' = \phi$
$\Rightarrow U \cap V = \phi$ $\therefore$ X is regular

⇒Completely Regular:- A topological space (X, τ) is said to be completely regular if for any closed set in X and a point $x \in X$ s.t $x \notin A$ there is a continuous function $f: X \to [0,1]$ s.t $f(x) = 0$ & $f(A) = 1$



Note $f(A) = 1$ means every element of set A is mapped onto 1 under f i.e $f(y) = 1 \ \forall \ y \in A$

Theorem:- Every Completely regular space is Regular.

Proof Let X be a completely regular space.

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To prove X is regular.
Let $x \in X$ and A be a closed set in X s.t $x \notin A$.
As X is completely regular so there exist a continuous function $f: X \to [0,1]$ s.t $f(x) = 0$, $f(A) = 1$
Let $U = [0, \frac{1}{2}[$ and $V =]\frac{1}{2}, 1]$

Then U and V are open sets in $[0,1]$
Since f is continuous, Therefore inverse image of each open set is open
So $f^{-1}(U)$ and $f^{-1}(V)$ are open in X
As $0 \in U \Rightarrow f^{-1}(0) \in f^{-1}(U)$
$\Rightarrow x \in f^{-1}(U)$

Also $1 \in V \Rightarrow f(A) \in V \Rightarrow A \subset f^{-1}(V)$
Further
As $U \cap V = \phi \Rightarrow f^{-1}(U \cap V) = f^{-1}(\phi)$
$\Rightarrow f^{-1}(U) \cap f^{-1}(V) = \phi$
So X is regular
Hence every completely regular is regular.

Theorem: Every subspace of completely regular space is completely regular.

Proof Let X be a completely regular space and Y be a subspace of X.
To prove Y is completely regular.
Let $x \in Y$ and A be a closed set in Y s.t $x \notin A$.

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As A is closed in Y and Y is a subspace of X so $\exists$ a closed set B (say) in X s.t $A = B \cap Y$

As $x \in Y \subseteq X \Rightarrow x \in X$

As $x \notin A \Rightarrow x \notin B$

As $x \in X$ and B is closed set in X s.t $x \notin B$ and X is completely regular so $\exists$ a continuous function $f : X \to [0,1]$ s.t $f(x) = 0$, $f(B) = 1$

Define $g : Y \to [0,1]$ by $g(y) = f(y), \forall y \in Y$

then g is restriction of f on Y. As f is continuous so g is also continuous

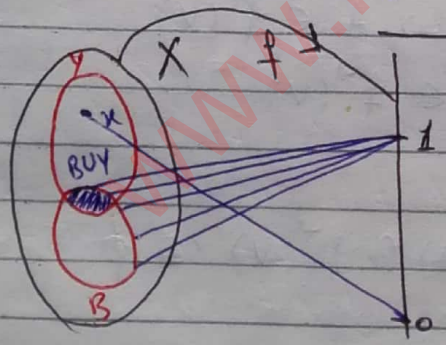
As $x \in Y \Rightarrow g(x) = f(x) = 0 \Rightarrow g(x) = 0$

As $A \subseteq Y \Rightarrow g(A) = f(A)$

$= f(B \cap Y) = 1$

$\Rightarrow g(A) = 1$

$\therefore Y$ is completely regular.



جیسا کہ $f(B)=1$ ہے یعنی B کا ہر Point
1 پر map ہوتا ہے تو جیسا کہ
$B \cap Y \subseteq B$ ہے اس لیے $B \cap Y$ کا ہر Pt
بھی 1 پر map ہوگا۔

$\Rightarrow T_{3\frac{1}{2}}$ - Space:- A completely regular T_1- space is called $T_{3\frac{1}{2}}$ - space or Tychnoff - space.

Theorem:- Every Tychnoff space is Housdorff space (T_2- space)

Proof:- Let X be Tychnoff space (i.e X be

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a completely regular T_1-space.)

To prove X is T_2-space.

Let $x, y \in X$ s.t $x \neq y$

As X is T_1-space then each singlton subset of X is closed (Theorem)

$\Rightarrow \{y\}$ is closed in X

Let $A = \{y\}$

So $x \in X$ and A is closed set s.t $x \notin A$

As X is completely regular, so there exist a continuous function $f: X \to [0,1]$ s.t $f(x) = 0$ and $f(A) = 1$

Let $U = [0, \frac{1}{2}[$ and $V =]\frac{1}{2}, 1]$

Then U and V are open sets in $[0,1]$

As f is continuous so $f^{-1}(U)$ and $f^{-1}(V)$ are open in X.

As $0 \in U \Rightarrow f(x) \in U \quad (\because f(x) = 0 \in U$

$\Rightarrow x \in f^{-1}(U)$ *

As $1 \in V \Rightarrow f(A) \in V \Rightarrow A \in f^{-1}(V)$

$\Rightarrow \{y\} \in f^{-1}(V) \Rightarrow y \in f^{-1}(V)$

As $U \cap V = \phi$

$\Rightarrow f^{-1}(U \cap V) = f^{-1}(\phi) \Rightarrow f^{-1}(U) \cap f^{-1}(V) = \phi$

$\Rightarrow X$ is T_2-space.

Theorem: Every Tychnoff space is T_3-space.

Proof: Let X be a Tychnoff space

To prove X is T_3-space

As X is Tychnoff space so X is T_1-space. We have to prove

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only that X is regular.
As X is Tychnoff space i.e X is completely regular.

To prove X is regular.
Let $x \in X$ and A be a closed set in X s.t $x \notin A$. As X is completely regular so there exist a continuous function

$f: X \longrightarrow [0,1]$ s.t

$f(x) = 0$ & $f(A) = 1$

Let $U = [0, \frac{1}{2}[$ and $V =]\frac{1}{2}, 1]$. Then U and V are open sets in $[0,1]$
As f is continuous, so $f^{-1}(U)$ and $f^{-1}(V)$ are open sets in X.

As $f(x) = 0 \in U \Rightarrow f(x) \in U$

$\Rightarrow x \in f^{-1}(U)$

Also $1 \in V \Rightarrow f(A) \in V \Rightarrow A \subseteq f^{-1}(V)$

Now $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$

$= f^{-1}(\phi)$

$\Rightarrow f^{-1}(U) \cap f^{-1}(V) = \phi$

Also X is T_1-space. So X is regular T_1-space

$\Rightarrow$ X is T_3-space.

Theorem:- For any topological space X if $C(X, \mathbb{R})$ separates the points of X then X is T_2-space.

Proof Let $x, y \in X$ s.t $x \neq y$ and $f \in C(X, \mathbb{R})$

i.e $f: X \longrightarrow \mathbb{R}$ is continuous. Then by

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given condition $f(x) \neq f(y)$

Suppose $f(x) < f(y)$

As $f(x), f(y) \in \mathbb{R}$ and $f(x) < f(y)$

so there exist a real number r

s.t $f(x) < r < f(y)$

Take $U = \{u : u \in X, f(u) < r\} = f^{-1}(-\infty, r)$

$V = \{v : v \in X, f(v) > r\} = f^{-1}(r, \infty)$

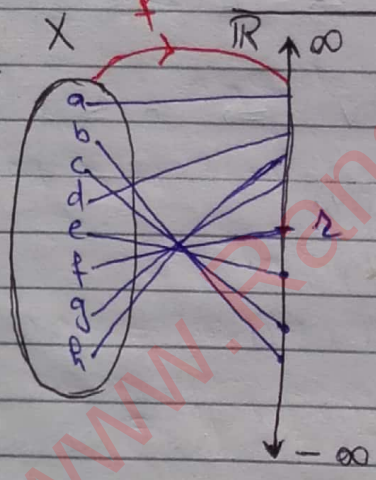
As $(-\infty, r)$ and (r, ∞) are open in $\mathbb{R}$, so U and V are open in X

Also note that

$x \in U$, $y \in V$ & $U \cap V = \phi$

$\Rightarrow X$ is T_2 - space.

Explanation



$U = \{b, c, e\}$ $\because f(u) < r$

$V = \{a, d, g, h\}$ $\because f(v) > r$

Then $f^{-1}(-\infty, r) = U$ &

$f^{-1}(r, \infty) = V$

★ Seprate کرنے کا مطلب یہ ہے کہ X کے مختلف Points کے ★
against ℝ میں مختلف images میں گی (i.e $x \neq y \Rightarrow f(x) \neq f(y)$)

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⇒ **Normal Space:-** A topological space (X, τ) is said to be normal space if for any two closed disjoint subsets A and B of X there exist open sets U & V s.t

$$A \subseteq U \quad , B \subseteq V$$
$$\text{and} \quad U \cap V = \phi$$

Theorem:- Every discrete space with atleast two points is normal.

Proof:-

Let X be a discrete space with atleast two points.

To prove X is normal

As X is discreate, so each subset of X is closed as well as open.

Let A and B are two closed subsets of X s.t $A \cap B = \phi$

Put $U = A$ and $V = B$

Then U and V are open sets in X because X is discreate.

Also $A \subseteq U \quad , B \subseteq V$ and $U \cap V = \phi$

$\Rightarrow$ X is normal.

Theorem:- Closed ~~Every~~ subspace of a normal space is normal.

Proof:-

Let X be a normal space and Y be a closed subspace of X.

To prove Y is normal

Let A and B be two closed sets in Y s.t $A \cap B = \phi$

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As A and B are closed in Y and Y is subspace of X so ∃ two closed subsets A_1 and B_1 in X s.t

$A = A_1 \cap Y$, $B = B_1 \cap Y$

As $A \cap B = \phi \Rightarrow A_1 \cap B_1 = \phi$

As A_1 and B_1 are closed sets in X, $A_1 \cap B_1 = \phi$ and X is normal so ∃ two open sets U_1 and V_1 in X s.t $A_1 \subseteq U_1$, $B_1 \subseteq V_1$

and $U_1 \cap V_1 = \phi$

Put $U = U_1 \cap Y$, $V = V_1 \cap Y$

Then U and V are open sets in Y

As $A_1 \subseteq U_1 \Rightarrow A_1 \cap Y \subseteq U_1 \cap Y$

$\Rightarrow A \subseteq U$

Also $B_1 \subseteq V_1 \Rightarrow B_1 \cap Y \subseteq V_1 \cap Y$

$\Rightarrow B \subseteq V$

and $U \cap V = (U_1 \cap Y) \cap (V_1 \cap Y)$

$= (U_1 \cap V_1) \cap Y$ (distributive law)

$= \phi \cap Y$

$= \phi$

$\Rightarrow U \cap V = \phi$

$\Rightarrow$ Y is normal space.

Theorem: Every metric space is normal space.

Proof: Let (X, d) be a metric space.

To prove X is normal

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Let A and B be two disjoint closed sets in X. Let $a \in A$ then

$$d(a,B) = \inf_{b \in B} d(a,b)$$

As $A \cap B = \phi$ and $a \in A$

$\Rightarrow a \notin B$

$\Rightarrow d(a,B) > 0$.

Let $r_a = d(a,B)$

Similarly $b \in B$ and $b \notin A$ ($\because A \cap B = \phi$

$\Rightarrow d(b,A) > 0$

Put $r_b = d(b,A)$

Now consider the open balls

$$B\left(a, \frac{r_a}{3}\right), \quad B\left(b, \frac{r_b}{3}\right)$$

Put $U = \bigcup_{a \in A} B\left(a, \frac{r_a}{3}\right)$, $V = \bigcup_{b \in B} B\left(b, \frac{r_b}{3}\right)$

Then U and V are open sets and $A \subseteq U$ and $B \subseteq V$

Now we prove $U \cap V = \phi$

Suppose $U \cap V \neq \phi$

Let $x \in U \cap V \Rightarrow x \in U$ and $x \in V$

As $x \in U = \bigcup_{a \in A} B\left(a, \frac{r_a}{3}\right)$

$\Rightarrow x \in B\left(a_1, \frac{r_{a_1}}{3}\right)$ for some $a_1 \in A$

Also $x \in V = \bigcup_{b \in B} B\left(b, \frac{r_b}{3}\right)$

$\Rightarrow x \in B\left(b_1, \frac{r_{b_1}}{3}\right)$ (for some $b_1 \in B$)

As $x \in B\left(a_1, \frac{r_{a_1}}{3}\right) \Rightarrow d(x,a_1) < \frac{r_{a_1}}{3}$

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Also $x \in B(b_1, \frac{r_{b_1}}{3})$

$\Rightarrow d(x, b_1) < \frac{r_{b_1}}{3}$

As $a_1 \in A$ and $b_1 \in B$ and $A \cap B = \phi$

$\Rightarrow a_1 \neq b_1 \Rightarrow d(a_1, b_1) > 0$

Let $r = d(a_1, b_1)$

Clearly $r_{a_1} < r$, $r_{b_1} < r$

Now $r = d(a_1, b_1) \leq d(a_1, x) + d(x, b_1)$

$< \frac{r_{a_1}}{3} + \frac{r_{b_1}}{3}$

$< \frac{r}{3} + \frac{r}{3} = \frac{2r}{3}$

$\Rightarrow r < \frac{2r}{3}$

which is a contradiction. So our supposition is wrong $\therefore U \cap V = \phi$

So X is normal.

T_4- Space:- A normal T_1- space is called T_4- space.

Theorem Every T_4- space is regular.

Proof:

Let X be a T_4- space.

To prove X is regular.

Let $x \in X$ and A be a closed set in X s.t $x \notin A$.

As X is T_1 space therefor each singlton set in X is closed

$\Rightarrow \{x\}$ is closed in X.

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Therefore $\{x\}$ and A are closed sets in X.
As X is normal then $\exists$ two open sets U and V s.t

$\{x\} \subseteq U$ and $A \subseteq V$ and $U \cap V = \phi$

$\Rightarrow x \in U$, $A \subseteq V$ and $U \cap V = \phi$

$\Rightarrow X$ is regular.

Imp

Theorem: Every metric space is completely regular.

Proof

Let (X, d) be a metric space.

To prove X is completely regular.
Let $x \in X$ and A be a closed subset of X s.t $x \notin A$

To prove there is a continuous function $f: X \longrightarrow [0,1]$ s.t

$f(x) = 0$, $f(A) = 1$

Define a function $g: X \longrightarrow \mathbb{R}$ by

$g(y) = d(y, B)$

where B is another closed set in X containing 'x' s.t $A \cap B = \phi$

clearly $g(x) = d(x, B) = 0$

and $g(A) = d(A, B) > 0$ $(\because A \cap B = \phi)$

$x \in B$
$d(x,B) = \inf_{b \in B} d(x,b)$
$\exists\ d(x,x) = 0$

Now for any $\varepsilon > 0$ choose $\delta = \varepsilon$ s.t $d(y, y') < \delta$

i.e $|y - y'| < \delta$

If $|x-a| < \delta$
$|f(x) - f(a)| < \varepsilon$
$\Rightarrow f$ is continuous

Now $|g(y) - g(y')| = |d(y, B) - d(y', B)|$

$\leq d(y, y') < \delta = \varepsilon$

$d(x,A) - d(y,A) \leq d(x,y)$

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$\Rightarrow |g(y) - g(y')| < \varepsilon$

So g is continuous.

Now define a function $f: X \to [0,1]$ by $f(y) = \frac{1}{k} g(y)$ where $k = g(A)$

Then f is continuous because g is continuous

Further $f(x) = \frac{1}{k} g(x)$

$= \frac{1}{k}(0)$

$\Rightarrow f(x) = 0$

and $f(A) = \frac{1}{k} g(A) = \frac{1}{k} \times k$

$\Rightarrow f(A) = 1 \quad \because g(A) = k$

$\Rightarrow X$ is completely regular.

Theorem: A topological space (X, τ) is normal iff for any closed set A and open set U s.t $A \subseteq U$, there is at least one open set 'V' containing A s.t

$$A \subseteq V \subseteq \bar{V} \subseteq U.$$

Proof

Given X is normal and A be a closed set in X with an open set 'U' containing 'A'

As $A \subseteq U \Rightarrow A \cap U' = \phi$

As U is an open set so U' is closed

As A and U' are two disjoint closed sets in X and X is normal

So $\exists$ two open sets V and V_1 s.t

$A \subseteq V$ and $U' \subseteq V_1$

and $V \cap V_1 = \phi$

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As $V \cap V_1 = \phi \Rightarrow V \subseteq V_1'$

Also $A \subseteq V$ and $U' \subseteq V_1$

$\Rightarrow V_1' \subseteq (U')' \Rightarrow V_1' \subseteq U$

As $V \subseteq V_1'$

$\Rightarrow V_1'$ is closed superset of V

But $\bar{V}$ = Intersection of all closed supersets of V

$\Rightarrow \bar{V} \subseteq V_1'$

$\Rightarrow V \subseteq \bar{V} \subseteq V_1' \subseteq U$

$\Rightarrow A \subseteq V \subseteq \bar{V} \subseteq V_1' \subseteq U$

$\Rightarrow A \subseteq V \subseteq \bar{V} \subseteq U$

Conversely given that for any closed set A in X and for any open set 'U' containing A, there is open set V s.t

$$A \subseteq V \subseteq \bar{V} \subseteq U$$

To prove X is normal

Let A_1 and A_2 be two disjoint closed sets in X i.e $A_1 \cap A_2 = \phi$

$\Rightarrow A_1 \subseteq A_2'$

As A_2 is closed, so A_2' is an open set containing A_1. Then by given condition there is an open set V s.t

$$A_1 \subseteq V \subseteq \bar{V} \subseteq A_2'$$

Note that $A_1 \subseteq V$

Also $\bar{V} \subseteq A_2' \Rightarrow (A_2')' \subseteq (\bar{V})'$

$\Rightarrow A_2 \subseteq (\bar{V})' = U$ (say)

$\Rightarrow A_2 \subseteq U$

As $\bar{V}$ is closed $\Rightarrow (\bar{V})'$ is an open set

i.e U is an open set

As $V \subseteq \bar{V}$

$\Rightarrow V \cap (\bar{V})' = \phi$

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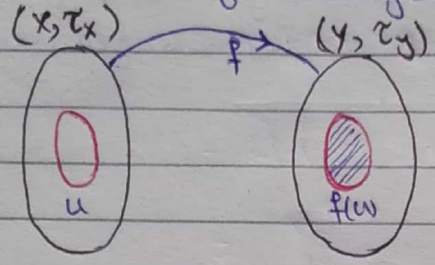
$\Rightarrow V \cap U = \phi \Rightarrow X$ is normal

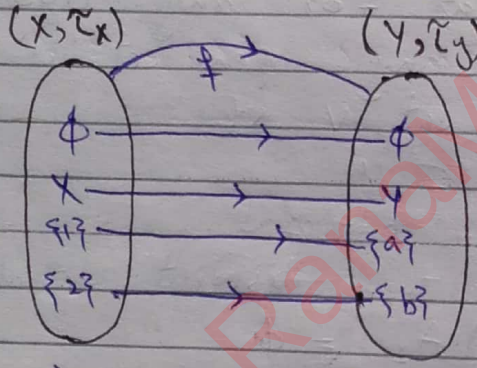
⇒ **Open Function OR Open Mapping:-** A function 'f' is said to be an open function if image of open set in open.



If $U \in \tau_X$
$\Rightarrow f(U) \in \tau_Y$
$\Rightarrow f$ is open.

Example:- $X = \{1, 2\}$, $\tau_X = \{\phi, X, \{1\}, \{2\}\}$

$Y = \{a, b\}$, $\tau_Y = \{\phi, Y, \{a\}, \{b\}\}$



$f(\phi) = \phi$
$f(X) = Y$
$f(\{1\}) = \{a\}$
$f(\{2\}) = \{b\}$

As each open set in X is mapped on to open set in Y
$\Rightarrow f$ is open function

⇒ **Closed Function OR Closed Mapping:-** A function 'f' is said to be closed if image of each closed set is closed.

Theorem: A closed continuous image of a normal space is normal.

Proof:

Let X be a normal space and

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$Y = f(X)$ is its closed continuous image

To prove 'Y' is normal

Let A_1 and B_1 be two disjoint closed sets in Y

As f is continuous, so inverse image of each closed set is closed so $f^{-1}(A_1)$ and $f^{-1}(B_1)$ are closed sets in X.

Let $A = f^{-1}(A_1)$, $B = f^{-1}(B_1)$

Now

$$A \cap B = f^{-1}(A_1) \cap f^{-1}(B_1)$$

$$= f^{-1}(A_1 \cap B_1) \quad = f^{-1}(\phi)$$

$$A \cap B = \phi$$

So A and B are disjoint closed sets in X

As X is normal so $\exists$ two open sets U and V s.t

$$A \subseteq U \quad , \quad B \subseteq V$$

and $U \cap V = \phi$

As U and V are open sets in X so U' and V' are closed sets in X.

As f is closed function

$\Rightarrow f(U')$ and $f(V')$ are closed sets in Y

Put $U_1 = Y \setminus f(U') = [f(U')]'$

$V_1 = Y \setminus f(V') = [f(V')]'$

Then U_1 and V_1 are open sets in Y

Next we show That

$A_1 \subseteq U_1$

Let $x \in A_1 = f(A)$

$\Rightarrow x \in f(A) \Rightarrow f^{-1}(x) \in A \subseteq U$

$\Rightarrow f^{-1}(x) \in U \Rightarrow f^{-1}(x) \notin U'$

$\Rightarrow x \notin f(U') \Rightarrow x \in [f(U')]' = U_1$

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$\Rightarrow A_1 \subseteq U_1$

Similarly we can prove that

$B_1 \subseteq V_1$

Now

$$U_1 \cap V_1 = [f(U)]' \cap [f(V')]'$$

$$= [f(U') \cup f(V')]'$$

$$= [f(U' \cup V')]' = [f(X)]'$$

$$= Y'$$

$$= \phi$$

$\Rightarrow U_1 \cap V_1 = \phi$

$\Rightarrow Y = f(X)$ is normal.

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CHAPTER #3

COMPACTNESS.

⇒ **Open Cover:-** Let (X, τ) be a topological space. A system $\{U_\alpha : \alpha \in I\}$ of open sets in X is said to be open cover for X if

$$X = \bigcup_{\alpha \in I} U_\alpha$$

⇒ **Compact:-** A topological space (X, τ) is said to be compact if every open cover for X has a finite subcover.

Theorem:- Let X be an infinite set with cofinite topology then X is compact.

Proof

Let $C = \{U_\alpha : \alpha \in I\}$ be an open cover for X

i.e $X = \bigcup_{\alpha \in I} U_\alpha$

Let $U_\beta \in C$

$\Rightarrow U_\beta$ is an open set $(\because U_\beta \in \tau_c)$

$\Rightarrow U_\beta'$ is finite

Let $U_\beta' = \{x_1, x_2, x_3, \cdots, x_n\}$

As C is an open cover for X so for each $x_i \in U_\beta'$ there is a $U_{\alpha_i} \in C$ s.t

$x_i \in U_{\alpha_i} \Rightarrow \{x_i\} \subseteq U_{\alpha_i}$

$$\Rightarrow \bigcup_{i=1}^{n} \{x_i\} \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$$

$$\Rightarrow U_\beta' \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$$

$$\Rightarrow U_\beta \cup U_\beta' \subseteq U_\beta \cup \left(\bigcup_{i=1}^{n} U_{\alpha_i}\right) \Rightarrow X \subseteq U_\beta \cup \left(\bigcup_{i=1}^{n} U_{\alpha_i}\right) \subseteq X$$

$$\Rightarrow X = U_\beta \cup \left(\bigcup_{i=1}^{n} U_{\alpha_i}\right)$$

$\Rightarrow \{U_\beta, U_{\alpha_i}, i = 1, 2, 3, \cdots n\}$ is a finite subcover for X $\Rightarrow$ X is Compact.

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⇒ **Finite Intersection Property:-** Let (X, τ) be a topological space and $C = \{U_\alpha : \alpha \in I\}$ be collection of some subsets of X, Then C is said to have finite intersection property, if each finite subcollection of C has non-empty intersection.

Example:- $X = \mathbb{N}$, $\tau = \tau_c$

$$C = \{\{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}, \cdots\}$$

Then clearly intersection of any finite subcollection of C has non-empty intersection. So C satisfies finite intersection property.

$C_1 = \{\{1\}, \{1,2,3\}, \{1,2,3,4\}\} \Rightarrow \cap C_1 = \{1\} \neq \phi$

$C_2 = \{\{1\}, \{1,2\}, \{1,2,3\}, \cdots \{1,2,\cdots 100\}\} \subseteq C$

$\therefore \cap C_2 = \{1\} \neq \phi$

Theorem A topological space (X, τ) is compact iff every collection of closed sets in X which satisfies finite intersection property has non-empty intersection.

Proof

Let X be a compact and let $\{U_\alpha : \alpha \in I\}$ be collection of closed sets which satisfies finite intersection property.

To prove $\bigcap_{\alpha \in I} U_\alpha \neq \phi$

Suppose that $\bigcap_{\alpha \in I} U_\alpha = \phi$

$$\Rightarrow \left(\bigcap_{\alpha \in I} U_\alpha\right)' = \phi' \Rightarrow \bigcup_{\alpha \in I} U_\alpha' = X$$

Now $\{U_\alpha : \alpha \in I\}$ is the collection of closed sets

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so $\{U_\alpha' : \alpha \in I\}$ is a collection of open sets with $\bigcup_{\alpha \in I} U_\alpha' = X \Rightarrow \{U_\alpha' : \alpha \in I\}$ is an open cover for X

As X is compact. So this open cover has a finite sub cover say $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$

$$\text{i.e } X = \bigcup_{i=1}^{n} U_{\alpha_i}$$

$$\Rightarrow X' = \left(\bigcup_{i=1}^{n} U_{\alpha_i}\right)' \Rightarrow \phi = \bigcap_{i=1}^{n} (U_{\alpha_i})'$$

$$\Rightarrow \phi = \bigcap_{i=1}^{n} U_{\alpha_i}$$

$\Rightarrow \{U_{\alpha_1}, U_{\alpha_2}, U_{\alpha_3}, \ldots, U_{\alpha_n}\}$ is a finite subcollection of $\{U_\alpha : \alpha \in I\}$ with empty intersection

$\Rightarrow \{U_\alpha : \alpha \in I\}$ does not satisfies finite intersection property.

Which is a contradiction. So our supposition is wrong and hence

$$\bigcap_{\alpha \in I} U_\alpha \neq \phi$$

Conversely let in a topological space (X, τ) each collection of closed sets in X which satisfies finite intersection property has non empty intersection.

To prove X is compact

Let $\{U_\alpha : \alpha \in I\}$ be an open cover for X

$$\text{i.e } \bigcup_{\alpha \in I} U_\alpha = X$$

$$\Rightarrow \left(\bigcup_{\alpha \in I} U_\alpha\right)' = X' \Rightarrow \bigcap_{\alpha \in I} U_\alpha' = \phi$$

$\Rightarrow \{U_\alpha' : \alpha \in I\}$ is a collection of closed sets with empty intersection then by given hypothesis $\{U_\alpha' : \alpha \in I\}$ does not satisfies

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finite intersection property. Then there exist a finite subcollection $\{U_{\alpha_1}', U_{\alpha_2}', \ldots, U_{\alpha_n}'\}$ with empty intersection

$$\Rightarrow \bigcap_{i=1}^{n} U_{\alpha_i}' = \phi \Rightarrow \left(\bigcap_{i=1}^{n} U_{\alpha_i}'\right)' = \phi'$$

$$\Rightarrow \bigcup_{i=1}^{n} U_{\alpha_i} = X$$

$\Rightarrow \{U_{\alpha_1}, U_{\alpha_2}, U_{\alpha_3}, \ldots, U_{\alpha_n}\}$ is a finite open subcover for X. $\Rightarrow X$ is compact

---

Theorem Every closed subspace of a compact space is compact.

Proof

Let X be a compact space and Y be closed subspace of X.

To prove Y is compact

Let $\{U_\alpha : \alpha \in I\}$ be an open cover for Y

As U_α is an open set in Y and Y is a subspace of X so

$U_\alpha = V_\alpha \cap Y$ where V_α is an open set in X

Now

$$U_\alpha = V_\alpha \cap Y \Rightarrow U_\alpha \subseteq V_\alpha$$

$$\Rightarrow \bigcup_{\alpha\in I} U_\alpha \subseteq \bigcup_{\alpha\in I} V_\alpha \Rightarrow Y \subseteq \bigcup_{\alpha\in I} V_\alpha$$

$$\Rightarrow X = Y \cup Y' \subseteq \bigcup_{\alpha\in I} V_\alpha \cup Y' \subseteq X$$

$$\Rightarrow X = \bigcup_{\alpha\in I} V_\alpha \cup Y'$$

As Y is closed in $X \Rightarrow Y'$ is an open set in $X \Rightarrow \{Y', V_\alpha : \alpha \in I\}$ is an open cover for X.

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As X is compact so this open cover has a finite subcover say $\{Y', V_{\alpha_1}, V_{\alpha_2}, \cdots, V_{\alpha_n}\}$

$$\therefore X = \bigcup_{i=1}^{n} V_{\alpha_i} \cup Y'$$

Now $Y \subseteq X = \bigcup_{i=1}^{n} V_{\alpha_i} \cup Y' \Rightarrow Y \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$ $\left[\because Y \not\subseteq Y'\right.$

$$\Rightarrow Y \cap Y \subseteq \left(\bigcup_{i=1}^{n} V_{\alpha_i}\right) \cap Y$$

$$Y \subseteq \bigcup_{i=1}^{n} (V_{\alpha_i} \cap Y) \quad \text{distributive law}$$

$$Y \subseteq \bigcup_{i=1}^{n} U_{\alpha_i} \subseteq Y \quad \left\{ U_{\alpha_i} \in \tau \right. \rightarrow \bigcup_{i=1}^{n} U_{\alpha_i} \in \tau \subseteq Y$$

$$\Rightarrow Y = \bigcup_{i=1}^{n} U_{\alpha_i}$$

$\Rightarrow \{U_{\alpha_1}, U_{\alpha_2}, \cdots, U_{\alpha_n}\}$ is a finite subcover for Y $\Rightarrow$ Y is compact.

$\blacksquare$

Theorem: Every compact T_2-space is normal

Proof: Let X be a compact T_2-space.

To prove X is normal

Let C_1 and C_2 be two disjoint closed sets in X. Let $x \in C_1 \Rightarrow x \notin C_2$

Further as X is compact and C_2 be a closed sub(space) of X so C_2 is also compact.

As X is a T_2-space and $x \in X$ a C_2 be a compact subspace of X s.t $x \notin C_2$ then by a well known theorem $\exists$ two open sets U_x and V_x s.t

$$x \in U_x \text{ & } C_2 \in V_x \text{ and}$$

$$U_x \cap V_x = \phi$$

As $x \in U_x \Rightarrow \{x\} \subseteq U_x$

$$\Rightarrow \bigcup_{x \in C_1} \{x\} \subseteq \bigcup_{x \in C_1} U_x \Rightarrow C_1 \subseteq \bigcup_{x \in C_1} U_x$$

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$\Rightarrow C_1 = \left(\bigcup_{x \in C_1} U_x\right) \cap C_1$ $\left(\because A \subseteq B \Rightarrow A = A \cap B\right.$

$= \bigcup_{x \in C_1} (U_x \cap C_1)$ distributive law

$\Rightarrow \{U_x \cap C_1 : x \in C_1\}$ is an open cover for C_1

As C_1 is compact so this open cover has a finite subcover

let $\{U_{x_1} \cap C_1, U_{x_2} \cap C_1, \cdots, U_{x_n} \cap C_1\}$ is a subcover for C_1 i.e

$C_1 = \bigcup_{i=1}^{n} (U_{x_i} \cap C_1) \Rightarrow C_1 \subseteq \bigcup_{i=1}^{n} U_{x_i}$

Put $\bigcup_{i=1}^{n} U_{x_i} = U$ & $V = \bigcap_{i=1}^{n} V_x$

Then U and V are open sets. Further

$C_1 \subseteq U$ & $C_2 \subseteq V$

Now we prove that $U \cap V = \phi$

Suppose $U \cap V \neq 0$ then $\exists\ z \in U \cap V$

$\Rightarrow z \in U$ and $z \in V$

$\Rightarrow z \in \bigcup_{i=1}^{n} U_{x_i}$ for some i and

$z \in \bigcap_{i=1}^{n} V_{x_i}$ for all i $(\because U_x \cap V_x = \phi)$

$\Rightarrow U_{x_i} \cap V_{x_i} \neq \phi$ so our supposition is wrong Hence $U \cap V = \phi$

$\Rightarrow X$ is normal

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Theorem: Continuous image of a compact space is compact.

Proof:

Let X be a compact space and $f: X \to Y$ be a continuous function.

To prove $f(X)$ is compact.

Let $\{U_\alpha : \alpha \in I\}$ be an open cover for $f(X)$

i.e $f(X) = \bigcup_{\alpha \in I} U_\alpha$

As U_α is open in $f(X)$ and f is continuous so $f^{-1}(U_\alpha)$ is an open set in X

Also U_α is open in $f(X)$ and $f(X)$ is subspace of Y so $\exists$ an open set V_α in Y s.t

$$U_\alpha = V_\alpha \cap f(X) \qquad [\because f(X) \subseteq Y]$$

$$\Rightarrow U_\alpha \subseteq V_\alpha \Rightarrow \bigcup_{\alpha \in I} U_\alpha \subseteq \bigcup_{\alpha \in I} V_\alpha$$

$$\Rightarrow f(X) \subseteq \bigcup_{\alpha \in I} V_\alpha \Rightarrow X \subseteq f^{-1}\left(\bigcup_{\alpha \in I} V_\alpha\right)$$

$$\Rightarrow X \subseteq \bigcup_{\alpha \in I} f^{-1}(V_\alpha) \subseteq X$$

$$\Rightarrow X = \bigcup_{\alpha \in I} f^{-1}(V_\alpha)$$

As V_α is open in Y and f is continuous so $f^{-1}(V_\alpha)$ is open in X

$\Rightarrow \{f^{-1}(V_\alpha) : \alpha \in I\}$ is an open cover for X

As X is compact so $\exists$ a finite subcover say $\{f^{-1}(V_1), f^{-1}(V_2), \ldots, f^{-1}(V_n)\}$ of $\{f^{-1}(V_\alpha) : \alpha \in I\}$

i.e $X = \bigcup_{i=1}^{n} f^{-1}(V_i) \Rightarrow X = f^{-1}\left(\bigcup_{i=1}^{n} V_i\right)$

$$\Rightarrow f(X) \subseteq \bigcup_{i=1}^{n} (V_i) \Rightarrow f(X) \subseteq \left(\bigcup_{i=1}^{n} V_i\right) \cap f(X)$$

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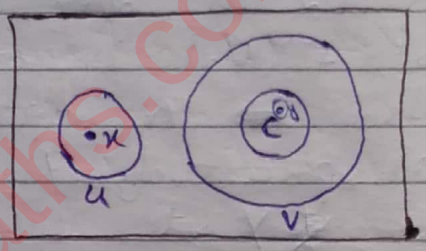
$\Rightarrow f(X) \subseteq \bigcup_{i=1}^{n} (V_i \cap f(X)$ distributive law

$\Rightarrow f(X) \subseteq \bigcup_{i=1}^{n} U_i \subseteq f(X)$

$\Rightarrow f(X) = \bigcup_{i=1}^{n} U_i \Rightarrow f(X)$ is compact

Theorem: In a T_2-space any point and a disjoint compact subspace of X can be seperated by open sets in the sence they have disjoint nbhds

Proof: Let $x \in X$ and C be a compact subspace of X s.t $x \notin C$



Let $y \in C$ then $y \neq x \ \forall y \in C$

As $x, y \in X$ and X is a T_2-space so $\exists$ open set U_x and V_y s.t

$x \in U_x$, $y \in V_y$ and $U_x \cap V_y = \phi$ Supposition *

As $y \in V_y \Rightarrow \{y\} \subseteq V_y$

$\Rightarrow \bigcup_{y \in C} \{y\} \subseteq \bigcup_{y \in C} V_y \Rightarrow C \subseteq \bigcup_{y \in C} V_y$

$\Rightarrow C \cap C \subseteq \left(\bigcup_{y \in C} V_y\right) \cap C$

$\Rightarrow C \subseteq \bigcup_{y \in C} (V_y \cap C)$

$\Rightarrow C \subseteq \bigcup_{y \in C} (V_y \cap C) \subseteq C \Rightarrow C = \bigcup_{y \in C} (V_y \cap C)$

$\Rightarrow \{V_y \cap C : y \in C\}$ is an open cover for C. As C is compact so this open cover has a finit subcover

Let $\{V_1 \cap C, V_2 \cap C, \ldots, V_n \cap C\}$

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i.e $C = \bigcup_{i=1}^{n} (V_i \cap C)$

$\Rightarrow C = \left(\bigcup_{i=1}^{n} V_i\right) \cap C \Rightarrow C \subseteq \bigcup_{i=1}^{n} V_i$

Put $U = \bigcap_{i=1}^{n} U_i$ & $V = \bigcup_{i=1}^{n} V_i$

Then U and V are open sets s.t
$x \in U$ and $C \subseteq V$

To prove $U \cap V = \phi$

Suppose $U \cap V \neq \phi \Rightarrow z \in U \cap V$

$\Rightarrow z \in U$ and $z \in V$

$\Rightarrow z \in \bigcap_{i=1}^{n} U_i$ and $z \in \bigcup_{i=1}^{n} V_i$

$\Rightarrow z \in U_i$ for all i & $z \in V_i$ for some i

$\Rightarrow z \in U_i \cap V_i$ for some i

$\Rightarrow U_i \cap V_i \neq \phi$ for some i

which is a contradiction. So our supposition is wrong. Hence $U \cap V = \phi$

$\Rightarrow$

Theorem:- Compact subspace of a T_2-space is closed.

Proof

Let X be T_2-space and C be a compact subspace of X. To prove C is closed

We have to prove C' is an open set.

If $C' = \phi \Rightarrow C'$ is an open set.

If $C' \neq \phi$

Let $x \in C' \Rightarrow x \notin C$

As we know that in a T_2-space any point and a disjoint compact subspace of X can be separated by open sets then there exist two open sets say

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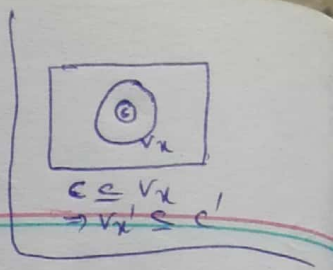

U_x and V_x st
$x \in U_x$ and $C \subseteq V_x$ and $U_x \cap V_x = \phi$

$\{x\} \subseteq U_x \Rightarrow$ [illegible]

As $C \subseteq V_x \Rightarrow V_x' \subseteq C'$

$\Rightarrow \bigcup_{x \in C'} \{x\} \subseteq \bigcup_{x \in C'} U_x$

Also $U_x \subseteq V_x' \subseteq C'$
$\bigcup U_x \subseteq \bigcup_{x \in C'} V_x' \subseteq C'$

$C' \subseteq \bigcup_{x \in C'} U_x \subseteq C'$? $\Rightarrow C' = \bigcup_{x \in C'} U_x$ $\rightarrow \bigcup_{x \in C'} U_x \subseteq C'$

$\Rightarrow C'$ is open set $\Rightarrow C$ is closed.

Homeomorphism:- A function $f: X \to Y$ is said to be Homeomorphism if
(i) f is bijective (ii) f is continuous
(iii) f is open.

Theorem:- A 1-1 continuous mapping from a compact space X onto a T_2-space Y is Homeomorphism.

Proof: Let $f: X \to Y$ be a 1-1 and continuous mapping from compact space X onto T_2-space Y.

Since f is 1-1 and onto
$\Rightarrow f$ is bijective.

Also f is continuous. To prove f is Homeomorphism. We just prove that f is open mapping.

Let G be an open set in X
$\Rightarrow G'$ is closed in X

A closed subspace of a compact space is compact $\Rightarrow G'$ is compact.

Also we know that continuous image

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of a compact space is compact.
So $f(G')$ is compact subspace of Y.
As compact subspace of T_2-space is closed
$\Rightarrow f(G')$ is closed subspace of Y.

As $f(G') = f(X - G)$
$= f(X) - f(G)$
$= Y - f(G)$ $\quad \because f$ is onto $\Rightarrow f(X) = Y$
$= [f(G)]'$

As $f(G')$ is closed $\Rightarrow [f(G)]'$ is closed
$\Rightarrow f(G)$ is open
$\Rightarrow f$ is an open mapping
$\Rightarrow f$ is Homeomorphism

Theorem:- A topological space X is comapact iff every class of closed sets with empty intersection has a finite sub class with empty intersection.

Proof Let X is compact and let $\{C_\alpha : \alpha \in I\}$ be a class of closed sets in X with $\bigcap_{\alpha \in I} C_\alpha = \phi \Rightarrow \left(\bigcap_{\alpha \in I} C_\alpha\right)' = \phi'$

$\Rightarrow \bigcup_{\alpha \in I} C_\alpha' = X$

As C_α is closed $\Rightarrow C_\alpha'$ is open
$\Rightarrow \{C_\alpha' : \alpha \in I\}$ is an open cover for X.

As X is compact so $\exists$ a finite subcover say $\{C_{\alpha_1}', C_{\alpha_2}', \cdots C_{\alpha_n}'\}$ of $\{C_\alpha' : \alpha \in I\}$

i.e $\bigcup_{i=1}^{n} C_{\alpha_i}' = X \Rightarrow \left(\bigcup_{i=1}^{n} C_{\alpha_i}'\right)' = X'$

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$$\Rightarrow \bigcap_{i=1}^{n} C_{\alpha_i} = \phi$$

Conversely suppose in a topological space X each class $\{C_\alpha : \alpha \in I\}$ of closed sets with empty intersection has a finite sub class with empty intersection.

To prove X is compact.

Let $\{U_\alpha : \alpha \in I\}$ be an open cover for X

i.e $X = \bigcup_{\alpha \in I} U_\alpha \Rightarrow X' = \left(\bigcup_{\alpha \in I} U_\alpha\right)'$

$\Rightarrow \phi = \bigcap_{\alpha \in I} U_\alpha'$. As U_α is open then U_α' is closed set so $\{U_\alpha' : \alpha \in I\}$ is class of closed sets with empty intersection so by given condition there exist a finite sub class say $\{U_{\alpha_1}', U_{\alpha_2}', \ldots, U_{\alpha_n}'\}$ s.t

$$\bigcap_{i=1}^{n} U_{\alpha_i}' = \phi \Rightarrow \left(\bigcap_{i=1}^{n} U_{\alpha_i}'\right)' = \phi'$$

$$\Rightarrow \bigcup_{i=1}^{n} U_{\alpha_i} = X$$

$\Rightarrow \{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$ is a finite subcover for X

$\Rightarrow X$ is compact.

Theorem: Every compact subspace of $\mathbb{R}$ is closed and bounded.

Proof: Let C be a compact space of $\mathbb{R}$

To prove C is closed and bounded.

As $\mathbb{R}$ is T_2-space (Every m.s is T_2-space) and C is compact subspace of $\mathbb{R}$

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As compact subspace of a T_2-space is closed
$\Rightarrow$ C being compact subspace of $\mathbb{R}$ (T_2-space) is closed.

Now we prove C is bounded.
Let $U_k = B(0, k)$, $k \in \mathbb{N}$
Then $\{U_k : k \in \mathbb{N}\}$ is an open cover for $\mathbb{R}$
So $U_k \cap C$ is an open set in C.
let $\{U_k \cap C : k \in \mathbb{N}\}$ be an open cover for C. As C is compact so this open cover has a finite subcover.
Let $\{U_{k_1} \cap C, U_{k_2} \cap C, \ldots, U_{k_n} \cap C\}$ be a finite subcover for C.

$$\Rightarrow C = \bigcup_{i=1}^{n} (U_{k_i} \cap C)$$

$$\Rightarrow C = \left(\bigcup_{i=1}^{n} U_{k_i}\right) \cap C \quad \text{distributive law.}$$

$$\Rightarrow C \subseteq \bigcup_{i=1}^{n} U_{k_i}$$

Let $n^* = \max(k_1, k_2, k_3, \ldots, k_n)$ then
$\bigcup_{i=1}^{n} U_{k_i} \subseteq U_{n^*} \Rightarrow C \subseteq U_{n^*}$

$\Rightarrow$ C is bounded because U_{n^*} is bounded
Hence compact subspace of $\mathbb{R}$ is closed and bounded.

Theorem:- A continuous real valued function defined on a compact space is bounded and attains its bounds.

Proof: Let $f: X \to \mathbb{R}$ be a continuous and real valued function from a compact space X to R. So $f(X)$ is compact subspace of $\mathbb{R}$ (∵ Continuous image of a compact space is compact

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As we know that compact subspace of $\mathbb{R}$ is closed and bounded

$\Rightarrow f(X)$ is closed and bounded.

As $f(X)$ is bounded $\Rightarrow f$ is bounded.

So $\sup f(X)$ and $\inf f(X)$ exists.

Let $a = \sup f(X)$ & $b = \inf f(X)$

As supremum & infimum are limit points and limit points of a closed set belongs to set

$\Rightarrow a, b \in f(X)$

So f attains its bounds.

Countably Compact Space:- A topological space (X, τ) is said to be countably compact space if every countable open cover for X has a finite subcover.

Remark:- Every compact space is countably compact.

Theorem:- A topological space X is countably compact iff every countable collection of closed sets in X which satisfies finite intersection property has non empty intersection.

Proof Suppose X is countably compact

Let $\{U_n : n \in \mathbb{N}\}$ be the collection of closed sets which satisfies finite intersection property.

To prove $\bigcap_{n \in \mathbb{N}} U_n \neq \phi$

Suppose $\bigcap_{n \in \mathbb{N}} U_n = \phi$

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$$\Rightarrow \left(\bigcap_{n\in\mathbb{N}} U_n\right)' = \phi' \Rightarrow \bigcup_{n\in\mathbb{N}} U_n' = X$$

As U_n is closed $\Rightarrow U_n'$ is open
$\Rightarrow \{U_n' : n\in\mathbb{N}\}$ is a countable open cover for X.
As X is countably compact space so this open cover have a finite subcover say $\{U_1', U_2', \ldots, U_n'\}$ then

$$X = \bigcup_{i=1}^{n} U_n' \Rightarrow X' = \left(\bigcup_{i=1}^{n} U_n'\right)'$$

$$\Rightarrow \phi = \bigcap_{i=1}^{n} U_n$$

$\Rightarrow \{U_n : n\in\mathbb{N}\}$ does not satisfy finite intersection property. Which is a contradiction. So our supposition is wrong.

$$\text{So } \bigcap_{n\in\mathbb{N}} U_n \neq \phi$$

Conversely suppose that for every countable collection of closed sets which satisfies finite intersection property has non-empty intersection.

To prove X is countably compact.
Suppose X is not countably compact.
Let $\{U_n : n\in\mathbb{N}\}$ be a countable open cover for X which has no finite subcover.
$\therefore \{U_1, U_2, U_3, \ldots U_n\}$ be a finite subfamily such that $\bigcup_{i=1}^{n} U_i \neq X_n$

$$\Rightarrow \left(\bigcup_{i=1}^{n} U_i\right)' \neq X' \Rightarrow \bigcap_{i=1}^{n} U_i' \neq \phi$$

Since $\{U_n : n\in\mathbb{N}\}$ is an open cover for X
i.e $X = \bigcup_{n\in\mathbb{N}} U_n$

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Every uncountable set contains (atleast) a countable subset

$\Rightarrow X' = \left(\bigcup_{n\in\mathbb{N}} U_n\right)' \Rightarrow X' = \bigcap_{n\in\mathbb{N}} U_n' \Rightarrow \phi = \bigcap_{n\in\mathbb{N}} U_n'$

$\Rightarrow \{U_n' : n \in \mathbb{N}\}$ be a countable collection of closed sets which satisfies finite intersection property with

$$\bigcap_{n\in\mathbb{N}} U_n' = \phi$$

which is a contradiction. So our supposition is wrong. Hence X is countably compact.

Theorem:- Let X be a topological space then any infinite subset of X has a limit point if & only if every countably infinite subset of X has a limit point.

Proof:- Let us suppose in a topological space X every infinite subset of X has limit point then trivially every countable infinite subset of X has a limit point.

Conversely suppose every countabe infinite subset of X has the limit point.

To prove every infinite subset of X has a limit point.

Let "A" be an infinite subset of X then by a well known theorem of set theory A has a countable infinite subset B. Then by the hypothesis B has the limit pt say x.

$\Rightarrow U \cap B \setminus \{x\} \neq \phi$

$\Rightarrow U \cap A \setminus \{x\} \neq \phi \quad \because B \subseteq A$

$\Rightarrow$ x is also the limit point of A.

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$\Rightarrow$ A has a limit point.

$\Rightarrow$ "Bolzano Weierstrass" Property or B.W Property:- A space X is said to satisfy the bolzano weierstrass property iff every infinite subset of X has a limit point in X.

Theorem:- Let X be a countably compact space then every infinite subset of X has a limit point.

Proof: Let A be an infinite subset of X. To prove "A" has a limit point. Suppose A has no limit point. Let $B = \{x_1, x_2, \ldots, x_n, \ldots\}$ be a countable infinite subset of X. Then by a well known theorem B has no limit pt. Now consider the subset $C_n = \{x_n, x_{n+1}, \ldots\}, n \in \mathbb{N}$ then for every C_n, $D(C_n) \subseteq C_n$

$\Rightarrow D(C_n) \subseteq C_n \Rightarrow C_n$ is closed

$\Rightarrow \{C_n : n \in \mathbb{N}\}$ be a family of closed sets which satisfies finite intersection property

Also note that

$$\bigcap_{n=1}^{\infty} C_n = \phi$$

$\Rightarrow$ X is not countably compact by previous theorem. Which is a contradiction. So our supposition is wrong. Hence every infinite subset of X has a limit pt.

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Important

Theorem Let "X" & "Y" be two topological spaces then a function $f: X \to Y$ is continuous iff for any subset A of X

$$f(\bar{A}) \subseteq \overline{f(A)}$$

Proof Given f is continuous.

To prove $f(\bar{A}) \subseteq \overline{f(A)}$

As $f(A) \subseteq \overline{f(A)}$ ($\because A \subseteq \bar{A}$ always True

$\Rightarrow A \subseteq f^{-1}(\overline{f(A)})$

As $\overline{f(A)}$ is closed and f is continuous Then $f^{-1}(\overline{f(A)})$ is closed in X [$\because$ inverse img of each closed set is closed]

$\Rightarrow f^{-1}(\overline{f(A)})$ is closed superset of A

$\Rightarrow \bar{A} \subseteq f^{-1}(\overline{f(A)}) \Rightarrow f(\bar{A}) \subseteq \overline{f(A)}$

Conversely suppose

$$f(\bar{A}) \subseteq \overline{f(A)}$$

To prove f is continuous.

Let C be a closed set in Y.

To prove $f^{-1}(C)$ is closed in X.

Let $A = f^{-1}(C)$

$\Rightarrow f(\bar{A}) = f(\overline{f^{-1}(C)}) \subseteq \overline{f(f^{-1}(C))}$

$\Rightarrow f(\bar{A}) \subseteq \bar{C} = C$ ($\because$ C is closed

$\Rightarrow f(\bar{A}) \subseteq C \Rightarrow f^{-1}(f(\bar{A})) \subseteq f^{-1}(C)$

$\Rightarrow \bar{A} \subseteq f^{-1}(C) \Rightarrow \bar{A} \subseteq A$

but $A \subseteq \bar{A} \Rightarrow \bar{A} = A \Rightarrow A$ is closed

$\Rightarrow f^{-1}(C)$ is closed $\Rightarrow f$ is continuous

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⇒ **Sequentially Compact:-** A space X is said to be sequentially compact iff every sequence in X has a convergent sequence

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Theorem:- Every sequentially compact metric space is countably compact.

Proof

Let X be a sequentially compact space.

To prove X is countably compact.

Let $C = \{U_1, U_2, \ldots, U_n, \ldots\}$ is a countable open cover for X.

Suppose X is not countably compact then this open cover has no finite subcover. i.e for every finite subcollection $\{U_1, U_2, \ldots, U_n\}$ of C

$\bigcup_{k=1}^{n} U_k$ does not contains all the points of X. i.e $X \neq \bigcup_{k=1}^{n} U_k$

Let $x_n \in X$ s.t $x_n \notin \bigcup_{k=1}^{n} U_k$

So we have the sequence $\{x_n\}$ in X but X is sequentially compact so this sequence has a convergent subsequence. Say $\{x_{n_k}\}$ s.t $x_{n_k} \longrightarrow x \in X = \bigcup_{i \in N} U_i = U_c$

$\Rightarrow x \in U_p$ for some $U_p \in C$

but for sufficiently large 'k' $x_{n_k} \in U_p$ which is a contradiction. So our supposition is wrong. So X is countably compact.

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Theorem A metric space is sequentially compact iff it satisfies B.W property.

Proof)

Given X is sequentially compact

To prove X satisfies B.W property.

Let A be an infinite subset of X. Now choose $\{x_n\}$ be a sequence in A.

As $A \subseteq X \Rightarrow \{x_n\}$ is a sequence in X.

As X is sequentially compact so this sequence has a convergent subsequence.

Let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}$ s.t $x_{n_k} \longrightarrow x \in X$

Let B be the set of points of the subsequence $\{x_{n_k}\}$ then x is the limit point of B.

As $B \subseteq A \Rightarrow x$ is the limit point of A.

Hence X satisfies B.W property.

Conversely suppose X satisfies B.W property.

To prove X is sequentially compact.

Let $\{x_n\}$ be a sequence in X. If a point x in $\{x_n\}$ repeated infinitely many times then $\{x, x, x, \ldots\} \longrightarrow x$ is a convergent sequence of $\{x_n\}$. If no point repeated infinitely many times.

Let A be the set of points of the sequence then by given condition A has a limit point.

So sequence has a limit point say x

Then we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ s.t $x_{n_k} \longrightarrow x$

$\Rightarrow$ X is sequentially compact.

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Theorem:- Every Compact metric space is sequentially compact.

Proof) Let X be a compact metric space.
To prove X is sequentially compact.
For this we prove X satisfies B.W property
"Because a metric space is sequentially compact iff it satisfies B.W property"
Let A be an infinite subset of X. To prove A has limit point.
Suppose A has no limit point. Then there exist an open ball $B(x, r_x)$ which does not contain any point of A different from x
$\Rightarrow \{B(x, r_x) : x \in X\}$ is an open cover for X
As X is compact so this open cover has a finite subcover say
$\{B(x_1, r_{x_1}), B(x_2, r_{x_2}), \cdots, B(x_n, r_{x_n})\}$
As A may contain only centres of these open balls $\Rightarrow$ A is finite
which is a contradiction. because A is a infinite set.
So our supposition is wrong.
Hence A has a limit point.
$\Rightarrow$ X satisfies B.W property
$\Rightarrow$ X is sequentially compact.

Rough

$$A \subseteq X \subseteq \bigcup_{i=1}^{n} B(x_i, r_{x_i})$$

$$\Rightarrow A \subseteq \bigcup_{i=1}^{n} B(x_i, r_{x_i})$$

$$= \{B(x_1, r_{x_1}), B(x_2, r_{x_2}), \cdots, B(x_n, r_{x_n})\}$$

$$A \subseteq \{x_1, x_2, \cdots, x_n\}$$

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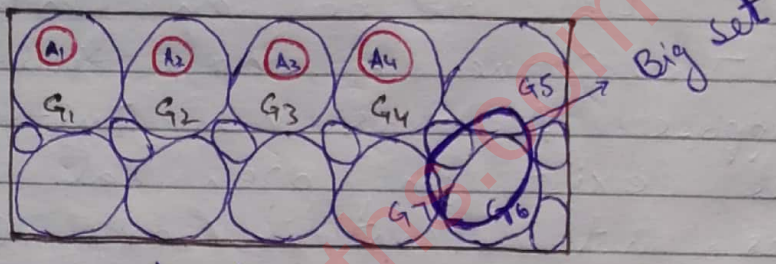
⇒ **Lebesgue Number :-** Let (X,d) be a metric space and $\{G_i\}$ be an open cover for X. A real number $a > 0$ is called Lebesgue number for this open cover $\{G_i\}$ if each subset of X whose diameter is less than "a" is contained in at-least one G_i



⇒ **Big Set :-** A subset of X is called big set if it is not contained in any G_i.

⇒ **Lebesgue Covering Lemma :-**

* **Statement :-** In a sequentially compact metric space every open cover has a Lebesgue number.

Proof Let X be a sequentially compact metric space and $\{G_i\}$ be an open cover for X i.e

$$X = \bigcup_i G_i$$

To prove G_i has a lebesgue number.

Case I :- If big set does not exist.

choose any real number "a" > 0.

As big set does not exist so for every $A \subseteq X$ s.t $d(A) < a$ there is an open set G_i s.t $A \subseteq G_i$

⇒ "a" is a Lebesgue number for $\{G_i\}$

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Case II :- If big sets exists then let $\acute{a}$ be the greatest lower bound of diameters of these big sets then $0 < \acute{a} \leq \infty$

(i) If $\acute{a} = \infty$ then any +ve real number $a > 0$ is the Lebesgue number for the open cover $\{G_i\}$. since for $a > 0$

Let $A \subseteq X$ s.t $d(A) < a < \acute{a}$

$\Rightarrow d(A) < \acute{a}$ $\Rightarrow$ A is not big set

Then there exist atleast one G_i s.t $A \subseteq G_i$

$\Rightarrow$ a is the Lebesgue number for $\{G_i\}$

(ii) If $0 < \acute{a} < \infty$ then choose a real number "a" s.t $0 < a < \acute{a}$ $\Rightarrow a < \acute{a}$

then by condition (i) a is Lebesgue number for $\{G_i\}$

Case III :- If $\acute{a} = 0$ then there exist a big set B_n s.t $d(B_n) < \frac{1}{n}$

Now choose $x_n \in B_n$

$\Rightarrow \{x_n\}$ is a sequence in X

As X is sequentially compact so this sequence has a convergent subsequence $\{x_{n_k}\}$ s.t $x_{n_k} \longrightarrow x \in X = \bigcup_i G_i$

$\Rightarrow x \in \bigcup_i G_i$ $\Rightarrow x \in G_{i_0}$ for some i_0

As G_{i_0} is an open set so there exist an open ball $B(x, r)$ s.t

$x \in B(x, r) \subseteq G_{i_0}$

Let $B(x, r/2)$ be the cocentric open ball to $B(x, r)$ then as x_{n_k} is a convergent sequence and $x_{n_k} \longrightarrow x \in B(x, r/2)$.

So $B(x, r/2)$ contains all the pts of the sequence except a finite number of pts

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Let n_0 be the +ve integer s.t $\frac{1}{n_0} < \frac{r}{2}$ and $x_{n_0} \in B(x, r/2)$

We claim that $B_{n_0} \subseteq B(x, r) \subseteq G_{i_0}$

Let $y \in B_{n_0}$

By Triangle inequality

$d(y, x) \leq d(y, x_{n_0}) + d(x_{n_0}, x)$ ——— ①

As $y, x_{n_0} \in B_{n_0}$ $\quad \because$ As $x_n \in B_n$

$\Rightarrow d(y, x_{n_0}) < d(B_{n_0}) < \frac{1}{n_0}$

$\Rightarrow d(y, x_{n_0}) < \frac{1}{n_0} < \frac{r}{2}$ $\quad (\because \frac{1}{n_0} < \frac{r}{2}$

$\Rightarrow d(y, x_{n_0}) < r/2$

Also $x_{n_0} \in B(x, r/2)$

$\Rightarrow d(x_{n_0}, x) < r/2$

Therefore ① $\Rightarrow d(y, x) < \frac{r}{2} + \frac{r}{2} = r$

$\Rightarrow d(y, x) < r$

$\Rightarrow y \in B(x, r)$

$\Rightarrow B_{n_0} \subseteq B(x, r) \subseteq G_{i_0}$

$\Rightarrow B_{n_0} \subseteq G_{i_0}$

$\Rightarrow B_{n_0}$ is not a big set. which is a contradiction. So this case is not possible. Hence $0 < a' \leq \infty$.

And in this case $\{G_i\}$ has a Lebesgue number.

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01-04-13

ایپسلان نیٹ سیٹ

⇒ **ε-Net set:-** Let (X,d) be a metric space a subset A of X is called an ε-net set if A is finite and $X = \bigcup_{a \in A} B(a,\varepsilon)$

⇒ **Totally Bounded Set:-** A metric space (X,d) is said to be totally bounded if it has an ε-net set for each $\varepsilon > 0$

Theorem:- A sequentially compact metric space is totally bounded.

Proof) Let (X,d) be a sequentially compact metric spae.

To prove X is totally bounded

Let $\varepsilon > 0$ and $a_1 \in X$

If $B(a_1,\varepsilon)$ contains all the points of X then $\{a_1\}$ is an ε-net set.

If there exist some points out side the open ball $B(a_1,\varepsilon)$. Let a_2 be the such a point i.e $a_2 \in X \setminus B(a_1,\varepsilon)$

If $X = B(a_1,\varepsilon) \cup B(a_2,\varepsilon)$

Then $\{a_1, a_2\}$ is an ε-net set.

If not then again choose a point $a_3 \in X \setminus B(a_1,\varepsilon) \cup B(a_2,\varepsilon)$.

If $X = B(a_1,\varepsilon) \cup B(a_2,\varepsilon) \cup B(a_3,\varepsilon)$

then $\{a_1, a_2, a_3\}$ is an ε-net set.

If not then continuing this process we have

So we union

$B(a_1,\varepsilon) \cup B(a_2,\varepsilon) \cup \cdots \cup B(a_n,\varepsilon)$ —①

will contains necessarily all the points

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of X. If not then the sequence $\{a_1, a_2, \ldots, a_n\}$ has no convergent subsequence.
$\Rightarrow$ X is not ~~necessarily~~ sequentially compact.
which is a contradiction.
So union ① will contain all the points of X and Hence. $\{a_1, a_2, \ldots, a_n\}$ is an ε-net set
$\Rightarrow$ X is totally bounded.

Theorem:- Every sequentially compact metric space is compact.

Proof:-

Let X be a sequentially compact metric space.

To prove X is compact.

Let $\{G_i\}$ be an open cover for X.

i.e $X = \bigcup_i G_i$

As $\{G_i\}$ is an open cover for X and X is sequentially compact so by Lebesgue covering lemma $\{G_i\}$ has a lebesgue number say a

Put $\varepsilon = a/3 > 0$

As every sequentially compact metric space is totally bounded so X is totally bounded.

Then for $\varepsilon > 0$ it has an ε-net set

$A = \{x_1, x_2, x_3, \ldots, x_n\}$

$\therefore X = \bigcup_{k=1}^{n} B(a_k, \varepsilon)$

Now $d(B(a_k, \varepsilon)) \leq 2\varepsilon \quad [\because d(\overline{B}(x, r) \leq 2r)]$

$\Rightarrow d(B(a_k, \varepsilon)) < \frac{2a}{3} < a$

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$\Rightarrow d(B(a_k, \varepsilon)) < a$

Then by definition of Lebesgue number

$B(a_k, \varepsilon) \subseteq G_{i_k}$

where $G_{i_k} \in \{G_i\}$

$\Rightarrow \bigcup_{k=1}^{n} B(a_k, \varepsilon) \subseteq \bigcup_{k=1}^{n} G_{i_k}$

$\Rightarrow X \subseteq \bigcup_{k=1}^{n} G_{i_k} \subseteq X$

$\Rightarrow X = \bigcup_{k=1}^{n} G_{i_k}$

$\Rightarrow \{G_{i_k} : 1 \leq k \leq n\}$ is a finite subcover for X. Hence X is compact.

Theorem:- Every totally bounded metric space is bounded.

Proof: Let (X, d) be a totally bounded metric space. To prove X is bounded

As X is totally bounded then for $\varepsilon > 0$ X has an ε-net

Say $A = \{a_1, a_2, a_3, \cdots, a_n\}$

i.e $X = \bigcup_{i=1}^{n} B(a_i, \varepsilon)$

So then for every $x \in X = \bigcup_{i=1}^{n} B(a_i, \varepsilon)$

$\Rightarrow x \in B(a_i, \varepsilon)$ for some i

Similarly if $y \in X$

$\Rightarrow y \in B(a_j, \varepsilon)$ for some j

$\Rightarrow d(y, a_j) < \varepsilon$

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Now $d(x,y) \leq d(x,a_i) + d(a_i,y)$

$\leq d(x,a_i) + d(a_i,a_j) + d(a_j,y)$

$\Rightarrow d(x,y) < \varepsilon + d(a_i,a_j) + \varepsilon$

Taking Supremum

$\sup d(x,y) < 2\varepsilon + d(a_i,a_j)$

$\Rightarrow \delta(X) < 2\varepsilon + d(a_i,a_j)$

As R.H.S is finite

$\Rightarrow$ L.H.S is finite

$\Rightarrow \delta(X)$ is finite

$\Rightarrow X$ is bounded.

Theorem:- A continuous function from a compact metric space is uniformly continuous.

Proof:- Let $f: X \to Y$ be a continuous mapping from a compact metric space X to metric space Y.

To prove f is uniformly continuous.

Let d_1 be metric space on X & d_2 be metric on Y.

Let $x \in X$ then $f(x) \in Y$

Now consider the open ball $B(f(x), \varepsilon/2)$ which is open set in Y.

As f is continuous $\therefore f^{-1}[B(f(x), \varepsilon/2)]$ is open in X

$\Rightarrow \{f^{-1}(B(f(x), \varepsilon/2)) : x \in X\}$ is open cover for X.

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As X is compact & every compact metric space is sequentially compact so X is sequentially compact. & in a sequentially compact metric space every open cover has a Lebesgue number. So this open cover has a Lebesgue number. say δ

Let $x_1, x_2 \in X$ s.t $d_1(x_1, x_2) < \delta$

diameter $\Rightarrow d(\{x_1, x_2\}) < \delta$

Then by condition of Lebesgue number

$\{x_1, x_2\} \subseteq f^{-1}[B(f(x^*), \varepsilon/2)]$

where $f^{-1}[B(f(x^*), \varepsilon/2)] \in$ open cover

$\Rightarrow f\{x_1, x_2\} \subseteq B(f(x^*), \varepsilon/2)$

$\Rightarrow \{f(x_1), f(x_2)\} \subseteq B(f(x^*), \varepsilon/2)$

$f(x_1) \in B(f(x^*), \varepsilon/2)$ & $f(x_2) \in B(f(x^*), \frac{\varepsilon}{2})$

$\Rightarrow d_2(f(x_1), f(x^*)) < \frac{\varepsilon}{2}$ and

$d_2(f(x_2), f(x^*)) < \frac{\varepsilon}{2}$

Now $d_2[f(x_1), f(x_2)] \leq d_2(f(x_1), f(x^*)) + d_2(f(x^*), f(x_2))$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow d_2[f(x_1), f(x_2)] < \varepsilon$

$\Rightarrow f$ is uniformly continuous

$f(A) = \{f(x) : x \in A\}$

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CHAPTER # 4

CONNECTED SPACES.

⇒ **Disconnected:-** A topological space (X, τ) is said to be disconnected if there exists non-empty open (or closed) sets A and B in X such that $A \cup B = X$ & $A \cap B = \phi$

e.g If $X = \{1,2,3,4\}$, $\tau = \{\phi, X, \{1,3\}, \{2,4\}\}$
Then (X, τ) is disconnected b/c we have $A = \{1,3\}$, $B = \{2,4\}$. A & B are open

$A \cup B = X$, $A \cap B = \phi$

⇒ **Connected Space:-** A topological space (X, τ) is said to be connected if it is not disconnected e.g if $X = \{1,2,3\}$, $\tau = \{\phi, X, \{2\}\}$ then (X, τ) is connected.

Remark:- For any set X if $\tau = \tau_I$ (Indiscrete). then (X, τ) is connected.

Theorem:- For any X with more than Points (X, τ_D) is disconnected.

Proof

Since $\tau = \tau_D$, so every subset of X is open (as well as closed)

Proper subset Let $A \subset X$ then $A' = X \setminus A \neq \phi$

Also A' is open ($\because$ A is also closed)

So we have two open sets A & $B = A'$ such that $A \cup B = X$ & $A \cap B = \phi$

So (X, τ_D) is disconnected

Hence proved.

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Theorem:- If X is infinite then (X, τ_c) is connected. (τ_c: cofinite)

Proof:- On the contrary suppose (X, τ_c) is disconnected.

Then $\exists$ two non-empty open sets (or closed sets) A & B s.t $A \cup B = X$ & $A \cap B = \phi$

Now as A & B are open & $\tau = \tau_c$

So A' & B' are finite.

Now $A \cap B = \phi$

$\Rightarrow (A \cap B)' = X \Rightarrow A' \cup B' = X$

$\Rightarrow X$ is finite (∵ union of two finite sets is finite) which is a contradiction

∴ X is infinite

So, our supposition is wrong

Hence (X, τ_c) is connected.

Theorem:- Continuous image of a connected space is connected.

Proof:-

Let X be a connected space & $f: X \to Y$ be a continuous function.

To prove $f(X)$ is connected

Suppose on the contrary that $f(X)$ is disconnected. Then there exist two open sets (non empty) A & B in $f(X)$ s.t

$A \cup B = f(X)$ & $A \cap B = \phi$

Now $A \cup B = f(X)$

Now as A & B are open in $f(X)$ & f is continuous function. So $f^{-1}(A)$ & $f^{-1}(B)$ are open in X.

Further $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B)$

$= f^{-1}(f(X)) = X$

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& $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\phi) = \phi$

$\Rightarrow$ X is disconnected

which is a contradiction b/c X is connected. So, our supposition is wrong

Hence $f(X)$ is connected.

Theorem: The space $\mathbb{Q}$ as subspace of $\mathbb{R}$ is disconnected.

Proof: Let r be any irrational number

Then, $]-\infty, r[$, $]r, \infty[$ are open in $\mathbb{R}$

$\Rightarrow]-\infty, r[\cap \mathbb{Q}$, $]r, \infty[\cap \mathbb{Q}$ are open sets in $\mathbb{Q}$

with $(]-\infty, r[\cap \mathbb{Q}) \cup (]r, \infty[\cap \mathbb{Q})$

$= (]-\infty, r[\cup]r, \infty[) \cap \mathbb{Q}$ By distributive law

$= (\mathbb{R} \setminus \{r\}) \cap \mathbb{Q}$

$= \mathbb{Q}$

& $(]-\infty, r[\cap \mathbb{Q}) \cap (]r, \infty[\cap \mathbb{Q})$

$= (]-\infty, r[\cap]r, \infty[) \cap \mathbb{Q}$

$= \phi \cap \mathbb{Q} = \phi$

$\Rightarrow \mathbb{Q}$ is disconnected.

Theorem: A topological space X is disconnected iff X contains a non empty subset A which is both open & closed.

Proof: Suppose X is disconnected & $A \neq \phi$ be a subset of X

To prove A is both open & closed

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As X is disconnected then there exist two open sets A & B s.t $A \cup B = X$ and $A \cap B = \phi$

Now A is open

Now as $A \cup B = X$ & $A \cap B = \phi$ so by Law of complements $B = A'$

Also B is open $\Rightarrow$ A' is open

$\Rightarrow$ A is closed

$\Rightarrow$ A is both open & closed.

Conversely suppose that for topological space X, there is a non empty subset A of X which is both open & closed.

To prove X is disconnected.

Let $B = A'$. As A is closed

$\Rightarrow$ A' is open $\Rightarrow$ B is open

$\Rightarrow$ A & B are open in X with [$\because$ also A is open]

$A \cup B = A \cup A' = X$ &

$A \cap B = A \cap A' = \phi$

$\Rightarrow$ X is disconnected.

Theorem: A space X is connected if & only if there does not exist a continuous surjective function from X to discrete two point space.

Proof:

Let X be connected

To prove there does not exist a continuous surjective function from X to discrete two point space $Y = \{a, b\}$

Suppose on the contrary that there exist a function $f: X \to Y = \{a, b\}$, Y

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is discrete, which is continuous & onto.
As Y is discrete so $\phi, \{a\}, \{b\}, \{a,b\}$ are open sets.

As f is continuous so $f^{-1}(\phi)$, $f^{-1}(\{a\})$, $f^{-1}(\{b\})$ & $f^{-1}(\{a,b\})$ all are open in X
Now as f is onto so $f(X) = Y \Rightarrow X = f^{-1}(Y)$

$$\Rightarrow X = f^{-1}(\{a,b\})$$
$$= f^{-1}(\{a\} \cup \{b\})$$
$$= f^{-1}(\{a\}) \cup f^{-1}(\{b\})$$

Further $f^{-1}(\{a\}) \cap f^{-1}(\{b\})$
$$= f^{-1}(\{a\} \cap \{b\}) = f^{-1}(\phi) = \phi$$

$\Rightarrow$ X is disconnected
which is contradiction $\because$ X is connected
So our supposition is wrong.
Hence, there does not exist a continuous function from X onto discrete two point space Y.

Conversely Suppose there does not exist a continuous function from X onto discrete two point space Y.

To prove X is connected.
Suppose on the contrary that X is not connected.
Then X is disconnected. So $\exists$ two non-empty open (or closed) sets A & B in X such that $A \cup B = X$ & $A \cap B = \phi$

Now define a function
$f: X \to Y = \{a,b\}$ by $f(A) = a$ & $f(B) = b$

$\Rightarrow A = f^{-1}(\{a\})$, $B = f^{-1}(\{b\})$

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Now as $Y = \{a, b\}$ is discreate so $\phi, \{a\}, \{b\}, \{a, b\}$ are open in Y.

Now $f^{-1}(\phi) = f^{-1}(\{a\} \cap \{b\})$

$= f^{-1}(\{a\}) \cap f^{-1}(\{b\})$

$= A \cap B = \phi$

$f^{-1}(Y) = f^{-1}(\{a\} \cup \{b\})$

$= f^{-1}(\{a\}) \cup f^{-1}(\{b\})$

$= A \cup B = X$

So f is continuous (∵ inverse image of each open set is open & here all four open sets of Y have open inverse images)

which is contradiction.

So our supposition is wrong

Hence X is connected.

Theorem:- A topological space X is disconnected iff there exist a continuous function from X onto discrete two point space.

Proof:- Above theorem with amendment

Theorem:- A topological space X is said to be connected iff every continuous function from X to discrete two point space Y reduces to a constant func

Proof Suppose X is connected

Then X has no proper subset which is both open & closed.

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Let $a \in Y$ as Y is discrete so $\{a\}$ is both open & closed

$\Rightarrow f^{-1}\{a\}$ is open & closed in X

$\Rightarrow f^{-1}\{a\}$ is not a proper subset of X

$\Rightarrow f^{-1}\{a\} = \phi$ or $f^{-1}\{a\} = X$

But $f^{-1}\{a\} \neq \phi$

$\Rightarrow f^{-1}\{a\} = X \Rightarrow f(X) = \{a\}$

$\Rightarrow f$ is constant function

Conversely suppose every continuous function f from X to discrete space Y reduces to a constant funct.

To prove X is connected

Suppose X is disconnected. Then there exist a constant function $f: X \to Y = \{a, b\}$ which is continuous & onto & Y is discrete

$\Rightarrow f(X) = Y \Rightarrow f$ is not constant which is contradiction.

So our supposition is wrong, & Hence X is connected.

Theorem:- Let X be disconnected space with disconnection $\{A, B\}$ and C is a connected subspace of X. Then either $C \subseteq A$ or $C \subseteq B$.

Proof)

Suppose on the contrary that $C \not\subseteq A$ & $C \not\subseteq B$. Then $C \cap A$ & $C \cap B$ are non empty open sets in C. $\because C$ is subspace so $C \cap A$ is open in C with

$$(C \cap A) \cup (C \cap B) = C \cap (A \cup B)$$
$$= C \cap X = C$$

& $$(C \cap A) \cap (C \cap B) = C \cap (A \cap B)$$
$$= C \cap \phi = \phi$$

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$\Rightarrow$ C is disconnected
A contradiction $\because$ C is connected.
So our Supposition is wrong
Hence $C \subseteq A$ or $C \subseteq B$

Theorem Let $X = \bigcup X_\alpha$ where each X_α is connected & $\bigcap_{\alpha \in I} X_\alpha \neq \phi$. Then, X is connected.

Proof Suppose X is disconnected then $\exists$ two non-empty open sets A & B in X such that $A \cup B = X$ & $A \cap B = \phi$
Now as $X = \bigcup_{\alpha \in I} X_\alpha$
$\Rightarrow$ For each $\alpha \in I$, $X_\alpha \subseteq X$.
As for each $\alpha \in I$, X_α is connected
So by well know theorem for each $\alpha \in I$
either $X_\alpha \subseteq A$ or $X_\alpha \subseteq B$
But as $\bigcap_{\alpha \in I} X_\alpha \neq \phi$

So $\bigcup_{\alpha \in I} X_\alpha \subseteq A$ or $\bigcup_{\alpha \in I} X_\alpha \subseteq B$

$\Rightarrow X \subseteq A$ or $X \subseteq B$

If $X \subseteq A \Rightarrow A = X \Rightarrow B = \phi$
If $X \subseteq B \Rightarrow B = X \Rightarrow A = \phi$
which is a contradiction. because both A & B are non empty.
So our suppo sition is wrong And Hence X is connected.

Theorem: Suppose X is connected and $x, y \in X$ such that $x \neq y$. To prove there is some connected subspace of X which contains both x & y. Then X itself

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Theorem:- A topological space X is connected iff for every pair of points in X there is some connected subspace of X which contains both.

Proof:- Suppose X is connected and $x, y \in X$ such that $x \neq y$. To prove there is some connected subspace of X which contains both x and y. Then X itself is the connected subspace of X which contains both x and y.

Conversely suppose in a topological space X, for every pair of points $x, y \in X$ such that $x \neq y$, there is some connected subspace of X which contains both x and y.

To prove X is connected.

Now let, $a \in X$ be some fixed point such that for $x \in X$, $a \neq x$. Then by hypothesis, there is a connected subspace $C_{a,x}$ of X s.t $a, x \in C_{a,x}$.

Then we have a collection $\{C_{a,x} : x \in X\}$ of connected subspaces of X such that

$$\bigcap_{x \in X} C_{a,x} \neq \phi \quad \& \quad \bigcup_{x \in X} C_{a,x} = X$$

Then by a well known theorem, X is connected.

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Theorem: Let C be a connected subspace of X and for some subset A of X, $C \subseteq A \subseteq \bar{C}$. Then A is connected. In particular $\bar{C}$ is connected.

Proof: Suppose on the contrary that A is disconnected. Then there exist two non-empty open sets U_1 & V_1 of A s.t

$$U_1 \cup V_1 = A \quad \& \quad U_1 \cap V_1 = \phi$$

As U_1 & V_1 are open in A & A is subspace of X then there exist two disjoint open sets U & V in X s.t

$$U_1 = U \cap A \quad \& \quad V_1 = V \cap A$$

Now $$C \subseteq A = U_1 \cup V_1 \subseteq U \cup V$$

$\Rightarrow C \subseteq U \cup V$ and C is connected and $U \cap V = \phi$ then by a well know theorem

$$C \subseteq U \quad \text{or} \quad C \subseteq V$$

Without any loss of generality, suppose

$$C \subseteq U$$

As $U \cap V = \phi \Rightarrow U \subseteq V'$

$\Rightarrow C \subseteq U \subseteq V' \Rightarrow C \subseteq V'$

As V is open $\Rightarrow V'$ is closed

So V' is the closed superset of C. But $\bar{C}$ is the smallest closed super set of C

So $\bar{C} \subseteq V'$

$\Rightarrow C \subseteq A \subseteq \bar{C}$ (Given)

$\Rightarrow C \subseteq A \subseteq \bar{C} \subseteq V' \Rightarrow A \subseteq V'$

$\Rightarrow A \cap V = \phi \Rightarrow V_1 = \phi$

which is a contradiction $\because V_1 \neq \phi$

So our supposition is wrong.

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Hence A is connected.
Now to prove $\bar{C}$ is connected
As $C \subseteq \bar{C} \subseteq \bar{C}$, so by the above argument $\bar{C}$ is connected.

Theorem: A subspace X of a real line $\mathbb{R}$ is connected iff X is an interval.

Proof: Suppose X is connected.
To prove X is an interval.
Suppose X is not an interval, then there exist x, y, z such that $x < y < z$ and $x, z \in X$ but $y \notin X$
Now $]-\infty, y[$ and $]y, \infty[$ are open in $\mathbb{R}$

$\Rightarrow]-\infty, y[\cap X$ & $]y, \infty[\cap X$ are open in X with

$$(]-\infty, y] \cap X) \cup (]y, \infty[\cap X)$$
$$= (]-\infty, y[\cup]y, \infty[) \cap X = (\mathbb{R} \setminus \{y\}) \cap X$$
$$= X$$

& $(]-\infty, y[\cap X) \cap (]y, \infty[\cap X) = (]-\infty, y[\cap]y, \infty[) \cap X$
$$= \phi \cap X = \phi$$

$\Rightarrow$ X is disconnected. Which is a contradiction. So our supposition is wrong
Hence X is an ~~open~~ interval.
Conversely suppose X is an interval
To prove X is connected. On the contrary suppose X is disconnected
Then there exist two non empty

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closed
open disjoint subsets A & B of X such that $A \cup B = X$ & $A \cap B = \phi$

Let $a \in A$ & $b \in B$

As $A \cap B = \phi \Rightarrow a \neq b$

Let $a < b$. put $y = \sup([a,b] \cap A)$

Then by the definition of supremum for every $\varepsilon > 0$ there is some point a' in A such that $y - \varepsilon < a'$

$\Rightarrow y - a' < \varepsilon \Rightarrow d(y, a') < \varepsilon$

$\Rightarrow a' \in B(y, \varepsilon)$

So every open ball with centre at y contains a point of A different from y

$\Rightarrow$ y is the limit point of A.

As A is also closed

So $y \in A$, similarly $y \in B \Rightarrow A \cap B \neq \phi$ which is a contradiction $\therefore A \cap B = \phi$

So our supposition is wrong

Hence X, is connected.

$\Rightarrow$ **Component:-** The maximal connected subspace of topological space X is called component of X. i.e a connected subspace of topological space X is called component of X if it is not contained in any other connected subspace of X.

Theorem Let X be a topological space then

(i) Each $x \in X$ is ~~connected~~ contained in exactly one component of X.

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(ii) Each connected subspace of X is contained in exactly one component of X.

(iii) Each connected subspace of X which is both open & closed is component of X

(iv) Every component of X is closed in X.

Proof) 1) Let $\gamma = \{C_\alpha : \alpha \in I$ and $x \in C_\alpha\}$ be a collection of connected subspaces of X which contains x.

Then, $\bigcap_{\alpha \in I} C_\alpha \neq \phi$

Then by a well known theorem,

$C = \cup\gamma = \cup_{\alpha \in I} C_\alpha$ is connected subspace of X and $x \in C$ and for every $\alpha \in I$, $C_\alpha \subseteq C$. This shows that C is component of X.

Now we show that C is the only component of X containing x. On the contrary let C^* be another component of X containing x.

Now as C^* is the component of X containing x and C is connected subspace of X. So $C \subseteq C^*$. Also as C^* is connected subspace of X containing x so $C^* \subseteq \gamma$

$\Rightarrow C^* \subseteq \cup\gamma = C$

$\Rightarrow C^* \subseteq C \Rightarrow C = C^*$

This shows that C is the only component of X containing x.

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(ii) Let A be a connected subspace of X and to prove A is contained only in one component of X. Let $\{C_\alpha : \alpha \in I\}$ be a collection of all connected subspaces of X containing A. Then

$\bigcap_{\alpha \in I} C_\alpha \neq \phi$ and $\bigcup_{\alpha \in I} C_\alpha = C$, which is connected subspace of X, Also $A \subseteq C$

$\Rightarrow$ C is connected subspace of X containing A. Also $C = \bigcup_{\alpha \in I} C_\alpha$ so C is such maximal connected subspace of X.

$\Rightarrow$ C is component of X containing A. Now we show C is the only component of X containing A.

For this, let C^* be another component of X containing A. Now, as C^* is maximal connected subspace of X containing A and C is connected subspace of X containing A. So $C \subseteq C^*$ Further also as C^* is connected subspace of X containing A. so $C^* \in \{C_\alpha : \alpha \in I\}$

$$\Rightarrow C^* \subseteq \bigcup_{\alpha \in I} C_\alpha = C$$

$\Rightarrow C^* \subseteq C \Rightarrow C = C^*$

Hence C is only component of X containing A.

(iii) Let A be a connected subspace of X which is both open & closed

To prove A is component of X.

Suppose A is not component of X. Then A is contained in exactly one component of X, say C. As C is

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component of X, $A \subseteq C$ and A is not component of X

$\Rightarrow$ A is proper subset of C. Then $A \cap C$ and $A' \cap C$ are both non-empty. Now as A is open in X, so $A \cap C$ is open in C. Also as A is closed in X $\Rightarrow$ A' is open in X

$\Rightarrow A' \cap C$ is open in C

$\Rightarrow A \cap C$ and $A' \cap C$ are open in C with

$$(A \cap C) \cup (A' \cap C) = (A \cup A') \cap C$$
$$= X \cap C$$
$$= C$$

& $$(A \cap C) \cap (A' \cap C) = (A \cap A') \cap C$$
$$= \phi \cap C$$
$$= \phi$$

$\Rightarrow$ C is disconnected. Which is a contradiction $\therefore$ C is component of X

So our supposition is wrong

Hence, A is component of X.

(iv) Let C be a component of X

To prove C is closed. For This we prove $C = \bar{C}$

Suppose $C \neq \bar{C}$

Now as $C \subseteq \bar{C}$ and $C \neq \bar{C}$

$\Rightarrow C \subset \bar{C}$

Now as C is connected, then by a well know theorem, $\bar{C}$ is connected

$\Rightarrow \bar{C}$ is connected subspace of X containing C $\Rightarrow$ C is not component of X. A contradiction $\therefore$ C is component

So our supposition is wrong. Hence $C = \bar{C}$

$\Rightarrow$ C is closed.

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X with discrete topology is called discrete space.

⇒ **Totally Disconnected:-** A topological space X is called totally disconnected if for each pair of component $x, y \in X$ we can form a disconnection $\{A, B\}$ of X such that $x \in A$ & $y \in B$

Theorem Every discrete space is totally disconnected.

Proof

Let X be a discrete space.
To prove X is totally disconnected
Let $x, y \in X$ s.t $x \neq y$.
Let $U = \{x\}$ and $V = X - \{x\}$
As X is discrete so U and V are open in X, Also clearly,
$x \in X$, $y \in V$, $U \cap V = \phi$ & $U \cup V = X$
⇒ X is totally disconnected.

Theorem Every totally disconnected is T_2-space.

Proof

Let X be a totally disconnected.
To prove X is T_2- space.
Let $x, y \in X$ s.t $x \neq y$. As X is totally disconnected so then there exist two open sets U and V in X s.t
$x \in U$, $y \in V$, $U \cap V = \phi$ & $U \cup V = X$
i.e we have two open sets U and V in X such that $x \in U$ & $y \in V$ and
$U \cap V = \phi$
⇒ X is T_2- space.

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Theorem:- A subspace $\mathbb{Q}$ of rationals in real line $\mathbb{R}$ is totally disconnected.

Proof:-

To prove $\mathbb{Q}$ is totally disconnected in $\mathbb{R}$. Let $r_1, r_2 \in \mathbb{Q}$ such that $r_1 \neq r_2$. Without any loss of generality, suppose $r_1 < r_2$

Now as by a well known theorem of calculas Here is an irrational number between every two rational numbers. There is an irrational number "t" such that $r_1 < t < r_2$. Now $]-\infty, t[$ and $]t, \infty[$ are two open sets in $\mathbb{R}$. Now as $\mathbb{Q}$ is subspace of $\mathbb{R}$. So

$U = \mathbb{Q} \cap]-\infty, t[$ and $V = \mathbb{Q} \cap]t, \infty[$ are open in $\mathbb{Q}$

Also $r_1 \in U$, $r_2 \in V$, $U \cap V = \phi$,

$U \cup V = \mathbb{Q}$

$\Rightarrow \mathbb{Q}$ is totally disconnected. proved.

Theorem:- The Components of totally disconnected space are its singlton subsets.

Proof:-

Let X be a totally disconnected space. To prove components of X are its singlton subsets.

For this we show that no two points subspace of X is connected

Let $x, y \in X$ such that $x \neq y$

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and $C = \{x, y\}$ be a subspace of X.
As X is totally disconnected and $x, y \in X$ such that $x \neq y$, so then there exist two open sets U and V in X such that
$x \in U$, $y \in V$, $U \cup V = X$, $U \cap V = \phi$
Now as U and V are open in X and C is subspace of X so
$U \cap C$ and $V \cap C$ are open in C.
Also, $x \in U$ and $x \in C \Rightarrow x \in U \cap C$
& $y \in V$, $y \in C \Rightarrow y \in V \cap C$

$$(U \cap C) \cup (V \cap C) = (U \cup V) \cap C$$
$$= X \cap C = C$$
$$(U \cap C) \cap (V \cap C) = (U \cap V) \cap C$$
$$= \phi \cap C = \phi$$

$\Rightarrow C$ is disconnected.

Theorem: If a T_2-space has an open base whose sets are also closed Then, X is totally disconnected.

Proof: Let $x, y \in X$ such that $x \neq y$. As X is T_2-space. so then, there exist two open sets U and V in X s.t
$x \in U$, $y \in V$ & $U \cap V = \phi$
Let $\mathcal{B}$ be an open base of X whose elements are also closed.
As $x \in U$, U is an open set, $\mathcal{B}$ is base, so then there is $B \in \mathcal{B}$ s.t
$x \in B \subseteq U$
Now as $B \subseteq U$ & $U \cap V = \phi$
so $B \cap V = \phi \Rightarrow V \subseteq B' = W$

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As $B \in \beta$, so B is closed
$\Rightarrow B'$ is open. Now B and W are two open sets in X with $x \in B$. As $y \notin B$
$\Rightarrow y \in B' = W \Rightarrow y \in W$
Also $B \cap W = B \cap B' = \phi$
$B \cup W = B \cup B' = X$
$\Rightarrow X$ is totally disconnected.

Paper
Theorem:- Let X and Y be two topological space, then a function $f: X \to Y$ is continuous iff for every subset A of X, $f(\bar{A}) \subseteq \overline{f(A)}$

Proof: Given, f is continuous & $A \subseteq X$
To prove: $f(\bar{A}) \subseteq \overline{f(A)}$
As $f(A) \subseteq \overline{f(A)} \Rightarrow A \subseteq f^{-1}(\overline{f(A)})$
As f is continuous and $\overline{f(A)}$ is closed in Y. Then $f^{-1}(\overline{f(A)})$ is closed in X [$\because f$ is continuous iff inverse image of each closed set is closed]
$\Rightarrow A \subseteq f^{-1}(\overline{f(A)})$ & $f^{-1}(\overline{f(A)})$ is closed subset of X
$\Rightarrow f^{-1}(\overline{f(A)})$ is the closed superset of A
But $\bar{A}$ is the smallest closed superset of A.
So $\bar{A} \subseteq f^{-1}(\overline{f(A)})$
$\Rightarrow f(\bar{A}) \subseteq \overline{f(A)}$

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Conversely suppose for any subset A of X,

$$f(\bar{A}) \subseteq \overline{f(A)}$$

To prove f is continuous

Let C be a closed set in Y. To prove f is continuous we have to prove $f^{-1}(C)$ is closed in X.

Let $A = f^{-1}(C)$

$\bar{A} = f^{-1}(C) \Rightarrow f(\bar{A}) = f(f^{-1}(C))$

$\subseteq \overline{f(f^{-1}(C))}$ [By given condition

$\Rightarrow f(\bar{A}) \subseteq \overline{(ff^{-1})(C)} = \overline{I(C)}$

$= \bar{C} = C$ [$\because$ C is closed

$\Rightarrow f(\bar{A}) \subseteq C \Rightarrow f^{-1}(f(\bar{A})) \subseteq f^{-1}(C)$

$\Rightarrow \bar{A} \subseteq A \quad (\because f^{-1}(C) = A)$

But $A \subseteq \bar{A} \Rightarrow A = \bar{A} \Rightarrow A$ is closed in X

$\Rightarrow f^{-1}(C)$ is closed in X.

$\Rightarrow f$ is continuous.

Definition:- Let X be a topological space and A & B are subsets of X. then A & B are said to be separated if and only if $A \cap \bar{B} = \phi$ & $\bar{A} \cap B = \phi$

Theorem:- Let X be a topological space and A, B are the subsets of X. if A & B are separated in X then $A \cup B$ is disconnected.

Proof: Let $Y = A \cup B$

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Now as, A & B are separated in X,
so $A \cap \bar{B} = \phi$ and $\bar{A} \cap B = \phi$
Now let $G = \bar{B}'$ & $H = \bar{A}'$
Then as $\bar{A}$ & $\bar{B}$ are closed
$\Rightarrow \bar{A}'$ & $\bar{B}'$ are open
$\Rightarrow$ H and G are open in X
$\Rightarrow Y \cap G$ and $Y \cap H$ are open in Y
[$\because$ Y is subspace

Now $A \cap \bar{B} = \phi \Rightarrow A \subseteq \bar{B}' \Rightarrow A \subseteq G$
Further as $B \subseteq \bar{B} \Rightarrow B \cap \bar{B}' = \phi$

Now $Y \cap G = (A \cup B) \cap G = (A \cap G) \cup (B \cap G)$ | $\because Y = A \cup B$
$\Rightarrow Y \cap G = A \cup (B \cap \bar{B}') = A \cup \phi$ [$\because G = \bar{B}'$ & $A \subseteq G$
$= \cancel{\phi}\ A$
Similarly $Y \cap H = B$

Now, $Y \cap G$ and $Y \cap H$ are open in Y with
$(Y \cap G) \cup (Y \cap H) = A \cup B = Y$
& $(Y \cap G) \cap (Y \cap H) = A \cap B = \phi$
$\Rightarrow$ Y is disconnected.

Theorem: Let G and H be the disconnection of a subset A of a topological space X then, show that $A \cap G$ and $A \cap H$ are separated.

Proof:
To prove $A \cap G$ and $A \cap H$ are separated.
i.e $(A \cap G) \cap \overline{(A \cap H)} = \phi$ & $\overline{(A \cap G)} \cap (A \cap H) = \phi$
First we prove, if $x \in D(A \cap G)$ then, $x \notin A \cap H$.
Suppose on the contrary

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$x \in D(A \cap G) \Rightarrow x \in A \cap H$

$\Rightarrow x \in A$ and $x \in H$

Now, $A \cap G \subseteq G$ and $A \cap G \subseteq A$

$\Rightarrow D(A \cap G) \subseteq D(G)$ & $D(A \cap G) \subseteq D(A)$

So $x \in D(A \cap G) \Rightarrow x \in D(G) \Rightarrow x \in G$ [$\because G$ is closed]

Now $x \in H$ & $x \in G \Rightarrow x \in G \cap H$

$\Rightarrow G \cap H \neq \phi$

Which is a contradiction. So, our supposition is wrong. Hence for $x \in D(A \cap G) \Rightarrow x \notin A \cap H$

$\Rightarrow D(A \cap G) \cap (A \cap H) = \phi$

Also, $(A \cap G) \cap (A \cap H) = A \cap (G \cap H)$

$= A \cap \phi = \phi$

$\Rightarrow [D(A \cap G) \cap (A \cap H)] \cup [(A \cap G) \cap (A \cap H)] = \phi$

$\Rightarrow [D(A \cap G) \cup (A \cap G)] \cap (A \cap H) = \phi$ · [$(X \cap Z) \cup (Y \cap Z) = (X \cup Y) \cap Z$]

$\Rightarrow \overline{(A \cap G)} \cap (A \cap H) = \phi$

Similarly, $(A \cap G) \cap \overline{(A \cap H)} = \phi$

$\Rightarrow (A \cap G)$ and $(A \cap H)$ are separated.

Theorem: Show that a topological space X is connected iff every non-empty proper subspace has a non-empty boundry.

Proof: We know that

i) A topological space X is disconnected iff it has a subset A which is both open an closed

ii) If (X, τ) is a topological space & $A \subseteq X$ then boundry of A is empty iff A is both open and closed.

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Now given X is connected and A is non empty proper subset of X.

To prove boundry of A is non-empty. Suppose boundry of A is empty. i.e $b(A) = \phi$ then by (ii) A is both open and closed, but by (i) X is disconnected.

Which is a contradiction. So our supposition is wrong and Hence,

$$b(A) \neq \phi$$

Conversely suppose in a topological space X every non-empty proper subset of X has non-empty boundry.

To prove X is connected

Suppose X is disconnected then by (i), there is subset A of X, which is both open and closed. Then by (ii)

$b(A) = \phi$. A contradiction b/c $b(A) \neq \phi$. So, our supposition is wrong. And Hence

X is connected.

Theorem: If X and Y are connected topological spaces, then $X \times Y$ is also connected.

Proof: Let $x \in X$ & $y \in Y$

Then $\{x\} \times Y$ and $X \times \{y\}$ are two topological spaces with $\{x\} \times Y \cong Y$ and $X \times \{y\} \cong X$

$\Rightarrow$ As X and Y are connected So $\{x\} \times Y$ and $X \times \{y\}$ are connected

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for all $x \in X$ and $y \in Y$

Also $(x,y) \in (\{x\} \times Y) \cap (X \times \{y\})$

$\Rightarrow (\{x\} \times Y) \cap (X \times \{y\}) \neq \phi$

$\Rightarrow (\{x\} \times Y) \cup (X \times \{y\})$ is connected [$\because$ The union of T.S is connected provided their intersection $\neq \phi$]

Furthermore

$$\bigcap_{x \in X} T_x \neq \phi \quad \text{and} \quad \bigcup_{x \in X} T_x = X \times Y$$

where $T_x = [(\{x\} \times Y) \cup (X \times \{y\})]$

$\Rightarrow X \times Y$ is connected.

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FUNCTIONAL **

NORMED SPACE

⇒ **Normed Space:-** Let X be a vector space over field F then a mapping $\|\cdot\| : X \to \mathbb{R}$ is norm if

(i) $\|x\| \geq 0$

(ii) $\|x\| = 0 \iff x = 0$

(iii) $\|\alpha x\| = |\alpha| \|x\| \quad \forall \alpha \in F, x \in X$

(iv) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$

Then the pair $(X, \|\cdot\|)$ is called norm space.

Example $X = \mathbb{R}$ and $\|\cdot\| : X \to \mathbb{R}$ by $\|x\| = |x|$ then show that $(X, \|\cdot\|)$ is a norm space.

Proof (i) As $|x| \geq 0 \Rightarrow \|x\| \geq 0$ ∵ by given

(ii) let $\|x\| = 0$

$\iff |x| = 0 \iff x = 0$

(iii) Let $\alpha \in F$ & $x \in X$

$\|\alpha x\| = |\alpha x| = |\alpha||x|$

$\Rightarrow \|\alpha x\| = |\alpha| \|x\| \quad [\because \|x\| = |x|]$

(iv) $\|x+y\| = |x+y|$

$\leq |x| + |y|$

$= \|x\| + \|y\|$

$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$

$\Rightarrow (X, \|\cdot\|)$ is a norm space.

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Example 2:- Let $X = \mathbb{C}$ & $\|\cdot\| : X \to \mathbb{R}$ by $\|z\| = |z|$

Solution

(i) As $|z| \geq 0 \Rightarrow \|z\| \geq 0$

(ii) Let $\|z\| = 0$

$\Leftrightarrow |z| = 0 \Leftrightarrow z = 0$

(iii) Let $\alpha \in F$ & $z \in X$

$\|\alpha z\| = |\alpha z| = |\alpha||z|$

$= |\alpha| \|z\|$ $[\because |z| = \|z\|$

(iv) $\|z + z_1\| = |z + z_1|$

$\leq |z| + |z_1| = \|z\| + \|z_1\|$

$\Rightarrow \|z + z_1\| \leq \|z\| + \|z_1\|$

$\Rightarrow (X, \|\cdot\|)$ is a norm space.

Example: $X = \mathbb{R}^n$ & $\|\cdot\| : X \to \mathbb{R}$ is defined by $\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$, $x \in \mathbb{R}^n$ where $x = (x_1, x_2, x_3, \ldots, x_n)$ Show that $(\mathbb{R}^n, \|\cdot\|)$ is norm space.

Solution

(i) As $|x_i|^2 \geq 0$

$\Rightarrow \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \geq 0 \Rightarrow \|x\| \geq 0$

(ii) Let $\|x\| = 0$

$\Leftrightarrow \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} = 0$

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$$\Leftrightarrow \sum_{i=1}^{n} |x_i|^2 = 0 \quad \Leftrightarrow |x_i|^2 = 0$$

$$\Leftrightarrow x_i = 0 \quad \forall\, i = 1, 2, 3, \cdots, n$$

$$\Leftrightarrow (x_1, x_2, x_3, \cdots, x_n) = 0$$

$$\Leftrightarrow x = 0$$

(iii) Let $\|\alpha x\| = \left(\sum_{i=1}^{n} |\alpha x_i|^2\right)^{1/2}$

$$= \left(\sum_{i=1}^{n} |\alpha|^2 |x_i|^2\right)^{1/2}$$

$$= \left[|\alpha|^2 \sum_{i=1}^{n} |x_i|^2\right]^{1/2}$$

$$= (|\alpha|^2)^{1/2} \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2}$$

$$= \alpha \|x\|$$

(iv) Let $x = (x_1, x_2, x_3, \cdots, x_n) \in \mathbb{R}^n$

$y = (y_1, y_2, y_3, \cdots, y_n) \in \mathbb{R}^n$

$$\|x+y\| = \left(\sum_{i=1}^{n} |x_i + y_i|^2\right)^{1/2}$$

$$\leq \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} + \left(\sum_{i=1}^{n} |y_i|^2\right)^{1/2}$$ | By Minkowski's inequality

$$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$$

$\Rightarrow (\mathbb{R}^n, \|\cdot\|)$ is normed space.

Example: $X = \mathbb{C}^n$ & $\|\cdot\| : X \to \mathbb{R}$ is defined by $\|z\| = \left(\sum_{i=1}^{n} |z_i|^2\right)^{1/2}$, $z \in \mathbb{C}^n$ where $z = (z_1, z_2, z_3, \cdots, z_n)$ Show that $(\mathbb{C}^n, \|\cdot\|)$

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is norm space.

Solution

(i) As $|z_i|^2 \geq 0$

$\Rightarrow \left(\sum_{i=1}^{n} |z_i|^2\right)^{1/2} \geq 0 \Rightarrow \|z\| \geq 0$

(ii)

Let $\|z\| = 0$

$\Leftrightarrow \left(\sum_{i=1}^{n} |z_i|^2\right)^{1/2} = 0$

$\Leftrightarrow \sum_{i=1}^{n} |z_i|^2 = 0 \Leftrightarrow |z_i|^2 = 0$

$\Leftrightarrow z_i = 0$ for $i = 1, 2, 3, \cdots, n$

$\Leftrightarrow (z_1, z_2, z_3, \cdots, z_n) = 0$

$\Rightarrow z = 0$

(iii) Let $\|\alpha z\| = \left(\sum_{i=1}^{n} |\alpha z_i|^2\right)^{1/2}$

$= \left(\sum_{i=1}^{n} |\alpha|^2 |z_i|^2\right)^{1/2}$

$= \left(|\alpha|^2 \sum_{i=1}^{n} |z_i|^2\right)^{1/2}$

$= (|\alpha|^2)^{1/2} \left(\sum_{i=1}^{n} |z_i|^2\right)^{1/2}$

$\|\alpha z\| = |\alpha| \|z\|$

(iv) Let $z = (z_1, z_2, z_3, \cdots z_n) \in \mathbb{C}^n$

& $z_0 = (z_{01}, z_{02}, z_{03}, \cdots, z_{0n}) \in \mathbb{C}^n$

$\|z + z_0\| = \left(\sum_{i=1}^{n} |z_i + z_{0i}|^2\right)^{1/2}$

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$$\|z+z_0\| \leq \left(\sum_{i=1}^{n}|z_i|^2\right)^{1/2} + \left(\sum_{i=1}^{n}|z_{0i}|^2\right)^{1/2}$$

$$\Rightarrow \|z+z_0\| \leq \|z\| + \|z_0\|$$

$\Rightarrow (\mathbb{C}^n, \|\cdot\|)$ is normed space.

Example: $X = C[a,b]$, $\|\cdot\| : X \to F$ by $\|f\| = \sup |f(x)|$ $x \in [a,b]$ show that $(X, \|\cdot\|)$ is normed space.

Solution (i) $|f(x)| \geq 0 \Rightarrow \sup_{x\in[a,b]} |f(x)| \geq 0$

$$\Rightarrow \|f\| \geq 0$$

(ii) Let $\|f\| = 0$

$$\Leftrightarrow \sup_{x\in[a,b]} |f(x)| = 0 \quad \forall\, x \in [a,b]$$

$$\Leftrightarrow |f(x)| = 0 \quad \forall\, x \in [a,b] \Leftrightarrow f(x) = 0$$

$$\Leftrightarrow f = 0 \Leftrightarrow o(x) = 0$$

(iii) $$\|\alpha f\| = \sup_{x\in[a,b]} |(\alpha f)(x)|$$

$$= \sup_{x\in[a,b]} |\alpha(f(x))|$$

$$= \sup_{x\in[a,b]} |\alpha||f(x)| = |\alpha| \sup_{x\in[a,b]} |f(x)|$$

$$\|\alpha f\| = |\alpha|\|f\|$$

(iv) $$\|f+g\| = \sup_{x\in[a,b]} |(f+g)(x)|$$

$$= \sup_{x\in[a,b]} |f(x)+g(x)|$$

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$$\Rightarrow \|f+g\| \leq \sup_{x\in[a,b]} \{|f(x)| + |g(x)|\}$$

$$= \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |g(x)|$$

$$= \|f\| + \|g\|$$

$$\Rightarrow \|f+g\| \leq \|f\| + \|g\|$$

$\Rightarrow (X, \|\cdot\|)$ is normed space.

Examples Let $X = \ell^\infty$ and $\|\cdot\| : X \to F$ be defined by $\|x\| = \sup_{i=1}^{n} |x_n|$, where $x = \{x_n\} \in \ell^\infty$ show that $(\ell^\infty, \|\cdot\|)$ is a normed space.

Solution

(i) As $|x_n| \geq 0$

$$\Rightarrow \sup_{i=1}^{n} |x_n| \geq 0$$

$$\Rightarrow \|x_n\| \geq 0$$

(ii) Let $\|x_n\| = 0$

$$\Leftrightarrow \sup_{i=1}^{n} |x_n| = 0$$

$$\Leftrightarrow |x_n| = 0 \quad \Leftrightarrow \quad x_n = 0$$

$$\Leftrightarrow x = \{x_n\} = \{0\} = 0$$

$$\Leftrightarrow x = 0$$

(iii) Let $\alpha \in F$ and $x = \{x_n\} \in \ell^\infty$

then $\alpha x = \{\alpha x_n\}$

Now $\|\alpha x\| = \sup_{i=1}^{n} |\alpha x_n|$

$$= \sup_{i=1}^{n} |\alpha| |x_n|$$

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$\Rightarrow \|\alpha x\| = |\alpha| \sup_{i=1}^{n} |x_n|$

$= |\alpha| \|x_n\|$

(iv) Let $x = \{x_n\}$ and $y = \{y_n\} \in \ell^\infty$

$x + y = \{x_n + y_n\}$

Now $\|x+y\| = \sup_{i=1}^{n} |x_n + y_n|$

$\leq \sup_{i=1}^{n} \{|x_n| + |y_n|\}$

$= \sup_{i=1}^{n} |x_n| + \sup_{i=1}^{n} |y_n|$

$= \|x\| + \|y\|$

$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$

So ℓ^∞ is a norm space.

Note:- Functional is a function whose domain is set of function and whose range is sub set of real no.

ℓ^∞:- Space of all bounded sequences of real numbers.

ℓ^p:- Space of all sequences of real numbers on X.

Example: Let $X = \ell^p$ and $p \geq 1$ and $\|\cdot\| : X \to F$ is defined by $\|x\| = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ where $x = \{x_n\} \in \ell^p$. show that $(\ell^p, \|\cdot\|)$ is norm space.

Solution

(i) As $|x_i|^p \geq 0 \Rightarrow \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \geq 0$

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$\Rightarrow \quad \|x\| \geq 0$

(ii) Let $\|x\| = 0$

$\Leftrightarrow \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} = 0$

$\Leftrightarrow |x_i| = 0 \quad \Leftrightarrow \quad x_i = 0$

$\Leftrightarrow x = \{x_i\} = \{0\} = 0$

$\Leftrightarrow x = 0$

(iii) Let $\alpha \in F$ and $x = \{x_n\} \in \ell^p$ where $\alpha x = \{\alpha x_n\}$

So $\|\alpha x\| = \left(\sum_{i=1}^{n} |\alpha x_i|^p\right)^{1/p}$

$= \left(\sum_{i=1}^{n} |\alpha|^p |x_i|^p\right)^{1/p}$

$= \left(|\alpha^p|^{1/p}\right)\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$

$= |\alpha| \|x\|$

(iv) Let $x = \{x_n\}$ and $y = \{y_n\} \in \ell^p$

then $x + y = \{x_n\} + \{y_n\} = \{x_n + y_n\}$

Now $\|x+y\| = \left(\sum_{i=1}^{n} (x_i + y_i)^p\right)^{1/p}$

$\leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}$ By Min... Inequality

$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$

So $(\ell^p, \|\cdot\|)$ is a norm space.

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Theorem:- Every normed space is metric space.

Proof:

Let $(X, \|\cdot\|)$ is a normed space

To prove X is metric space.

Define $d: X \times X \longrightarrow \mathbb{R}$ by

$$d(x,y) = \|x-y\|$$

(i) $\|x-y\| \geq 0$ $\because$ X is norm space

$\Rightarrow d(x,y) \geq 0$ $\because$ by definition

(ii) Let $d(x,y) = 0$

$\Leftrightarrow \|x-y\| = 0 \Leftrightarrow x-y = 0$

$\Leftrightarrow x = y$

(iii) $d(x,y) = \|x-y\|$

$= \|-1(y-x)\|$

$= |-1| \|y-x\| = \|y-x\|$

$\Rightarrow d(x,y) = d(y,x)$

(iv) $d(x,y) = \|x-y\|$

$= \|x-z+z-y\|$

$\leq \|x-z\| + \|z-y\|$

$d(x,y) \leq d(x,y) + d(z,y)$

$\Rightarrow (X, d)$ is a metric space.

→ Converse of above theorem is not true i.e every metric space may or may not be normed space.

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Examples Let $X = \mathbb{N}$

Define $d: X \times X \to R$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Then d is metric & (X,d) is metric space. But X is not a normed space because $\mathbb{N}$ is not vector.

Note:- The metric $d(x,y) = \|x-y\|$ is called a metric induced by norm space.

Theorem:- Show that a metric d induced by normed on a norm space satisfied the followings

(i) $d(x+a, y+a) = d(x,y)$

(ii) $d(\alpha x, \alpha y) = |\alpha| \, d(x,y)$, $x, y \in X$ & $\alpha \in F$

Proof (i) Metric induced by norm is given by $d(x,y) = \|x-y\| \quad \forall\, x, y \in X$

Now $d(x+a, y+a) = \|(x+a)-(y+a)\|$

$= \|x+a-y-a\|$

$= \|x-y\|$

$\Rightarrow d(x+a, y+a) = d(x,y)$

(ii) $d(\alpha x, \alpha y) = \|\alpha x - \alpha y\|$

$= \|\alpha(x-y)\|$

$= |\alpha| \, \|x-y\|$

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$$\Rightarrow d(\alpha x, \alpha y) = |\alpha| d(x,y)$$

Theorem: Let X be a norm space & $x, y \in X$ then $|\|x\| - \|y\|| \leq \|x-y\|$

Solution:

As $x = x - y + y$

$$\Rightarrow \|x\| = \|(x-y)+y\|$$

$$\Rightarrow \|x\| \leq \|x-y\| + \|y\|$$

$$\Rightarrow \|x\| - \|y\| \leq \|x-y\| \quad \text{---(1)}$$

Interchanging x & y in (1)

$$\|y\| - \|x\| \leq \|y-x\|$$

$$\Rightarrow -\{\|x\| - \|y\|\} \leq \|x-y\| \quad \text{---(2)}$$

from (1) & (2)

$$|\|x\| - \|y\|| \leq \|x-y\|$$

⇒ Uniform Continuity:- The norm function from $X \to \mathbb{R}$ is uniformly continuous iff for every $\varepsilon > 0$ there is a $\delta > 0$ s.t

$$|\|x\| - \|y\|| < \varepsilon \quad \text{whenever } \|x-y\| < \delta$$

Theorem: Let X be a norm space then a function $f: X \to \mathbb{R}$ is defined by $f(x) = \|x\|$ is uniformly continuous or norm is uniformly continuous function.

Proof: Let $\varepsilon > 0$ & chose $\delta = \varepsilon$

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$f: X \to Y$ is continuous at x whenever $x_n \to x$
$\Rightarrow f(x_n) \to f(x)$

Now for $x, y \in X$ s.t
$\|x - y\| < \delta$

Then $|\|x\| - \|y\|| \leq \|x - y\| < \delta = \varepsilon$

$\Rightarrow |\|x\| - \|y\|| < \varepsilon$

$\Rightarrow f$ is uniformly continuous.

Theorem: Let X be a norm space we
(i) Define $f: X \times X \to X$ by $f(x,y) = x+y$
(ii) $g: F \times X \to X$ by $g(\alpha, x) = \alpha x$ then show that f and g are continuous.

Proof:
(i) Let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $x_n \to x$ and $y_n \to y$
then $\|x_n - x\| \to 0$ & $\|y_n - y\| \to 0$ as $n \to \infty$

Now $\|f(x_n, y_n) - f(x,y)\| = \|(x_n + y_n) - (x+y)\|$
$= \|x_n + y_n - x - y\|$
$= \|(x_n - x) + (y_n - y)\|$
$\leq \|x_n - x\| + \|y_n - y\| \to 0$ as $n \to \infty$

$\Rightarrow \|f(x_n, y_n) - f(x,y)\| \to 0$ as $n \to \infty$

$\Rightarrow f(x_n, y_n) \to f(x,y)$

there f is continuous mapping.
Hence f is continuous.

(ii) Let $\{\alpha_n\}$ and $\{x_n\}$ be sequence in F & X respectively such that

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$\alpha_n \to \alpha$ and $x_n \to x$

$\Rightarrow |\alpha_n - \alpha| \to 0$ & $\|x_n - x\| \to 0$ as $n \to \infty$

Now $\|g(\alpha_n, x_n) - g(\alpha, x)\| = \|\alpha_n x_n - \alpha x\|$

$= \|\alpha_n x_n - \alpha x_n + \alpha x_n - \alpha x\|$

$= \|(\alpha_n - \alpha)x_n + \alpha(x_n - x)\|$

$\leq \|(\alpha_n - \alpha)x_n\| + \|\alpha(x_n - x)\|$

$= |\alpha_n - \alpha| \|x_n\| + |\alpha| \|x_n - x\|$

$\to 0$ as $n \to \infty$

$\Rightarrow \|g(\alpha_n, x_n) - g(\alpha, x)\| \to 0$

$\Rightarrow g(\alpha_n, x_n) \to g(\alpha, x)$

$\Rightarrow g$ is continuous.

Closed unit Ball:- Let X be a norm space then the set denoted and defined by $\overline{B}_1(0) = \{x : x \in X \wedge \|x\| \leq 1\}$ is called closed unit ball

Convex Set:- Let X be a normed space and $M \subseteq X$ then M is said to be convex if for all $x, y \in M$ and $\alpha \in]0,1[$, $\alpha x + (1-\alpha)y \in M$

Theorem Prove that a closed unit ball in a normed space X is convex.

Proof Let $\overline{B}_1(0) = \{x : x \in X \wedge \|x\| \leq 1\}$ be a closed unit ball in a normed space X.

To prove $\overline{B}_1(0)$ is convex.

Let $x, y \in \overline{B}_1(0)$ & $\alpha \in]0,1[$

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VECTOR SPACE

As $x, y \in \overline{B}_1(0)$

So $\|x\| \leq 1$ & $\|y\| \leq 1$

Now $\|\alpha x + (1-\alpha)y\| \leq \|\alpha x\| + \|(1-\alpha)y\|$

$= |\alpha|\|x\| + |1-\alpha|\|y\|$

$\leq |\alpha| 1 + |1-\alpha| 1$

$= \alpha + 1 - \alpha \quad [\because 1-\alpha > 0$

$= 1$

$\Rightarrow \|\alpha x + (1-\alpha)y\| \leq 1$

$\Rightarrow \|\alpha x + (1-\alpha)y\| \in \overline{B}_1(0)$

$\Rightarrow \overline{B}_1(0)$ is a convex set.

Vector Space:- Let $V \neq \phi$ & F be the field then

(i) $\forall \alpha \in F$ and $x \in V \Rightarrow \alpha x \in V$

(ii) $(V, +)$ is abelian group

(iii) $\forall \alpha, \beta \in F$ and $x \in V \Rightarrow (\alpha+\beta)x = \alpha x + \beta x$

(iv) $\forall \alpha \in F$ and $x, y \in V$

$\Rightarrow \alpha(x+y) = \alpha x + \alpha y$

(v) $\alpha, \beta \in F$ & $x \in V \Rightarrow (\alpha\beta)x = \alpha(\beta x)$

(vi) $\forall x \in V$ and for $1 \in F$

$1 \cdot x = x$ Then V is called vector space over the field F.

Example:- $V = \mathbb{R}^2$, $F = \mathbb{R}$

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$\alpha(x, y) = (\alpha x, \alpha y)$ then V is vector space over $\mathbb{R}$.

Solution: (i) $(x_1, y_1), (x_2, y_2) \in V$

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$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in \mathbb{R}^2$

$\Rightarrow V = \mathbb{R}^2$ is closed w.r.t "+"

(ii) $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in V = \mathbb{R}^2$

Now $\{(x_1, y_1) + (x_2, y_2)\} + (x_3, y_3) = [(x_1 + x_2), (y_1 + y_2)] + (x_3, y_3)$

$= [(x_1 + x_2) + x_3, (y_1 + y_2) + y_3]$

$= [x_1 + (x_2 + x_3), y_1 + (y_2 + y_3)]$

$= (x_1, y_1) + [x_2 + x_3, y_2 + y_3]$

$= (x_1, y_1) + \{(x_2, y_2) + (x_3, y_3)\}$

$\Rightarrow$ Associative law holds in $V = \mathbb{R}^2$ under "+"

(iii) $(0,0) \in V = \mathbb{R}^2$ acts as identity element of $\mathbb{R}^2$.

Let $(x, y) \in \mathbb{R}^2 \Rightarrow (x, y) + (0, 0) = (x + 0, y + 0)$

$= (x, y)$

$\Rightarrow (0, 0)$ is identity

(iv) Let $(x, y) \in \mathbb{R}^2$ then $\exists\ (-x, -y) \in \mathbb{R}^2$ s.t

$(x, y) + (-x, -y) = (x + (-x), y + (-y))$

$= (0, 0)$

$\Rightarrow (-x, -y)$ is inverse of x, y

$\Rightarrow$ Inverses exist in $\mathbb{R}^2$

$\Rightarrow \mathbb{R}^2$ is a group w.r.t "+"

(v) $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$= (x_2 + x_1, y_2 + y_1)$

$= (x_2, y_2), (x_1, y_1)$

$\rightarrow$ Commutative law under addition holds in $\mathbb{R}^2$

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$\Rightarrow (\mathbb{R}^2, +)$ is an abelian group.

2) $\alpha, \beta \in F = \mathbb{R}$

$v = (x, y) \in V = \mathbb{R}^2$

$(\alpha+\beta)\cdot v = (\alpha+\beta)(x, y)$

$= [(\alpha+\beta)x, (\alpha+\beta)y]$

$= [\alpha x + \beta x, \alpha y + \beta y]$

$= [\alpha x, \alpha y] + [\beta x, \beta y]$

$= \alpha(x, y) + \beta(x, y)$

$= \alpha v + \beta v$

3) $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2) \in V = \mathbb{R}^2$

& $\alpha \in F = \mathbb{R}$

$\alpha(v_1 + v_2) = \alpha\{(x_1, y_1) + (x_2, y_2)\}$

$= \alpha\{(x_1 + x_2), (y_1 + y_2)\}$

$= \alpha(x_1 + x_2, y_1 + y_2)$

$= \alpha(x_1 + x_2), \alpha(y_1 + y_2)$

$= (\alpha x_1 + \alpha x_2, \alpha y_1 + \alpha y_2)$

$= \alpha(x_1, y_1) + \alpha(x_2, y_2)$

$= \alpha v_1 + \alpha v_2$

4) $\alpha, \beta \in F = \mathbb{R}$, $v = (x, y) \in V = \mathbb{R}^2$

$(\alpha\beta)v = \alpha\beta(x, y)$

$= [\alpha\beta(x), (\alpha\beta)y]$

$= [\alpha(\beta x), \alpha(\beta y)]$

$= \alpha[\beta(x, y)]$

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$\Rightarrow (\alpha\beta)V = \alpha(\beta V)$

5) $1 \in F = \mathbb{R}$, $V = (x,y) \in \mathbb{R}^2 = V$

$1 \cdot V = 1(x,y)$

$= (1 \cdot x, 1 \cdot y) = (x,y)$

$\Rightarrow 1 \cdot V = V$

$\Rightarrow \mathbb{R}^2$ is a vector space over $\mathbb{R} = F$

⇒ Linear Combination:-

$x_1, x_2, x_3, \ldots, x_n \in V$ & $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \in F$

$\Rightarrow \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \ldots + \alpha_n x_n$ in V is called linear combination of $x_1, x_2, x_3, \ldots, x_n$

⇒ Spanning Set:-

If $S \subseteq V$ over F then the set of all linear combination of finite number of elements of S is called linear span of S or spanning set of S. It is denoted by $\langle S \rangle$ or $L(S)$

⇒ Linearly dependent & Linearly Independent

If $x_1, x_2, \ldots, x_n \in V(F)$ then for any choice of scalers $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \in F$

(i) If $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0 \Rightarrow \alpha_i = 0 \ \forall i$, $1 \le i \le n$

then $x_1, x_2, \ldots, x_n$ are called linearly indepe

(ii) If $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0 \Rightarrow \alpha_i \neq 0$ for some i , $1 \le i \le n$

then $x_1, x_2, x_3, \ldots, x_n$ are called linearly dependent.

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Examples Let $V=\mathbb{R}^2$ & $F=\mathbb{R}$ we define $(x_1,y_1)+(x_2,y_2)=(x_1+x_2, y_1+y_2)$ & $\alpha(x,y)=(\alpha x, \alpha y)$ then

i) $x=(1,-1)$, $y=(2,3)$

$\Rightarrow \alpha(1,-1)+\beta(2,3)=(0,0)$

$\Rightarrow (\alpha,-\alpha)+(2\beta,3\beta)=(0,0)$

$\Rightarrow (\alpha+2\beta, -\alpha+3\beta)=(0,0)$

$\Rightarrow \alpha+2\beta=0$ ——— ① & $-\alpha+3\beta=0$ ——— ②

from ① & ② $\alpha=0$ & $\beta=0$

$\Rightarrow x$ & y are linearly independent.

ii) Let $x=(1,2)$ & $y=(3,6)$

Let $\alpha x+\beta y=0$

$\Rightarrow \alpha(1,2)+\beta(3,6)=(0,0)$

$\Rightarrow (\alpha+3\beta, 2\alpha+6\beta)=(0,0)$

$\Rightarrow [(\alpha+3\beta), (2\alpha+6\beta)]=0$

$\Rightarrow \alpha+3\beta=0$ ——— ③ & $2\alpha+6\beta=0$

$\Rightarrow \alpha+3\beta=0$ ——— ④

As both equations are same so x & y are linearly dependent.

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⇒ **Linearly Independent Lemma:-** Let $\{x_1, x_2, x_3, \cdots, x_n\}$ be a linearly independent set of vectors in a norm space X. Then for any real number $c > 0$ & for every choice of scalers $\alpha_1, \alpha_2, \cdots, \alpha_n$.

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\| \geq c\{|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|\}$$

Proof: To prove $\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\| \geq c\{|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|\}$ ——— ①

Let $S = \{|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|\}$

If $S = 0$ then $\alpha_1 = 0, \alpha_2 = 0, \cdots, \alpha_n = 0$

⇒ Inequality ① holds

If $S \neq 0$ then $S > 0$

Dividing both sides of ① by S

$$\Rightarrow \frac{1}{S}\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\| \geq c$$

$$\Rightarrow \left\|\frac{\alpha_1 x_1}{S} + \frac{\alpha_2 x_2}{S} + \cdots + \frac{\alpha_n x_n}{S}\right\| \geq c \quad \{\because \|\alpha x\| = |\alpha|\|x\|$$

Let $\beta_j = \frac{\alpha_j}{S}$, $j = 1, 2, \cdots, n$

$$\Rightarrow \|\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n\| \geq c \quad \text{——— ②}$$

Equation ① & ② are same. So to prove ① we prove ②

As $\beta_j = \frac{\alpha_j}{S} \Rightarrow |\beta_j| = \frac{|\alpha_j|}{S}$

$$= \sum_{j=1}^{n} |\beta_j| = \sum_{j=1}^{n} \frac{|\alpha_j|}{S} = \frac{\sum_{j=1}^{n} |\alpha_j|}{S} = \frac{S}{S}$$

$$\Rightarrow \sum_{j=1}^{n} |\beta_j| = 1 \quad \left\{\because \sum_{j=1}^{n} |\alpha_j| = |\alpha_1| + \cdots + |\alpha_n| = S\right\}$$

Suppose inequality ② is wrong then there exist a sequence $\{y_m\}$ s.t $\|y_m\| \longrightarrow 0$ and $y_m = \beta_1^m x_1 + \beta_2^m x_2 + \cdots + \beta_n^m x_n$

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Also $\sum_{j=1}^{n} |\beta_j^m| = 1$

Now $\sum_{j=1}^{m} |\beta_j^m| = 1$

$\Rightarrow |\beta_j^m| \leq 1 \quad \forall\ j,\ j = 1, 2, \cdots, n$

for fixed j say $j = 1$

$|\beta_1^m| \leq 1$

$\Rightarrow |\beta_1^m|$ is a bounded sequence.

Then By B.W property this sequence has a convergent subsequence say $\{\gamma_1^m\}$ s.t $\gamma_1^m \to \beta_1$ then then $|y_m|$ has a subsequence $\{y_{(m,1)}\}$ and

$$y_{(m,1)} = \gamma_1^m x_1 + \beta_2^m x_2 + \beta_3^m x_3 + \cdots + \beta_n^m x_n$$

Continuing in this way after n-steps we have a subsequence $\{y_{(m,n)}\}$ of $\{y_m\}$ & $y_{(m,n)} \to \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n = y$

As norm is continuous function so

$\|y_{(m,n)}\| \longrightarrow \|y\|$

As $\sum_{j=1}^{n} |\beta_j| = 1$ so not all β_j's are zero. As $\{x_1, x_2, \cdots, x_n\}$ is Linearly Independent set of vectors and all β_j's are not equal to zero so $y \neq 0$

Now as $\|y_m\| \longrightarrow 0$ and $\{y_{(m,n)}\}$ is a subsequence of $\{y_m\}$ so $\|y\| = 0$ $\therefore y_{(m,n)} \to 0$

$\Rightarrow y = 0$

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which is a contradiction. So our supposition is wrong. Hence inequality ② holds.

$\Rightarrow$ Inequality ① holds

$\{y_m\}$ بہت سے sequences کا ایک sequence ہے۔ اور جب ان بہت سے sequences میں سے ایک sequence کا سب sequence γ_1^m ہوگا تو جب ہم اس sequence کو اس کے sub sequence سے تبدیل کریں گے تو $\{y_m\}$ کی ٹرمز میں تبدیلی آنے کی وجہ سے جو نیا sequence (sequence کا sequences) آئے گا اس کو $\{y_{(m,1)}\}$ کا نام دیا ہے۔

⇒ Basis of a Vector Space:- If $V(F)$ is a vector space and $S \neq 0$ be a subspace of V then S is said to be basis for V if (i) S is linearly independent (ii) $\langle S \rangle = V$ (i.e every $v \in V$ is linear combination of finite number of elements of S)

⇒ Definition:- If S is the basis of V, then order of S is called the dimension of V. If $O(S)$ is finite then V is called finite dimentional.

Example:- Let $V = \mathbb{R}^2$ and $S = \{(1,0), (0,1)\}$

Let $\alpha(1,0) + \beta(0,1) = 0$

$\Rightarrow \alpha(1,0) + \beta(0,1) = (0,0)$

$\Rightarrow (\alpha, 0) + (0, \beta) = (0,0)$

$\Rightarrow (\alpha + 0, 0 + \beta) = (0,0)$

$\Rightarrow \alpha = 0, \beta = 0$

$\Rightarrow S$ is linearly independent

Now Let $v = (x, y) \in V$ then

$v = (x, y) = (x + 0, 0 + y)$

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$\Rightarrow V = (x,y) = (x,0) + (0,y)$

$= x(1,0) + y(0,1) \in \langle S \rangle$

$\Rightarrow V \subseteq \langle S \rangle \Rightarrow \langle S \rangle = V$

$\Rightarrow S$ is basis for V &

dim of $V = 2$

Note:- $\langle S \rangle \subseteq V$ is obvious.

Theorem: Let Y be a finite dimensional subspace of a norm space X. Then Y is complete.

Proof:

Since Y is finite dimentional. So Y has a finite basis. Let $\{e_1, e_2, \cdots e_n\}$ be a basis for Y. To prove Y is complete.

Let $\{x^{(m)}\}$ be the cauchy sequence in Y then for every $\varepsilon > 0$ $\exists$ a natural number $n_0 \in \mathbb{N}$ s.t $\|x^{(m)} - x^{(p)}\| < \varepsilon$ whenever $m, p \geq n_0$

$\Rightarrow \varepsilon > \|x^{(m)} - x^{(p)}\| \quad \forall \quad m, p \geq n_0$

Now as $\{x^{(m)}\}$ is a sequence in Y, so then $\exists$ scalers $\alpha_1^{(m)}, \alpha_2^{(m)}, \cdots, \alpha_n^{(m)} \in F$ s.t

$$x^{(m)} = \alpha_1^{(m)} e_1 + \alpha_2^{(m)} e_2 + \cdots + \alpha_n^{(m)} e_n$$

Similarly $x^{(p)} = \alpha_1^{(p)} e_1 + \alpha_2^{(p)} e_2 + \cdots + \alpha_n^{(p)} e_n$

Now $\|x^{(m)} - x^{(p)}\| = \|\alpha_1^{(m)} e_1 + \alpha_2^{(m)} e_2 + \cdots + \alpha_n^{(m)} e_n - \alpha_1^{(p)} e_1 - \alpha_2^{(p)} e_2 - \cdots - \alpha_n^{(p)} e_n\|$

$= \|(\alpha_1^{(m)} - \alpha_1^{(p)}) e_1 + (\alpha_2^{(m)} - \alpha_2^{(p)}) e_2 + \cdots + (\alpha_n^{(m)} - \alpha_n^{(p)}) e_n\|$

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$$\Rightarrow \|x^{(m)} - x^{(p)}\| \geq c\left[|\alpha_1^{(m)} - \alpha_1^{(p)}| + |\alpha_2^{(m)} - \alpha_2^{(p)}| + \cdots + |\alpha_n^{(m)} - \alpha_n^{(p)}|\right] \quad \because \text{By L.H.L}$$

$$\Rightarrow \|x^{(m)} - x^{(p)}\| \geq c \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i^{(p)}|$$

$$\Rightarrow \varepsilon > \|x^{(m)} - x^{(p)}\| \geq c \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i^{(p)}| \quad \forall\, m,p \geq n_0$$

$$\Rightarrow \varepsilon > c \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i^{(p)}| \quad \forall\, m,p \geq n_0$$

$$\Rightarrow \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i^{(p)}| < \varepsilon/c \quad \forall\, m,p \geq n_0$$

$$\Rightarrow |\alpha_i^{(m)} - \alpha_i^{(p)}| < \varepsilon/c \quad \forall\, m,p \geq n_0$$

$\Rightarrow \alpha_i^{(p)}$ is a cauchy sequene in $F = \mathbb{R}$

Since $\mathbb{R}$ is complete so $\exists\ \alpha_i \in \mathbb{R}$ s.t $\alpha_i^{(p)} \longrightarrow \alpha_i$

Put $x = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n \in Y$

Now $\|x^{(m)} - x\| = \|(\alpha_1^{(m)} - \alpha_1)e_1 + (\alpha_2^{(m)} - \alpha_2)e_2 + (\alpha_3^{(m)} - \alpha_3)e_3 + \cdots + (\alpha_n^{(m)} - \alpha_n)e_n\|$

$$\leq |\alpha_1^{(m)} - \alpha_1|\|e_1\| + |\alpha_2^{(m)} - \alpha_2|\|e_2\| + \cdots + |\alpha_n^{(m)} - \alpha_n|\|e_n\|$$

$$\leq |\alpha_1^{(m)} - \alpha_1|k + |\alpha_2^{(m)} - \alpha_2|k + \cdots + |\alpha_n^{(m)} - \alpha_n|k$$

where $k = \operatorname{Max}\|e_i\|$

$$= k \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i| = k(\varepsilon/c) = \varepsilon'$$

$$\Rightarrow \|x^{(m)} - x\| = \varepsilon'$$

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$\Rightarrow x^{(m)} \longrightarrow x \in Y$

$\Rightarrow Y$ is complete.

⇒ Equivalent Norms:- Two norms $\|\cdot\|_1$ & $\|\cdot\|_2$ on a linear space X are equivalent if $\exists$ non-zero reals a and b s.t $a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$

Theorem Any two norms defined on a finite dimentional norm space X are equivalent.

Proof:

Let X be a finite dimentional norm space and $\{e_1, e_2, \cdots, e_n\}$ be a finite basis for X.

Let $\|\cdot\|_1$ & $\|\cdot\|_2$ be two normed on X. Let $x \in X$ then $\exists$ scalers $\alpha_1, \alpha_2, \cdots, \alpha_n$ s.t $x = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n$

$$\|x\|_1 = \|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n\|_1$$

$$\geq c\{|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|\}, c > 0 \{\text{By L-4L}\}$$

Let $\xi = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|$

$\therefore \|x\|_1 \geq c\xi$ or $c\xi \leq \|x\|_1$

$$\Rightarrow \xi \leq \frac{1}{c}\|x\|_1 \quad \text{——} (1)$$

Also $\|x\|_2 = \|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n\|_2$

$$\leq |\alpha_1|\|e_1\|_2 + |\alpha_2|\|e_2\|_2 + \cdots + |\alpha_n|\|e_n\|_2$$

Let $k = \max_{i=1}^{n} \|e_i\|_2$

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$\therefore \|x\|_2 \leq k\{|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|\}$

$= k\{|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|\}$

$\Rightarrow \|x\|_2 \leq k s \Rightarrow \frac{1}{k}\|x\|_2 \leq s$ ——— (2)

from (1) & (2)

$\frac{1}{k}\|x\|_2 \leq s \leq \frac{1}{c}\|x\|_1$

$\Rightarrow \frac{1}{k}\|x\|_2 \leq \frac{1}{c}\|x\|_1$

$\Rightarrow \|x\|_2 \leq \frac{k}{c}\|x\|_1$ ——— (3)

Now consider $\|x\|_2 = \|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n\|_2$

$\Rightarrow \|x\|_2 \geq c'\{|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|\}$, $c' > 0$ [By I-I-L

$\Rightarrow \|x\|_2 \geq c' s$ ——— (4)

$\|x\|_1 = \|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n\|_1$

$\leq |\alpha_1|\|e_1\|_1 + |\alpha_2|\|e_2\|_1 + \cdots + |\alpha_n|\|e_n\|_1$

Let $k' = \max_{i=1} \|e_i\|_1$

$\Rightarrow \|x\|_1 \leq k'|\alpha_1| + k'|\alpha_2| + \cdots + k'|\alpha_n|$

$= k'\{|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|\}$

$\Rightarrow \|x\|_1 \leq k' s \Rightarrow \frac{1}{k'}\|x\|_1 \leq s$ ——— (5)

from (4) & (5)

$\frac{1}{k'}\|x\|_1 \leq s \leq \frac{1}{c'}\|x\|_2 \Rightarrow \frac{c'}{k'}\|x\|_1 \leq \|x\|_2$ ——— (6)

from (3) & (6)

$\frac{c'}{k'}\|x\|_1 \leq \|x\|_2 \leq \frac{k}{c}\|x\|_1$

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$\|\cdot\|$ ↳ represent the element of vector space.

Let $\frac{c'}{k'} = a$ and $\frac{k}{c} = b$ then

$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$$

Hence $\|\cdot\|_1$ & $\|\cdot\|_2$ are equivalent norms.

Theorem:- Prove that equivalent norms define same topology on a linear space.

Proof:- Let $\|\cdot\|_1$ & $\|\cdot\|_2$ be two equivalent norms on a linear space X. Let τ_1 and τ_2 be the topologies w.r.t $\|\cdot\|_1$ & $\|\cdot\|_2$. As $\|\cdot\|_1$ & $\|\cdot\|_2$ are equivalent norms then $\exists$ $a, b \in \mathbb{R}^+$ s.t

$$a\|\cdot\|_1 \leq \|\cdot\|_2 \leq b\|\cdot\|_1$$

To prove $\tau_1 = \tau_2$

Let $U_1 \in \tau_1 \Rightarrow U_1$ is an open set w.r.t $\|\cdot\|_1$. Then for each $x \in U_1$ $\exists$ an open ball $b_1(x, r)$ s.t

$x \in b_1(x, r) \subset U_1$ [Def of open set

where $b_1(x, r) = \{y \mid y \in X \wedge \|y - x\|_1 < r\}$

Let $r' = ar$

we claim that $B_2(x, r') \subseteq B_1(x, r)$

where $B_2(x, r') = \{y \mid y \in X \wedge \|y - x\|_2 < r'\}$

Let $y \in B_2(x, r')$

As $\|\cdot\|_1$ & $\|\cdot\|_2$ are equivalent norms $\therefore \|x - y\|_1 \leq \frac{1}{a}\|x - y\|_2$

$\leq \frac{1}{a} r' = \frac{1}{a}(ar)$

$a\|\cdot\| \leq \|\cdot\|_2$

$\Rightarrow a\|x - y\|_1 \leq \|x - y\|_2$

$\Rightarrow \|x - y\|_1 \leq \frac{1}{a}\|x - y\|_2$

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$\Rightarrow \|x-y\|_1 < r$

$\Rightarrow y \in B_1(x,r) \quad \therefore B_2(x,r') \subseteq B_1(x,r) \subseteq U_1$

$\Rightarrow x \in B_2(x,r') \subseteq U_1$

$\Rightarrow U_1$ is open w.r.t $\|\cdot\|_2$ i.e $U_1 \in \tau_2$

$\therefore \tau_1 \subseteq \tau_2$ ——— (1)

Now Let $U_2 \in \tau_2 \Rightarrow U_2$ is open w.r.t $\|\cdot\|_2$ then for each $x \in U_2$ $\exists$ an open ball $B_2(x,r)$ s.t $x \in B_2(x,r) \subseteq U_2$

where $B_2(x,r) = \{y / y \in X \wedge \|y-x\|_2 < r\}$

take $r' = r/b$

we claim that $B_1(x,r') \subseteq B_2(x,r)$

where $B_1(x,r') = \{y : y \in X \wedge \|y-x\|_1 < r'\}$

Let $y \in B_1(x,r') \Rightarrow \|y-x\|_1 < r'$

As $\|\cdot\|_1$ & $\|\cdot\|_2$ are equivalent norms

$\Rightarrow \|\cdot\|_2 \leq b\|\cdot\|_1$

$\Rightarrow \|y-x\|_2 \leq b\|y-x\|_1 < br' = b(r/b)$

$\Rightarrow \|y-x\|_2 < r$

$\Rightarrow y \in B_2(x,r) \quad \therefore B_1(x,r') \subseteq B_2(x,r) \subseteq U_2$

$\Rightarrow x \in B_1(x,r') \subseteq U_2$

$\Rightarrow U_2$ is open w.r.t $\|\cdot\|_1 \quad \therefore U_2 \in \tau_1$

$\Rightarrow \tau_2 \subseteq \tau_1$ ——— (2)

from (1) & (2) $\tau_1 = \tau_2$

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Theorem: Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be the two equivalent norms on a normed space X. Then prove that every cauchy sequence w.r.t $\|\cdot\|_1$ is also cauchy sequence w.r.t $\|\cdot\|_2$ and conversely.

Proof: Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms Then there exist $a, b \in \mathbb{R}^+$ s.t

$$a\|\cdot\|_1 \leq \|\cdot\|_2 \leq b\|\cdot\|_1$$

Let $\{x_n\}$ be a cauchy sequence w.r.t $\|\cdot\|_1$ then for $\varepsilon > 0$ there exist a +ve integer n_0 s.t $\|x_m - x_n\|_1 < \varepsilon \quad \forall \ m, n \geq n_0$

Now $\|\cdot\|_1$ & $\|\cdot\|_2$ are equivalent

$$\therefore \|x_m - x_n\|_2 \leq b\|x_m - x_n\|_1 < b\varepsilon \quad \forall \ m, n \geq n_0$$

$$\Rightarrow \|x_m - x_n\|_2 < b\varepsilon = \varepsilon' \quad \forall \ m, n \geq n_0$$

$\Rightarrow \{x_n\}$ is cauchy sequence w.r.t $\|\cdot\|_2$

Converse, Let $\{x_n\}$ be a cauchy sequence w.r.t $\|\cdot\|_2$, Then for $\varepsilon > 0$ there exist a +ve integer n_0 s.t

$$\|x_m - x_n\|_2 < \varepsilon \quad \forall \ m, n \geq n_0$$

Now $\|\cdot\|_1$ & $\|\cdot\|_2$ are equivalent so

$$\|x_m - x_n\|_1 \leq \frac{1}{a}\|x_m - x_n\|_2$$

$$< \frac{1}{a}\varepsilon = \varepsilon'$$

$$\Rightarrow \|x_m - x_n\|_1 < \varepsilon' \quad \forall \ m, n \geq n_0$$

So $\{x_n\}$ is cauchy sequence w.r.t $\|\cdot\|_1$

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⇒ "F" Riez Lemma:- Let Y & Z be subspaces of a norm space X. Let Y is closed proper subspace of Z then for any real number $\theta \in]0,1[$ $\exists$ a point $z \in Z$ s.t $\|z\| = 1$ and $\|z - y\| \geq \theta \ \forall \ y \in Y$.

Proof: As Y is closed subspace of Z. So $\exists$ a point $u \in Z$ s.t $u \notin Y$

Put $a = \inf \|u - y\|$, $y \in Y$

then $a > 0$ and $a \leq \|u - y\| \ \forall \ y \in Y$

Also by definition of infimum for $\varepsilon > 0$ $\exists$ a point $y_0 \in Y$ s.t

$\|u - y_0\| < a + \varepsilon$ for some $y_0 \in Y$

Put $a + \varepsilon = a/\theta$, $\theta \in]0,1[$

then $\|u - y_0\| < a/\theta$

$\because u \in Z, y_0 \in Y \subseteq Z$
$\Rightarrow u, y_0 \in Z \Rightarrow u - y_0 \in Z$
$\because Z$ is V.S so $c(u - y_0) \in Z$

Let $z = c(u - y_0) \in Z$

where $c = ? \dfrac{1}{\|u - y_0\|}$

$$\|z\| = \|c(u - y_0)\| = |c| \|u - y_0\|$$

$$= \frac{1}{\|u - y_0\|} \|u - y_0\| = 1$$

$\Rightarrow \|z\| = 1$

Now $\|z - y\| = \|c(u - y_0) - y\| = \|c[(u - y_0) - c^{-1} y]\|$

$= c \|u - (y_0 + c^{-1} y)\|$

As $y_0, y \in Y$ and Y is a subspace

$\therefore y_0 + c^{-1} y \in Y$ [By Property of subspace of vector space

$\therefore \|z - y\| = c\|u - (y_0 + c^{-1} y)\| \geq ca$

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$$\Rightarrow \|z-y\| \geq ca = \frac{1}{\|u-y_0\|} a$$

$$\Rightarrow \|z-y\| \geq \frac{a}{\|u-y_0\|} \quad \text{———} \quad (1)$$

Further $\|u-y_0\| < \frac{a}{\theta} \Rightarrow \theta < \frac{a}{\|u-y_0\|}$

So (1) becomes

$$\|z-y\| \geq \frac{a}{\|u-y_0\|} > \theta$$

$$\Rightarrow \|z-y\| > \theta, \quad \theta \in]0,1[$$

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BANACH SPACE ★

★ A complete norm space is called Banach space.

Theorem:- Prove that $\mathbb{R}^n$ is a Banach space

Proof:-

$\mathbb{R}^n$ is a normed space [Already Proved]

Let $\{x^{(m)}\}$ be a cauchy sequence in $\mathbb{R}^n$ then for $\varepsilon > 0$ there exist a +ve integer n_0 s.t $\|x^{(m)} - x^{(n)}\| < \varepsilon \quad \forall\ m, n \geq n_0$

$$\Rightarrow \left(\sum_{i=1}^{n} \left| x_i^{(m)} - x_i^{(n)} \right|^2 \right)^{1/2} < \varepsilon \quad \text{---} (1)$$

$$\Rightarrow \left| x_i^{(m)} - x_i^{(n)} \right| < \varepsilon \quad \forall\ m, n \geq n_0$$

$\Rightarrow \{x_i^{(m)}\}$ is cauchy sequence in $\mathbb{R}$

As $\mathbb{R}$ is complete then $x_i^{(n)} \longrightarrow x_i \in \mathbb{R}$ as $n \to \infty$

Put $x = (x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n$

As $n \to \infty$ (1) $\Rightarrow$

$$\left(\sum_{i=1}^{n} \left| x_i^{(m)} - x \right|^2 \right)^{1/2} < \varepsilon \quad \forall\ m \geq n_0$$

$$\|x_i^{(m)} - x\| < \varepsilon \quad \forall\ m \geq n_0$$

$\Rightarrow x_i^{(m)} \longrightarrow x \in \mathbb{R}^n \therefore \mathbb{R}^n$ is complete

$\Rightarrow \mathbb{R}^n$ is Banach space.

Theorem:- Prove that $\mathbb{C}$ is Banach space.

Proof:-

$\mathbb{C}$ is normed space [Already Proved]

Let $\{z_m\}$ be a cauchy sequence in $\mathbb{C}$ then $\exists$ a +ve integer n_0 s.t

$\|z_m - z_n\| < \varepsilon \quad \forall\ m, n \geq n_0$ ---- (1)

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As $z_m = x_m + iy_m$ & $z_n = x_n + iy_n$

So eqn ① becomes

$$\|x_m + iy_m - (x_n + iy_n)\| < \varepsilon \quad \text{——— ②}$$

$$\Rightarrow \|(x_m - x_n) + i(y_m - y_n)\| < \varepsilon$$

$$\Rightarrow |(x_m - x_n) + i(y_m - y_n)| < \varepsilon \quad [\because \|z\| = |z|$$

$$\Rightarrow \sqrt{(x_m - x_n)^2 + (y_m - y_n)^2} < \varepsilon$$

$$\Rightarrow |x_m - x_n| < \varepsilon \ \& \ |y_m - y_n| < \varepsilon \quad \forall m, n \geq n_0$$

$\Rightarrow \{x_n\}$ & $\{y_n\}$ are cauchy sequence in $\mathbb{R}$. As $\mathbb{R}$ is complete so

$x_n \to x \in \mathbb{R}$ & $y_n \to y \in \mathbb{R}$ when $n \to \infty$

Define $x + iy \in \mathbb{C}$

As $n \to \infty$ ② $\Rightarrow$

$$\|(x_m + iy_m) - (x + iy)\| < \varepsilon \quad \forall \ m \geq n_0$$

$$\Rightarrow \|z_m - z\| < \varepsilon \quad \forall \ m \geq n_0$$

$\Rightarrow z_m \to z \in \mathbb{C} \Rightarrow \mathbb{C}$ is complete

$\Rightarrow \mathbb{C}$ is Banach space.

Question: Prove that ℓ^p is a Banach space:-

Proof: ℓ^p is norm space (Already Proved)

Let $\{x^{(m)}\}$ be a cauchy sequence in ℓ^p then for $\varepsilon > 0$ there exist a +ve integer "n_0" s.t

$$\|x^{(m)} - x^{(n)}\| < \varepsilon \quad \forall \ m, n \geq n_0$$

$$\Rightarrow \left(\sum_{i=1}^{n} |x_i^{(m)} - x_i^{(n)}|^p\right)^{1/p} < \varepsilon \quad \text{——— ①}$$

$$\Rightarrow |x_i^{(m)} - x_i^{(n)}| < \varepsilon \quad \forall \ m, n \geq n_0$$

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$\Rightarrow \{x_i^{(n)}\}$ is a cauchy sequence in $\mathbb{R}$

As $\mathbb{R}$ is complete

$\therefore x_i^{(n)} \longrightarrow x_i \in \mathbb{R}$ as $n \longrightarrow \infty$

Define $x = \{x_1, x_2, \cdots\} = \{x_i\}$

As $n \to \infty$ ① $\Rightarrow$

$$\left(\sum_{i=1}^{n} |x_i^{(m)} - x|^p\right)^{1/p} < \varepsilon \quad \forall\ m \geq n_0$$

$$\Rightarrow \|x^{(m)} - x\| < \varepsilon \quad \forall\ m \geq n_0$$

$$\Rightarrow x^{(m)} \longrightarrow x$$

Now $x = x^{(m)} - (x^{(m)} - x) \in l^p$

$\Rightarrow l^p$ is complete

$\Rightarrow l^p$ is Banach space.

Question: Prove that l^∞ is a Banach space.

Proof:

l^∞ is norm space (Already proved)

Let $\{x^{(m)}\}$ be a cauchy sequence in l^∞ then for $\varepsilon > 0$ $\exists$ a +ve integer n_0 s.t

$$\|x^{(m)} - x^{(n)}\| < \varepsilon \quad \forall\ m, n \geq n_0$$

$$\Rightarrow \sup_{i=1}^{n} |x_i^{(m)} - x_i^{(n)}| < \varepsilon \quad \forall\ m, n \geq n_0$$

——— ①

$$\Rightarrow |x_i^{(m)} - x_i^{(n)}| < \varepsilon \quad \forall\ m, n \geq n_0$$

$\Rightarrow \{x_i^{(n)}\}$ is a cauchy sequence in $\mathbb{R}$

As $\mathbb{R}$ is complete then

$x_i^{(n)} \longrightarrow x_i \in \mathbb{R}$ as $n \longrightarrow \infty$

Define $x = \{x_1, x_2, \cdots\} = \{x_i\}$

As $n \to \infty$ then eqn ① become

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$$\sup_{i=1}^{n} |x_i^{(m)} - x_i| < \varepsilon \quad \forall\ m \geq n_0$$

$$\Rightarrow \|x^{(m)} - x\| < \varepsilon \quad \forall\ m \geq n_0$$

$$\Rightarrow x^{(m)} \longrightarrow x$$

Now $x = x^{(m)} - (x^{(m)} - x) \in \ell^\infty$ [$\because \ell^\infty$ is vector space]

Therefore ℓ^∞ is complete.

Hence ℓ^∞ is Banach space.

* * *

Question: Prove that $C[a,b]$ is a Banach space.

Proof:

(i) $C[a,b]$ is norm space.

(ii) Let $\{f_m\}$ be a cauchy sequence in $C[a,b]$ then for $\varepsilon > 0$ $\exists$ a +ve integer n_0 s.t

$$\|f_m - f_n\| < \varepsilon \quad \forall\ m, n \geq 0$$

$$\Rightarrow \sup_{i=1}^{n} |(f_m - f_n)(x)| < \varepsilon \qquad \text{---------- (1)}$$

$$\Rightarrow |(f_m - f_n)(x)| < \varepsilon \quad \forall\ m, n \geq n_0$$

$$\Rightarrow |f_m(x) - f_n(x)| < \varepsilon \quad \forall\ m, n \geq 0$$

$\Rightarrow \{f_n(x)\}$ is a cauchy sequence in $\mathbb{R}$. As $\mathbb{R}$ is complete

$\Rightarrow f_n(x) \longrightarrow f(x) \in \mathbb{R}$ as $n \to \infty$

As f_n is continuous on $[a,b]$

$\therefore f$ " " " "

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$\Rightarrow f \in C[a,b]$

As $n \to \infty$ ① $\Rightarrow$

$$\sup_{x \in [a,b]} |(f_m - f)(x)| < \varepsilon \quad \forall \; m \geq n_0$$

$$\Rightarrow \|f_m - f\| < \varepsilon \quad \forall \; m \geq n_0$$

$$\Rightarrow f_m \to f \in C[a,b]$$

$\therefore C[a,b]$ is complete

Hence $C[a,b]$ is Banach space.

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Theorem:- A subspace Y of Banach space X is complete iff Y is closed in X.

Proof:

Suppose Y is complete.

To prove Y is closed in X

Let $\{x_n\}$ be a ~~cauchy~~ sequence in Y s.t $x_n \to x$. As every convergent sequence is cauchy so $\{x_n\}$ is cauchy sequence in Y. As Y is complete

Therefore $x_n \to x \in Y$

$\Rightarrow Y$ is closed.

Conversely suppose Y is closed.

To prove Y is complete

Let $\{x_n\}$ be a cauchy sequence in Y & $Y \subseteq X$. Therefore $\{x_n\}$ is cauchy sequence in X. As X is Banach space

Therefore $x_n \to x \in X$ ($\because X$ is complete)

As $\{x_n\}$ is sequence in Y & $x_n \to x$ & Y is closed $\therefore x \in Y$

$\Rightarrow Y$ is complete.

$\Rightarrow$ Pseudo Metric:- Let X be a non-empty set then a function $d: X \times X \to \mathbb{R}$ is called Pseudo metric if

(i) $d(x,y) \geq 0 \quad \forall\ x, y \in X$

(ii) If $x = y$ then $d(x,y) = 0$

(iii) $d(x,y) = d(y,x)$

(iv) $d(x,y) \leq d(x,z) + d(z,y)$

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Theorem:- Show that every metric space is Pseudo metric But Converse in not true.

Proof:- Let (X, d) be a metric space.
To prove X is Pseudo metric
(i) $d(x,y) \geq 0 \ \forall x,y \in X$ $\because$ X is metric space.
(ii) $d(x,y) = 0 \iff x = y$ $\because$ X is " "
So in particular
If $x = y$ then $d(x,y) = 0$
(iii) $d(x,y) = d(y,x)$ $\because$ X is Metric spa
(iv) $d(x,y) \leq d(x,z) + d(z,y)$ $\because$ X " " "
So all the properties of Pseudo metric are satisfied Therefore (X, d) is Pseudo metric. But converse in not true in general.

Counter Example:-
Let X be a non-empty set define $d: X \times X \to \mathbb{R}$ by $d(x,y) = 0 \ \forall x, y \in X$. Then d is Pseudo metric but not metric
e.g if $x = y$ then $d(x,y) = 0$
But if $d(x,y) = 0 \not\Rightarrow x = y$

Theorem:- Prove that every finite dimentional subspace Y of norm space X is closed.

Proof:- Let Y be a finite dimentional subspace of a norm space X.
As finite dimentional subspace of a norm space is complete then Y is complete
As complete subspace of a norm space is closed ~~complete~~ then Y is closed in X

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Theorem: Prove that every finite dimentional subspace of a Banach space is Banach space.

Proof: Let X be a Banach space and Y be a finite dimentional subspace of X.

To prove Y is Banach space.

As every finite dimentional subspace of a norm space X is closed So Y is closed in X.

"As a subspace is complete iff it is closed" Therefore Y is complete. $\Rightarrow$ Y is Banach space.

Therefore every finite dimentional subspace of a Banach space is Banach space.

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HILBERT SPACE ★★

⇒ **Inner Product Space:-** Let V be a vector space then a mapping $\langle \cdot,\cdot \rangle : V \times V \longrightarrow F$ is called inner product if

(i) $\langle x,x \rangle \geq 0 \quad \forall\ x \in X$ [X mean V.S]

(ii) $\langle x,x \rangle = 0 \iff x = 0$

(iii) $\langle x+y, z \rangle = \langle x,z \rangle + \langle y,z \rangle$

(iv) $\langle \alpha x, y \rangle = \alpha \langle x,y \rangle$

(v) $\langle x,y \rangle = \overline{\langle y,x \rangle}$

Then pair $(V, \langle \cdot,\cdot \rangle)$ is called inner product space.

Example:- Let $X = \mathbb{R}^n$ and $\langle \cdot,\cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $\langle x,y \rangle = \sum_{i=1}^{n} x_i y_i$ where $x = (x_1, x_2, x_3, \cdots, x_n)$ & $y = (y_1, y_2, y_3, \cdots, y_n) \in \mathbb{R}^n$ show that $\mathbb{R}^n$ is inner product space.

Solution (i) As $\langle x,x \rangle = \sum_{i=1}^{n} x_i x_i$

$$= \sum_{i=1}^{n} x_i^2 \geq 0$$

$$\Rightarrow \langle x,x \rangle \geq 0 \quad \forall\ x \in \mathbb{R}^n$$

(ii) Let $\langle x,x \rangle = 0$

$$\iff \sum_{i=1}^{n} x_i^2 = 0$$

$$\iff x_i = 0 \quad \forall\ i = 1, 2, \cdots, n$$

$$\iff x = (x_1, x_2, \cdots, x_n) = (0, 0, \cdots, 0) = 0$$

$$\iff x = 0$$

(iii) Let $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$ & $z = (z_1, z_2, \cdots, z_n) \in \mathbb{R}^n$

Now $\langle x+y, z \rangle = \sum_{i=1}^{n} (x_i + y_i) z_i$

$$= \sum_{i=1}^{n} (x_i z_i + y_i z_i)$$

$x+y = (x_1+y_1, x_2+y_2, \cdots, x_n+y_n)$

$\langle x+y, z \rangle = (x_1+y_1)z_1 + (x_2+y_2)z_2 + \cdots + (x_n+y_n)z_n$

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iv) $\langle \alpha x, y \rangle = \sum_{i=1}^{n} (\alpha x_i) y_i$

$= \alpha \sum_{i=1}^{n} x_i y_i$

$= \alpha \langle x, y \rangle$

$\Rightarrow \langle x+y, z \rangle = \sum_{i=1}^{n} x_i z_i + \sum_{i=1}^{n} y_i z_i$

(iv) $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$

$= \sum_{i=1}^{n} y_i x_i = \overline{\sum_{i=1}^{n} y_i x_i}$

$= \overline{\langle y, x \rangle}$

Real جب $\overline{z} = z$ ہو تو z حقیقی ہوتا ہے

$\Rightarrow \mathbb{R}^n$ is an inner product space.

Example:- Let $X = \mathbb{C}^n$ and $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to F$ by defined by $\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$ where $x = (x_1, x_2, \cdots, x_n)$ & $y = (y_1, y_2, \cdots, y_n) \in \mathbb{C}^n$ show that $\mathbb{C}^n$ is inner product space

Solution

(i) As $\langle x, x \rangle = \sum_{i=1}^{n} x_i \overline{x_i}$

$= \sum_{i=1}^{n} x_i^2 \geq 0 \quad \because z\overline{z} = |z|^2$

$\Rightarrow \langle x, x \rangle \geq 0$

(ii) Let $\langle x, x \rangle = 0$

$\Leftrightarrow \sum_{i=1}^{n} x_i^2 = 0$

$\Leftrightarrow x_i = 0 \quad \forall i = 1, 2, \cdots, n$

$\Leftrightarrow x = (x_1, x_2, \cdots, x_n) = (0, 0, \cdots, 0) = 0$

$\Leftrightarrow x = 0$

(iii) Let $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$ & $z = (z_1, z_2, \cdots, z_n)$

Now $\langle x+y, z \rangle = \sum_{i=1}^{n} (x_i + y_i) \overline{z_i}$

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$$\Rightarrow \langle x+y, z\rangle = \sum_{i=1}^{n}(x_i+y_i)\bar{z}_i$$
$$= \sum_{i=1}^{n} x_i\bar{z}_i + \sum_{i=1}^{n} y_i\bar{z}_i$$
$$= \langle x, z\rangle + \langle y, z\rangle$$

(iv) $\langle \alpha x, y\rangle = \sum_{i=1}^{n}(\alpha x_i)\bar{y}_i = \alpha\sum_{i=1}^{n} x_i\bar{y}_i$

$$= \alpha\langle x, y\rangle$$

(v) $\langle x, y\rangle = \sum_{i=1}^{n} x_i\bar{y}_i$

$$= \sum_{i=1}^{n} \bar{y}_i x_i = \overline{\sum_{i=1}^{n} y_i\bar{x}_i}$$
$$= \overline{\langle y, x\rangle}$$

$\Rightarrow (\mathbb{C}^n, \langle\cdot,\cdot\rangle)$ is an inner product space.

Some Properties of Inner Product Space:-

(i) $\langle x, y+z\rangle = \overline{\langle y+z, x\rangle}$ [∵ Property v

$= \overline{\langle y, x\rangle + \langle z, x\rangle}$ [∵ Property iii]

$= \overline{\langle y, x\rangle} + \overline{\langle z, x\rangle}$

$= \overline{\overline{\langle x, y\rangle}} + \overline{\overline{\langle x, z\rangle}}$

$= \langle x, y\rangle + \langle x, z\rangle$

$\Rightarrow \langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$

(ii) $\langle \alpha x + \beta y, z\rangle = \langle \alpha x, z\rangle + \langle \beta y, z\rangle$

$= \alpha\langle x, z\rangle + \beta\langle y, z\rangle$

(iii) $\langle x, \alpha y\rangle = \overline{\langle \alpha y, x\rangle} = \overline{\alpha\langle y, x\rangle}$

$= \bar{\alpha}\,\overline{\langle y, x\rangle} = \bar{\alpha}\langle x, y\rangle$

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⇒ Cauchy Schwarz's Inequality:- OR Caudry Bunyakaybi Inequality:- In an inner product space X for all $x, y \in X$ $|\langle x, y\rangle| \leq \|x\| \|y\|$

Proof

$\forall \ \alpha \in F$

$$0 \leq \|x - \alpha y\|^2 = \langle x - \alpha y, x - \alpha y\rangle \quad \left[\because \|x\|^2 = \langle x, x\rangle\right.$$

$$= \langle x, x - \alpha y\rangle + \langle -\alpha y, x - \alpha y\rangle$$

$$= \langle x, x\rangle + \langle x, -\alpha y\rangle + -\alpha \langle y, x - \alpha y\rangle$$

$$= \langle x, x\rangle + \langle x, -\alpha y\rangle - \alpha\{\langle y, x\rangle - \bar{\alpha}\langle y, y\rangle\}$$

choose $\bar{\alpha} = \dfrac{\langle y, x\rangle}{\|y\|^2}$ ——— (*)

$$\Rightarrow 0 \leq \|x\|^2 - \bar{\alpha}\langle x, y\rangle - \alpha\{\langle y, x\rangle - \bar{\alpha}\langle y, y\rangle\}$$

$$\Rightarrow 0 \leq \|x\|^2 - \frac{\langle y, x\rangle\langle x, y\rangle}{\|y\|^2} - \alpha\langle y, x\rangle + \alpha\bar{\alpha}\|y\|^2 \quad [\text{by } (*)$$

$$\Rightarrow 0 \leq \|x\|^2 - \frac{\langle y, x\rangle\langle x, y\rangle}{\|y\|^2} - \alpha\langle y, x\rangle + \alpha\frac{\langle y, x\rangle}{\|y\|^2}\|y\|^2$$

$$\Rightarrow 0 \leq \|x\|^2 - \frac{\langle y, x\rangle\langle x, y\rangle}{\|y\|^2} - \alpha\langle y, x\rangle + \alpha\langle y, x\rangle$$

$$\Rightarrow 0 \leq \|x\|^2\|y\|^2 - \langle x, y\rangle\langle y, x\rangle$$

$$\Rightarrow \langle x, y\rangle\langle y, x\rangle \leq \|x\|^2\|y\|^2$$

$$\Rightarrow \langle x, y\rangle\overline{\langle x, y\rangle} \leq \|x\|^2\|y\|^2$$

$$\Rightarrow |\langle x, y\rangle|^2 \leq (\|x\|\|y\|)^2$$

$$\Rightarrow |\langle x, y\rangle| \leq \|x\|\|y\| \quad \text{proved.}$$

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Equation is true for roots e.g $x^2+7x+9=0$ is equation
Identity is true for all number e.g $(a+b)^2 = a^2+b^2+2ab$ is identity.

⇒ Parallelogram Law:-

For any two vectors x, y of inner product space V

$$\|x+y\|^2 + \|x-y\|^2 = 2\{\|x\|^2 + \|y\|^2\}$$

Proof L.H.S $\|x+y\|^2 + \|x-y\|^2$ $\quad \because \|x\| = \sqrt{\langle x,x\rangle}$

$= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$

$= \langle x, x+y\rangle + \langle y, x+y\rangle + \langle x, x-y\rangle + \langle -y, x-y\rangle$

$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle + \langle x,x\rangle + \langle x,-y\rangle + \langle -y,x\rangle + \langle -y,-y\rangle$

$= \|x\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|y\|^2 + \|x\|^2 - \langle x,y\rangle - \langle y,x\rangle + \|y\|^2$ $\quad [\because \bar{1} = -1]$

$= 2\|x\|^2 + 2\|y\|^2$

$= 2\{\|x\|^2 + \|y\|^2\}$

$\Rightarrow \|x+y\|^2 + \|x-y\|^2 = 2\{\|x\|^2 + \|y\|^2\}$

⇒ Polarization Identity:-

$$\text{Re}\langle x,y\rangle = \frac{1}{4}\{\|x+y\|^2 - \|x-y\|^2\}$$

$$\text{Img}\langle x,y\rangle = \frac{1}{4}\{\|x+iy\|^2 - \|x-iy\|^2\}$$

$z+\bar{z} = 2\text{Re}(z)$

Proof (i) R.H.S $= \frac{1}{4}\{\|x+y\|^2 - \|x-y\|^2\}$

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Theorem: Every Inner product space is norm space.

Proof: Let V be an inner product space

Define $\|\cdot\| : V \longrightarrow F$ by $\|x\| = \{\langle x, x\rangle\}^{1/2}$

(i) As $\langle x, x\rangle \geq 0$

$\Rightarrow \{\langle x, x\rangle\}^{1/2} \geq 0 \Rightarrow \|x\| \geq 0$

(ii) Let $\|x\| = 0$

$\Leftrightarrow \{\langle x, x\rangle\}^{1/2} = 0 \Leftrightarrow \langle x, x\rangle = 0$

$\Leftrightarrow x = 0$

(iii) Let $\alpha \in F$

$$\|\alpha x\| = \{\langle \alpha x, \alpha x\rangle\}^{1/2}$$

$$= \{\alpha\bar{\alpha}\langle x, x\rangle\}^{1/2}$$

$$= \{|\alpha|^2 \langle x, x\rangle\}^{1/2} \qquad \because z\bar{z} = |z|^2$$

$$\Rightarrow \|\alpha x\| = |\alpha| \|x\|$$

(iv) $\|x+y\| = \{\langle x+y, x+y\rangle\}^{1/2}$

$$\|x+y\|^2 = \{\langle x+y, x+y\rangle\}$$

$$= \{\langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle\}$$

$$= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2$$

$$\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2 \qquad \because Re(z) \leq |z|$$

$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \qquad [\because \text{By C.S inequality}$$

$$\Rightarrow \|x+y\|^2 \leq (\|x\| + \|y\|)^2$$

$$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$$

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ℓ^p :- Space of all bounded sequences defined on real number X

$\Rightarrow (V, \|\cdot\|)$ is a normed space.
Hence every inner product space is norm space.

Note:- Converse of above theorem is not true in general i.e norm space need not to be an inner product space.

Counter Example:- Let x^p, $p > 2$

Let $x = (1,1,0,0,0,\cdots)$, $y = (1,-1,0,0,0) \in X$

$\|x\| = 2^{1/p}$, $\|y\| = 2^{1/p}$

$\|x+y\| = 2$, $\|x-y\| = 2$

Now $\|x+y\|^2 + \|x-y\|^2 = 4+4$

$= 8$

$2\{\|x\|^2 + \|y\|^2\} = 2\{2^{2/p} + 2^{2/p}\}$

$= 4 \cdot 2^{2/p}$

$\Rightarrow \|x+y\|^2 + \|x-y\|^2 \neq 2\{\|x\|^2 + \|y\|^2\}$

$\Rightarrow$ Parallelogram is not satisfied
$\Rightarrow \ell^p$ is not an inner product space.

Example 7:- $X = \ell^p$ and $p \geq 1$ $\|\cdot\| : X \to F$ defined by $\|\cdot\|_{\ell^p} = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ where $x = \{x_n\} \in \ell^p$ show that $(\ell^p, \|\cdot\|)$ is a norm space.

Solution (Already Proved)

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⇒ Appalonius Inequality:-

Theorem:- Let V be an inner product space then for any $x, y, z \in V$

$$\|z-x\|^2 + \|z-y\|^2 = \frac{1}{2}\|x-y\|^2 + 2\|z-\frac{1}{2}(x+y)\|^2$$

Proof: Since V is an inner product space

So by Parallelogram law

$$\|x'+y'\|^2 + \|x'-y'\|^2 = 2\{\|x'\|^2 + \|y'\|^2\} \quad \forall\ x', y' \in V$$

Put $x' = z-x$, $y' = z-y$

$$\Rightarrow \|z-x+z-y\|^2 + \|z-x-z+y\|^2 = 2\{\|z-x\|^2 + \|z-y\|^2\}$$

$$\|2z-(x+y)\|^2 + \|y-x\|^2 = 2\|z-x\|^2 + 2\|z-y\|^2$$

$$4\|z-\frac{1}{2}(x+y)\|^2 + \|y-x\|^2 = 2\|z-x\|^2 + 2\|z-y\|^2$$

$$\Rightarrow 2\|z-\frac{1}{2}(x+y)\|^2 + \frac{1}{2}\|x-y\|^2 = \|z-x\|^2 + \|z-y\|^2$$

$$\Rightarrow \|z-x\|^2 + \|z-y\|^2 = \frac{1}{2}\|x-y\|^2 + 2\|z-\frac{1}{2}(x+y)\|^2$$

Theorem: Let V be an

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⇒ **Hilbert Space:-** A complete inner product space is called Hilbert space.
e.g. $\mathbb{R}^n$ and $\mathbb{C}^n$ are Hilbert spaces.

Definition:- Let V be an inner product space and $x, y \in V$ then x is orthogonal to y if $\langle x, y \rangle = 0$
If x is orthogonal to y we denote it as $x \perp y$.

Definition:- Let V be an inner product space and $x \in V$ and $A \subseteq V$ then x is said to be orthogonal to A if
$\langle x, a \rangle = 0 \quad \forall \; a \in A$
e.g. if $A = \{a_1, a_2, \ldots a_n\}$ then
$\langle x, a_1 \rangle = 0 \;,\; \langle x, a_2 \rangle = 0 \;, \ldots\ldots, \langle x, a_n \rangle = 0$

⇒ **Pathagorian Theorem:-** Let V be an inner product space and $x, y \in V$ if $x \perp y$ then

~~$\|x+y\|^2 = \langle x+y, x+y \rangle$~~

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

Proof:- $\|x+y\|^2 = \langle x+y, x+y \rangle$

$= \langle x, x+y \rangle + \langle y, x+y \rangle$

$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$

Since $x \perp y$ therefore

$\langle x, y \rangle = 0 \quad \& \quad \langle y, x \rangle = 0$

$\Rightarrow \|x+y\|^2 = \langle x, x \rangle + \langle y, y \rangle$

$\Rightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2$

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Question:- Let V be an inner product space and $u, v \in V$ such that $\langle x, u \rangle = \langle x, v \rangle$ then $u = v$.

Solution:

Given $\langle x, u \rangle = \langle x, v \rangle$

$\Rightarrow \langle x, u \rangle - \langle x, v \rangle = 0$

$\Rightarrow \langle x, u - v \rangle = 0$

This result is true for all $x \in V$

As $u, v \in V$ and V is an inner product space therefor $u - v \in V$

So in particular for $x = u - v$

$\langle u - v, u - v \rangle = 0$

$\Rightarrow u - v = 0 \Rightarrow u = v$

⇒ Direct Sum:- A vector space X is said to be direct sum of two subspaces Y and Z if each $x \in X$ has a unique representation of the form $x = y + z$, $y \in Y$ & $z \in Z$

⇒ Anhilator:- Let H be a Hilbert space and Y be a closed subspace of H then Anhilator of Y or orthogonal complement of Y is denoted & defined as

$Y^{\perp} = \{x : x \in H \wedge x \perp Y\}$

Theorem:- Let Y be a closed subspace of a Hilbert space H then H is the direct sum of Y and $Z = Y^{\perp}$ i.e $H = Y \oplus Y^{\perp}$

Proof: Let $x \in H$. As Y is closed subspace of complete space H and we know

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that "closed subspace of a complete space is complete" so Y is complete subspace of H then by a theorem "If Y is complete subspace of an inner product space X then for $x \in X$ there exist a unique $y \in Y$ such that $z = x - y$ is orthogonal to Y"

$\Rightarrow z \in Y^\perp \Rightarrow x - y \in Y^\perp$

$\Rightarrow x - y = z$ for some $z \in Y^\perp$

$\Rightarrow x = y + z$

Now we prove uniqueness

Suppose $x = y_1 + z_1$, $x = y_2 + z_2$ where $y_1, y_2 \in Y$ and $z_1, z_2 \in Y^\perp$

$\Rightarrow y_1 + z_1 = y_2 + z_2$

$\Rightarrow y_1 - y_2 = z_2 - z_1$ (Additive inverse)

As $y_1, y_2 \in Y$ and Y is subspace

$\Rightarrow y_1 - y_2 \in Y$ and $z_2 - z_1 \in Y^\perp$

$\Rightarrow y_1 - y_2 = z_2 - z_1 \in Y \cap Y^\perp = \{0\}$

$\Rightarrow y_1 - y_2 = 0$, $z_2 - z_1 = 0$

$\Rightarrow y_1 = y_2$, $z_2 = z_1$

So representation is unique

So $H = Y \oplus Y^\perp$

Theorem: Let A and B be the subsets of a Hilbert space H then

(i) $A \subset A^{\perp\perp}$ (ii) $A \subseteq B \Rightarrow B^\perp \subseteq A^\perp$

(iii) $(A \cup B)^\perp = A^\perp \cap B^\perp$ & $A^\perp \cup B^\perp \subset (A \cap B)^\perp$

(iv) $A^\perp = A^{\perp\perp\perp}$ (v) $A \cap A^\perp \subseteq \{0\}$

(vi) $A^\perp$ is a closed subspace of H.

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$A^{\perp} = \{x : x \in H \wedge x \perp A\}$

$A^{\perp\perp} = \{y : y \in H \wedge y \perp A^{\perp}\}$

Proof (i) Let $x \in A$ then

$\langle x, y \rangle = 0 \quad \forall y \in A^{\perp}$

$\Rightarrow x \perp A^{\perp} \Rightarrow x \in A^{\perp\perp}$

$\therefore A \subseteq A^{\perp\perp}$

(ii) Given $A \subseteq B$ To prove $B^{\perp} \subseteq A^{\perp}$

Let $x \in B^{\perp} \Rightarrow x \perp B$

$\Rightarrow \langle x, y \rangle = 0 \quad \forall y \in B$

As $A \subseteq B \quad \therefore \langle x, y \rangle = 0 \quad \forall y \in A$

$\Rightarrow x \perp A \Rightarrow x \in A^{\perp}$

$\therefore B^{\perp} \subseteq A^{\perp}$

(iii) As $A \subseteq A \cup B$

$\Rightarrow (A \cup B)^{\perp} \subseteq A^{\perp}$

As $B \subseteq A \cup B$

$\Rightarrow (A \cup B)^{\perp} \subseteq B^{\perp}$

As $(A \cup B)^{\perp} \subseteq A^{\perp}$ and $(A \cup B)^{\perp} \subseteq B^{\perp}$

$\Rightarrow (A \cup B)^{\perp} \subseteq A^{\perp} \cap B^{\perp}$ ——— (3)

Now let $x \in A^{\perp} \cap B^{\perp}$

$\Rightarrow x \in A^{\perp}$ and $x \in B^{\perp}$

$\Rightarrow x \perp A$ and $x \perp B$

$\Rightarrow \langle x, a \rangle = 0 \quad \forall a \in A, \langle x, b \rangle = 0 \quad \forall b \in B$

$\langle x, y \rangle = 0 \quad \forall y \in A \cup B$

$\Rightarrow x \perp (A \cup B) \Rightarrow x \in (A \cup B)^{\perp}$

$\Rightarrow A^{\perp} \cap B^{\perp} \subseteq (A \cup B)^{\perp}$ ——— (4)

From (3) & (4)

$(A \cup B)^{\perp} = A^{\perp} \cap B^{\perp}$

(iv) As $A \cap B \subseteq A \Rightarrow A^{\perp} \subseteq (A \cap B)^{\perp}$

As $A \cap B \subseteq B \Rightarrow B^{\perp} \subseteq (A \cap B)^{\perp}$

As $A^{\perp} \subseteq (A \cap B)^{\perp}$ & $B^{\perp} \subseteq (A \cap B)^{\perp}$

$\Rightarrow A^{\perp} \vee B^{\perp} \subseteq (A \cap B)^{\perp}$

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(iv) As $A \subseteq A^{\perp\perp}$

then $A^{\perp\perp\perp} \subseteq A^{\perp}$ ——— ①

As $A^{\perp} \subseteq (A^{\perp})^{\perp\perp} \Rightarrow A^{\perp} \subseteq A^{\perp\perp\perp}$ ——— ②

from ① & ②

$A^{\perp} = A^{\perp\perp\perp}$

(v) Let $x \in A \cap A^{\perp}$

$\Rightarrow x \in A$ and $x \in A^{\perp}$

$\Rightarrow \langle x, x \rangle = 0 \Rightarrow x = 0$

$\Rightarrow x \in \{0\}$

So $A \cap A^{\perp} \subseteq \{0\}$

(vi) Let $y, z \in A^{\perp}$ and $a, b \in F$

then for any $x \in A$

$\langle y, x \rangle = 0$, $\langle z, x \rangle = 0$

$\langle ay + bz, x \rangle = a\langle y, x \rangle + b\langle z, x \rangle$

$= 0 + 0 = 0$

$\Rightarrow ay + bz \in A^{\perp}$

So $A^{\perp}$ is a subspace of H

Let y be the limit point of $A^{\perp}$ then there exist a sequence $\{y_n\}$ in $A^{\perp}$ s.t

$\lim_{n \to \infty} y_n = y$

As $\langle y_n, x \rangle = 0 \ \forall \ x \in A$ then

$0 = \lim_{n \to \infty} \langle y_n, x \rangle$

$0 = \langle \lim_{n \to \infty} y_n, x \rangle$

$0 = \langle y, x \rangle \Rightarrow y \in A^{\perp}$

So $A^{\perp}$ is a closed subspace of H.

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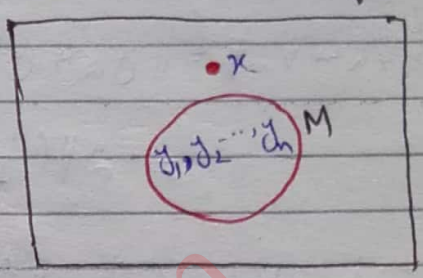
⇒ **Definition:-** Let V be an inner product space and $x \in V$, $M \subseteq V$ then

$\delta = D$ = Distance b/w x and M

$= \inf_{\bar{y} \in M} \|x - \bar{y}\|$



Let $a_1 = \|x - y_1\|$, $a_2 = \|x - y_2\|$

$\cdots$, $a_n = \|x - y_n\|$

$\delta = \inf\{a_1, a_2, \cdots, a_n\}$

⇒ **Convex Set:-** Let X be a linear space (vector space) and $A \subseteq X$ then A is called convex set if for $x, y \in A$ and $\alpha, \beta \in R^+$ with $\alpha + \beta = 1$ then $\alpha x + \beta y \in A$

⇒ **Minimizing Vector Theorem:-** Let X be an inner product space and M be a non-empty convex subset of X which is complete then for $x \in X$ there exist unique $y \in M$ s.t

$$\delta = \inf_{\bar{y} \in M} \|x - \bar{y}\| = \|x - y\|$$

Proof Let $\delta = \inf_{\bar{y} \in M} \|x - \bar{y}\|$

Then by definition of infimum there exist a sequence $\{y_n\}$ in M s.t

$$\delta = \lim_{n \to \infty} \|x - y_n\|$$

we have to prove $\{y_n\}$ is a cauchy sequence in M

Let $v_n = y_n - x$

then $\|v_n\| = \|y_n - x\| = \|x - y_n\|$

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$v_n + v_m = y_n - x + y_m - x$

$= y_n + y_m - 2x$

$\|v_n + v_m\| = \|y_n + y_m - 2x\|$

$= 2\|(\frac{1}{2}y_n + \frac{1}{2}y_m) - x\|$

$\frac{1}{2}\|v_n + v_m\| = \|(\frac{1}{2}y_n + \frac{1}{2}y_m) - x\|$

As $y_n, y_m \in M$ and M is convex $\Rightarrow \frac{1}{2}y_n + \frac{1}{2}y_m \in M$

So $\|\frac{1}{2}(y_n + y_m) - x\| \geq \delta$ ($\because \delta$ is infimum)

$\Rightarrow \frac{1}{2}\|v_n + v_m\| \geq \delta$

$\Rightarrow \|v_n + v_m\| \geq 2\delta$ ——— ①

Now $v_n - v_m = y_n - x - y_m + x$

$= y_n - y_m$

or $\|y_n - y_m\|^2 = \|v_n - v_m\|^2$

$= 2\|v_n\|^2 + 2\|v_m\|^2 - \|v_n + v_m\|^2$

(By Parallelogram Law)

$= 2(\delta_n)^2 + 2(\delta_m)^2 - \|v_n + v_m\|^2$

when $n \to \infty$

$\|y_n - y_m\|^2 \geq 2\delta^2 + 2\delta^2 - (2\delta)^2$ [$\because \delta_n \to \delta$ as]

using ①

$= 4\delta^2 - 4\delta^2$

$= 0$

$\delta = \inf_{y \in M} \|x - y\|$

$\delta \leq \delta_1 \leq \delta_2 \leq \delta_3 \leq \cdots$

$\delta_1 = \|x - y_1\|$

$\delta_2 = \|x - y_2\|$

$\{y_1, y_2, \cdots\} = y_n$

$s.t \|x - y_n\| \to \delta$

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$\therefore \|y_n - y_m\| \to 0$ as $n \to \infty$

$\Rightarrow \{y_n\}$ is cauchy sequence in M

As M is complete so this cauchy sequece converge to some point y

so $y_n \longrightarrow y \in M$

$$\|x - y\| \geq \delta \quad \text{---------} (1)$$

Now $\|x - y\| = \|x - y_n + y_n - y\|$

$= \|(x - y_n) + (y_n - y)\|$

$\leq \|x - y_n\| + \|y_n - y\|$

$\|x - y\| \leq \delta_n + \|y_n - y\|$

As $n \to \infty$

$\|x - y\| \leq \delta + 0 \quad (\because y_n \to y \ \& \ \delta_n \to \delta$

$\Rightarrow \|x - y\| \leq \delta \quad \text{---------} (2)$

From (1) & (2)

$\|x - y\| = \delta$

Uniqueness:- Let $y, y_0 \in M$

s.t $\delta = \|x - y\|$ and $\delta = \|x - y_0\|$

To Prove $y = y_0$

Now $\|y - y_0\|^2 = \|y - x + x - y_0\|^2$

$= \|(y - x) + (x - y_0)\|^2$

$= 2\|y - x\|^2 + 2\|x - y_0\|^2$

$- \|(y - x) - (x - y_0)\|^2$ By Paralelogram

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$$\|y-y_0\|^2 = 2\|y-x\|^2 + 2\|x-y_0\|^2 - \|y+y_0-2x\|^2$$

$$= 2\|y-x\|^2 + 2\|x-y_0\|^2 - 4\left\|\tfrac{1}{2}(y+y_0)-x\right\|^2$$

$$\leq 2\delta^2 + 2\delta^2 - 4\delta^2$$

$$\Rightarrow \|y-y_0\|^2 \leq 0$$

$$\Rightarrow \|y-y_0\| = 0$$

$$\Rightarrow y = y_0$$

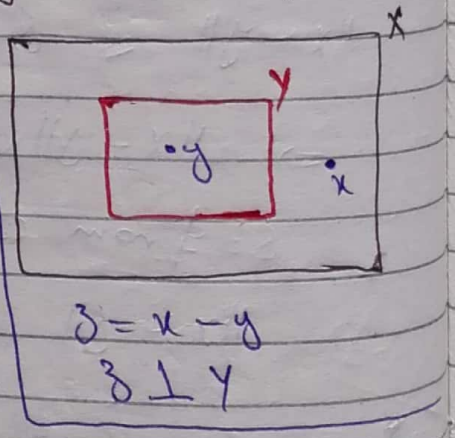
This prove the uniqueness of "y"

Theorem 1 Let Y be a complete subspace of an innerproduct space X then for $x \in X$ there exist a unique $y \in Y$ such that $z = x - y$ is orthogonal to Y

Proof Let $x \in X$

As Y is complete subspace of X so $\exists$ unique $y \in Y$ s.t

$\|x-y\| = \delta$ [By minimizing vector theorem]



Let $z = x - y$

we have to prove $z \perp Y$

Suppose it is not true then $\exists$ atleast one $y_1 \in Y$ s.t

$$\langle z, y_1 \rangle \neq 0$$

Let $\langle z, y_1 \rangle = \beta$

Let $\alpha \in F$

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$$\|z-\alpha y_1\|^2 = \langle z-\alpha y_1, z-\alpha y_1\rangle$$

$$= \langle z, z\rangle + \langle z, -\alpha y_1\rangle + \langle -\alpha y_1, z\rangle + \langle -\alpha y_1, -\alpha y_1\rangle$$

$$= \langle z, z\rangle - \alpha\langle y_1, z\rangle + \alpha\bar{\alpha}\langle y, y_1\rangle - \bar{\alpha}\langle z, y_1\rangle$$

$$= \langle z, z\rangle - \bar{\alpha}\langle z, y_1\rangle - \alpha\{\langle y_1, z\rangle - \bar{\alpha}\langle y_1, y\rangle\}$$

$$= \|z\|^2 - \bar{\alpha}\langle z, y_1\rangle - \alpha\{\langle y_1, z\rangle - \bar{\alpha}\|y_1\|^2\} \quad \text{---------- (1)}$$

Now choose $\bar{\alpha}$ in such a way that the term with in bracket became zero

$$\bar{\alpha} = \frac{\langle y_1, z\rangle}{\|y_1\|^2}$$

So eqn (1) becomes

$$\|z-\alpha y_1\|^2 = \|z\|^2 - \frac{\langle y_1, z\rangle\langle z, y_1\rangle}{\|y_1\|^2}$$

$$= \|z\|^2 - \frac{\langle y_1, z\rangle\overline{\langle y_1, z\rangle}}{\|y_1\|^2}$$

$$= \|z\|^2 - \frac{|\langle y_1, z\rangle|^2}{\|y_1\|^2}$$

$$\Rightarrow \|z-\alpha y_1\|^2 < \|z\|^2 = \|x-y\|^2$$

$$\Rightarrow \|z-\alpha y_1\|^2 < \delta^2$$

$$\Rightarrow \|z-\alpha y_1\|^2 < \delta \quad (\delta \text{ is infimum}$$

which is not true because

$$\|z-\alpha y_1\| = \|x-y-\alpha y_1\|$$

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$\Rightarrow \|z - \alpha y_1\| = \|x - (y + \alpha y_1)\|$

$= \|x - y_2\| \geq \delta$

where $y_2 = y + \alpha y_1$

So our supposition is wrong

$\therefore z \perp Y$

Theorem:- If Y is the closed subspace of a Hilbert space H then $Y = Y^{\perp\perp}$ or $Y = (Y^{\perp})^{\perp}$

Proof: Let $x \in Y$, for $y \in Y^{\perp}$

$\Rightarrow y \perp Y$

As $x \in Y$ and $y \in Y^{\perp}$

$\therefore \langle x, y \rangle = 0 \quad \forall\ y \in Y^{\perp}$

$\Rightarrow x \in Y^{\perp\perp}$

So $Y \subseteq Y^{\perp\perp}$ ——①

Now Let $x \in Y^{\perp\perp} \subseteq H$

$\Rightarrow x \in H = Y \oplus Y^{\perp}$ (b/c H is Hilbert space)

$\Rightarrow x = y + z$ where $y \in Y$, $z \in Y^{\perp}$

As $y \in Y \subseteq Y^{\perp\perp}$ (using Previous theorem)

$\Rightarrow y \in Y^{\perp\perp}$

As $x, y \in Y^{\perp\perp}$ and $Y^{\perp\perp}$ is subspace of H so $x - y \in Y^{\perp\perp}$

$\Rightarrow z \in Y^{\perp\perp} \Rightarrow z \perp Y^{\perp}$

Also $z \in Y^{\perp}$

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$\Rightarrow z \perp z \Rightarrow \langle z, z \rangle = 0$

$\Rightarrow z = 0$

So $x = y + z \Rightarrow x = y + 0$

$\Rightarrow x = y \in Y \Rightarrow x \in Y$

$\therefore Y^{\perp\perp} \subseteq Y$ ——— (2)

from (1) & (2) $Y^{\perp\perp} = Y$

⇒ Orthogonal Set:- Let X be an inner product space and $M \subseteq X$ then M is called orthogonal set if for $x, y \in M$

$\langle x, y \rangle = 0$ if $x = y$

$\neq 0$ if $x \neq y$

⇒ Orthonormal Set:- Let X be an innerproduct space and $M \subseteq X$ then M is called orthonormal set if for $x, y \in M$

$\langle x, y \rangle = 0$ if $x \neq y$

$= 1$ if $x = y$

Theorem:- An orthonormal set in an inner product space X is linearly independent.

Proof: Let $S = \{e_1, e_2, \cdots, e_n\}$ be an orthonormal set. To prove S is linearly independent.

Let $\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n = 0$

where $\alpha_1, \alpha_2, \cdots, \alpha_n \in F$

Consider $\langle 0, e_j \rangle = 0$

$\langle \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n, e_j \rangle = 0$

$\Rightarrow \langle \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_{j-1} e_{j-1} + \alpha_j e_j + \alpha_{j+1} e_{j+1} + \cdots + \alpha_n e_n, e_j \rangle = 0$

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$$\Rightarrow \alpha_1\langle e_1, e_j\rangle + \alpha_2\langle e_2, e_j\rangle + \cdots + \alpha_{j-1}\langle e_{j-1}, e_j\rangle + \alpha_j\langle e_j, e_j\rangle + \alpha_{j+1}\langle e_{j+1}, e_j\rangle + \cdots + \langle e_n, e_j\rangle = 0$$

$\because \{e_1, e_2, \cdots, e_n\}$ is orthonormal set

so

$$\langle e_j, e_j\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\therefore 0 + 0 + 0 + \cdots + 0 + \alpha_j(1) + 0 + \cdots + 0 = 0$$

$$\Rightarrow \alpha_j = 0 \quad \text{for } j = 1, 2, \cdots, n$$

$$\Rightarrow \alpha_1 = 0, \alpha_2 = 0, \cdots, \alpha_n = 0$$

$\Rightarrow S$ is linearly independent.

Theorem: For a non-empty subspace M of H (Hilbert space) then span of M is dence in H iff $M^\perp = \{0\}$

Proof: Let V = span of M

Suppose span is dence in H i.e

$$\bar{V} = H$$

To prove $M^\perp = \{0\}$

Let $x \in M^\perp \subseteq H \Rightarrow x \in \bar{V}$

then there exist a sequence $\{x_n\}$ in V s.t $x_n \longrightarrow x$

As $x \in M^\perp$ so $x \perp M$

$\Rightarrow x \perp V$

As $x \perp V$ and $\{x_n\}$ is a sequence in V $\therefore \langle x_n, x\rangle = 0$

As $x \longrightarrow x$ and $x \longrightarrow x$

and $\langle \cdot, \cdot \rangle$ is continuous

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$\because \langle x_n, x \rangle \longrightarrow \langle x, x \rangle$

As $\langle x_n, x \rangle = 0$

$\Rightarrow \langle x, x \rangle = 0 \quad \because x_n \longrightarrow x$

$\Rightarrow x = 0$

As x is arbitrary. So every element of $M^{\perp}$ is zero

$\Rightarrow M^{\perp} = \{0\}$

Conversely given

$M^{\perp} = \{0\}$

To prove $\bar{V} = H$

For this $x \in V^{\perp} \Rightarrow x \perp V$

$\Rightarrow x \perp M$

$\Rightarrow x \in M^{\perp} = \{0\} \Rightarrow x = 0$

$\Rightarrow V^{\perp} = \{0\}$

$(\bar{V})^{\perp} = \{0\}$

As $\bar{V}$ is closed subspace of H so

$H = \bar{V} \oplus (\bar{V})^{\perp}$ (By using theorem)

$H = \bar{V} \oplus \{0\}$

$H = \bar{V}$

$\Rightarrow V$ is dence in H

Assignment:- Show that if $V^{\perp} = \{0\}$ then

$(\bar{V})^{\perp} = \{0\}$

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Theorem:- The relation of "being equivallent" to among the norms that be defined on a linear space N is equivallence relation.

Proof) **Reflexive:-** For any norm $\|\cdot\|$ on N the condition $\alpha\|x\| \leq \|x\| \leq b\|x\|$ $\forall x \in N$ is satistisfied for $\alpha = b = 1$

Hence $\|\cdot\| \sim \|\cdot\|$

Symmetric:- Let $\|\cdot\|_1 \sim \|\cdot\|_2$ then $\exists$ real number $a > 0$, $b > 0$ s.t

$$a\|\cdot\|_2 \leq \|\cdot\|_1 \leq b\|\cdot\|_2$$

$$\Rightarrow a\|\cdot\|_2 \leq \|\cdot\|_1 \;\&\; \|\cdot\|_1 \leq b\|\cdot\|_2$$

$$\Rightarrow \|\cdot\|_2 \leq \frac{1}{a}\|\cdot\|_1 \;\&\; \frac{1}{b}\|\cdot\|_1 \leq \|\cdot\|_2$$

$$\Rightarrow \frac{1}{b}\|\cdot\|_1 \leq \|\cdot\|_2 \leq \frac{1}{a}\|\cdot\|_1$$

$\Rightarrow \|\cdot\|_2 \sim \|\cdot\|_1 \Rightarrow$ "$\sim$" is symmetric

Transitive:-

Let $\|\cdot\|_1 \sim \|\cdot\|_2$ & $\|\cdot\|_2 \sim \|\cdot\|_3$

Then $\exists$ real numbers $a > 0$, $b > 0$, $a_1 > 0$, $b_1 > 0$ s.t

$$a\|x\|_2 \leq \|x\|_1 \leq b\|\cdot\|_2$$

$$\& \; a_1\|x\|_3 \leq \|x\|_2 \leq b_1\|x\|_3 \quad \forall x \in N$$

Now $a_1\|x\|_3 \leq \|x\|_2 \;\&\; \|x\|_2 \leq b_1\|x\|_3$

$$aa_1\|x\|_3 \leq a\|x\|_2 \;\&\; b\|x\|_2 \leq bb_1\|x\|_3$$

Now $aa_1\|x\|_3 \leq a\|x\|_2 \leq \|x\|_1 \leq b\|x\|_2 \leq bb_1\|x\|_3$

$$\Rightarrow aa_1\|x\|_3 \leq \|x\|_1 \leq bb_1\|x_3\|$$

let $aa_1 = a_2$ & $bb_1 = b_2$

$$\Rightarrow a_2\|x\|_3 \leq \|x\|_1 \leq b_2\|x\|_3$$

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$\Rightarrow \|x\|_1 \sim \|x\|_2 \Rightarrow$ '$\sim$' is Transitive
$\Rightarrow$ '$\sim$' is an equivallence relation
$\Rightarrow \|\cdot\|$ is an equivallence relation

Topological Property:- A property P of topological space is said to be topological property if whenever a space X has the property P, all the spaces which are Homomorphic to H have the property P.

Theorem:- Compactness is a topological property.

Proof: Suppose $f: X \to Y$ be a Homomorphism. Let X is compact.

To prove $Y = f(X)$ is compact.

Let $\{U_\alpha : \alpha \in I\}$ be an open cover for $f(X) = Y$.

As f is Homomorphism so f is continuous then $V_\alpha = f^{-1}(U_\alpha)$ is open in X [$\because$ Inverse image of open set is open]

$\Rightarrow \{V_\alpha = f^{-1}(U_\alpha) : \alpha \in I\}$ is an open cover for X. Since X is compact so $\exists$ a subcover say $\{V_{\alpha_1}, V_{\alpha_2}, \cdots, V_{\alpha_n}\}$ s.t

$$X = \bigcup_{i=1}^{n} V_{\alpha_i} \Rightarrow X = \bigcup_{i=1}^{n} V_{\alpha_i}$$

$$\Rightarrow f(X) = f\left(\bigcup_{i=1}^{n} V_{\alpha_i}\right) \subseteq \bigcup_{i=1}^{n} f(V_{\alpha_i}) = \bigcup_{i=1}^{n} U_{\alpha_i} \subseteq f(X)$$

Hence $f(X) = \bigcup_{i=1}^{n} U_{\alpha_i}$

$$\Rightarrow Y = \bigcup_{i=1}^{n} U_{\alpha_i}$$ So Y is compact.

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Theorem The Homomorphic Image of a compact space is compact.

Proof) Let $f: X \to Y$ be a Homomorphism from topological space X to topological space Y.

To prove $f(X)$ is compact.

Rough

$$f\left(\bigcup_{i=1}^{n} V_{\alpha_i}\right) = f(V_{\alpha_1} \cup V_{\alpha_2} \cup \cdots \cup V_{\alpha_n})$$
$$= f(V_{\alpha_1}) \cup f(V_{\alpha_2}) \cup \cdots \cup f(V_{\alpha_n})$$
$$= \bigcup_{i=1}^{n} f(V_{\alpha_i})$$

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