Riemannian Geometry 12-9-14

A triplet $(\underline{t}, \underline{n}, \underline{b})$ is called moving Trihedron.

Locus of a point whose position vector is represented as a function of single parameter is called curve, if two parameters then it is called surface.

Derivative is the rate of change of dependent variable w.r.t independent variable.

volume of cylinder $= \pi r^2 h$

" " Sphere $= \frac{4}{3}\pi r^3$

" " Cone $= \frac{1}{3}\pi r^2 h$

union of line = plane

" " Curve = Surface

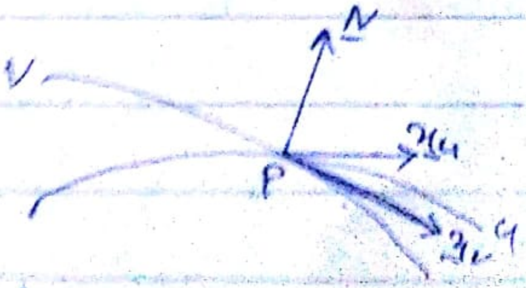
First fundamental form 12-9-14

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 \longrightarrow ①$$

Where

$$E = \underline{x}_u \cdot \underline{x}_u \,,\; F = \underline{x}_u \cdot \underline{x}_v \,,\; G = \underline{x}_v \cdot \underline{x}_v$$

Now we can express eq ① in terms of metric tensor as



$$ds^2 = g_{ab}\, dx^a dx^b$$

②

$$= g_{1b}\, dx^1 dx^b + g_{2b}\, dx^2 dx^b$$

$$= g_{11}\, dx^1 dx^1 + g_{21}\, dx^2 dx^1 + g_{12}\, dx^1 dx^2$$

$$+ g_{22}\, dx^2 dx^2$$

Let $x^1 = u$, $x^2 = v$

$$= g_{11}(du)^2 + (g_{12} + g_{21})\, du dv + g_{22}(dv)^2 \longrightarrow ②$$

Comparing eq. ① & eq. ②, we have

$g_{11} = E$, $g_{12} = g_{21} = F$, $g_{22} = G$

Then we have

$$g_{ab} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

$$= \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

Q7 We consider a plane in cartesian coordinates then find g_{ab} ?

$x_u = \hat{i}$, $x_v = \hat{j}$

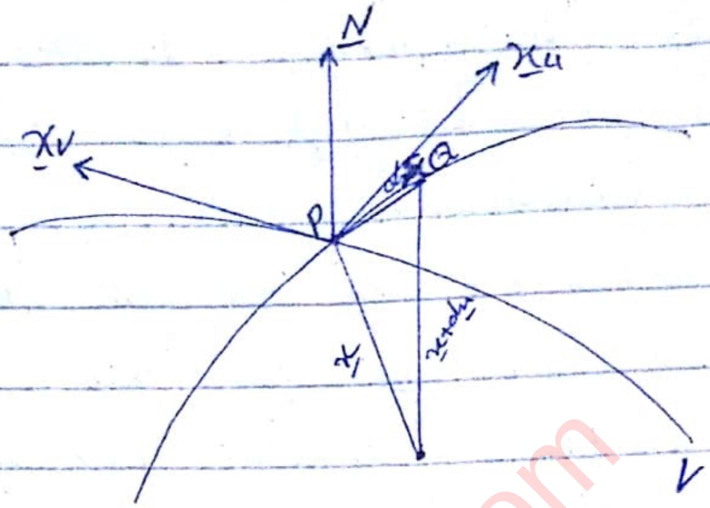
$E = \hat{i}\cdot\hat{i}$, $F = i\cdot j$, $G = j\cdot j$

$\Rightarrow$ $E = 1$, $F = 0$, $G = 1$

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

First Fundamental Form

15-9-14



$$\underline{x} = \underline{x}(u,v)$$

$$d\underline{x} = \underline{x}_u\, du + \underline{x}_v\, dv$$

$$= \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

If ds is the distance b/w two neighbouring points P & Q on the surface then,

$$ds^2 = d\underline{x} \cdot d\underline{x}$$

$$= (\underline{x}_u\, du + \underline{x}_v\, dv)(\underline{x}_u du + \underline{x}_v\, dv)$$

$$= \underline{x}_u \cdot \underline{x}_u\, du^2 + \underline{x}_u \cdot \underline{x}_v\, du\,dv + \underline{x}_v \cdot \underline{x}_u\, dv\,du + \underline{x}_v \cdot \underline{x}_v\, dv^2$$

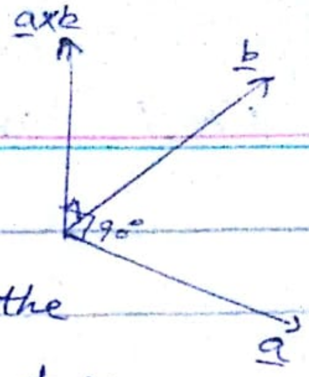
$$= \underline{x}_u \cdot \underline{x}_u\, du^2 + 2\,\underline{x}_u \cdot \underline{x}_v\, du\,dv + \underline{x}_v \cdot \underline{x}_v\, dv^2$$

Let $\underline{x}_u \cdot \underline{x}_u = E$, $\underline{x}_u \cdot \underline{x}_v = F$, $\underline{x}_v \cdot \underline{x}_v = G$

$$\Rightarrow ds^2 = E du^2 + 2F du\,dv + G dv^2$$

which is called First fundamental form due to the fact that there involve the first derivative quantities.

④



Normal:-

The normal to the surface at any point is the perpendicular to the parametric curves passing through that point.

If $\underline{x}_u$ & $\underline{x}_v$ are the unit tangents to the respective parametric curve, then we can write

$$\underline{N} = \frac{\underline{x}_u \times \underline{x}_v}{|\underline{x}_u \times \underline{x}_v|}$$

Obviously,

$$\underline{N} \cdot \underline{x}_u = 0 = \underline{x}_v \cdot \underline{N}$$

The tendency of turning of curve is called curvature of curve.

Normal Curvature:-

It is denoted by K_n and defined as,

$$K_n = \underline{t}' \cdot \underline{N}$$

As $\underline{t} \cdot \underline{N} = 0$

Diff. w.r.t s.

$$\underline{t}' \cdot \underline{N} + \underline{t} \cdot \underline{N}' = 0$$

$$\Rightarrow \underline{t}' \cdot \underline{N} = - \underline{t} \cdot \underline{N}'$$

$$\Rightarrow K_n = - \underline{t} \cdot \underline{N}'$$

⑤

$$\Rightarrow \kappa_n = -\underline{t}\cdot\underline{N}'$$
$$= -\frac{d\underline{x}}{ds}\cdot\frac{d\underline{N}}{ds} = -\frac{d\underline{x}\cdot d\underline{N}}{ds^2} \longrightarrow$$

As in general;

$$\underline{x} = \underline{x}(u,v) \quad \& \quad \underline{N} = \underline{N}(u,v)$$

$$d\underline{x} = \underline{x}_u\,du + \underline{x}_v\,dv, \quad d\underline{N} = \underline{N}_u\,du + \underline{N}_v\,dv$$

Then

$$ds_n^2 = d\underline{x}\cdot d\underline{N} = \underline{x}_u\cdot\underline{N}_u\,du^2 + \underline{x}_u\cdot\underline{N}_v\,du\,dv + \underline{x}_v\cdot\underline{N}_u\,du\,dv + \underline{x}_v\cdot\underline{N}_v\,dv^2$$

$$= \underline{x}_u\cdot\underline{N}_u\,du^2 + (\underline{x}_u\cdot\underline{N}_v + \underline{x}_v\cdot\underline{N}_u)\,du\,dv + \underline{x}_v\cdot\underline{N}_v\,dv^2$$

We know that

$$\underline{x}_u\cdot\underline{N} = 0 \quad \& \quad \underline{x}_v\cdot\underline{N} = 0$$

Diff. w.r.t. u & v, we have

$$\underline{x}_{uu}\cdot\underline{N} + \underline{x}_u\cdot\underline{N}_u = 0 \longrightarrow (i)$$

$$\underline{x}_{uv}\cdot\underline{N} + \underline{x}_u\cdot\underline{N}_v = 0 \longrightarrow (ii)$$

$$\underline{x}_{vu}\cdot\underline{N} + \underline{x}_v\cdot\underline{N}_u = 0 \longrightarrow (iii)$$

$$\underline{x}_{vv}\cdot\underline{N} + \underline{x}_v\cdot\underline{N}_v = 0 \longrightarrow (iv)$$

From (i)

$$\underline{x}_u\cdot\underline{N}_u = -\underline{x}_{uu}\cdot\underline{N} = e$$

from (iv) $\underline{x}_v\cdot\underline{N}_v = -\underline{x}_{vv}\,\underline{N} = g$

from (ii) & (iii)

$$\underline{x}_u\cdot\underline{N}_v = \underline{x}_v\cdot\underline{N}_u = -\underline{x}_{uv}\cdot\underline{N} = f$$

⑥

$$k_n = -\frac{d\underline{r}\cdot d\underline{N}}{ds^2}$$

neglecting -ive sign

$$= \frac{e\,du^2 + 2f\,du\,dv + g\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}$$

We can also write,

$$k_n = \frac{ds_n^2}{ds^2}, \text{ where}$$

$$ds_n^2 = e\,du^2 + 2f\,du\,dv + g\,dv^2$$

Which is known as second fundamental form because second ordered derivatives are used.

Q:- $x(u,v) = (Au + Qu + \alpha, Bu + bv + \beta, Cu + cv + \gamma)$ Find first and second fundamental form.

Sol:-

$$x(u,v) = (Au + av + \alpha, Bu + bv + \beta, Cu + cv + \gamma)$$

We know that

$$\underline{x}_u \cdot \underline{x}_u = E, \quad \underline{x}_u \cdot \underline{x}_v = F, \quad \underline{x}_v \cdot \underline{x}_v = G$$

$$\underline{N} = \frac{\underline{x}_u \times \underline{x}_v}{|x_u \times x_v|}$$

$$\underline{x}_u \cdot \underline{N}_u = e, \quad \underline{x}_u \cdot \underline{N}_v = f, \quad \underline{x}_v \cdot \underline{N}_v = g$$

$$k_n = -\frac{e\,du^2 + 2f\,du\,dv + g\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}$$

$\underline{x}_u = (A, B, C)$

$\underline{x}_v = (a, b, c)$

$E = \underline{x}_u \cdot \underline{x}_u = A^2 + B^2 + C^2$

$F = \underline{x}_u \cdot \underline{x}_v = aA + bB + cC$

$G = \underline{x}_v \cdot \underline{x}_v = a^2 + b^2 + c^2$

$$\underline{x}_u \times \underline{x}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A & B & C \\ a & b & c \end{vmatrix}$$

$$= \hat{i}(Bc - Cb) - \hat{j}(Ac - Ca) + \hat{k}(Ab - Ba)$$

$$= (Bc - Cb, Ca - Ac, Ab - Ba)$$

$$|\underline{x}_u \times \underline{x}_v| = \sqrt{(Bc - Cb)^2 + (Ca - Ac)^2 + (Ab - Ba)^2}$$

$$\underline{N} = \frac{(Bc - Cb, Ca - Ac, Ab - Ba)}{\sqrt{(Bc - Cb)^2 + (Ca - Ac)^2 + (Ab - Ba)^2}}$$

Now

$$\underline{N}_u = (0, 0, 0) = \underline{N}_v$$

&

$$\underline{x}_u \cdot \underline{N}_u = \underline{x}_u \cdot \underline{N}_v = \underline{x}_v \cdot \underline{N}_v = 0$$

$$\kappa_n = -\frac{0du^2 + 2(0)dudv + 0dv^2}{(A^2 + B^2 + C^2)du^2 + 2(Aa + Bb + Cc)dudv + (a^2 + b^2 + c^2)dv^2}$$

$$\kappa_n = 0$$

Principal Directions & principal curvature:-

16-9-14

The directions on a surface along which the normal curvature attain it's extreme values are called principal directions. The extreme values of normal curvature are denoted by κ_1 & κ_2 called principal curvature.

We know that

$$\kappa_n = \frac{e\,du^2 + 2f\,du\,dv + g\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}$$

To transform it into single parameter we divide the denominator and numerator by du^2 of R.H.S

$$\kappa_n = \frac{e + 2f\left(\frac{dv}{du}\right) + g\left(\frac{dv}{du}\right)^2}{E + 2F\left(\frac{dv}{du}\right) + G\left(\frac{dv}{du}\right)^2}$$

Put $\frac{dv}{du} = \lambda$

$$\kappa_n = \frac{e + 2f\lambda + g\lambda^2}{E + 2F\lambda + G\lambda^2} \longrightarrow ①$$

To find extreme values, we Diff. eq. ① w.r.t λ.

$$\frac{d\kappa_n}{d\lambda} = \frac{(E+2F\lambda+G\lambda^2)(2f+2g\lambda) - (e+2f\lambda+g\lambda^2)(2F+2G\lambda)}{(E+2F\lambda+G\lambda^2)^2}$$

$$\frac{d\kappa_n}{d\lambda} = \frac{\begin{matrix}2Ef+4Ff\lambda+2Gf\lambda^2+2Eg\lambda+4Fg\lambda^2+2gG\lambda^3 \\ -2eF-4Ff\lambda-2Fg\lambda^2-2eG\lambda-4fG\lambda^2-2gG\lambda^3\end{matrix}}{(E+2F\lambda+G\lambda^2)^2}$$

$$\frac{d\kappa_n}{d\lambda} = \frac{2Ef-2fG\lambda^2+2Eg\lambda+2Fg\lambda^2-2eF-2eG\lambda}{(E+2F\lambda+G\lambda^2)^2}$$

After adding and subtracting $2fF\lambda$.

$$\frac{d\kappa_n}{d\lambda} = \frac{2\left[\begin{matrix}Ef+fF\lambda+Eg\lambda+Fg\lambda^2 \\ -eF-fF\lambda-eG\lambda-fG\lambda^2\end{matrix}\right]}{(E+2F\lambda+G\lambda^2)^2}$$

$$= \frac{2\left[f(E+F\lambda)+g\lambda(E+F\lambda)-\{F(e+f\lambda)+G\lambda(e+f\lambda)\}\right]}{(E+2F\lambda+G\lambda^2)^2}$$

$$= \frac{2\left[(E+F\lambda)(f+g\lambda)-(e+f\lambda)(F+G\lambda)\right]}{(E+2F\lambda+G\lambda^2)^2}$$

For extreme values we put $\frac{d\kappa_n}{d\lambda}=0$

$$\Rightarrow (E+F\lambda)(f+g\lambda)-(e+f\lambda)(F+G\lambda)=0$$

$$\Rightarrow (E+F\lambda)(f+g\lambda)=(e+f\lambda)(F+G\lambda)$$

$$\left.\begin{array}{l}\dfrac{e+f\lambda}{f+g\lambda} = \dfrac{E+F\lambda}{F+G\lambda} \rightarrow (i) \\ \text{or} \\ \dfrac{f+g\lambda}{e+f\lambda} = \dfrac{F+G\lambda}{E+F\lambda} \rightarrow (ii)\end{array}\right\} \rightarrow ②$$

Now eq. ① can be written as

$$K_n = \frac{e+f\lambda+f\lambda+g\lambda^2}{E+F\lambda+F\lambda+G\lambda^2}$$

$$= \frac{e+f\lambda+\lambda(f+g\lambda)}{E+F\lambda+\lambda(F+G\lambda)} \longrightarrow ③$$

$$= \frac{(e+f\lambda)\left[1+\dfrac{\lambda(f+g\lambda)}{(e+f\lambda)}\right]}{(E+F\lambda)\left[1+\dfrac{\lambda(F+G\lambda)}{(E+F\lambda)}\right]}$$

using (ii) here,

$$K_n = \frac{(e+f\lambda)\,\cancel{\left[1+\dfrac{(f+g\lambda)}{(e+f\lambda)}\lambda\right]}}{(E+F\lambda)\,\cancel{\left[1+\dfrac{(f+g\lambda)}{(e+f\lambda)}\lambda\right]}}$$

$$K_1 = \frac{e+f\lambda}{E+F\lambda}$$

Now using eq. ③ we can also write as:

⑪

$$\kappa_n = \frac{(f+g\lambda)\left[\frac{e+f\lambda}{f+g\lambda}+\lambda\right]}{(F+G\lambda)\left[\frac{E+F\lambda}{F+G\lambda}+\lambda\right]}$$

Using eq ② here we get.

$$\kappa_n = \frac{(f+g\lambda)\cancel{\left[\frac{e+f\lambda}{f+g\lambda}+\lambda\right]}}{(F+G\lambda)\cancel{\left[\frac{e+f\lambda}{f+g\lambda}+\lambda\right]}}$$

$$\kappa_n = \frac{f+g\lambda}{F+G\lambda}$$

18-9-14

To obtain the value of λ, we solve the eq ②.

$$\frac{e+f\lambda}{f+g\lambda} = \frac{E+F\lambda}{F+G\lambda}$$

$$eF+eG\lambda+\cancel{fF\lambda}+fG\lambda^2 = Ef+Eg\lambda+\cancel{Ff\lambda}+Fg\lambda^2$$

$$fG\lambda^2-Fg\lambda^2+eG\lambda-Eg\lambda+eF-Ef=0$$

$$(fG-Fg)\lambda^2+(eG-Eg)\lambda+eF-Ef=0 \longrightarrow ④$$

Let λ_1 & λ_2 be two roots of this eq. then

$$\lambda_1+\lambda_2 = \frac{-b}{a} = \frac{Eg-eG}{fG-Fg} \longrightarrow (a)$$

$$\lambda_1 \cdot \lambda_2 = \frac{c}{a} = \frac{eF-Ef}{fG-Fg} \longrightarrow (b)$$

We know that

$$(\lambda_1-\lambda_2)^2=(\lambda_1+\lambda_2)^2-4\lambda_1\lambda_2$$

After putting the value of $\lambda_1+\lambda_2$ & $\lambda_1\lambda_2$ from ⓐ & ⓑ. we get

$$(\lambda_1-\lambda_2)^2=\left(\frac{Eg-eG}{fG-Fg}\right)^2-4\left(\frac{eF-Ef}{fG-Fg}\right)$$

$$(\lambda_1-\lambda_2)^2=\frac{(Eg-eG)^2}{(fG-Fg)^2}-4\frac{eF-Ef}{fG-Fg}$$

$$(\lambda_1-\lambda_2)^2=\frac{(Eg-eG)^2-4(eF-Ef)(fG-Fg)}{(fG-Fg)^2}$$

$$(\lambda_1-\lambda_2)=\frac{\sqrt{(Eg-eG)^2-4(eF-Ef)(fG-Fg)}}{fG-Fg} \longrightarrow (c)$$

Adding (a) and (c).

$$2\lambda_1=\frac{Eg-eG}{fG-Fg}+\frac{\sqrt{(Eg-eG)^2-4(eF-Ef)(fG-Fg)}}{fG-Fg}$$

$$\lambda_1=\frac{(Eg-eG)+\sqrt{(Eg-eG)^2-4(eF-Ef)(fG-Fg)}}{2(fG-Fg)}$$

$$ⓐ\Rightarrow \lambda_2=\frac{Eg-eG}{fG-Fg}-\lambda_1$$ gives the corresponding value of λ_2

Putting the value of λ_1 we get

$$\lambda_2 = \frac{Eg-eG}{fG-Fg} - \frac{Eg-eG+\sqrt{(Eg-eG)^2-4(eF-Ef)(fG-Fg)}}{2(fG-Fg)}$$

$$\lambda_2 = \frac{(Eg-eG)-\sqrt{(Eg-eG)^2-4(eF-Ef)(fG-Fg)}}{2(fG-Fg)}$$

Now,

Subtituting $\lambda = \frac{dv}{du}$ in eq ④ and then multiplying du^2, we get.

$$(fG-Fg)\,dv^2+(eG-Eg)\,du\,dv+(eF-Ef)\,du^2=0$$

Multiplying by negative.

$$(Fg-fG)\,dv^2+(Eg-eG)\,du\,dv+(Ef-eF)\,du^2=0.$$

We can write this eq. in determinant form as;

$$\begin{vmatrix} dv^2 & -du\,dv & du^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0 \longrightarrow ⑤$$

The max. and min. values of λ are given as;

$du=0$ $(dv\neq 0)$ & $dv=0$ $(du\neq 0)$

respectively.

For $du=0$ eq-⑤ becomes.

$$\begin{vmatrix} dv^2 & 0 & 0 \\ E & F & G \\ e & f & g \end{vmatrix} = 0$$

$$dv^2(Fg - fG) = 0$$

$$\Rightarrow dv \neq 0, \quad \boxed{Fg - fG = 0}$$

Now for $dv=0$ eq-⑤ becomes.

$$\begin{vmatrix} 0 & 0 & du^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0$$

$$\Rightarrow du^2(Ef - Fe) = 0$$

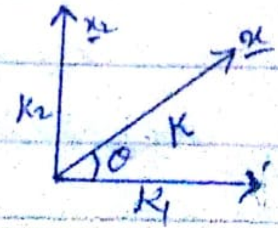
$$\Rightarrow du \neq 0, \quad \boxed{Ef - Fe = 0}$$

Euler's Theorem

23-9-14

If K is the curvature along any direction and K_1, K_2 are the extreme values the euler's theorem states that

$$K = K_1 \cos^2\theta + K_2 \sin^2\theta$$



Where θ is the angle b/w directions ~~and~~ K & K_1.

Note:- We can define a curve on a surface But converse is not true.

Q:- $\underline{x}(u,v) = (\sin u \cos v, \sin u \sin v, \cos u)$

Sol:-

$$x_u = (\cos u \cos v, \cos u \sin v, -\sin u)$$

& $$x_v = (-\sin u \sin v, \sin u \cos v, 0)$$

$$E = x_u \cdot x_u = \cos^2 u \cos^2 v + \cos^2 u \sin^2 v + \sin^2 u$$

$$\Rightarrow \quad E = \cos^2 u + \sin^2 u$$

$$\Rightarrow \boxed{E = 1}$$

$$F = x_u \cdot x_v = -\sin u \cancel{\sin v} \cos u \cos v + \sin u \cancel{\sin v} \cos u \cos v + 0$$

$$\Rightarrow \quad \boxed{F = 0}$$

$$G = x_v \cdot x_v = \sin^2 u \sin^2 v + \sin^2 u \cos^2 v + 0$$

$$\Rightarrow \boxed{G = \sin^2 u}$$

$$x_u \times x_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos u \cos v & \cos u \sin v & -\sin u \\ -\sin u \sin v & \sin u \cos v & 0 \end{vmatrix}$$

$$= \hat{i}(\sin^2 u \cos v) - \hat{j}(-\sin^2 u \sin v) + \hat{k}(\cos u \cos^2 v \sin u + \cos u \sin^2 v \sin u)$$

$$= \hat{i}(\sin^2 u \cos v) + \hat{j}(\sin^2 u \sin v) + \hat{k}(\sin u \cos u)$$

$$|x_u \times x_v| = \sqrt{\sin^4 u \cos^2 v + \sin^4 u \sin^2 v + \sin^2 u \cos^2 u}$$

$$= \sqrt{\sin^4 u + \sin^2 u \cos^2 u}$$

$$= \sqrt{\sin^2 u} = \sin u$$

$$\underline{N} = \frac{\underline{x}_u \times \underline{x}_v|}{|\underline{x}_u \times \underline{x}_v|}$$

$$\underline{N} = \frac{1}{\sin u}\left(\cos v \sin^2 u\, \hat{i} + \sin^2 u \sin v\, \hat{j} + \sin u \cos u\, \hat{k}\right)$$

$$\underline{N} = (\cos v \sin u, \sin u \sin v, \cos u)$$

$$\underline{N}_u = (\cos v \cos u, \cos u \sin v, -\sin u)$$

&

$$\underline{N}_v = (-\sin u \sin v, \sin u \cos v, 0)$$

$$e = \underline{x}_u \cdot \underline{N}_u = (\cos u \cos v, \cos u \sin v, -\sin u).$$
$$(\cos u \cos v, \cos u \sin v, -\sin u)$$

$$\Rightarrow e = (\cos^2 u \cos^2 v + \cos^2 u \sin^2 v + \sin^2 u)$$

$$\Rightarrow e = (\cos^2 u + \sin^2 u)$$

$$\boxed{e = 1}$$

$$f = \underline{x}_u \cdot \underline{N}_v = (\cos u \cos v, \cos u \sin v, -\sin u).$$
$$(-\sin u \sin v, \sin u \cos v, 0)$$

$$\Rightarrow f = (\cancel{-\sin u \sin v \cos u \cos v} + \cancel{\sin u \sin v \cos u \cos v} + 0)$$

$$\Rightarrow \boxed{f = 0}$$

$$g = \underline{x}_v \cdot \underline{N}_v = (-\sin u \sin v, \sin u \cos v, 0).$$
$$(-\sin u \sin v, \sin u \cos v, 0)$$

$$g = (\sin^2 u \sin^2 v + \sin^2 u \cos^2 v)$$

$$\boxed{g = \sin^2 u}$$

$$k_n = -\frac{e\,du^2 + 2f\,du\,dv + g\,du^2}{E\,du^2 + 2F\,du\,dv + G\,du^2}$$

$$k_n = -\left(\frac{1\,du^2 + 2(0)\,du\,dv + (\sin^2)du^2}{1\,du^2 + 2(0)\,du\,dv + \sin^2 du^2}\right)$$

$$k_n = \frac{-(\cancel{du^2 + \sin^2 du^2})}{\cancel{du^2 + \sin^2 du^2}}$$

$$\boxed{k_n = -1}$$

Q:- $\underline{x}(u,v) = (v\cos u, v\sin u, v)$

Sol:- $\underline{x}_u = (-v\sin u, v\cos u, 0)$

& $\underline{x}_v = (\cos u, \sin u, 1)$

$E = \underline{x}_u \cdot \underline{x}_u = v^2\sin^2 u + v^2\cos^2 u$

$\Rightarrow \boxed{E = v^2}$

$F = \underline{x}_u \cdot \underline{x}_v = -v\sin u\cos u + v\sin u\cos v$

$\Rightarrow \boxed{F = 0}$

$G = \underline{x}_v \cdot \underline{x}_v = \cos^2 u + \sin^2 u + 1$

$\Rightarrow \boxed{G = 2}$

$$\underline{x}_u \times \underline{x}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 1 \end{vmatrix}$$

$$= \hat{i}\, v\cos u - \hat{j}\,(-v\sin u) + \hat{k}\,(-v\sin^2 u - v\cos^2 u)$$

$$= (v\cos u\,,\, v\sin u,\, -v)$$

$$|\underline{x}_u \times \underline{x}_v| = \sqrt{v^2\cos^2 u + v^2\sin^2 u + v^2}$$

$$= \sqrt{2v^2}$$

$$= \sqrt{2}\, v$$

$$\underline{N} = \frac{\underline{x}_u \times \underline{x}_v}{|\underline{x}_u \times \underline{x}_v|}$$

$$= \frac{1}{\sqrt{2}\, v}(v\cos u, v\sin u, -v)$$

$$= \frac{1}{\sqrt{2}}(\cos u, \sin u, -1)$$

Now

$$\underline{N}_u = \frac{1}{\sqrt{2}}(-\sin u, \cos u, 0)$$

&

$$\underline{N}_v = \frac{1}{\sqrt{2}}(0, 0, 0)$$

$$\Rightarrow \underline{N}_v = (0, 0, 0)$$

$$e = \underline{x}_u \cdot \underline{N}_u = (-v\sin u, v\cos u, 0) \cdot \frac{1}{\sqrt{2}}(-\sin u, \cos u, 0)$$

$$\Rightarrow e = \frac{1}{\sqrt{2}}(v\sin^2 u + v\cos^2 u)$$

$$\Rightarrow \boxed{e = \frac{v}{\sqrt{2}}}$$

$f = \underline{x}_u \cdot \underline{N}_v = 0$

$g = \underline{x}_v \cdot \underline{N}_v = 0$

$$\kappa_n = -\frac{e\,du^2 + 2f\,du\,dv + g\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}$$

$$\kappa_n = -\frac{\frac{v}{\sqrt{2}}\,du^2 + 0 + 0}{v^2 du^2 + 0 + 2\,dv^2}$$

$$\boxed{\kappa_n = \frac{-v\,du^2}{\sqrt{2}\,(v^2 du^2 + 2dv^2)}}$$

25-09-14

Orthogonal vectors:→ $\vec{A}\cdot\vec{B} = 0$

Orthonormal vectors:→ $\hat{i}\cdot\hat{j} = 0$

Metric Tensor:→ $g_{ab} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = (E, F \vdots F\ G)$

Orthogonality Condition

Two vectors ξ^i & η^j are said to be orthogonal if

$$g_{ij}\,\xi^i \eta^j = 0$$

To check Orthogonality:-

Let $\xi^i = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$ & $\eta^j = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$

Consider

$$g_{ij}\xi^i\eta^j = (E\ \ F : F\ \ G)\begin{pmatrix}1\\ \lambda_1\end{pmatrix}\begin{pmatrix}1\\ \lambda_2\end{pmatrix}$$

$$= (E + F\lambda_1 : F + G\lambda_1)\begin{pmatrix}1\\ \lambda_2\end{pmatrix}$$

$$= E + F\lambda_1 + F\lambda_2 + G\lambda_1\lambda_2$$

$$= E + (\lambda_1 + \lambda_2)F + G\lambda_1\lambda_2 \longrightarrow ①$$

We know that

$$\lambda_1 + \lambda_2 = \frac{Eg - eG}{fG - gF} \longrightarrow ②$$

&

$$\lambda_1\lambda_2 = \frac{eF - fE}{fG - gF} \longrightarrow ③$$

Using ② and ③ in ① we have

$$g_{ij}\xi^i\eta^j = E + \left(\frac{Eg - eG}{fG - gF}\right)F + \left(\frac{eF - fE}{fG - gF}\right)G$$

$$= \frac{f\cancel{EG} - g\cancel{EF} + g\cancel{EF} - e\cancel{GF} + e\cancel{GF} - f\cancel{EG}}{fG - gF}$$

$$= 0$$

$\Rightarrow$ ξ^i & η^j are orthogonal.

Gaussian & Mean Curvature

If κ_1 & κ_2 are the extreme values of the principal curvature the guassian curvature is defined as,

$$\kappa_{G} = \kappa_1 \kappa_2$$ (Product of the principal curvature)

and the mean curvature is defined as

$$\kappa_m = \frac{\kappa_1 + \kappa_2}{2}$$

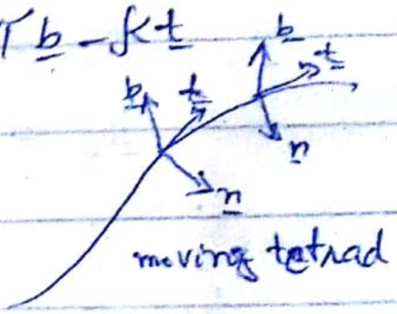
S.F. formulae

$\underline{t}' = \kappa \underline{n}$, $\underline{b}' = -\tau \underline{n}$, $\underline{n}' = \tau \underline{b} - \kappa \underline{t}$

$(\underline{b}, \underline{t}, \underline{n})$ are orthonomal.
→ for curve.

$(\underline{x}_u, \underline{x}_v, \underline{N})$ → For surface.
→ In general they are not orthogonal.

because we don't know $\underline{x}_u$ & $\underline{x}_v$ are $\perp$ or not.



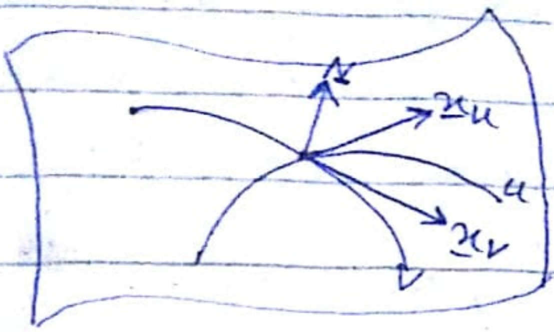
We can also write S.F formulae as

$$\underline{t}' = 0\underline{t} + \kappa \underline{n} + 0\underline{b}$$

$$\underline{n}' = -\kappa \underline{t} + 0\underline{n} + \tau \underline{b}$$

$$\underline{b}' = 0\underline{t} - \tau \underline{n} + 0\underline{b}$$

Gauss Eq — 29-9-14



let us denote.

$\underline{x}_u = \underline{x}_1$, $\underline{x}_v = \underline{x}_2$, $\underline{x}_{uv} = \underline{x}_{12} = \underline{x}_{21}$, $\underline{x}_{vv} = \underline{x}_{22}$

and write

$$\underline{x}_{11} = \alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2 + \alpha_3 \underline{N} \longrightarrow ①$$

$$\underline{x}_{12} = \underline{x}_{21} = \beta_1 \underline{x}_1 + \beta_2 \underline{x}_2 + \beta_3 \underline{N} \longrightarrow ②$$

$$\underline{x}_{22} = \gamma_1 \underline{x}_1 + \gamma_2 \underline{x}_2 + \gamma_3 \underline{N} \longrightarrow ③$$

Where α's, β's & γ's are constants.

$$g_{ij} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \qquad \bar{g}_{ij} = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

$$\Rightarrow \quad g^{ij} = \frac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \qquad \bar{g}^{ij} = \frac{1}{eg-f^2}\begin{pmatrix} g & -f \\ -f & e \end{pmatrix}$$

Taking dot product with $\underline{N}$ on both sides of eq. ①.

$$\underline{x}_{11} \cdot \underline{N} = \alpha_1 \underbrace{\cancel{\underline{x}_1 \cdot \underline{N}}}_{0} + \alpha_2 \underbrace{\cancel{\underline{x}_2 \cdot \underline{N}}}_{0} + \alpha_3 \underline{N} \cdot \underline{N}$$

$$\underline{x}_{11} \cdot \underline{N} = \alpha_3 \qquad \because \underline{N} \cdot \underline{N} = 1$$

$$\Rightarrow \alpha_3 = \underline{x}_{11} \cdot \underline{N} = -e \longrightarrow ④$$

Taking dot product with N on both sides of eq. ②

$$\underline{x}_{21}\cdot\underline{N} = \beta_1 \underset{0}{\cancel{\underline{x}_1\cdot\underline{N}}} + \beta_2 \underset{0}{\cancel{\underline{x}_2\cdot\underline{N}}} + \beta_3\,\underline{N}\cdot\underline{N}$$

$$\Rightarrow \underline{x}_{21}\cdot\underline{N} = \beta_3$$

$$\Rightarrow \beta_3 = \underline{x}_{21}\cdot\underline{N} = -f \longrightarrow ⑤$$

Taking dot product with N on both sides of eq. ③

$$\underline{x}_{22}\cdot\underline{N} = \gamma_1\,\underline{x}_1\cdot\underline{N} + \gamma_2\,\underline{x}_2\cdot\underline{N} + \gamma_3\,\underline{N}\cdot\underline{N}$$

$$\underline{x}_{22}\cdot\underline{N} = \gamma_3$$

$$\Rightarrow \gamma_3 = \underline{x}_{22}\cdot\underline{N} = -g \longrightarrow ⑥$$

We know that

$$\underline{x}_1\cdot\underline{x}_1 = E \longrightarrow ⑦$$

Diff. eq 7 w.r.t 1.

$$\underline{x}_{11}\cdot\underline{x}_1 = \frac{1}{2}E_1 \longrightarrow ⑧$$

Again diff. eq. (7) w.r.t 2

$$\underline{x}_{12}\cdot\underline{x}_1 = \frac{1}{2}E_2 \longrightarrow ⑨$$

Also we know that

$$\underline{x}_1\cdot\underline{x}_2 = F \longrightarrow ⑩$$

Diff. eq. ⑩ w.r.t 1 & 2 we have

$$\underline{x}_{11}\cdot\underline{x}_2 + \underline{x}_1\cdot\underline{x}_{21} = F_1 \longrightarrow ⑪$$

&

$$\underline{x}_{12}\cdot\underline{x}_2 + \underline{x}_1\cdot\underline{x}_{22} = F_2 \longrightarrow ⑫$$

Also

$$\underline{x}_2 \cdot \underline{x}_2 = G \longrightarrow (13)$$

Diff. eq. (13) w.r.t. 1 & 2. we have

$$\underline{x}_{21} \cdot \underline{x}_2 = \frac{1}{2} G_1 \longrightarrow (14)$$

&

$$\underline{x}_{22} \cdot \underline{x}_2 = \frac{1}{2} G_2 \longrightarrow (15)$$

Let us denote; 30-9-14

$$\left.\begin{aligned}
\underline{x}_1 \cdot \underline{x}_{11} &= [1,11] = \frac{1}{2} g_{11,1} \\
\underline{x}_1 \cdot \underline{x}_{12} &= [1,12] = \frac{1}{2} g_{11,2} \\
\underline{x}_1 \cdot \underline{x}_{22} &= [1,22] = g_{21,2} - \frac{1}{2} g_{22,1} \\
\underline{x}_2 \cdot \underline{x}_{11} &= [2,11] = g_{21,2} - \frac{1}{2} g_{11,2} \\
\underline{x}_2 \cdot \underline{x}_{21} &= [2,21] = \frac{1}{2} g_{21,1} \\
\underline{x}_2 \cdot \underline{x}_{22} &= [2,22] = \frac{1}{2} g_{22,2}
\end{aligned}\right\} \longrightarrow (16)$$

These are called Christofall symboles of First kind.

Now taking dot product of eq. (1) with x_1 & x_2 we have

$$\underline{x}_1 \cdot \underline{x}_{11} = \alpha_1 \underline{x}_1 \cdot \underline{x}_1 + \alpha_2 \underline{x}_1 \cdot \underline{x}_2 + \alpha_3 \underbrace{\underline{x}_1 \cdot \underline{N}}_{0}$$

$$[1,11] = \alpha_1 E + \alpha_2 F \longrightarrow (17)$$

&

$$\underline{x}_2 \cdot \underline{x}_{11} = \alpha_1 F + \alpha_2 G$$

$$[2,11] = \alpha_1 F + \alpha_2 G \longrightarrow (18)$$

Multiplying eq. ⑰ with F & eq ⑱ with E we have

$$[2,11]E = \alpha_1 E\cancel{F} + \alpha_2 GE$$

$$\underline{\underset{-}{[1,11]}F = \underset{-}{\alpha_1}\cancel{E}F \underset{-}{+} \alpha_2 F^2}$$

$$[2,11]E - [1,11]F = \alpha_2 (GE - F^2)$$

$$\alpha_2 = \frac{[2,11]E - [1,11]F}{EG - F^2} \longrightarrow$$ ⑲

We can write this

$$\alpha_2 = [2,11]\frac{E}{EG-F^2} + [1,11]\left(\frac{-F}{EG-F^2}\right)$$

$$\Rightarrow \alpha_2 = [2,11]\, g^{22} + [1,11]\, g^{21} \longrightarrow (i)$$

Put eq. ⑲ in eq. ⑰.

$$[1,11] = \alpha_1 E + F\left[\frac{[2,11]E - [1,11]F}{EG - F^2}\right].$$

$$\alpha_1 E = [1,11] - \frac{[2,11]FE - [1,11]F^2}{EG - F^2}$$

$$\alpha_1 = \frac{1}{E}\ \frac{[1,11]EG - \cancel{[1,11]F^2} - [2,11]FE + \cancel{[1,11]F^2}}{EG - F^2}$$

$$\alpha_1 = \frac{1}{\cancel{E}} \cdot \frac{\cancel{E}\left[[1,11]G - [2,11]F\right]}{EG - F^2}$$

$$\alpha_1 = \frac{[1,11]G - [2,11]F}{EG - F^2} \longrightarrow$$ ⑳

We can write it as.

$$\alpha_1 = [1,11]\frac{G}{EG-F^2} + [2,11]\frac{-F}{EG-F^2}$$

$$\alpha_1 = [1,11]g^{11} + [2,11]g^{12} \longrightarrow xii$$

Similarly taking dot product of $\underline{x}_1$ & $\underline{x}_2$ with eq. ② . we have,

$$\underline{x}_1\cdot\underline{x}_{12} = \beta_1 \underline{x}_1\cdot\underline{x}_1 + \beta_2 \underline{x}_1\cdot\underline{x}_2 + \beta_3 \underset{0}{\cancel{\underline{x}_1\cdot\underline{N}}}$$

$$[1,12] = \beta_1 E + \beta_2 F \longrightarrow (21)$$

$$\& \ [2,12] = \beta_1 F + \beta_2 G \longrightarrow (22)$$

Multiplying F with eq.(21) & E with eq(22) and we have,

$$\begin{array}{l} [1,12]F = \beta_1 E\cancel{F} + \beta_2 F^2 \\ -[2,12]E \underset{-}{=} \beta_1 \cancel{EF} \underset{-}{+} \beta_2 EG \\ \hline [1,12]F - [2,12]E = -\beta_2(EG-F^2) \end{array}$$

$$\beta_2(EG-F^2) = [2,12]E - [1,12]F$$

$$\beta_2 = \frac{[2,12]E - [1,12]F}{EG-F^2}$$

$$\beta_2 = [2,12]\frac{E}{EG-F^2} + [1,12]\frac{-F}{EG-F^2}$$

$$\beta_2 = [2,12]g^{22} + [1,12]g^{21} \longrightarrow (iii)$$

xing G with eq.(21) & F with eq.(22)

$$[1,12]G = \beta_1 GE + \beta_2 \cancel{GF}$$

$$-[2,12]F \underset{-}{=} \beta_1 F^2 \underset{-}{+} \beta_2 \cancel{GF}$$

$$[1,12]G - [2,12]F = \beta_1 (GE - F^2)$$

$$\beta_1 = [1,12]\frac{G}{GE-F^2} + [2,12]\frac{-F}{GE-F^2}$$

$$\beta_1 = [1,12]\, g^{11} + [2,12]\, g^{12} \longrightarrow (iv)$$

Similarly taking dot product of x_1 & x_2 with eq.(3), we have

$$\underline{x}_1 \cdot \underline{x}_{22} = \gamma_1 \underline{x}_1 \cdot \underline{x}_1 + \gamma_2 \underline{x}_1 \cdot \underline{x}_2 + \gamma_3 \underline{x}_1 \cdot \underline{N}$$

$$[1,22] = \gamma_1 E + \gamma_2 F \longrightarrow (23)$$

$$\& \quad [2,22] = \gamma_1 F + \gamma_2 G \longrightarrow (24)$$

eq (23) and (24) $\Longrightarrow$

$$[2,22]E = \gamma_1 \cancel{EF} + \gamma_2 EG$$

$$-[1,22]F \underset{-}{=} \gamma_1 \cancel{EF} \underset{-}{+} \gamma_2 F^2$$

$$[2,22]E - [1,22]F = \gamma_2 (GE - F^2)$$

$$\gamma_2 = [2,22]\frac{E}{EG-F^2} + [1,22]\frac{-F}{EG-F^2}$$

$$\gamma_2 = [2,22]\, g^{22} + [1,22]\, g^{21} \longrightarrow (V)$$

also eq. ㉓ & ㉔ ⇒

$$[1,22]G = \gamma_1 EG + \cancel{\gamma_2 FG}$$

$$-[2,22]F = -\gamma_1 F^2 \pm \cancel{\gamma_2 FG}$$

$$[1,22]G - [2,22]F = \gamma_1 (EG - F^2)$$

$$\gamma_1 = [1,22]\frac{G}{EG-F^2} + [2,22]\frac{-F}{EG-F^2}$$

$$\gamma_1 = [1,22]\, g^{11} + [2,22]\, g^{12} \longrightarrow (\text{vi})$$

We can write eq. (i), (ii), (III), (IV), (v) & (VI) as

02-10-2014

$$(\text{ii}) \Rightarrow \alpha_1 = g^{1i}[i,11] \longrightarrow (\text{VII}) \qquad i = 1,2$$

$$(\text{i}) \Rightarrow \alpha_2 = g^{2i}[i,11] \longrightarrow (\text{VIII})$$

$$(\text{iv}) \Rightarrow \beta_1 = g^{1i}[i,12] \longrightarrow (\text{ix})$$

$$(\text{iii}) \Rightarrow \beta_2 = g^{2i}[i,12] \longrightarrow (\text{x})$$

$$(\text{vi}) \Rightarrow \gamma_1 = g^{1i}[i,22] \longrightarrow (\text{xi})$$

$$(\text{v}) \Rightarrow \gamma_2 = g^{2i}[i,22] \longrightarrow (\text{xii})$$

By combining eq. (VII) & eq. (VIII)

$$\Rightarrow \alpha_K = g^{Ki}[i,11] \longrightarrow (\text{XIII}) \qquad K = 1,2$$

$$(\text{ix}) \,\&\, (\text{x}) \Rightarrow \beta_K = g^{Ki}[i,12] \longrightarrow (\text{xiv})$$

$$(\text{xi}) \,\&\, (\text{xii}) \Rightarrow \gamma_K = g^{Ki}[i,22] \longrightarrow (\text{xv})$$

$$\alpha_K = \Gamma^K_{11}, \quad \beta_K = \Gamma^K_{12} = \Gamma^K_{21}, \quad \gamma_K = \Gamma^K_{22}$$

Eq. (1) can be written as

$$\underset{\sim}{x}_{11} = \alpha_1 \underset{\sim}{x}_1 + \alpha_2 \underset{\sim}{x}_2 + \alpha_3 \underset{\sim}{N}$$

$$\Rightarrow \quad \underset{\sim}{x}_{11} = \alpha_K \underset{\sim}{x}_K + \alpha_3 \underset{\sim}{N} \qquad K=1,2$$

$$= \Gamma^K_{11} \underset{\sim}{x}_K - e \underset{\sim}{N}$$

$$= \Gamma^K_{11} \underset{\sim}{x}_K - \tilde{g}_{11} \underset{\sim}{N} \longrightarrow (XVi)$$

Similarly ② & ③ gives

$$\underset{\sim}{x}_{12} = \Gamma^K_{12} \underset{\sim}{x}_K - \tilde{g}_{12} \underset{\sim}{N} \longrightarrow (Xvii)$$

&

$$\underset{\sim}{x}_{22} = \Gamma^K_{22} \underset{\sim}{x}_K - \tilde{g}_{22} \underset{\sim}{N} \longrightarrow (Xviii)$$

Combining (XVi), (Xvii) & (Xviii).

$$\underset{\sim}{x}_{ij} = \Gamma^K_{ij} \underset{\sim}{x}_K - \tilde{g}_{ij} \underset{\sim}{N} \qquad i,j,K=1,2$$

This is general form of equation which is called Guass eq.

14-10-14

Now we will derive Wiengarten eqs

We know that

$\underline{N}\cdot\underline{N}=1$

Diff. w.r.t 1 & 2.

$\underline{N}\cdot\underline{N}_1=0$

& $\underline{N}\cdot\underline{N}_2=0$

$\Rightarrow$ This shows that $\underline{N}_1$ & $\underline{N}_2$ are $\perp$ to $\underline{N}$. And also $\underline{N}$ is $\perp$ to the plane form by $\underline{x}_1$ & $\underline{x}_2$.

In other words $\underline{N}_1$ & $\underline{N}_2$ will lie in the plane of $\underline{x}_1$ & $\underline{x}_2$ and hence can be written as (the vectors $\underline{N}_1$ & $\underline{N}_2$ written as linear combination of tangent vectors $\underline{x}_1$ & $\underline{x}_2$ as)

$$\underline{N}_1=P_1\underline{x}_1+P_2\underline{x}_2\longrightarrow①$$

$$\& \quad \underline{N}_2=q_1\underline{x}_1+q_2\underline{x}_2\longrightarrow②$$

Where P's & q's are constants.

Now taking dot product of eq. ① with $\underline{x}_1$ & $\underline{x}_2$.

$$\underline{N}_1\cdot\underline{x}_1=P_1\underline{x}_1\cdot\underline{x}_1+P_2\underline{x}_2\cdot\underline{x}_1$$

$$e=P_1E+P_2F\longrightarrow③$$

$$\& \quad \underline{N}_1\cdot\underline{x}_2=P_1\underline{x}_1\cdot\underline{x}_2+P_2\underline{x}_2\cdot\underline{x}_2$$

$$\Rightarrow \quad f = P_1 F + P_2 G \longrightarrow ④$$

Similarly, taking dot product of eq ② with x_1 & x_2.

$$\underline{N}_2 \cdot \underline{x}_1 = q_1\, \underline{x}_1 \cdot \underline{x}_1 + q_2\, \underline{x}_2 \cdot \underline{x}_1$$

$$f = q_1 E + q_2 F \longrightarrow ⑤$$

&

$$\underline{N}_2 \cdot \underline{x}_2 = q_1\, \underline{x}_1 \cdot \underline{x}_2 + q_2\, \underline{x}_2 \cdot \underline{x}_2$$

$$g = q_1 F + q_2 G \longrightarrow ⑥$$

③ & ④ $\Rightarrow$

$$eF = P_1 \cancel{EF} + P_2 F^2$$

$$\underline{\underset{-}{f}E \underset{-}{=} \underset{-}{P_1}\cancel{EF} \pm P_2 EG}$$

$$eF - fE = P_2 (F^2 - EG)$$

$$\Rightarrow \boxed{P_2 = \frac{fE - eF}{EG - F^2}}$$

also ③ & ④ $\Rightarrow$

$$eG = P_1 EG + P_2 \cancel{GF}$$

$$\underline{\underset{-}{f}F \underset{-}{=} \underset{-}{P_1} F^2 \pm P_2\, GF}$$

$$eG - fF = P_1 (EG - F^2)$$

$$\Rightarrow \boxed{P_1 = \frac{eG - fF}{EG - F^2}}$$

Now ⑤ & ⑥ ⇒

$$fF = \cancel{q_1 EF} + q_2 F^2$$

$$\underset{-}{}gE = \underset{-}{}\cancel{q_1 EF} \underset{-}{+} q_2 GE$$

$$fF - gE = q_2(F^2 - GE)$$

$$\Rightarrow \boxed{q_2 = \frac{gE - fF}{EG - F^2}}$$

Also ⑤ & ⑥ ⇒

$$fG = q_1 EG + \cancel{q_2 FG}$$

$$\pm gF = \pm q_1 F^2 \pm \cancel{q_2 FG}$$

$$fG - gF = q_1(EG - F^2)$$

$$\Rightarrow \boxed{q_1 = \frac{fG - gF}{EG - F^2}}$$

Now putting these values in eq ① & ②.

$$\text{eq ①} \Rightarrow \underline{N}_1 = \left(\frac{eG - fF}{EG - F^2}\right)\underline{x}_1 + \left(\frac{fE - eF}{EG - F^2}\right)\underline{x}_2$$

$$= \left[e\left(\frac{G}{EG - F^2}\right) + f\left(\frac{-F}{EG - F^2}\right)\right]\underline{x}_1 + \left[f\left(\frac{E}{EG - F^2}\right) + e\left(\frac{-F}{EG - F^2}\right)\right]\underline{x}_2$$

$$= (g^{11}\tilde{g}_{11} + g^{21}\tilde{g}_{12})\underline{x}_1 + (g^{22}\tilde{g}_{12} + g^{12}\tilde{g}_{11})\underline{x}_2$$

$$= g^{i1}\tilde{g}_{1i}\underline{x}_1 + g^{i2}\tilde{g}_{1i}\underline{x}_2 \qquad i=1,2$$

$$N_1 = g^{ij}\, g_{1i}\, x_j \rightarrow ⑦ \qquad j=1,2$$

eq. ② ⇒

$$N_2 = \frac{fG - gF}{EG-F^2}\, x_1 + \frac{gE - fF}{EG-F^2}\, x_2$$

$$= \left[\left(\frac{G}{EG-F^2}\right) f + \left(\frac{-F}{EG-F^2}\right) g\right] x_1$$

$$+ \left[\left(\frac{E}{EG-F^2}\right) g + \left(\frac{-F}{EG-F^2}\right) f\right] x_2$$

$$= \left(g^{11}\, \tilde{g}_{21} + g^{21}\, \tilde{g}_{22}\right) x_1$$

$$+ \left(g^{22}\, \tilde{g}_{22} + g^{12}\, \tilde{g}_{21}\right) x_2$$

$$= g^{i1}\, \tilde{g}_{2i}\, x_1 + g^{i2}\, \tilde{g}_{2i}\, x_2 \quad \{i=1,2]$$

$$N_2 = g^{ij}\, \tilde{g}_{2i}\, x_j \longrightarrow ⑧ \qquad \{j=1,2\}$$

Combining eq. ⑦ & eq. ⑧ finally we have

$$N_K = g^{ij}\, \tilde{g}_{Ki}\, x_j \qquad K=1,2$$

These are called weingarton's eq's.

Q:- 16-10-14

Find the christoffel symbles of
$x(u,v) = (\cos u \cosh v, \sin u \cosh v, \sinh v)$

Sol:-

$$\Gamma^{k}_{ij} = \frac{1}{2} g^{kl}(g_{il,j} + g_{lj,i} - g_{ij,l})$$

we have to find

$$\Gamma^{1}_{11}, \Gamma^{1}_{21} = \Gamma^{1}_{12}, \Gamma^{1}_{22}$$

$$\Gamma^{2}_{11}, \Gamma^{2}_{21} = \Gamma^{2}_{12}, \Gamma^{2}_{22}$$

$$\underline{x}_u = (-\sin u \cosh v, \cos u \cosh v, 0)$$

& $$\underline{x}_v = (\cos u \sinh v, \sin u \sinh v, \cosh v)$$

$$E = \underline{x}_u \cdot \underline{x}_u = \sin^2 u \cosh^2 v + \cos^2 u \cosh^2 v + 0$$

$$\Rightarrow \boxed{E = \cosh^2 v}$$

$$F = \underline{x}_u \cdot \underline{x}_v = -\sin u \cos u \cancel{\sinh v \cosh v} + \sin u \cancel{\cos u \sinh v \cosh v}$$

$$\Rightarrow \boxed{F = 0}$$

$$G = \underline{x}_v \cdot \underline{x}_v = \cos^2 u \sinh^2 v + \sin^2 u \sinh^2 v + \cosh^2 v$$

$$\Rightarrow \boxed{G = \sinh^2 v + \cosh^2 v}$$

$$g_{ij} = \begin{pmatrix} \cosh^2 v & 0 \\ 0 & \sinh^2 v + \cosh^2 v \end{pmatrix}$$

$$g^{ij} = \begin{pmatrix} 1/\cos^2 hv & 0 \\ 0 & \frac{1}{\sin^2 hv + \cos^2 hv} \end{pmatrix}$$

$$\Gamma^1_{11} = \frac{1}{2} g^{1l}(g_{1l,1} + g_{l1,1} - g_{11,l})$$

where $l = 1,2$

$$\Gamma^1_{11} = \frac{1}{2} g^{11}(g_{11,1} + \cancel{g_{11,1}} - \cancel{g_{11,1}}) + \frac{1}{2} \overset{0}{\cancel{g^{12}}}(g_{12,1} + g_{21,1} - g_{11,2})$$

$$\Rightarrow \Gamma^1_{11} = \frac{1}{2} g^{11} \cdot g_{11,1}$$

$$= \frac{1}{2} \cdot \left(\frac{1}{\cos^2 hv}\right) \cdot \overset{0}{\cancel{\frac{\partial}{\partial u}(\cos^2 hv)}}$$

$$\boxed{\Gamma^1_{11} = 0}$$

$$\Gamma^1_{12} = \frac{1}{2} g^{1l}(g_{1l,2} + g_{l2,1} - g_{12,l})$$

$$= \frac{1}{2} g^{11}(g_{11,2} + \cancel{g_{12,1}} - \cancel{g_{12,1}}) + \frac{1}{2} \underset{0}{\cancel{g^{12}}}(g_{12,2} + g_{22,1} - g_{12,2})$$

$$= \frac{1}{2} g^{11}(g_{11,2})$$

$$= \frac{1}{2} \cdot \frac{1}{\cos^2 hv} \cdot \frac{\partial}{\partial v}(\cos^2 hv)$$

$$= \frac{1}{\cancel{2}\cos^{\cancel{2}} hv} \cdot \cancel{2} \cos hv \sin hv$$

$$\boxed{\Gamma^1_{21} = \tan hv = \Gamma^1_{12}}$$

$$\Gamma^1_{22} = \frac{1}{2} g^{1l}\left(g_{2l,2} + g_{l2,2} - g_{22,l}\right)$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11}\left(\underset{0}{\cancel{g_{21,2}}} + \underset{0}{\cancel{g_{12,2}}} - g_{22,1}\right) + \frac{1}{2} \underset{0}{\cancel{g^{12}}}\left(g_{22,2} + g_{22,2} - g_{22,2}\right)$$

$$= -\frac{1}{2} \cdot g^{11} \cdot g_{22,1}$$

$$= -\frac{1}{2} \cdot \frac{1}{\cos^2 hv} \cdot \frac{\partial}{\partial u}\underset{0}{\cancel{\left(\sin^2 hv + \cos^2 hv\right)}}$$

$$\boxed{\Gamma^1_{22} = 0}$$

$$\Gamma^2_{11} = \frac{1}{2} g^{2l}\left(g_{1l,1} + g_{l1,1} - g_{11,l}\right)$$

$$\Gamma^2_{11} = \frac{1}{2} \underset{0}{\cancel{g^{21}}}\left(g_{11,1} + g_{11,1} - g_{11,1}\right) + \frac{1}{2} \cdot g^{22}\left(\underset{0}{\cancel{g_{12,1}}} + \underset{0}{\cancel{g_{21,1}}} - g_{11,2}\right)$$

$$= -\frac{1}{2} g^{22} \cdot g_{11,2}$$

$$= -\frac{1}{2} \cdot \frac{1}{\sin^2 hv + \cos^2 hv} \cdot \frac{\partial}{\partial v}\left(\cos^2 hv\right)$$

$$= \frac{-1}{\cancel{2}\left(\sin^2 hv + \cos^2 hv\right)} \cdot \cancel{2} \cos hv \sin hv$$

$$\boxed{\Gamma^2_{11} = \frac{-\sin hv \cdot \cos hv}{\sin^2 hv + \cos^2 hv}}$$

$$\Gamma^{2}_{12} = \frac{1}{2} g^{2l}\left(g_{1l,2} + g_{l2,1} - g_{12,l}\right)$$

$$= \frac{1}{2} \underset{0}{\cancel{g^{21}}}\left(g_{11,2} + g_{12,1} - g_{12,1}\right) + \frac{1}{2} g^{22}\left(\cancel{g_{12,2}} + g_{22,1} - \cancel{g_{12,2}}\right)$$

$$= \frac{1}{2} g^{22} \cdot g_{22,1}$$

$$= \frac{1}{2} \cdot \frac{1}{\sin^2 hv + \cos^2 hv} \cdot \frac{\partial}{\partial u}\underset{0}{\cancel{\left(\sin^2 hv + \cos^2 hv\right)}}$$

$$\boxed{\Gamma^{2}_{12} = 0}$$

$$\Gamma^{2}_{22} = \frac{1}{2} g^{2l}\left(g_{2l,2} + g_{l2,2} - g_{22,l}\right)$$

$$= \frac{1}{2} \underset{0}{\cancel{g^{21}}}\left(g_{21,2} + g_{12,2} - g_{22,1}\right) + \frac{1}{2} g^{22}\left(g_{22,2} + \cancel{g_{22,2}} - \cancel{g_{22,2}}\right)$$

$$= \frac{1}{2} g^{22} \cdot g_{22,2}$$

$$= \frac{1}{2} \cdot \frac{1}{\sin^2 hv + \cos^2 hv} \cdot \frac{\partial}{\partial v}\left(\sin^2 hv + \cos^2 hv\right)$$

$$= \frac{1}{2(\sin^2 hv + \cos^2 hv)} \cdot \left(2 \sinh v \cosh v + 2 \cosh v \sinh v\right)$$

$$= \frac{\overset{2}{\cancel{4}} \sinh v \cosh v}{\cancel{2}(\sin^2 hv + \cos^2 hv)}$$

$$\boxed{\Gamma^{2}_{22} = \frac{2 \sinh v \cosh v}{\sin^2 hv + \cos^2 hv}}$$

Q:- $x(u,v) = (v\cos u, v\sin u, v)$

Find christofell symboles.

Sol: $$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(g_{il,j} + g_{lj,i} - g_{ij,l}\right)$$

We have to find-

$$\Gamma^1_{11},\ \Gamma^1_{12} = \Gamma^1_{21},\ \Gamma^1_{22},\quad \Gamma^2_{11},\ \Gamma^2_{21} = \Gamma^2_{12},\ \Gamma^2_{22}$$

$$x_u = (-v\sin u, v\cos u, 0)$$

& $$x_v = (\cos u, \sin u, 1)$$

$$E = x_u \cdot x_u = v^2\sin^2 u + v^2\cos^2 u + 0$$

$$\Rightarrow \boxed{E = v^2}$$

$$F = x_u \cdot x_v = \cancel{-v\sin u\cos v} + \cancel{v\sin u\cos v}$$

$$\Rightarrow \boxed{F = 0}$$

$$G = x_v \cdot x_v = \cos^2 u + \sin^2 u + 1$$

$$\boxed{G = 2}$$

$$g_{ij} = \begin{pmatrix} v^2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$g^{ij} = \begin{pmatrix} 1/v^2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$\Rightarrow g_{21} = g^{12} = 0 = g^{21} = g_{12}$$

$$\Gamma^1_{11} = \frac{1}{2} g^{1l}(g_{1l,1} + g_{l1,1} - g_{11,l})$$

where $l = 1,2$

$$\Gamma^1_{11} = \frac{1}{2} g^{11}(g_{11,1} + \cancel{g_{11,1}} - \cancel{g_{11,1}}) + \frac{1}{2} \underset{0}{\cancel{g^{12}}}(g_{12,1} + g_{21,1} - g_{11,2})$$

$$= \frac{1}{2} g^{11} \cdot g_{11,1}$$

$$= \frac{1}{2} \left(\frac{1}{V^2}\right) \cdot \underset{0}{\cancel{\frac{\partial}{\partial u}(V^2)}}$$

$$\boxed{\Gamma^1_{11} = 0}$$

$$\Gamma^1_{12} = \frac{1}{2} g^{1l}(g_{1l,2} + g_{l2,1} - g_{12,l})$$

where $l = 1,2$

$$\Gamma^1_{12} = \frac{1}{2} g^{11}(g_{11,2} + \cancel{g_{12,1}} - \cancel{g_{12,1}}) + \frac{1}{2} \underset{0}{\cancel{g^{12}}}(g_{12,2} + g_{22,1} - g_{12,2})$$

$$\Gamma^1_{12} = \frac{1}{2} g^{11} \cdot g_{11,2}$$

$$= \frac{1}{2}\left(\frac{1}{V^2}\right) \cdot \frac{\partial}{\partial v}(V^2)$$

$$= \frac{1}{2V^2} \cdot 2V$$

$$\boxed{\Gamma^1_{12} = \frac{1}{V} = \Gamma^1_{21}}$$

$$\Gamma^1_{22} = \frac{1}{2} g^{1l}\left[g_{2l,2} + g_{l2,2} - g_{22,l} \right)$$

where $l = 1,2$

$$\Gamma^1_{22} = \frac{1}{2} g^{11}\left[\underset{0}{\cancel{g_{21,2}}} + \underset{0}{\cancel{g_{12,2}}} - g_{22,1}\right) + \frac{1}{2}\underset{0}{\cancel{g^{12}}}\left[g_{22,2} + g_{22,2} - g_{22,2}\right]$$

$$= -\frac{1}{2} g^{11} \cdot g_{22,1}$$

$$= -\frac{1}{2} \cdot \frac{1}{v^2} \cdot \left(\underset{0}{\cancel{\frac{\partial}{\partial u}(2)}}\right)$$

$$\boxed{\Gamma^1_{22} = 0}$$

$$\Gamma^2_{11} = \frac{1}{2} g^{2l}\left(g_{1l,1} + g_{l1,1} - g_{11,l}\right)$$

$$= \frac{1}{2}\underset{0}{\cancel{g^{21}}}\left(g_{11,1} + g_{11,1} - g_{11,1}\right) + \frac{1}{2} g^{22}\left[\underset{0}{\cancel{g_{12,1}}} + \underset{0}{\cancel{g_{21,1}}} - g_{11,2}\right)$$

$$= -\frac{1}{2} g^{22} \cdot g_{11,2}$$

$$= -\frac{1}{2} \cdot \frac{1}{2} \frac{\partial}{\partial v}(v^2)$$

$$= -\frac{1}{4} \cdot 2v$$

$$\boxed{\Gamma^2_{11} = \frac{-v}{2}}$$

$$\Gamma^2_{12} = \frac{1}{2} g^{2l}\left(g_{1l,2} + g_{l2,1} - g_{12,l}\right)$$

$$= \frac{1}{2} \underset{0}{\cancel{g^{21}}}\left(g_{11,2} + g_{12,1} - g_{12,1}\right) + \frac{1}{2} g^{22}\left(\cancel{g_{12,2}} + g_{22,1} - \cancel{g_{12,2}}\right)$$

$$= \frac{1}{2} g^{22} \cdot g_{22,1}$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \underset{0}{\cancel{\frac{\partial}{\partial u}(2)}}$$

$$\boxed{\Gamma^2_{12} = 0}$$

$$\Gamma^2_{22} = \frac{1}{2} g^{2l}\left(g_{2l,2} + g_{l2,2} - g_{22,l}\right)$$

$$= \frac{1}{2} \underset{0}{\cancel{g^{21}}}\left(g_{21,2} + g_{12,2} - g_{22,1}\right) + \frac{1}{2} g^{22}\left(g_{22,2} + \cancel{g_{22,2}} - \cancel{g_{22,2}}\right)$$

$$= \frac{1}{2} g^{22} \cdot g_{22,2}$$

$$= \frac{1}{2}\left(\frac{1}{2}\right) \cdot \left[\underset{0}{\cancel{\frac{\partial}{\partial v}(2)}}\right]$$

$$\boxed{\Gamma^2_{22} = 0}$$

*

Non zero christoffell symbols

20-10-14

$$g_{ij} = \begin{pmatrix} v^2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$g_{11,2} \neq 0$$

$$\Rightarrow \Gamma^2_{11}, \; \Gamma^1_{12} = \Gamma^1_{21},$$

111, 112, -221
-112, 221, 222

$$g_{22,2} \neq 0 \qquad \Gamma^2_{22} = 0$$

Gauss Godazzi Eq.

Gauss derived identities on the basis of the fact that

$$x_{112} = x_{121} \rightarrow ① \;\&\; x_{221} = x_{122} \rightarrow ②$$

We will use the eq. ① to derive the Gauss Godazzi eq.

We know that Gauss eq.

$$x_{ij} = \Gamma^k_{ij} x_k - \bar{g}_{ij} \underline{N} \longrightarrow ③$$

put $i=1, j=1$

$$x_{11} = \Gamma^k_{11} x_k - \bar{g}_{11} \underline{N}$$

$$\Rightarrow x_{11} = \Gamma^k_{11} x_k - e \underline{N}$$

Diff. w.r.t 2.

$$x_{112} = \Gamma^k_{11,2} x_k + \Gamma^k_{11} x_{k2} - e_{,2} \underline{N} - e \underline{N}_2 \longrightarrow ④$$

eq③ can be written as

$$x_{ij} = \Gamma^l_{ij} x_l - \bar{g}_{ij} \underline{N}$$

$$\Rightarrow \underline{x}_{K2} = \Gamma^{l}_{K2}\,\underline{x}_l - \tilde{g}_{K2}\,\underline{N}$$

Weingarton eq

$$\underline{N}_K = g^{ij}\cdot\tilde{g}_{Ki}\,\underline{x}_j$$

$K=2$

$$\Rightarrow \underline{N}_2 = g^{ij}\cdot\tilde{g}_{2i}\,\underline{x}_j$$

Using these equations in eq. ④.

$$\underline{x}_{1|2} = \Gamma^{K}_{11,2}\,\underline{x}_K + \Gamma^{K}_{11}\left(\Gamma^{l}_{K2}\,\underline{x}_l - \tilde{g}_{K2}\,\underline{N}\right) - e_{,2}\,\underline{N}$$
$$- e\, g^{ij}\,\tilde{g}_{2i}\,\underline{x}_j$$

Since j & K are dummy index, So we can replace as,

$$\underline{x}_{1|2} = \Gamma^{l}_{11,2}\,\underline{x}_l + \Gamma^{K}_{11}\Gamma^{l}_{K2}\,\underline{x}_l - \Gamma^{K}_{11}\,\tilde{g}_{K2}\,\underline{N} - e_{,2}\,\underline{N}$$
$$- e\, g^{il}\,\tilde{g}_{2i}\,\underline{x}_l$$

$$\underline{x}_{1|2} = \left(\Gamma^{l}_{11,2} + \Gamma^{K}_{11}\Gamma^{l}_{K2} - e\, g^{il}\,\tilde{g}_{2i}\right)\underline{x}_l - \left(\Gamma^{K}_{11}\,\tilde{g}_{K2} + e_{,2}\right)\underline{N}$$

Put $i=1,2$ $\quad\therefore \tilde{g}_{21} = f,\ \tilde{g}_{22} = g$

$$\underline{x}_{1|2} = \left(\Gamma^{l}_{11,2} + \Gamma^{K}_{11}\Gamma^{l}_{K2} - e\left(g^{1l} f + g^{2l} g\right)\right)\underline{x}_l$$
$$- \left(\Gamma^{K}_{11}\,\tilde{g}_{K2} + e_{,2}\right)\underline{N} \longrightarrow ⑤$$

Similarly eq③ becomes, when $i=1, j=2$

$$\underline{x}_{12} = \Gamma^{K}_{12}\,\underline{x}_K - \tilde{g}_{12}\,\underline{N}$$
$$= \Gamma^{K}_{12}\,\underline{x}_K - f\,\underline{N}$$

Diff. w.r.t. 1.

$$x_{121} = \Gamma^{k}_{12,1}\, x_k + \Gamma^{k}_{12}\, x_{k1} - f_{,1} N - f N_1 \longrightarrow ⑥$$

We know that

$$x_{ij} = \Gamma^{l}_{ij}\, x_l - \tilde{g}_{ij} N$$

$$\Rightarrow x_{k1} = \Gamma^{l}_{k1}\, x_l - \tilde{g}_{k1} N$$

&

$$N_k = g^{ij}\, \tilde{g}_{ki}\, x_j$$

$$\Rightarrow N_1 = g^{ij}\, \tilde{g}_{1i}\, x_j$$

Using these equations in eq. ⑥

$$x_{121} = \Gamma^{k}_{12,1}\, x_k + \Gamma^{k}_{12}\left(\Gamma^{l}_{k1} x_l - \tilde{g}_{k1} N\right) - f_{,1} N - f\left(g^{ij}\, \tilde{g}_{1i}\, x_j\right)$$

Since k & j are dummy index

So, we can replace as;

$$x_{121} = \Gamma^{l}_{12,1}\, x_l + \Gamma^{k}_{12}\Gamma^{l}_{k1}\, x_l - \Gamma^{k}_{12}\tilde{g}_{k1} N - f_{,1} N - f\, g^{il}\, \tilde{g}_{1i}\, x_l$$

$$x_{121} = \left(\Gamma^{l}_{12,1} + \Gamma^{k}_{12}\Gamma^{l}_{k1} - f\, g^{il}\, \tilde{g}_{1i}\right) x_l - \left(\Gamma^{k}_{12}\tilde{g}_{k1} + f_{,1}\right) N$$

After putting i=1,2

$$x_{121} = \Gamma^{l}_{12,1} + \Gamma^{k}_{12}\Gamma^{l}_{k1} - f\left(g^{1l} e + g^{2l} f\right) x_l - \left(\Gamma^{k}_{12}\tilde{g}_{k1} + f_{,1}\right) N \longrightarrow ⑦$$

After putting eq ⑤ & eq ⑦ in eq ① we have;

$$\left[\Gamma^{l}_{11,2}-\Gamma^{l}_{12,1}+\Gamma^{k}_{11}\Gamma^{l}_{k2}-\Gamma^{k}_{12}\Gamma^{l}_{k1}-e(\cancel{g^{1l}f}+g^{2l}g)+f(\cancel{eg^{1l}}+fg^{2l})\right]\underline{x}_l$$

$$+\left(\Gamma^{k}_{12}\tilde{g}_{k1}+f_{,1}-\Gamma^{k}_{11}\bar{g}_{k2}-e_{,2}\right)\underline{N}=0$$

Since $\underline{x}_1, \underline{x}_2$ & $\underline{N}$ are linearly independent. So coefficients of x_1, x_2 and $\underline{N}$ must be zero. Considering the co-efficients of x_1 we have

$$\Gamma^{1}_{11,2}-\Gamma^{1}_{12,1}+\Gamma^{k}_{11}\Gamma^{1}_{k2}-\Gamma^{k}_{12}\Gamma^{1}_{k1}-e\cancel{g^{11}f}-eg^{21}g$$

$$+\cancel{fe g^{11}}+f^{2}g^{21}=0$$

$$\Gamma^{1}_{11,2}-\Gamma^{1}_{12,1}+\Gamma^{k}_{11}\Gamma^{1}_{k2}-\Gamma^{k}_{12}\Gamma^{1}_{k1}-(eg-f^{2})\,g^{21}=0$$

$$(eg-f^{2})\,g^{21}=\Gamma^{1}_{11,2}-\Gamma^{1}_{12,1}+\Gamma^{k}_{11}\Gamma^{1}_{k2}-\Gamma^{k}_{12}\Gamma^{1}_{k1}$$

$$\frac{(eg-f^{2})(-F)}{EG-F^{2}}=\Gamma^{1}_{11,2}-\Gamma^{1}_{12,1}+\Gamma^{k}_{11}\Gamma^{1}_{k2}-\Gamma^{k}_{12}\Gamma^{1}_{k1}$$

$$\frac{eg-f^{2}}{EG-F^{2}}=\frac{-1}{F}\left\{\Gamma^{1}_{11,2}-\Gamma^{1}_{12,1}+\Gamma^{k}_{11}\Gamma^{1}_{k2}-\Gamma^{k}_{12}\Gamma^{1}_{k1}\right\}=K$$

Now considering the co-efficient of x_2, We have

$$\Gamma^{2}_{11,2}-\Gamma^{2}_{12,1}+\Gamma^{k}_{11}\Gamma^{2}_{k2}-\Gamma^{k}_{12}\Gamma^{2}_{k1}-ef\cancel{g^{12}}-eg\,g^{22}+\cancel{efg^{12}}+f^{2}g^{22}=0$$

$$\Gamma^2_{11,2} - \Gamma^2_{12,1} + \Gamma^k_{11}\Gamma^2_{k2} - \Gamma^k_{12}\Gamma^2_{k1} - (eg-f^2)\,g^{22} = 0$$

$$(eg-f^2)\,g^{22} = \Gamma^2_{11,2} - \Gamma^2_{12,1} + \Gamma^k_{11}\Gamma^2_{k2} - \Gamma^k_{12}\Gamma^2_{k1}$$

$$\frac{(eg-f^2)E}{EG-F^2} = \Gamma^2_{11,2} - \Gamma^2_{12,1} + \Gamma^k_{11}\Gamma^2_{k2} - \Gamma^k_{12}\Gamma^2_{k1}$$

$$\frac{eg-f^2}{EG-F^2} = \frac{1}{E}\left[\Gamma^2_{11,2} - \Gamma^2_{12,1} + \Gamma^k_{11}\Gamma^2_{k2} - \Gamma^k_{12}\Gamma^2_{k1}\right] = K$$

Now putting the co-efficient of $\underline{N}$ to be zero, we get,

$$\Gamma^k_{12}\tilde{g}_{k1} + f_{,1} - \Gamma^k_{11}\tilde{g}_{k2} - e_{,2} = 0 \qquad k=1,2$$

$$\Gamma^1_{12}\tilde{g}_{11} + \Gamma^2_{12}\tilde{g}_{21} + f_{,1} - \Gamma^1_{11}\tilde{g}_{12} - \Gamma^2_{11}\tilde{g}_{22} - e_{,2} = 0$$

$$\Gamma^1_{12}e + \Gamma^2_{12}f + f_{,1} - \Gamma^1_{11}f - \Gamma^2_{11}g - e_{,2} = 0$$

$$f_{,1} - e_{,2} + \Gamma^1_{12}e + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\,\Gamma^2_{11} = 0$$

(47) 23-10-14

Tensor

Tensor is the generalization of vector and scalar. Which obeys the coordinate transformation law.

(It can be reduced to vector as well as scalar)

OR

The quantity which remains invariant under co-ordinate transformation.

A tensor of rank 1 is called vector.

A tensor of rank 1 & valence $\begin{bmatrix}1\\0\end{bmatrix}$ is called **contravariant vector.**

A tensor of rank 1 & valence $\begin{bmatrix}0\\1\end{bmatrix}$ is called covariant vector.

A tensor of rank 2 & valence $\begin{bmatrix}1\\1\end{bmatrix}$ is called Mixed Tensor.

g_{ij} $\qquad$ g^{ij}

Rank = 2 , Valence $\begin{bmatrix}0\\2\end{bmatrix}$ $\qquad$ valence $\begin{bmatrix}2\\0\end{bmatrix}$, Rank = 2

Transformation law

The contravariant vector transform according as;

$$A^{\hat{a}} = \frac{\partial x^{\hat{a}}}{\partial x^{b}} A^{b}$$

and the covariant vector as

$$A_{\hat{a}} = \frac{\partial x^{b}}{\partial x^{\hat{a}}} A_{b}$$

Similarly the transformation law for a tensor of rank 2 and valence $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is given as,

$$A^{\hat{a}\hat{b}} = \frac{\partial x^{\hat{a}}}{\partial x^{c}} \cdot \frac{\partial x^{\hat{b}}}{\partial x^{d}} A^{cd}$$

Similarly the tensor $A_{\hat{a}\hat{b}}$ will be transform as

$$A_{\hat{a}\hat{b}} = \frac{\partial x^{c}}{\partial x^{\hat{a}}} \cdot \frac{\partial x^{d}}{\partial x^{\hat{b}}} A_{cd}$$

In the case of a mixed tensor of rank 2. Then the transformation law becomes

$$A^{\hat{a}}_{\hat{b}} = \frac{\partial x^{\hat{a}}}{\partial x^{c}} \cdot \frac{\partial x^{d}}{\partial x^{\hat{b}}} \cdot A^{c}_{d}$$

In general, for a tensor of rank $(k+l)$ and valance $\begin{bmatrix} k \\ l \end{bmatrix}$. The transformation law can be written as

$$A^{\hat{a} \cdots \hat{e}}_{\hat{d} \cdots \hat{f}} = \frac{\partial x^{\hat{a}}}{\partial x^{m}} \cdots \frac{\partial x^{\hat{e}}}{\partial x^{n}} \cdot \frac{\partial x^{p}}{\partial x^{\hat{d}}} \cdots \frac{\partial x^{q}}{\partial x^{\hat{f}}} A^{m \cdots n}_{p \cdots q}$$

Q:- Transform

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

into plane polar coordinates.

Solution:- $x = r\cos\theta$ $\quad \frac{\partial x}{\partial r} = \cos\theta$

$y = r\sin\theta$ $\quad \frac{\partial x}{\partial \theta} = -r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial y}{\partial \theta} = r\cos\theta$

Known will written as $\rightarrow x^a = (x,y)$

Unknown $\rightarrow x^{\hat a} = (r,\theta)$

The transformation is given as,

$$g_{\hat a\hat b} = \frac{\partial x^c}{\partial x^{\hat a}} \cdot \frac{\partial x^d}{\partial x^{\hat b}} g_{cd} \qquad a,b,c,d = 1,2$$

$$g_{\hat a\hat b} = \begin{pmatrix} g_{\hat1\hat1} & g_{\hat1\hat2} \\ g_{\hat2\hat1} & g_{\hat2\hat2} \end{pmatrix}$$

$$g_{\hat1\hat1} = \frac{\partial x^c}{\partial x^{\hat1}} \cdot \frac{\partial x^d}{\partial x^{\hat1}} g_{cd} \qquad c,d = 1,2$$

$$= \frac{\partial x^1}{\partial x^{\hat1}} \cdot \frac{\partial x^d}{\partial x^{\hat1}} g_{1d} + \frac{\partial x^2}{\partial x^{\hat1}} \cdot \frac{\partial x^d}{\partial x^{\hat1}} g_{2d}$$

$$= \frac{\partial x^1}{\partial x^{\hat1}} \cdot \frac{\partial x^1}{\partial x^{\hat1}} g_{11} + \frac{\partial x^1}{\partial x^{\hat1}} \cdot \frac{\partial x^2}{\partial x^{\hat1}} \overset{\nearrow 0}{g_{12}} + \frac{\partial x^2}{\partial x^{\hat1}} \cdot \frac{\partial x^1}{\partial x^{\hat1}} \overset{\nearrow 0}{g_{21}}$$

$$+ \frac{\partial x^2}{\partial x^{\hat1}} \cdot \frac{\partial x^2}{\partial x^{\hat1}} g_{22}$$

$$= \left(\frac{\partial x^1}{\partial x^{\hat1}}\right)^2 + \left(\frac{\partial x^2}{\partial x^{\hat1}}\right)^2 \qquad \because g_{11} = 1 = g_{22}$$

$$= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2$$

$$= \cos^2\theta + \sin^2\theta$$

$$= 1$$

$$g_{\hat1\hat2} = \frac{\partial x^c}{\partial x^{\hat1}} \cdot \frac{\partial x^d}{\partial x^{\hat2}} g_{cd}$$

$$= \frac{\partial x^1}{\partial x^{\hat1}} \cdot \frac{\partial x^d}{\partial x^{\hat2}} g_{1d} + \frac{\partial x^2}{\partial x^{\hat1}} \cdot \frac{\partial x^d}{\partial x^{\hat2}} g_{2d}$$

$$= \frac{\partial x^1}{\partial x^{\hat1}} \cdot \frac{\partial x^1}{\partial x^{\hat2}} g_{11} + \frac{\partial x^1}{\partial x^{\hat1}} \cdot \frac{\partial x^2}{\partial x^{\hat2}} \underset{0}{\cancel{g_{12}}} + \frac{\partial x^2}{\partial x^{\hat1}} \cdot \frac{\partial x^1}{\partial x^{\hat2}} \underset{0}{\cancel{g_{21}}}$$

$$+ \frac{\partial x^2}{\partial x^{\hat1}} \cdot \frac{\partial x^2}{\partial x^{\hat2}} g_{22}$$

$$= \left(\frac{\partial x}{\partial r}\right) \cdot \left(\frac{\partial x}{\partial \theta}\right) + \left(\frac{\partial y}{\partial r}\right)\left(\frac{\partial y}{\partial \theta}\right) \qquad g_{11} = g_{22} = 1$$

$$= (\cos\theta)(-r\sin\theta) + (\sin\theta)(r\cos\theta)$$

$$= -r\cancel{\sin\theta\cos\theta} + r\cancel{\sin\theta\cos\theta}$$

$$= 0 = g_{\hat2\hat1}$$

$$g_{\hat2\hat2} = \frac{\partial x^c}{\partial x^{\hat2}} \cdot \frac{\partial x^d}{\partial x^{\hat2}} g_{cd}$$

$$= \frac{\partial x^1}{\partial x^{\hat2}} \cdot \frac{\partial x^d}{\partial x^{\hat2}} g_{1d} + \frac{\partial x^2}{\partial x^{\hat2}} \cdot \frac{\partial x^d}{\partial x^{\hat2}} g_{2d}$$

$$= \frac{\partial x^1}{\partial x^{\hat2}} \cdot \frac{\partial x^1}{\partial x^{\hat2}} g_{11} + \frac{\partial x^1}{\partial x^{\hat2}} \cdot \frac{\partial x^2}{\partial x^{\hat2}} \overset{0}{\cancel{g_{12}}}$$

$$+ \frac{\partial x^2}{\partial x^{\hat2}} \cdot \frac{\partial x^1}{\partial x^{\hat2}} \underset{0}{\cancel{g_{21}}} + \frac{\partial x^2}{\partial x^{\hat2}} \cdot \frac{\partial x^2}{\partial x^{\hat2}} g_{22}$$

$$= \left(\frac{\partial x^1}{\partial x^{\hat2}}\right)^2 + \left(\frac{\partial x^2}{\partial x^{\hat2}}\right)^2 \qquad g_{11} = 1 = g_{22}$$

$$= \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2$$

$$= (-r\sin\theta)^2 + (r\cos\theta)^2$$

$$= r^2$$

So, $g_{\hat{a}\hat{b}} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

Q:— Transform 27-10-14

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

into spherical polar coordinates.

Solution:—

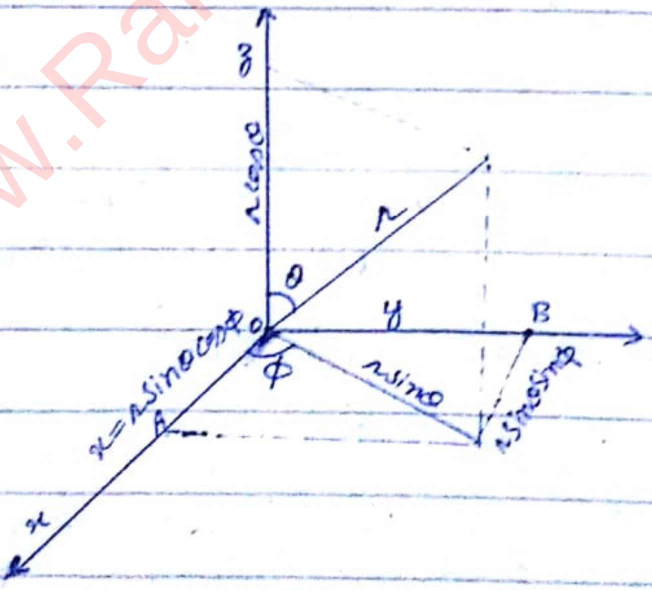
$$x^a = (x, y, z)$$

$$x^{\hat{a}} = (r, \theta, \phi)$$

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$



$$\frac{\partial x}{\partial r} = \sin\theta \cos\phi, \quad \frac{\partial x}{\partial \theta} = r \cos\theta \cos\phi, \quad \frac{\partial x}{\partial \phi} = -r \sin\theta \sin\phi$$

$$\frac{\partial y}{\partial r} = \sin\theta \sin\phi, \quad \frac{\partial y}{\partial \theta} = r \cos\theta \sin\phi, \quad \frac{\partial y}{\partial \phi} = r \sin\theta \cos\phi$$

$$\frac{\partial z}{\partial r} = \cos\theta, \quad \frac{\partial z}{\partial \theta} = -r \sin\theta, \quad \frac{\partial z}{\partial \phi} = 0$$

$$g_{\hat{a}\hat{b}} = \frac{\partial x^c}{\partial x^{\hat{a}}} \frac{\partial x^d}{\partial x^{\hat{b}}} g_{cd} \qquad a,b,c,d=1,2,3$$

$$g_{\hat{a}\hat{b}} = \begin{pmatrix} g_{\hat{1}\hat{1}} & g_{\hat{1}\hat{2}} & g_{\hat{1}\hat{3}} \\ g_{\hat{2}\hat{1}} & g_{\hat{2}\hat{2}} & g_{\hat{2}\hat{3}} \\ g_{\hat{3}\hat{1}} & g_{\hat{3}\hat{2}} & g_{\hat{3}\hat{3}} \end{pmatrix}$$

$$g_{\hat{1}\hat{1}} = \frac{\partial x^c}{\partial x^{\hat{1}}} \cdot \frac{\partial x^d}{\partial x^{\hat{1}}} g_{cd}$$

$$= \frac{\partial x^1}{\partial x^{\hat{1}}} \cdot \frac{\partial x^d}{\partial x^{\hat{1}}} g_{1d} + \frac{\partial x^2}{\partial x^{\hat{1}}} \cdot \frac{\partial x^d}{\partial x^{\hat{1}}} g_{2d} + \frac{\partial x^3}{\partial x^{\hat{1}}} \frac{\partial x^d}{\partial x^{\hat{1}}} g_{3d}$$

$$= \frac{\partial x^1}{\partial x^{\hat{1}}} \cdot \frac{\partial x^1}{\partial x^{\hat{1}}} g_{11} + \frac{\partial x^1}{\partial x^{\hat{1}}} \cdot \frac{\partial x^2}{\partial x^{\hat{1}}} \cancel{g_{12}}^{0} + \frac{\partial x^1}{\partial x^{\hat{1}}} \cdot \frac{\partial x^3}{\partial x^{\hat{1}}} \cancel{g_{13}}^{0}$$

$$+ \frac{\partial x^2}{\partial x^{\hat{1}}} \cdot \frac{\partial x^1}{\partial x^{\hat{1}}} \cancel{g_{21}}^{0} + \frac{\partial x^2}{\partial x^{\hat{1}}} \cdot \frac{\partial x^2}{\partial x^{\hat{1}}} g_{22} + \frac{\partial x^2}{\partial x^{\hat{1}}} \frac{\partial x^3}{\partial x^{\hat{1}}} \cancel{g_{23}}^{0}$$

$$+ \frac{\partial x^3}{\partial x^{\hat{1}}} \cdot \frac{\partial x^1}{\partial x^{\hat{1}}} \cancel{g_{31}}^{0} + \frac{\partial x^3}{\partial x^{\hat{1}}} \cdot \frac{\partial x^2}{\partial x^{\hat{1}}} \cancel{g_{32}}^{0} + \frac{\partial x^3}{\partial x^{\hat{1}}} \cdot \frac{\partial x^3}{\partial x^{\hat{1}}} g_{33}$$

$$= \left(\frac{\partial x^1}{\partial x^{\hat{1}}}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial x^{\hat{1}}}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial x^{\hat{1}}}\right)^2 g_{33}$$

$$g_{11} = g_{22} = g_{33} = 1$$

$$= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2$$

$$= \sin^2\theta \cos^2\phi + \sin^2\theta \sin^2\phi + \cos^2\theta$$

$$= \sin^2\theta + \cos^2\theta$$

$$= 1$$

Now $g_{\hat{2}\hat{2}} = \frac{\partial x^c}{\partial x^{\hat{2}}} \cdot \frac{\partial x^d}{\partial x^{\hat{2}}} g_{cd}$

$$= \frac{\partial x^1}{\partial x^{\hat{2}}} \cdot \frac{\partial x^1}{\partial x^{\hat{2}}} g_{11} + \frac{\partial x^2}{\partial x^{\hat{2}}} \cdot \frac{\partial x^2}{\partial x^{\hat{2}}} g_{22} + \frac{\partial x^3}{\partial x^{\hat{2}}} \cdot \frac{\partial x^3}{\partial x^{\hat{2}}} g_{33}$$

$$= \left(\frac{\partial x^1}{\partial x^{\hat{2}}}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial x^{\hat{2}}}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial x^{\hat{2}}}\right)^2 g_{33}$$

$$= \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 \qquad g_{11}=g_{22}=g_{33}=1$$

$$= r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\theta + r^2\sin^2\theta$$

$$= r^2\cos^2\theta + r^2\sin^2\theta$$

$$= r^2$$

Now

$$g_{\hat{3}\hat{3}} = \frac{\partial x^c}{\partial x^{\hat{3}}} \cdot \frac{\partial x^d}{\partial x^{\hat{3}}} g_{cd}$$

$$= \frac{\partial x^1}{\partial x^{\hat{3}}} \cdot \frac{\partial x^1}{\partial x^{\hat{3}}} g_{11} + \frac{\partial x^2}{\partial x^{\hat{3}}} \cdot \frac{\partial x^2}{\partial x^{\hat{3}}} g_{22} + \frac{\partial x^3}{\partial x^{\hat{3}}} \cdot \frac{\partial x^3}{\partial x^{\hat{3}}} g_{33}$$

$$= \left(\frac{\partial x^1}{\partial x^{\hat{3}}}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial x^{\hat{3}}}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial x^{\hat{3}}}\right)^2 g_{33}$$

$$= \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2 \qquad g_{11}=g_{22}=g_{33}=1$$

$$= r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi$$

$$= r^2\sin^2\theta$$

So,

$$g_{\hat{a}\hat{b}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

very short → Now for cylinderical polar coordinates....

$x = \rho\cos\theta$

$y = \rho\sin\theta$ $\qquad x^a = (x, y, z)$

$z = z$ $\qquad x^{\hat{a}} = (\rho, \theta, z)$

$\frac{\partial x}{\partial \rho} = \cos\theta$, $\frac{\partial y}{\partial \rho} = \sin\theta$, $\frac{\partial z}{\partial \rho} = 0$

$\frac{\partial x}{\partial \theta} = -\rho \sin\theta$, $\frac{\partial y}{\partial \theta} = \rho \cos\theta$, $\frac{\partial z}{\partial \theta} = 0$

$\frac{\partial x}{\partial z} = 0$, $\frac{\partial y}{\partial z} = 0$, $\frac{\partial z}{\partial z} = 1$

$$g_{11} = \left(\frac{\partial x^1}{\partial x^{\hat{1}}}\right)^2 + \left(\frac{\partial x^2}{\partial x^{\hat{1}}}\right)^2 + \left(\frac{\partial x^3}{\partial x^{\hat{1}}}\right)^2$$

$$= (\cos\theta)^2 + \sin^2\theta + 0$$

$$= 1$$

$$g_{\hat{2}\hat{2}} = \left(\frac{\partial x^1}{\partial x^{\hat{2}}}\right)^2 + \left(\frac{\partial x^2}{\partial x^{\hat{2}}}\right)^2 + \left(\frac{\partial x^3}{\partial x^{\hat{2}}}\right)^2$$

$$= (-\rho \sin\theta)^2 + (\rho \cos\theta)^2 + 0$$

$$= \rho^2$$

$$g_{\hat{3}\hat{3}} = \left(\frac{\partial x^1}{\partial x^{\hat{3}}}\right)^2 + \left(\frac{\partial x^2}{\partial x^{\hat{3}}}\right)^2 + \left(\frac{\partial x^3}{\partial x^{\hat{3}}}\right)^2$$

$$= (0)^2 + (0)^2 + 1$$

$$= 1$$

So,

$$g_{\hat{a}\hat{b}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Assignment

Q. Transform

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

into Hyper spherical coordinates.

Solution:

$$\omega = r\sin\theta\cos\phi\cos\psi$$

$$x = r\sin\theta\cos\phi\sin\psi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$x^a = (\omega, x, y, z)$$

$$x^{\hat{a}} = (r, \theta, \phi, \psi)$$

$$\frac{\partial\omega}{\partial r} = \sin\theta\cos\phi\cos\psi, \quad \frac{\partial x}{\partial r} = \sin\theta\cos\phi\sin\psi, \quad \frac{\partial y}{\partial r} = \sin\theta\sin\phi, \quad \frac{\partial z}{\partial r} = \cos\theta$$

$$\frac{\partial\omega}{\partial\theta} = r\cos\theta\cos\phi\cos\psi, \quad \frac{\partial x}{\partial\theta} = r\cos\theta\cos\phi\sin\psi, \quad \frac{\partial y}{\partial\theta} = r\cos\theta\sin\phi, \quad \frac{\partial z}{\partial\theta} = -r\sin\theta$$

$$\frac{\partial\omega}{\partial\phi} = -r\sin\theta\sin\phi\cos\psi, \quad \frac{\partial x}{\partial\phi} = -r\sin\theta\sin\phi\sin\psi, \quad \frac{\partial y}{\partial\phi} = r\sin\theta\cos\phi, \quad \frac{\partial z}{\partial\phi} = 0$$

$$\frac{\partial\omega}{\partial\psi} = -r\sin\theta\cos\phi\sin\psi, \quad \frac{\partial x}{\partial\psi} = r\sin\theta\cos\psi$$

$$\frac{\partial y}{\partial\psi} = 0 = \frac{\partial z}{\partial\psi}$$

$$g_{\hat{a}\hat{b}} = \begin{pmatrix} g_{\hat{1}\hat{1}} & g_{\hat{1}\hat{2}} & g_{\hat{1}\hat{3}} & g_{\hat{1}\hat{4}} \\ g_{\hat{2}\hat{1}} & g_{\hat{2}\hat{2}} & g_{\hat{2}\hat{3}} & g_{\hat{2}\hat{4}} \\ g_{\hat{3}\hat{1}} & g_{\hat{3}\hat{2}} & g_{\hat{3}\hat{3}} & g_{\hat{3}\hat{4}} \\ g_{\hat{4}\hat{1}} & g_{\hat{4}\hat{2}} & g_{\hat{4}\hat{3}} & g_{\hat{4}\hat{4}} \end{pmatrix}$$

$$g_{\hat{a}\hat{b}} = \frac{\partial x^c}{\partial x^{\hat{a}}} \cdot \frac{\partial x^d}{\partial x^{\hat{b}}} \cdot g_{cd} \qquad a,b,c,d = 1,2,3,4$$

$$g_{\hat{1}\hat{1}} = \frac{\partial x^c}{\partial x^{\hat{1}}} \cdot \frac{\partial x^d}{\partial x^{\hat{1}}} g_{cd} \qquad c,d = 1,2,3,4$$

After putting the values of,

$$g_{12} = g_{13} = g_{14} = g_{21} = g_{23} = g_{24} = g_{31} = g_{32} = g_{34} = g_{41} = g_{42} = g_{43} = 0$$

& $g_{11} = g_{22} = g_{33} = g_{44} = 1$

$$g_{\hat{1}\hat{1}} = \frac{\partial x^1}{\partial x^{\hat{1}}} \cdot \frac{\partial x^1}{\partial x^{\hat{1}}} + \frac{\partial x^2}{\partial x^{\hat{1}}} \cdot \frac{\partial x^2}{\partial x^{\hat{1}}} + \frac{\partial x^3}{\partial x^{\hat{1}}} \cdot \frac{\partial x^3}{\partial x^{\hat{1}}} + \frac{\partial x^4}{\partial x^{\hat{1}}} \cdot \frac{\partial x^4}{\partial x^{\hat{1}}}$$

$$= \left(\frac{\partial x^1}{\partial x^{\hat{1}}}\right)^2 + \left(\frac{\partial x^2}{\partial x^{\hat{1}}}\right)^2 + \left(\frac{\partial x^3}{\partial x^{\hat{1}}}\right)^2 + \left(\frac{\partial x^4}{\partial x^{\hat{1}}}\right)^2$$

$$= \left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2$$

$$= \sin^2\theta\cos^2\phi\cos^2\psi + \sin^2\theta\cos^2\phi\sin^2\psi + \sin^2\theta\sin^2\phi + \cos^2\theta$$

$$= \sin^2\theta\cos^2\phi\underbrace{(\cos^2\psi + \sin^2\psi)}_{1} + \sin^2\theta\sin^2\phi + \cos^2\theta$$

$$= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta$$

$$= \sin^2\theta\underbrace{(\cos^2\phi + \sin^2\phi)}_{1} + \cos^2\theta$$

$$= \sin^2\theta + \cos^2\theta$$

$$= 1$$

$$g_{\hat{2}\hat{2}} = \frac{\partial x^c}{\partial x^{\hat{2}}} \cdot \frac{\partial x^d}{\partial x^{\hat{2}}} g_{cd} \qquad c,d = 1,2,3,4$$

After putting the values

$$= \left(\frac{\partial x^1}{\partial x^{\hat{2}}}\right)^2 + \left(\frac{\partial x^2}{\partial x^{\hat{2}}}\right)^2 + \left(\frac{\partial x^3}{\partial x^{\hat{2}}}\right)^2 + \left(\frac{\partial x^4}{\partial x^{\hat{2}}}\right)^2$$

$$= \left(\frac{\partial w}{\partial \theta}\right)^2 + \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2$$

$$= r^2\cos^2\theta\cos^2\phi\cos^2\psi + r^2\cos^2\theta\cos^2\phi\sin^2\psi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta$$

$$= r^2\cos^2\theta\cos^2\phi(\cos^2\psi + \sin^2\psi) + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta$$

$$= r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta$$

$$= r^2\cos^2\theta\,(\cos^2\phi + \sin^2\phi) + r^2\sin^2\theta$$

$$= r^2\cos^2\theta + r^2\sin^2\theta$$

$$= r^2(\cos^2\theta + \sin^2\theta)$$

$$= r^2$$

$$g_{\hat{3}\hat{3}} = \frac{\partial x^c}{\partial x^{\hat{3}}} \cdot \frac{\partial x^d}{\partial x^{\hat{3}}} g_{cd} \qquad c,d = 1,2,3,4$$

After putting the values.

$$g_{\hat3\hat3} = \left(\frac{\partial x^1}{\partial x^{\hat3}}\right)^2 + \left(\frac{\partial x^2}{\partial x^{\hat3}}\right)^2 + \left(\frac{\partial x^3}{\partial x^{\hat3}}\right)^2 + \left(\frac{\partial x^4}{\partial x^{\hat3}}\right)^2$$

$$= \left(\frac{\partial w}{\partial \phi}\right)^2 + \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2$$

$$= r^2 \sin^2\theta \sin^2\phi \cos^2\psi + r^2 \sin^2\theta \sin^2\phi \sin^2\psi + r^2 \sin^2\theta \cos^2\phi + 0$$

$$= r^2 \sin^2\theta \sin^2\phi \underbrace{(\cos^2\psi + \sin^2\psi)}_{1} + r^2 \sin^2\theta \cos^2\phi$$

$$= r^2 \sin^2\theta \sin^2\phi + r^2 \sin^2\theta \cos^2\phi$$

$$= r^2 \sin^2\theta \underbrace{(\sin^2\phi + \cos^2\phi)}_{1}$$

$$= r^2 \sin^2\theta$$

$$g_{\hat4\hat4} = \frac{\partial x^c}{\partial x^{\hat4}} \cdot \frac{\partial x^d}{\partial x^{\hat4}} \cdot g_{cd} \qquad c, d = 1, 2, 3, 4$$

After putting the values,

$$g_{\hat4\hat4} = \left(\frac{\partial x^1}{\partial x^{\hat4}}\right)^2 + \left(\frac{\partial x^2}{\partial x^{\hat4}}\right)^2 + \left(\frac{\partial x^3}{\partial x^{\hat4}}\right)^2 + \left(\frac{\partial x^4}{\partial x^{\hat4}}\right)^2$$

$$= \left(\frac{\partial w}{\partial \psi}\right)^2 + \left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial y}{\partial \psi}\right)^2 + \left(\frac{\partial z}{\partial \psi}\right)^2$$

$$= r^2 \sin^2\theta \cos^2\phi \sin^2\psi + r^2 \sin^2\theta \cos^2\phi \cos^2\psi + 0 + 0$$

$$= r^2 \sin^2\theta \cos^2\phi (\sin^2\psi + \cos^2\psi)$$

$$= r^2 \sin^2\theta \cos^2\phi$$

So,

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta\cos^2\phi \end{pmatrix}.$$

13-11-

13-11-14

Separable Space:-

A space Ω is said to be separable if there exists a countably infinite subspace of it whose closure is entire space.

Connected Space:-

A space Ω is said to be connected if there does not exist $A, B \subset \Omega$ such that

$$A \cup B = \Omega \quad \text{and} \quad A \cap \bar{B} = \bar{A} \cap B = \Phi$$

Housdorff Space:-

A space Ω is said to be housdorff if $\forall\ x, y \in \Omega$ such that $x \neq y$ $\exists$ neighbourhoods $\eta_1(x), \eta_2(y)$ such that

$$\eta_1(x) \cap \eta_2(y) = \phi$$

Manifold:-

The manifold is a generalization of usual surface on which we perform differential

OR

A manifold M_n of dimension n, is a separable, connected & housdorff space with a homeomorphism (similar shape) from each element of it's open ball cover into R^n (R^n is euclidan n dim)

Compact Manifold.

A manifold is said to be compact if there exists a finite open cover.

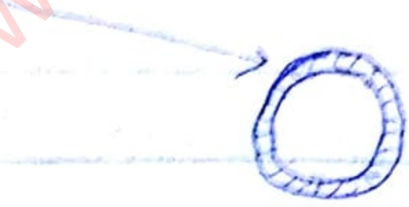
'Para compact Manifold:'

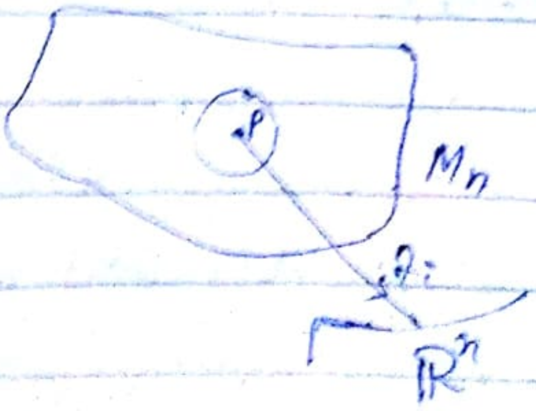
A manifold is said to be para compact if there is a finite refinement of it.

Example:-

Circle, Sphere, torus etc

Torus ⇒ like tube of cycle.





Homeomorphism:-

A function $f: X \to Y$ b/w two topological spaces (X, T_X) & (Y, T_Y) is called a homeomorphism if it has the following properties:

(i):- f is bijective (1-1 & onto)

(ii):- f is continuous.

(iii):- The inverse function f^{-1} is continuous (f is an open mapping)

We say that X & Y are homeomorphic.

"Coordinate Patch"

20-11-14

Let the open cover by $\{U_i\}_{i \in I}$, where I is some index set. If "I" is finite then the open cover is said to be finite. If I is countable then the open cover is said to has a locally finite refinement. Thus in these cases, the space is compact or paracompact. If I is uncountable & there is no choice of open cover where I can become countable, the space is called non-compact. Each U_i is called coordinate patch.

Coordinatization:-

The homeomorphism $f_i : U_i \to \mathbb{R}^n$ is called coordinatization. It is defined as

$$f_i(P) = (x^1, x^2, x^3, \ldots, x^n)$$
$$= x^i \quad \because i = 1, 2, 3, \ldots n$$

Here x^i are called coordinates of P in $\mathbb{R}^n$.

The pair (U_i, f_i) is called coordinate chart.

The collection of all charts i.e $\{(U_i, f_i)\}_{i \in I}$ is called "Atlas".

Since U_i are open & also $\bigcup_{i \in I} U_i = M_n$, it is obvious that $\forall\ u_i\ \exists\ U_j$ such that

$$U_i \cap U_j \neq \phi$$

Let us asume that $P \in U_i \cap U_j$ & let f_i & f_j for the respective coordinatization such that

$$f_i(P) = x^a \quad \& \quad f_j(P) = x^{\hat{a}}$$

Thus we have two sets of coordinates for the same point P

to be able to deal with these we must be able to convert from one set of coordinates to other.

This is where the fact that the homeomorphism is bijective is needed. Due to this bijectiveness property $\exists$ f_i^{-1} such that $(f_i \circ f_i^{-1})(P) = P$.

Now we consider

$$(f_j \circ f_i^{-1}) \circ f_i(P) = f_j \circ (f_i^{-1} \circ f_i)(P)$$

∵ Associative property

$$\Rightarrow (f_j \circ f_i^{-1}) x^a = f_j(P)$$
$$= x^{\hat{a}}$$

This shows that $f_j \circ f_i^{-1}$ is the mapping which transform x^a to $x^{\hat{a}}$.

Assignment:- Find out the mapping which transform $x^{\hat{a}}$ to x^a:

Let $P \in U_i \cap U_j$

Let f_i & f_j for the respective coordinatigation such that

$$f_i(P) = x^a \quad \& \quad f_j(P) = x^{\hat{a}}$$

To be able to deal with these we must be able to convert from one set of coordinates to other. This is where the fact that the homeomorphism is bijective is needed, because of this property of bijectiveness $\exists$ $f^{-1}{}_j$ such that

$$(f_j^{-1} \circ f_j)(P) = P$$

Now we consider

$$(f_i \circ f_j^{-1}) \circ f_j(P) = f_i \circ (f_j^{-1} \circ f_j)(P)$$

$$\Rightarrow \quad (f_i \circ f_j^{-1})\, x^{\hat{a}} = f_i(P)$$

$$\Rightarrow (f_i \circ f_j^{-1})\, x^{\hat{a}} = x^a$$

This shows that $f_i \circ f_j^{-1}$ is the mapping which transform $x^{\hat{a}}$ to x^a.

"Differentiable Manifold"

A manifold is said to be differentiable if the homeomorphism is differentiable. A differentiable manifold is called as: "diffeomorphism"

If the homeomorphism is K-time differentiable then the manifold is called C^K manifold.

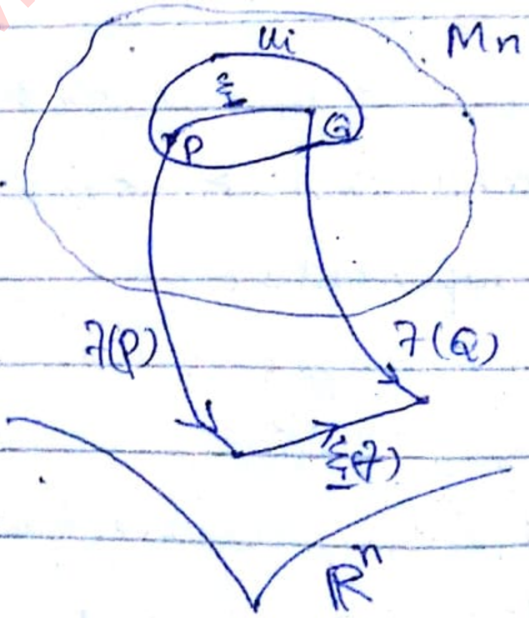
An finitly differentiable manifold is called C^∞-manifold.

If the homeomorphism is analytic & not only infinitly differentiable then the manifold is called C^ω-manifold.

A C°-manifold is not differentiable in usual sense but it can be diffentiable in the sense of generalized function.

Example:— $|x|$ is not differentiable at $x=0$ but all other points it is differentiable.

Derivation:-



Let us consider a differentiable manifold M_n. A derivation ξ is a mapping, given as

$$\xi : M_n \longrightarrow M_n$$

such that

$$\xi : P \longrightarrow Q$$

where $P, Q \in U_i$

It is not possible to deal properly with derivation taking . . points such that one belongs to coordinate patch & other lies out side it (or in other coordinate patch)

24-11-14

Now we discuss the coordinatization of derivation, denoted by $\underline{\xi}(f)$ instead of $f(\underline{\xi})$ such that

$$\underline{\xi}(f) : f(P) \longrightarrow f(Q)$$

If we write $f(P) = x^a$ & $f(Q) = y^a$ and treat the operator $\underline{\xi}(f)$ as an additive operator, then

$$\underline{\xi}(f) = y^a - x^a$$

So that

$$\underline{\xi}(f) + f(P) = y^a - x^a + x^a$$
$$= y^a = f(Q)$$

Thus coordinatization of the derivation is exactly the same as the components of a vector in Eucledian geometry.

Now we can define the addition of two derivations $\underline{\xi}$ & η as,

$$(\underline{\xi}_P + \underline{\eta}_P)(f) = \underline{\xi}_P(f) + \eta_P(f)$$

and scalar multiplication as;

$$(\lambda \underline{\xi}_P)(f) = \lambda \underline{\xi}_P(f) \; , \quad \lambda \in \mathbb{R}$$

Thus the set of derivations at P forms a vector space. It is denoted D_P

There is a complete vector space of derivation at each point of manifold. The collection of all such spaces $\forall p \in M_n$ called tangent bundle. We will not be dealing with the entire collection but at a given point p.

Consequently, we will write $\underset{-}{\xi}$ instead of $\underset{-}{\xi}_p$ and D instead of D_p.

"Dual Derivation"

We define dual derivation, denoted by $\underset{-}{\alpha}$, as a mapping

$$\underset{-}{\alpha} : D \longrightarrow \mathbb{R}$$

$$\text{i.e } \underset{-}{\alpha} : \underset{-}{\xi} \longrightarrow r \in R$$

It is generally written as

$$\underset{-}{\alpha} \cdot \underset{-}{\xi} = \underset{-}{\xi} \cdot \underset{-}{\alpha} = r \in R$$

Again we define the addition of two dual derivations as;

$$(\underset{-}{\alpha} + \underset{-}{\beta}) \cdot \underset{-}{\xi} = \underset{-}{\alpha} \cdot \underset{-}{\xi} + \underset{-}{\beta} \cdot \underset{-}{\xi}$$

$$= r_1 + r_2 \qquad r_1, r_2 \in R$$

$$= r \in R$$

The scalar multiplication as;

$$(\lambda \underline{\alpha}) \cdot \underline{\xi} = \lambda (\underline{\alpha} \cdot \underline{\xi}) \qquad \lambda \in \mathbb{R}$$
$$= \lambda (r_1) \qquad r_1 \in \mathbb{R}$$
$$= r \in \mathbb{R}$$

Thus the set of dual derivation forms a vector space over the field $\mathbb{R}$. It is denoted by D^*.

It is worth mentioning here that we can define a complex manifold with $\mathbb{R}$ everywhere replaced by $\mathbb{C}$.

The vector space D can be multiply together. e.g $D \times D$ is a space where elements are of the form (ξ, η) such that $\xi, \eta \in D$.

This product is not a vector space itself as linearity is not hold.

Assignment:- Show that the cross product is not linear.

25-11-14

Tensor Product

A vector space can be defined from these spaces, by defining a product which conserve linearity.

This product is called Tensor product denoted by $D \otimes D$ & defined as

$$D \otimes D = D \times D / C$$

where $C = \{\lambda(\xi,\eta) = (\lambda\xi, \lambda\eta)$ s.t $\xi, \eta \in D, \lambda \in R\}$

[Assignment:- Show that $D \otimes D$ is linear.]

Similarly we can defined $D \otimes D^*$, $D^* \otimes D^*$ etc. In general, a vector space V_l^K of valance $\begin{bmatrix} K \\ l \end{bmatrix}$ and rank $(k+l)$ can be defined as

$$V_l^K = \underbrace{D \otimes D \otimes \cdots \otimes D}_{K\text{-times}} \otimes \overbrace{D^* \otimes \cdots \otimes D^*}^{l\text{-times}}$$

Where there are k-derivations and l-dual derivations.

An element belonging to V_l^K is called a tensor of valance $\begin{bmatrix} K \\ l \end{bmatrix}$ & Rank $(k+l)$

The tensor are defined as manifold.

"Contraction"

A contraction of a tensor is an operator or operation which reduces the rank by 2 & valance by $\begin{bmatrix}1\\1\end{bmatrix}$ by letting one of the dual derivation act on one of the derivation.

$$T^{ab}_{ade} = A^{b}_{de}$$

The manipulation of tensors is greatly simplified with the help of abstract index notation. The space of derivations along with an index label $(\mathcal{D}, a)$ is written as T^{a}, so that $(\underline{\xi}, a) = \xi^{a}$ and called the space of contravariant vectors. It can be written with any other label e.g $\xi^{b} \in T^{b}$, $\xi^{c} \in T^{c}$ etc.

It is noted that we have only change the label while the derivation remains the same. Similarly the space of covariant vector is defined by,

Abstract index notation $\longrightarrow$ $T_{\underline{a}} = (\mathcal{D}^{*}, a)$ give the elements like α_{a} , β_{a} etc.

Generally, the space of tensors of rank $(k+l)$ and valance $\begin{bmatrix}k\\l\end{bmatrix}$ is defined as

$$\mathcal{T}^{\underline{a}\cdots\underline{c}}_{\underline{d}\cdots\underline{f}} = \mathcal{T}^{\underline{a}} \otimes \cdots \otimes \mathcal{T}^{\underline{c}} \otimes \mathcal{T}_{\underline{d}} \otimes \cdots \otimes \mathcal{T}_{\underline{f}}$$

where $(\{a, \ldots, c\}) = k$, $(\{d, \ldots, f\}) = l$

Note:- A scalar is a tensor of rank 0 and valance $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

"Index Substitution"

We denotes the index subtitution as $\delta^{\underline{a}}_{\underline{b}}$ and define as a tensor which changes the index label without changing the derivation (or dual derivation) given as

$$\delta^{\underline{a}}_{\underline{b}} : \mathcal{T}^{\underline{b}} \longrightarrow \mathcal{T}^{\underline{a}}$$

s.t $\quad \delta^{\underline{a}}_{\underline{b}}(\xi^{\underline{b}}) = \xi^{\underline{a}} \in \mathcal{T}^{\underline{a}}$

Similarly $\quad \delta^{\underline{a}}_{\underline{b}} : \mathcal{T}_{\underline{a}} \longrightarrow \mathcal{T}_{\underline{b}}$

s.t $\quad \delta^{\underline{a}}_{\underline{b}}(\alpha_{\underline{a}}) = \alpha_{\underline{b}} \in \mathcal{T}_{\underline{b}}$

For a mixed tensor

$$\delta^{\underline{a}}_{\underline{b}} : \mathcal{T}^{\underline{b}}_{\underline{a}} \longrightarrow \mathcal{T}$$

Where $\mathcal{T}$ is the space of scalar function.

Thus $(\delta^a_{\underline{b}} \xi^{\underline{b}})\alpha_{\underline{a}} = \xi^{\underline{a}}\alpha_{\underline{a}} = \underline{\xi}\cdot\underline{\alpha}$ which is scalar. We see that δ just replace one label by another.

Following this notation the components of $\xi^{\underline{a}}$ in a coordinate system, previously written as $\underline{\xi}(q)$ will be written as ξ^a.

If $\underline{\xi}: P \to Q$ and $q(P) = x^a$, $q(Q) = y^a$

then
$$\underline{\xi}(q) = \xi^a = y^a - x^a$$

So, the coordinatization is then written as $\delta^a_{\underline{a}}$, Thus
$$\delta^a_{\underline{a}} : T^{\underline{a}} \longrightarrow R^n$$
Such that $\delta^a_{\underline{a}}(\xi^{\underline{a}}) = \xi^a \in R^n$

We can also define the inverse of coordinatization as $\delta^{\underline{a}}_a : R^n \longrightarrow T^{\underline{a}}$

such that $\delta^{\underline{a}}_a(\xi^a) = \xi^{\underline{a}} \in T^{\underline{a}}$

$\boxed{Imp}$ It is to be noted here that $\xi^{\underline{a}}$ is defined on manifold while $\xi^a\,(a=1,2,\cdots,n)$ is defined in R^n. Obviously $\delta^{\underline{a}}_a$ plays the role of basis vector. (e.g $\underline{\xi} = \xi^a \underline{e}_a$)

As we can write,

$$\alpha_a \xi^a = \alpha_{\underline{a}} \xi^{\underline{a}} \checkmark$$

$$= \alpha_{\underline{a}} (\xi^a \delta^{\underline{a}}_a) \checkmark$$

$$\alpha_a \xi^a = (\alpha_{\underline{a}} \delta^{\underline{a}}_a) \xi^a \checkmark$$

$$\boxed{\alpha_{\underline{a}} \delta^{\underline{a}}_a = \alpha_a}$$

Similarly,

$$\boxed{\alpha_{\underline{a}} = \delta^a_{\underline{a}} \alpha_a}$$

Thus we have

$$\delta^{\underline{a}}_a : T_{\underline{a}} \longrightarrow \mathbb{R}^n$$

$$\& \quad \delta^a_{\underline{a}} : \mathbb{R}^n \longrightarrow T_{\underline{a}} \text{ Respectivly.}$$

"Affine Connection" 27-11-14

The concept of derivation needs to be generalized in the abstract space (manifold). In particular we need to generalize, the gradiant operator ∇ ($\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$) for a curved space. Let us write this gradiant operator as $\nabla_{\underline{a}}$ and require that for any scalar function say, f the affine connection is given as $\delta^{\underline{a}}_a \nabla_{\underline{a}} f = f_{,a} = \frac{\partial f}{\partial x^a}$ (where "," denotes partial derivative).

However, there is no guaranty that $\nabla_{\underline{a}}$ acting on some tensor give its partial derivative.

It is required that it must satisfied the usual differentiation rules. i.e $\underline{\nabla}\cdot(\underline{A}+\underline{B}\times\underline{C}) = \underline{\nabla}\underline{A}+(\nabla\underline{B})\times\underline{C}$
$+\underline{B}\times(\underline{\nabla}\,\underline{C})$,

Further it should not act on the index substitution i.e

$$\nabla_{\underline{a}}\,(\delta^{b}_{\underline{c}}) = 0$$

It is clear that

$$\nabla_{\underline{a}} f(p) = \nabla_{\underline{a}}\, x^{a} = \frac{\partial x^{a}}{\partial x^{\underline{a}}} = \delta^{a}_{\underline{a}}$$

which is basis vector. for covariant.

As we know that,

$$\underline{\xi}(f(P)) = \xi^{a} = \xi^{\underline{a}}\,\delta^{a}_{\underline{a}}$$

$$= \xi^{\underline{a}}\,\nabla_{\underline{a}}\, x^{a}$$

$$\underline{\xi}(f(P)) = \xi^{\underline{a}}\,\nabla_{\underline{a}} f(P)$$

$$\Rightarrow \underline{\xi} = \xi^{\underline{a}}\,\nabla_{\underline{a}}$$

Thus we can replace derivation $\underline{\xi}$ by the contraction of contravariant vector $\xi^{\underline{a}}$ with the affine connection $\nabla_{\underline{a}}$.

If the derivation lies in the intersection of two coordinate patch U_i & U_j with component ξ^a & $\xi^{\hat{a}}$

then $$\xi^{\hat{a}} = \delta^{\hat{a}}_a \xi^a = \delta^{\hat{a}}_a \delta^a_a \xi^a$$

$$= \delta^{\hat{a}}_a \xi^a$$

$\Rightarrow$ $\delta^{\hat{a}}_a$ is transformation matrix from x^a-frame to $x^{\hat{a}}$-frame.

Similarly $$\xi^a = \delta^a_{\hat{a}} \xi^{\hat{a}}$$

$\Rightarrow$ $\delta^a_{\hat{a}}$ is transformation matrix from $x^{\hat{a}}$-frame to x^a-frame

(S) Consider the operator (for scaler function)

$$\frac{\partial}{\partial x^{\hat{a}}} = \delta^a_{\hat{a}} \nabla_a \longrightarrow ①$$

Also we know that

$$\frac{\partial}{\partial x^a} = \delta^a_a \nabla_a \longrightarrow ②$$

By chain rule

$$\frac{\partial}{\partial x^{\hat{a}}} = \frac{\partial x^a}{\partial x^{\hat{a}}} \cdot \frac{\partial}{\partial x^a} \longrightarrow ③$$

Computing ① & ② in ③ we have

$$\delta^a_{\hat{a}} \nabla_a = \frac{\partial x^a}{\partial x^{\hat{a}}} \cdot \left(\delta^a_a \nabla_a\right)$$

Metric Tensor. 1-12-14

Def. 1):- The metric tensor is defined by a mapping given by $g_{\underline{ab}} : T^{ab} \longrightarrow R$ such that it define the length square of a vector as

$$g_{\underline{ab}}\ \xi^a \xi^b = |\underline{\xi}|^2 = \xi^2 \in R$$

Def. 2):- We can also define metric tensor by an other way as $g_{\underline{ab}} : T^a \longrightarrow T_{\underline{b}}$

i-e $$g_{\underline{ab}} (\xi^a) = \xi_{\underline{b}}$$

This means that corresponding to every contravariant vector metric tensor assigns a covariant vector

Def. 3):- The third way of defining metric tensor as a quantity which appears in the first fundamental form, relating the arc length (ds^2) to $d\underline{x}$ (where $x = x(u,v)$ is a surface) as

$$ds^2 = d\underline{x} \cdot d\underline{x}$$

$$ds^2 = g_{ab}\, dx^a dx^b$$

If ξ^a is a constant vector then

$$\nabla_{\underline{c}} (\xi^a) = 0 \longrightarrow ①$$

then $$\nabla_{\underline{c}} (\xi_{\underline{b}}) = 0 \longrightarrow ②$$

$$\nabla_c (g_{ab}\, \xi^a) = 0 \qquad \text{By ①.}$$

$$\Rightarrow \nabla_c (g_{ab})\, \xi^a + g_{ab}\, \cancel{\nabla_c (\xi^a)}^{0} = 0$$

$$\Rightarrow \nabla_c (g_{ab})\, \xi^a = 0$$

as $\xi^a \neq 0 \Rightarrow \nabla_c (g_{ab}) = 0$

(Affine connection acting on metric tensor gives us zero).

The inverse of metric tensor g_{ab} is denoted by, g^{ab} & we have

$$g^{ac}\, g_{bc} = \delta^a_b$$

we know that $\boxed{g_{ab}\, \xi^b = \xi_a}$

Multiplying by g^{ac} on both sides

$$g^{ac}\, g_{ab}\, \xi^b = g^{ac}\, \xi_a$$

$$\delta^c_b\, \xi^b = g^{ac}\, \xi_a$$

$$\xi^c = g^{ac}\, \xi_a$$

$$\Rightarrow \boxed{g^{ac}\, \xi_a = \xi^c}$$

It is required that at every point on manifold there exists the metric tensor as well as its inverse.

Q:- Prove that $\nabla_c (g^{ab}) = 0$

Ans:- We know that $\nabla_c (\delta^a_d) = 0$

$$\Rightarrow \nabla_c (g^{ab}\, g_{bd}) = 0$$

$$g^{ab}\cancelto{0}{\nabla_c(g_{bd})} + g_{bd}\nabla_c(g^{ab}) = 0$$

$$g_{bd}\nabla_c(g^{ab}) = 0$$

as $g_{bd} \neq 0$ so $\nabla_c(g^{ab}) = 0$

"Covariant Derivative"

Let us now consider the action of affine connection ∇_b on a genuine vector ξ^a, and have

$$\nabla_b(\xi^a) = \nabla_b(\delta^a_a\,\xi^a) \longrightarrow (1)$$

It is noted that ξ^a are the components of ξ^a but are scalar quantities.

Multiply ① by δ^b_b on both sides.

$$\delta^b_b\,\nabla_b(\xi^a) = \delta^b_b\,\nabla_b\left(\delta^a_a\,\xi^a\right)$$

$$= \delta^b_b\,\delta^a_a\,\nabla_b(\xi^a) + \{\delta^b_b\,\nabla_b(\delta^a_a)\}\,\xi^a$$

$$= \delta^a_a\,\nabla_b(\xi^a) + (\delta^b_b\,\nabla_b\,\delta^a_a)\,\xi^a$$

$$= \delta^a_a\,\xi^a_{,b} + (\delta^b_b\,\nabla_b\,\delta^a_a)\,\xi^a$$

Again multiplying by δ^c_a on both sides...

$$\delta^c_a\,\delta^b_b\,\nabla_b\,\xi^a = \delta^c_a\,\delta^a_a\,\xi^a_{,b} + (\delta^c_a\,\delta^b_b\,\nabla_b\,\delta^a_a)\,\xi^a$$

$$\boxed{\xi^c_{;b} = \xi^c_{,b} + \Gamma^c_{ab}\,\xi^a} \longrightarrow ②$$

Where Γ^c_{ab} are called connection symbols.

where ";" denotes a covariant derivative & "," is the partial derivative.

The extra term on right hand side of eq. ② comes from the derivative of the basis vector.

Since cartesion basis vector $\hat{i}, \hat{j}, \hat{k}$ are constant vectors. So all the connection symbols becomes zero. However there are many spaces in which the cartesian coordinates can not be used.

In non-cartesian coordinates (curvilinear coordinates) the all connection symboles are not zero as the basis vectors are not constant.

It is obvious that $\nabla_{\underline{b}}(\delta^c_a) = 0$ 8-12-14

xing $\delta^{\underline{b}}_b$ $\qquad \delta^{\underline{b}}_b \nabla_{\underline{b}}(\delta^c_a) = 0$

$$\delta^{\underline{b}}_b \nabla_{\underline{b}}(\delta^c_{\underline{a}} \delta^{\underline{a}}_a) = 0$$

$$\delta^{\underline{a}}_a \delta^{\underline{b}}_b \nabla_{\underline{b}}(\delta^c_{\underline{a}}) + \delta^c_{\underline{a}} \delta^{\underline{b}}_b \nabla_{\underline{b}}(\delta^{\underline{a}}_a) = 0$$

$$\delta^{\underline{a}}_a \delta^{\underline{b}}_b \nabla_{\underline{b}}(\delta^c_{\underline{a}}) + \Gamma^c_{ab} = 0$$

$$\Rightarrow \delta^{\underline{a}}_a \delta^{\underline{b}}_b \nabla_{\underline{b}}(\delta^c_{\underline{a}}) = -\Gamma^c_{ab}.$$

Now $\qquad \nabla_{\underline{b}}(\alpha_{\underline{a}}) = \nabla_{\underline{b}}(\delta^c_{\underline{a}} \alpha_c)$.

$$\nabla_b(\alpha_a) = \delta_a^c \nabla_b(\alpha_c) + (\nabla_b(\delta_a^c))\alpha_c$$

x ing By $\delta_a^a \delta_b^b$

$$\delta_a^a \delta_b^b \nabla_b(\alpha_a) = \delta_a^c \delta_a^a \delta_b^b \nabla_b(\alpha_c) + \{\delta_a^a \delta_b^b \nabla_b(\delta_a^c)\}\alpha_c$$

$$\boxed{\alpha_{a;b} = \alpha_{a,b} - \Gamma_{ab}^c \alpha_c}$$

Thus for a mixed tensor T_b^a we have

$$\boxed{T^a_{b;c} = T^a_{b,c} + \Gamma^a_{cd} T^d_b - \Gamma^d_{bc} T^a_d}$$

In general for a tensor of rank $(k+l)$ & valance $\begin{bmatrix}k\\l\end{bmatrix}$ we have

$$T^{a\cdots c}_{d\cdots f;p} = T^{a\cdots c}_{d\cdots f,p} + \Gamma^a_{p\alpha} T^{\alpha\cdots c}_{d\cdots f} + \cdots + \Gamma^c_{p\alpha} T^{a\cdots\alpha}_{d\cdots f} - \Gamma^\alpha_{dp} T^{a\cdots c}_{\alpha\cdots f} - \cdots - \Gamma^\alpha_{pf} T^{a\cdots c}_{d\cdots\alpha}$$

Example:- 4-12-14

Find covariant derivative of $A^a = \begin{pmatrix} r\theta \\ r/\theta \end{pmatrix}$ w.r.t $g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ on plane polar coordinates.

Sol:- We know that $x^a = (\overset{1}{r}, \overset{2}{\theta})$

$$A^a_{;b} = A^a_{,b} + \Gamma^a_{bc} A^c$$

$$\& \quad \Gamma^c_{ab} = \frac{1}{2} g^{cd}(g_{bd,a} + g_{ad,b} - g_{ab,d})$$

$\boxed{\Gamma^1_{22} = -r}$, $\boxed{\Gamma^2_{12} = \frac{1}{r}}$

$A^1_{;1} = A^1_{,1} + \Gamma^1_{1c} A^c$ $\quad g_{22,1} = 2r$

$= A^1_{,1} + \underset{0}{\cancel{\Gamma^1_{11}}} A^1 + \underset{0}{\cancel{\Gamma^1_{12}}} A^2$ $\quad 0 \neq \Gamma^1_{22}, \Gamma^2_{12} \neq 0$

$= \theta$

$A^1_{;2} = A^1_{,2} + \Gamma^1_{2c} A^c$

$= A^1_{,2} + \underset{0}{\cancel{\Gamma^1_{21}}} A^1 + \Gamma^1_{22} A^2$

$= r + (-r)\left(\frac{r}{\theta}\right)$

$= \frac{\theta r - r^2}{\theta} = r - \frac{r^2}{\theta}$

$A^2_{;1} = A^2_{,1} + \Gamma^2_{1c} A^c$

$= A^2_{,1} + \underset{0}{\cancel{\Gamma^2_{11}}} A^1 + \Gamma^2_{12} A^2$

$= \frac{1}{\theta} + \left(\frac{1}{r}\right) \frac{r}{\theta}$

$= \frac{2}{\theta}$

$A^2_{;2} = A^2_{,2} + \Gamma^2_{2c} A^C$

$= A^2_{,2} + \Gamma^2_{21} A^1 + \underset{0}{\cancel{\Gamma^2_{22}}} A^2$

$= -\frac{r}{\theta^2} + \frac{1}{r}(r\theta)$

$= \frac{\theta^3 - r}{\theta^2}$

Q. find covariant derivative of

$$T^a_b = \begin{pmatrix} r & \theta & \phi \\ r\theta & \theta\phi & r\phi \\ 0 & \phi & r\theta \end{pmatrix}$$

in spherical polar coordinates i.e

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

9-12-14

$$\Gamma^{c}_{ab} = \delta^{c}_{\underline{a}}\, \delta^{\underline{b}}_{b}\, \nabla_{\underline{b}}\, \delta^{\underline{a}}_{a}$$

Connection symbols

Q:- Whether connection symbols transform like a tensor or not?

Ans:- We know that

$$\Gamma^{c}_{ab} = \delta^{c}_{\underline{a}}\, \delta^{\underline{b}}_{b}\, \nabla_{\underline{b}}\, \delta^{\underline{a}}_{a} \longrightarrow ①$$

Now consider the connection symbols in $\hat{x}$ - frame.

$$\Gamma^{\hat{c}}_{\hat{a}\hat{b}} = \delta^{\hat{c}}_{\underline{a}}\, \delta^{\underline{b}}_{\hat{b}}\, \nabla_{\underline{b}}(\delta^{\underline{a}}_{\hat{a}})$$

$$= \delta^{\hat{c}}_{\underline{a}}\, \delta^{\underline{b}}_{\hat{b}}\, \nabla_{\underline{b}}(\delta^{\underline{a}}_{a}\, \delta^{a}_{\hat{a}})$$

$$= \delta^{a}_{\hat{a}}\, \delta^{\hat{c}}_{\underline{a}}\, \delta^{\underline{b}}_{\hat{b}}\, \nabla_{\underline{b}}(\delta^{\underline{a}}_{a}) + \underbrace{\delta^{\underline{a}}_{a}\, \delta^{\hat{c}}_{\underline{a}}}\, \underbrace{\delta^{\underline{b}}_{\hat{b}}\, \nabla_{\underline{b}}}(\delta^{a}_{\hat{a}})$$

$$= \delta^{a}_{\hat{a}}\, \underbrace{\delta^{\hat{c}}_{\underline{a}}\, \delta^{\underline{b}}_{\hat{b}}}\, \delta^{\underline{a}}_{c}\, \delta^{b}_{\underline{b}}\, \underbrace{\delta^{c}_{\underline{a}}\, \delta^{\underline{b}}_{b}\, \nabla_{\underline{b}}(\delta^{\underline{a}}_{a})} + \delta^{\hat{c}}_{a}\, \nabla_{\hat{b}}(\delta^{a}_{\hat{a}})$$

$$= \delta^{a}_{\hat{a}}\, \delta^{\hat{c}}_{c}\, \delta^{b}_{\hat{b}}\, \Gamma^{c}_{ab} + \frac{\partial x^{\hat{c}}}{\partial x^{a}} \cdot \frac{\partial}{\partial x^{\hat{b}}}\left(\frac{\partial x^{a}}{\partial x^{\hat{a}}}\right)$$

$$= \delta^{a}_{\hat{a}}\, \delta^{\hat{c}}_{c}\, \delta^{b}_{\hat{b}}\, \Gamma^{c}_{ab} + \frac{\partial x^{\hat{c}}}{\partial x^{a}} \left(\frac{\partial^2 x^{a}}{\partial x^{\hat{b}}\partial x^{\hat{a}}}\right) \longrightarrow ②$$

The eq. ② shows that the connection symbols don't transform as tensor.

In general, coordinates system are not constant. So, $\frac{\partial x^{\hat{c}}}{\partial x^{a}} \neq 0$

$f(x^a) = Ax^a + B$

So, $\dfrac{\partial^2 x^a}{\partial x^{\hat a}\partial x^{\hat b}} = 0$

The eq. ②, shows that the second term on R.H.S will disappear if the transformation from $x^{\hat a}$ – frame to x^a frame is linear.

Then obviously, $\Gamma^{\hat c}_{\hat a\hat b} = \delta^{\hat c}_c\, \delta^a_{\hat a}\, \delta^b_{\hat b}\, \Gamma^c_{ab}$ which mean the connection symbols will transform as tensor.

Torsion Tensor

11-12-14

Now, we define a quantity with the help of connection symbols,

$$\star \quad T^c_{ab} = \Gamma^c_{ab} - \Gamma^c_{ba} \longrightarrow ③$$

Now, we will check whether the quantity T^c_{ab} is a tensor or not.

Consider,

$$T^{\hat c}_{\hat a\hat b} = \Gamma^{\hat c}_{\hat a\hat b} - \Gamma^{\hat c}_{\hat b\hat a}$$

Using eq ②.

$$T^{\hat c}_{\hat a\hat b} = \delta^{\hat c}_c\, \delta^a_{\hat a}\, \delta^b_{\hat b}\, \Gamma^c_{ab} + \underbrace{\frac{\partial x^{\hat c}}{\partial x^a}\frac{\partial^2 x^a}{\partial x^{\hat a}\partial x^{\hat b}}}_{a\leftrightarrow b} - \delta^{\hat c}_c\, \delta^a_{\hat a}\, \delta^b_{\hat b}\, \Gamma^c_{ba} - \frac{\partial x^{\hat c}}{\partial x^b}\frac{\partial^2 x^b}{\partial x^{\hat b}\partial x^{\hat a}}$$

$$= \delta^{\hat c}_c\, \delta^a_{\hat a}\, \delta^b_{\hat b}\left(\Gamma^c_{ab} - \Gamma^c_{ba}\right) + \frac{\partial x^{\hat c}}{\partial x^b}\frac{\partial^2 x^b}{\partial x^{\hat b}\partial x^{\hat a}} - \frac{\partial x^{\hat c}}{\partial x^b}\frac{\partial^2 x^b}{\partial x^{\hat b}\partial x^{\hat a}}$$

$$= \delta^{\hat c}_c\, \delta^a_{\hat a}\, \delta^b_{\hat b}\, T^c_{ab} + \frac{\partial x^{\hat c}}{\partial x^b}\left(\frac{\partial^2 x^b}{\partial x^{\hat b}\partial x^{\hat a}} - \frac{\partial^2 x^b}{\partial x^{\hat b}\partial x^{\hat a}}\right)$$

$$\Rightarrow T^{\hat{c}}_{\hat{a}\hat{b}} = \delta^{\hat{c}}_{c}\,\delta^{a}_{\hat{a}}\,\delta^{b}_{\hat{b}}\,T^{c}_{ab}$$

This shows that T^{c}_{ab} is a tensor quantity and is called torsion tensor.

In General Relativity we will consider only these spaces which are torsion free. i.e $T^{c}_{ab} = 0$

eq. ③ $\Rightarrow \Gamma^{c}_{ab} - \Gamma^{c}_{ba} = 0$

$$\Rightarrow \Gamma^{c}_{ab} = \Gamma^{c}_{ba}$$

That is, the connection symbols become symetric w.r.t lower indexis & then called christofall symbols.

Hints

Q:- Find the christofall symbols of

(i):- Plane polar coordinates.

$$x^{a} = (r, \theta)$$

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \qquad g_{22,1} = 2r$$

Non-zero christofall symbols will be

$$\Gamma^{1}_{22} \;\&\; \Gamma^{2}_{12} = \Gamma^{2}_{21}$$

Sphere of unity radius on S^2.

$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$

(ii):- On the surface of a sphere of radius "a".

$x^a = (\theta, \phi)$

$g_{ab} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{pmatrix}$ $\qquad g_{22,1} = a^2\sin 2\theta$

Non-zero christoffall symbols will be,

Γ^1_{22} & $\Gamma^2_{12} = \Gamma^2_{21}$

(iii) Spherical polar coordinates.

$x^a = (r, \theta, \phi)$

$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$ $\qquad g_{22,1} = 2r$

$g_{33,1} = 2r\sin^2\theta$

$g_{33,2} = r^2\sin 2\theta$

Non-zero christoffall symbols will be

$\Gamma^1_{22}, \Gamma^2_{12}, \Gamma^1_{33}, \Gamma^3_{13}, \Gamma^2_{33}, \Gamma^3_{23}$

(iv):- Cylindenical polar coordinates

$x^a = (\rho, \theta, z)$

$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\qquad g_{22,1} = 2\rho$

Non zero christoffall symbols will be

$\Gamma^1_{22}, \Gamma^2_{12}$

Q:- Work out covariant derivative of

(i):- $A_{ab} = \begin{pmatrix} r & r/\theta \\ \theta/r & \theta \end{pmatrix}$

in plan polar coordinates.

(ii):- $T^a_b = \begin{pmatrix} r\cos\theta & r\sin\phi & r \\ \cos\theta\sin\phi & \sin\theta\sin\phi & 1 \\ \cos\phi & r\cos\phi & 0 \end{pmatrix}$

In spherical polar coordinates.

"Christofell Symbols by using Covariant derivative"

In general relativity, we require that covariant derivative of the metric tensor vanishes at every point of the manifold.

$$g_{ab;c} = 0 \Rightarrow g_{ab,c} - \Gamma^d_{ac} g_{db} - \Gamma^d_{bc} g_{da} = 0 \longrightarrow ①$$

Similarly

$$g_{cb;a} = 0 \Rightarrow g_{cb,a} - \Gamma^d_{ba} g_{dc} - \Gamma^d_{ca} g_{db} = 0 \longrightarrow ②$$

$$g_{ac;b} = 0 \Rightarrow g_{ac,b} - \Gamma^d_{cb} g_{da} - \Gamma^d_{ab} g_{dc} = 0 \longrightarrow ③$$

Now eq (2)+(3)−(1), we have

$$g_{cb,a} + g_{ac,b} - g_{ab;c} - \Gamma^d_{ba} g_{dc} - \cancel{\Gamma^d_{ca} g_{db}} - \cancel{\Gamma^d_{cb} g_{da}} - \Gamma^d_{ab} g_{dc} + \cancel{\Gamma^d_{ac} g_{db}} + \cancel{\Gamma^d_{bc} g_{da}} = 0$$

$$2\Gamma^d_{ab} g_{dc} = g_{cb,a} + g_{ac,b} - g_{ab,c}$$

$$g_{dc}\Gamma^d_{ab} = \frac{1}{2}(g_{cb,a} + g_{ac,b} - g_{ab,c})$$

Multiplying by g^{ec}

$$g^{ee} g_{dc}\Gamma^d_{ab} = \frac{1}{2} g^{ec}(g_{cb,a} + g_{ac,b} - g_{ab,c})$$

$$\delta^e_d \Gamma^d_{ab} = \frac{1}{2} g^{ec}(g_{cb,a} + g_{ac,b} - g_{ab,c})$$

$$\Gamma^{e}_{ab} = \frac{1}{2} g^{ec}(g_{cb,a} + g_{ac,b} - g_{ab,c})$$

$e \longleftrightarrow c$

$$\Gamma^{c}_{ab} = \frac{1}{2} g^{ce}(g_{eb,a} + g_{ae,b} - g_{ab,e})$$

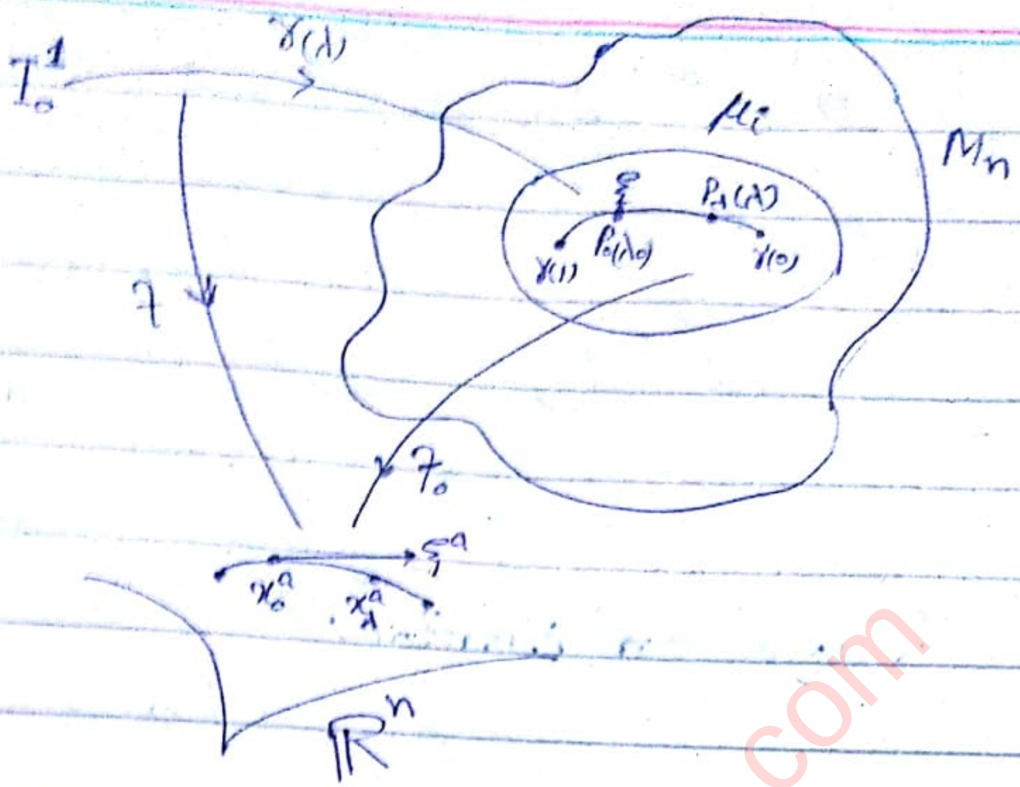
"Curve on Manifold"

It is necessary to generalize the concept of space curve in an arbitrary manifold. A curve is denoted by "γ" is defined as $\gamma : [0,1] \longrightarrow M_n$ and is given by $\gamma(\lambda) = P \in M_n \quad \forall \lambda \in [0,1]$

The $\gamma(0)$ is the initial/starting point of the curve & $\gamma(1)$ is the final/ending point of the curve.

"A manifold is said to be Arcwise Connected if there exist a curve for every pair of point of manifold."

The λ is called parameter of the curve. We can change the parameter so that choosen parameter has no infinite value at the starting or ending points (or both).



Let us consider a point on some curve γ, parameterized by λ. Let $P_o(\lambda_o)$ and $P_\lambda(\lambda)$ be two neighbouring points on this curve γ. Let there be a derivation $\underset{\sim}{\xi}$ at point P_o which is tangent to the curve at P_o. Let we consider a coordinatization ϕ from open (or coordinate) patch containing the curve γ to $\mathbb{R}^n$ & assume that

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$$\phi_o(P_o) = x_o^a \quad \text{and} \quad \phi_o(P_\lambda) = x_\lambda^a$$

Let us consider another mapping "ϕ" such that

$$\phi_{P_o}(\lambda) = \phi_o(P_\lambda) \longrightarrow ①$$

Now we define the tangent vector ξ^a

$\delta^a_{\underline{a}} = \frac{\partial x^a}{\partial x^{\underline{a}}}$

to the coordinatized curve in $\mathbb{R}^n$ at $\gamma_0(P_0)$, as usual, by

$$\xi^a = \left.\frac{dx^a}{d\lambda}\right|_{x^a = x^a_0} = \left.\frac{d\gamma_{P_0}(\lambda)}{d\lambda}\right|_{\lambda = \lambda_0}$$

$$= \lim_{\lambda \to \lambda_0} \frac{\gamma_{P_0}(\lambda) - \gamma_{P_0}(\lambda_0)}{\lambda - \lambda_0} \longrightarrow ②$$

Also, we know that

$$\xi^a = \xi^{\underline{a}} \delta^a_{\underline{a}} = \xi^{\underline{a}} \nabla_{\underline{a}} x^a \Big|_{x^a = x^a_0}$$

$$= \xi^{\underline{a}} \nabla_{\underline{a}} \gamma_0(P_0)$$

$$= \xi^{\underline{a}} \nabla_{\underline{a}} \gamma_{P_0}(\lambda) \Big|_{\lambda = \lambda_0} \longrightarrow ③$$

Comparing ② & ③ we have,

$$\xi^{\underline{a}} \nabla_{\underline{a}} = \frac{d}{d\lambda}$$

$$\Rightarrow \underline{\xi} = \frac{d}{d\lambda} \qquad \because \underline{\xi} = \xi^{\underline{a}} \nabla_{\underline{a}}$$

This is the tangent vector at P_0 on the manifold. If we use arc length parameter instead of λ. Then the tangent $\underline{\xi}$ becomes a unit tangent vector. This derivative is called Intrinsic Derivative and denoted by $D_{\underline{\xi}}$ (derivative along $\underline{\xi}$).

When we take intrinsic derivative of a tensor $\underline{T}$. We define

$$\underset{\underline{\xi}}{\mathcal{D}}(\underline{T}) = \underline{\xi}(\underline{T}) = \xi^a \nabla_a(\underline{T})$$

If $\underline{T} = T^{\underline{b}\cdots \underline{d}}_{\underline{h}\cdots \underline{m}}$ then

$$\underset{\underline{\xi}}{\mathcal{D}}(\underline{T}) = \xi^a \nabla_{\underline{a}} T^{\underline{b}\cdots d}_{\underline{h}\cdots \underline{m}}$$

In components form,

$$\underset{\underline{\xi}}{\mathcal{D}}(T)^{b\cdots d}_{h\cdots m} = \xi^a T^{b\cdots d}_{h\cdots m;a}$$

"Lie Derivative"

Lie was a mathematician.

Now we define Lei Derivative w.r.t $\underline{\xi}$ as a linear operator denoted by $\underset{\underline{\xi}}{\mathcal{L}}$. which operates according to Leibnitz rule. Further it acts as $\frac{d}{d\lambda}$. It becomes intrinsic derivative when acting on a scalar function.

i.e

$$\underset{\underline{\xi}}{\mathcal{L}}(f) = \underset{\underline{\xi}}{\mathcal{D}}(f)$$

Then obviously;

$$\mathcal{L}_{\underline{\xi}}[\eta(f)] = \frac{d}{d\lambda}[\eta(f)] = \underline{\xi}[\eta(f)] \longrightarrow ①$$

Where η is a derivation and f is the coordinatization.

By Leibnitz rule,

$$\mathcal{L}_{\underline{\xi}}[\eta(f)] = \mathcal{L}_{\underline{\xi}}(\eta)\, f + \eta\, \mathcal{L}_{\underline{\xi}}(f)$$

$$\Rightarrow [\mathcal{L}_{\underline{\xi}}(\eta)]\, f = \mathcal{L}_{\underline{\xi}}[\eta(f)] - \eta\, \mathcal{L}_{\underline{\xi}}(f)$$

$\because \underline{\xi} = \xi^a \nabla_a$

$\eta = \eta^b \nabla_{\underline{b}}$

Put ① here

$$= \underline{\xi}[\eta(f)] - \eta\, [\underline{\xi}(f)]$$

$$= \xi^a \nabla_{\underline{a}}[\eta^b \nabla_{\underline{b}}(f)] - \eta^a \nabla_{\underline{a}}[\xi^b \nabla_{\underline{b}}(f)]$$

$$= \xi^a\{(\nabla_{\underline{a}}\eta^b)\nabla_{\underline{b}}(f) + \eta^b \nabla_{\underline{a}}\nabla_{\underline{b}} f\}$$

$$-\eta^a[(\nabla_{\underline{a}}\xi^b)\nabla_{\underline{b}}(f) + \xi^b \nabla_{\underline{a}}\nabla_{\underline{b}} f]$$

$$= \xi^a(\nabla_{\underline{a}}\eta^b)\nabla_{\underline{b}}(f) + \xi^a\eta^b \nabla_{\underline{a}}\nabla_{\underline{b}} f$$

$$-\eta^a(\nabla_{\underline{a}}\xi^b)\nabla_{\underline{b}}(f) - \xi^b\eta^a \nabla_{\underline{a}}\nabla_{\underline{b}} f$$

$\uparrow a \longleftrightarrow b$

$$= \xi^a(\nabla_{\underline{a}}\eta^b)\nabla_{\underline{b}}(f) - \eta^a(\nabla_{\underline{a}}\xi^b)\nabla_{\underline{b}}(f)$$

$$+\xi^a\eta^b \nabla_{\underline{a}}\nabla_{\underline{b}} f - \xi^a\eta^b \nabla_{\underline{b}}\nabla_{\underline{a}} f$$

$$= \left[\xi^a(\nabla_a \eta^b) - \eta^a(\nabla_a \xi^b)\right]\nabla_b(f)$$

$$+ \xi^a \eta^b (\nabla_a \nabla_b - \nabla_b \nabla_a) f$$

$$= \left[\xi^a(\nabla_a \eta^b) - \eta^a(\nabla_a \xi^b)\right]\nabla_b(f) + \xi^a \eta^b \cancelto{0}{\Delta_{ab}} f$$

$$\Rightarrow \left[\mathcal{L}_\xi \eta\right] f = \left[\xi^a(\nabla_a \eta^b) - \eta^a(\nabla_a \xi^b)\right]\nabla_b f$$

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$$\Rightarrow \left[\mathcal{L}_\xi \eta\right] = \left[\xi^a(\nabla_a \eta^b) - \eta^a(\nabla_a \xi^b)\right]\nabla_b$$

$$\Rightarrow \mathcal{L}_\xi \eta = A^b \nabla_b$$

$$= \underline{A}$$

where

$$A^b = \xi^a(\nabla_a \eta^b) - \eta^a(\nabla_a \xi^b)$$

Which shows that the action of Lie Derivative Operator on a derivation yeilds another derivation.

Also we can write,

$$\left[\mathcal{L}_\xi \eta\right]^b = \xi^a(\nabla_a \eta^b) - \eta^a(\nabla_a \xi^b)$$

In component form;

$$\left[\mathcal{L}_\xi \eta\right]^b = \xi^a \eta^b_{;a} - \eta^a \xi^b_{;a}$$

Using formula of covariant derivative, We have

$$= \xi^a(\eta^b_{,a} + \Gamma^b_{ca}\eta^c) - \eta^a(\xi^b_{,a} + \Gamma^b_{ca}\xi^c)$$

$a \leftrightarrow c$

$$= \xi^a \eta^b_{,a} + \Gamma^b_{ca}\xi^a \eta^c - \eta^a \xi^b_{,a} - \Gamma^b_{ca}\xi^c \eta^a$$

$$= \xi^a \eta^b_{,a} + \cancel{\Gamma^b_{ca}\xi^a\eta^c} - \eta^a \xi^b_{,a} - \cancel{\Gamma^b_{ac}\xi^a\eta^c}$$

$$= \xi^a \eta^b_{,a} - \eta^a \xi^b_{,a}$$

↑ Lie derivative of a contravariant vector.

Let us see what comes out from the Lie derivative of a dual derivation $\underline{\alpha_c}$ (or covariant vector). Let us assume that

$$\alpha_c \eta^c = F \text{ (some scalar function)}$$

Then

$$\mathcal{L}_{\underline{\xi}}[F] = \mathcal{D}_{\underline{\xi}}[F] = \underline{\xi}[F]$$

$$= \xi^a \nabla_{\underline{a}}[F]$$

$$= \xi^a \nabla_{\underline{a}}(\alpha_{\underline{c}} \eta^c)$$

$$= \xi^a (\nabla_{\underline{a}} \alpha_{\underline{c}}) \eta^c + \xi^a (\nabla_{\underline{a}} \eta^c) \alpha_{\underline{c}} \longrightarrow ①$$

Also we can write,

$$\mathcal{L}_{\underline{\xi}}[F] = \mathcal{L}_{\underline{\xi}}[\alpha_c \eta^c]$$

$$= \alpha_{\underline{c}}[\mathcal{L}_{\underline{\xi}} \eta^c] + [\mathcal{L}_{\underline{\xi}} \alpha_{\underline{c}}] \eta^c$$

$$\Rightarrow [\mathcal{L}_{\underline{\xi}} \alpha_{\underline{c}}] \eta^c = \mathcal{L}_{\underline{\xi}}[F] - \alpha_{\underline{c}}[\mathcal{L}_{\underline{\xi}} \eta^c]$$

By putting eq. ①, we have

$$[\underset{\xi}{\mathcal{L}}\,\alpha_c]\eta^c = \xi^a(\nabla_a\alpha_c)\eta^c + \xi^a(\nabla_a\eta^c)\alpha_c$$

$$-\alpha_c(\xi^a\nabla_a\eta^c - \eta^a\nabla_a\xi^c)$$

$$= \xi^a(\nabla_a\alpha_c)\eta^c + \cancel{\xi^a(\nabla_a\eta^c)\alpha_c}$$

$$-\cancel{\xi^a(\nabla_a\eta^c)\alpha_c} + \alpha_c\eta^a(\nabla_a\xi^c)$$

$$a \leftrightarrow c$$

$$= \xi^a(\nabla_a\alpha_c)\eta^c + \alpha_a(\nabla_c\xi^a)\eta^c$$

$$[\underset{\xi}{\mathcal{L}}\,\alpha_c]\cancel{\eta^c} = [\xi^a(\nabla_a\alpha_c) + \alpha_a(\nabla_c\xi^a)]\cancel{\eta^c}$$

$$\Rightarrow \underset{\xi}{\mathcal{L}}\,\alpha_c = \xi^a(\nabla_a\alpha_c) + \alpha_a(\nabla_c\xi^a)$$

in component form $\underset{\xi}{\mathcal{L}}[\alpha_c] = \xi^a\alpha_{c;a} + \alpha_a\,\xi^a_{;c}$

In general for a mixed tensor of rank $(b+e)$ and valance $\begin{bmatrix} b \\ e \end{bmatrix}$ the Lie derivative can be written as

$$\underset{\xi}{\mathcal{L}}\,T^{a\ldots c}_{d\ldots f} = \xi^p\nabla_p T^{a\ldots c}_{d\ldots f} - T^{p\ldots c}_{d\ldots f}\nabla_p\xi^a - \cdots$$

$$- T^{a\ldots p}_{d\ldots f}\nabla_p\xi^c + T^{a\ldots c}_{p\ldots f}\nabla_d\xi^p + \cdots$$

$$\cdots + T^{a\ldots c}_{d\ldots p}\nabla_f\xi^p$$

In component form;

* as on pg 97

$$\underset{\xi}{\mathcal{L}}\, T^{a\cdots c}_{d\cdots q} = \xi^{p}\, T^{a\cdots c}_{d\cdots q,p} - T^{p\cdots c}_{d\cdots q}\, \xi^{a}_{,p} - \cdots - T^{a\cdots p}_{d\cdots q}\, \xi^{c}_{,p}$$

$$+ T^{a\cdots c}_{p\cdots q}\, \xi^{p}_{,d} + \cdots + T^{a\cdots c}_{d\cdots p}\, \xi^{p}_{,q}$$

Hint

Q:- Work out the Lie derivative of

$$T_{ab} = \begin{pmatrix} pr & rq & pq \\ p^2 & r^2 & q^2 \\ p/r & r/q & q/p \end{pmatrix} \text{ along } \xi^{a} = \begin{pmatrix} p-q \\ q-r \\ r-p \end{pmatrix}$$

Here $x^{a} = (p, q, r)$

Sol:-

$$A_{ab} = \left[\underset{\xi}{\mathcal{L}}\, T\right]_{ab} = T_{ab,c}\,\xi^{c} + T_{cb}\,\xi^{c}_{,a} + T_{ac}\,\xi^{c}_{,b}$$

$$A_{11} = T_{11,c}\,\xi^{c} + T_{c1}\,\xi^{c}_{,1} + T_{1c}\,\xi^{c}_{,1}$$

Put $c = 1, 2, 3$

$$A_{11} = T_{11,1}\,\xi^{1} + \cancelto{0}{T_{11,2}}\,\xi^{2} + T_{11,3}\,\xi^{3} + T_{11}\,\xi^{1}_{,1} + T_{21}\,\cancelto{0}{\xi^{2}_{,1}}$$

$$+ T_{31}\,\xi^{3}_{,1} + T_{11}\,\xi^{1}_{,1} + T_{12}\,\cancelto{0}{\xi^{2}_{,1}} + T_{13}\,\xi^{3}_{,1}$$

$$= T_{11,1}\,\xi^{1} + T_{11,3}\,\xi^{3} + 2\,T_{11}\,\xi^{1}_{,1} + (T_{31} - T_{13})\,\xi^{3}_{,1}$$

$$= r(p-q) + p(r-p) + 2rp(1) + \left(\frac{p}{r} - pq\right)(-1)$$

$$= pr - qr + pr - p^2 + 2pr - \frac{p}{r} + pq$$

$$= 4pr - qr + pq - p^2 - \frac{p}{r}$$

$$A_{12} = T_{12,c}\,\xi^c + T_{c2}\,\xi^c_{,1} + T_{1c}\,\xi^c_{,2}$$

$$= \cancelto{0}{T_{12,1}}\,\xi^1 + T_{12,2}\,\xi^2 + T_{12,3}\,\xi^3 + T_{12}\,\xi^1_{,1}$$

$$+ T_{22}\,\cancelto{0}{\xi^2_{,1}} + T_{32}\,\xi^3_{,1} + T_{11}\,\xi^1_{,2} + T_{12}\,\xi^2_{,2}$$

$$+ T_{13}\,\cancelto{0}{\xi^3_{,2}}$$

$$= T_{12,2}\,\xi^2 + T_{12,3}\,\xi^3 + T_{12}\,\xi^1_{,1} + T_{32}\,\xi^3_{,1}$$

$$+ T_{11}\,\xi^1_{,2} + T_{12}\,\xi^2_{,2}$$

$$= r(q-r) + q(r-p) + rq(1) + \frac{r}{q}(-1)$$

$$+ pr(-1) + rq(1)$$

$$= \overset{\checkmark}{rq} - r^2 + \overset{\checkmark}{rq} - qp + \overset{\checkmark}{rq} - \frac{r}{q} - rp + \overset{\checkmark}{rq}$$

$$= 4rq - qp - rp - r^2 - \frac{r}{q}$$

$$A_{13} = T_{13,c}\,\xi^c + T_{c3}\,\xi^c_{,1} + T_{1c}\,\xi^c_{,3}$$

$$= T_{13,1}\,\xi^1 + T_{13,2}\,\xi^2 + \cancelto{0}{T_{13,3}}\,\xi^3 + T_{13}\,\xi^1_{,1} + T_{23}\,\cancelto{0}{\xi^2_{,1}}$$

$$+ T_{33}\,\xi^3_{,1} + T_{11}\,\cancelto{0}{\xi^1_{,3}} + T_{12}\,\xi^2_{,3} + T_{13}\,\xi^3_{,3}$$

$$= q(p-q) + p(q-r) + pq(1) + \frac{q}{p}(-1) + rq(-1)$$

$$+ pq(1)$$

$$= pq - q^2 + pq - pr + pq - \frac{q}{p} - rq + pq$$

$$= 4pq - pr - qr - \frac{q}{p} - q^2$$

next do yourself.

"Parallel Transport & Lie Transport" 19-11-14

We shall use derivative operators (Intrinsic or Lie) in generalized Taylor series theorem. We know that the Taylor's series for a function of one variable is given as

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2 f''(a)}{2!} + \cdots$$

Put $a+h = \lambda$ and $a = \lambda_0$, then

$$f(\lambda) = f(\lambda_0) + \frac{(\lambda-\lambda_0)}{1!} f'(\lambda_0) + \frac{(\lambda-\lambda_0)^2}{2!} f''(\lambda_0) + \cdots$$

where $\frac{d}{d\lambda} = D$

$$= D^0 f(\lambda_0) + \frac{(\lambda-\lambda_0)}{1!} D^1 f(\lambda_0) + \frac{(\lambda-\lambda_0)^2}{2!} D^2 f(\lambda_0) + \cdots$$

$$= \left[D^0 + \frac{(\lambda-\lambda_0)}{1!} D + \frac{(\lambda-\lambda_0)^2}{2!} D^2 + \cdots\right] f(\lambda_0)$$

$$= \left[\sum_{n=0}^{\infty} \frac{(\lambda-\lambda_0)^n}{n!} D^n\right] f(\lambda_0)$$

$$= e^{(\lambda-\lambda_0)D} f(\lambda_0) \qquad \because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= \left\{\exp\left[(\lambda-\lambda_0)D\right]\right\} f(\lambda_0)$$

Similarly, for a function of several variables, we can write as

$$f(x^a) = \left[\exp\left[(x^a - x_0^a)\frac{\partial}{\partial x^a}\right]\right] f(x_0^a) \longrightarrow ①$$

Now we are able to define parallel transport and Lie transport.

A tensor will be parallely transported along a curve with tangent vector ξ^a when we replace partial derivative operator in eq ① with intrinsic derivative operator i-e D_{ξ}.

Then eq. ① becomes,

$$\overset{\parallel}{T}{}^{a\cdots c}_{d\cdots f}(x^a) = \left[\exp\left(x^a - x^a_0\right) D_{\xi}\right] T^{a\cdots c}_{d\cdots f}(x^a_0)$$

Similarly, the tensor $\underline{T}$ will be Lei transported if we replace partial derivative operator in eq ① with Lei derivative operator i-e $\mathcal{L}_{\xi}$

$$\overset{\mathcal{L}}{T}{}^{a\cdots c}_{d\cdots f}(x^a) = \left[\exp\left(x^a - x^a_0\right) \mathcal{L}_{\xi}\right] T^{a\cdots c}_{d\cdots f}(x^a_0)$$

A tensor is said to be constant if its derivative is zero. It is obvious that it may be constant w.r.t one derivative, and not w.r.t another. We say that it is invariant under parallel transport if it's intrinsic derivative is zero. i-e

$$\overset{\parallel}{\underline{T}}(x^a) = \underline{T}(x^a_0) \Longleftrightarrow \boxed{D_{\xi}(\underline{T}) = 0}$$

In component form we can write

$$\overset{\shortparallel}{T}{}^{a \cdots c}_{d \cdots f}(x^a) = T^{a \cdots c}_{d \cdots f}(x_o^a)$$

Similarly it is said to be invariant under Lei transport it

$$\overset{\mathcal{L}}{\underline{T}}(x^a) = \underline{T}(x_o^a) \Leftrightarrow \underset{\underline{\xi}}{\mathcal{L}}(\underline{T}) = 0$$

In component form.

$$\overset{\mathcal{L}}{T}{}^{a \cdots c}_{d \cdots f}(x^a) = T^{a \cdots c}_{d \cdots f}(x_o^a)$$

Geodesic & Geodesic Equation

The shortest available path between two points on the manifold is called Geodesic. This is the generalization of Euclidean Theorem that states "The shortest path between two points is a straight line"

Consider the curve such that the intrinsic derivative of the tangent vector to this curve is zero.

i.e The tangent vector is parallely transported along the curve.

In other words we say that

the derivative of the vector remain same.

It is, therefore, the straightest available path in the manifold. Thus

$$\underset{\underline{\xi}}{D}(\underline{\xi}) = 0 \Rightarrow \underline{\xi}(\underline{\xi}) = 0$$

$$\xi^a \nabla_{\underline{a}} (\xi^b) = 0$$

In component form

$$\xi^a \xi^b_{;a} = 0$$

$$\Rightarrow \xi^a \left(\xi^b_{,a} + \Gamma^b_{ab} \xi^b \right) = 0 \longrightarrow ①$$

$$\xi^a \xi^c_{,a} + \Gamma^c_{ab} \xi^a \xi^b = 0$$

We know that

$$\xi^a = \xi^a \delta^a_{\underline{a}} = \underline{\xi^a \nabla_{\underline{a}}}\, x^a$$

$$= \frac{d}{ds} x^a = \dot{x}^a \longrightarrow ②$$

Now consider

$$\xi^a \xi^b_{;a} = \frac{dx^a}{ds} \frac{d}{dx^a} \xi^b$$

$$= \frac{d}{ds}\left(\frac{dx^b}{ds}\right)$$

$$= \frac{d^2 x^b}{ds^2} = \ddot{x}^b$$

Thus eq. ① become

$$\boxed{\ddot{x}^b + \Gamma^b_{ac} \dot{x}^a \dot{x}^b = 0}$$

Geodesic eq.

"Geodesic Eq. By using Euler-Lagrange Equation." 12-01-2015

We can also derive the Geodesic equation by using the Euler-Lagrange eq. given as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \left(\frac{\partial L}{\partial q_s}\right) = 0$$

q_s are generalized coordinates & $\dot{q}_s$ are generalized velocities 'L' is Lagrangian.

s shows dimensions.

$$\text{Lagrangian} = L = \underset{K.E}{T} - \underset{P.E}{V}$$

Euler Langrange eq. is responsible for motion of particle under Kinetic and potential energy of particle

or

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \left(\frac{\partial L}{\partial q_s}\right) = 0 \longrightarrow ①$$

We know that

$$ds^2 = g_{ab}\, dx^a dx^b$$

Dividing by ds^2.

$$1 = g_{ab}\, \dot{x}^a \dot{x}^b$$

Taking "1" as lagrangian i.e

$$L = 1 = g_{ab}(x^d)\, \dot{x}^a \dot{x}^b \longrightarrow ②$$

Now eq. ① can be written as

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^d}\right) - \left(\frac{\partial L}{\partial x^d}\right) = 0 \longrightarrow ③$$

from eq. ②.

$$\frac{\partial L}{\partial x^d} = g_{ab,d}\, \dot{x}^a \dot{x}^b \longrightarrow ④$$

& $$\frac{\partial L}{\partial \dot{x}^d} = g_{ab}\left[\dot{x}^a \frac{\partial \dot{x}^b}{\partial \dot{x}^d} + \frac{\partial \dot{x}^a}{\partial \dot{x}^d}\dot{x}^b\right]$$

$$= g_{ab}\dot{x}^a \delta^b_d + g_{ab}\delta^a_d \dot{x}^b$$

$$= g_{ad}\dot{x}^a + g_{db}\dot{x}^b \longrightarrow ⑤$$

Diff. eq.⑤ w.r.t s.

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^d}\right) = g_{ad}\ddot{x}^a + g_{db}\ddot{x}^b + \frac{d}{ds}(g_{ad})\dot{x}^a + \frac{d}{ds}(g_{db})\dot{x}^b \longrightarrow ⑥$$

Now consider

$$\frac{d}{ds}\left[g_{db}(x^c)\right] = \frac{\partial}{\partial x^a}(g_{db})\frac{dx^a}{ds}$$

$$= g_{db,a}\dot{x}^a$$

Similarly

$$\frac{d}{ds}\left[g_{ad}(x^c)\right] = \frac{\partial}{\partial x^b}(g_{ad})\frac{dx^b}{ds} = g_{ad,b}\dot{x}^b$$

So, eq.⑥ becomes,

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^d}\right) = g_{ad}\ddot{x}^a + g_{db}\ddot{x}^b + g_{ad,b}\dot{x}^a\dot{x}^b + g_{db,a}\dot{x}^a\dot{x}^b \longrightarrow ⑦$$

Use eq④ and eq⑦ in eq③.

We have

$$g_{ad}\ddot{x}^a + g_{db}\ddot{x}^b + g_{ad,b}\dot{x}^a\dot{x}^b + g_{db,a}\dot{x}^a\dot{x}^b - g_{ab,d}\dot{x}^a\dot{x}^b = 0$$

$$g_{ad}\ddot{x}^a + g_{db}\ddot{x}^b + \left[g_{ad,b} + g_{db,a} - g_{ab,d}\right]\dot{x}^a\dot{x}^b = 0$$

Multiplying by $\frac{1}{2}g^{cd}$.

$$\frac{1}{2} g^{cd} g_{ad}\ddot{x}^a + \frac{1}{2} g^{cd} g_{bd}\ddot{x}^b + \frac{1}{2} g^{cd}\left[g_{ad,b} + g_{db,a} - g_{ab,d}\right]\dot{x}^a \dot{x}^b = 0$$

$$\frac{1}{2}\delta^c_a \ddot{x}^a + \frac{1}{2}\delta^c_b \ddot{x}^b + \Gamma^c_{ab}\dot{x}^a \dot{x}^b = 0$$

$$\frac{1}{2}\ddot{x}^c + \frac{1}{2}\ddot{x}^c + \Gamma^c_{ab}\dot{x}^a \dot{x}^b = 0.$$

$$\boxed{\ddot{x}^c + \Gamma^c_{ab}\dot{x}^a \dot{x}^b = 0}$$

Example:- Work out the geodesic eq. on a sphere of radius "a". 13-01-2015

Sol:- We know that the metric tensor for a sphere of radius 'a' is given by

$$g_{ab} = \begin{bmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{bmatrix}$$

$$x^a = (\theta, \phi)$$

For non-zero christofall symboles.

$$g_{22,1} = 2a^2\sin\theta\cos\theta$$

$$\Rightarrow \Gamma^1_{22}, \Gamma^2_{12}$$

We know that

$$\Gamma^c_{ab} = \frac{1}{2} g^{ce}\left[g_{ae,b} + g_{eb,a} - g_{ab,e}\right]$$

Now

$$\Gamma^1_{22} = \frac{1}{2} g^{1e}\left[g_{2e,2} + g_{e2,2} - g_{22,e}\right]$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11}\left[\cancel{g_{21,2}}^{\,0} + \cancel{g_{12,2}}^{\,0} - g_{22,1}\right] + \frac{1}{2}\cancel{g^{12}}^{\,0}\left[g_{22,2} + g_{22,2} - g_{22,2}\right]$$

$$\Gamma^1_{22} = -\frac{1}{2} g^{11} g_{22,1}$$

$$\Gamma^1_{22} = -\frac{1}{\cancel{2}}\left(\frac{1}{\cancel{a^2}}\right)\left(\cancel{2}\cancel{a^2}\sin\theta\cos\theta\right)$$

$$\boxed{\Gamma^1_{22} = -\sin\theta\cos\theta}$$

Now

$$\Gamma^2_{12} = \frac{1}{2} g^{2e}\left[g_{e2,1} + g_{1e,2} - g_{12,e}\right]$$

$$\Gamma^2_{12} = \frac{1}{2}\cancel{g^{21}}^{\,0}\left[g_{12,1} + g_{11,2} - g_{12,1}\right] + \frac{1}{2} g^{22}\left[g_{22,1} + \cancel{g_{12,2}} - \cancel{g_{12,2}}\right]$$

$$\Gamma^2_{12} = \frac{1}{2} g^{22} g_{22,1}$$

$$\Gamma^2_{12} = \frac{1}{\cancel{2}}\left(\frac{1}{\cancel{a^2}\sin^2\theta}\right)\cdot\left(\cancel{2}\cancel{a^2}\sin\theta\cos\theta\right)$$

$$\boxed{\Gamma^2_{12} = \cot\theta}$$

We know Geodesic eq

$$\ddot{x}^c + \Gamma^c_{ab}\dot{x}^a\dot{x}^b = 0$$

For $c=1$

$$\ddot{x}^1 + \Gamma^1_{ab}\dot{x}^a\dot{x}^b = 0$$

$$\ddot{x}^1 + \Gamma^1_{22}\dot{x}^2\dot{x}^2 = 0$$ because $\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{21} = 0$

$$\ddot{\theta} - \sin\theta\cos\theta\,(\dot{\phi}\,\dot{\phi}) = 0$$

$$\ddot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0 \longrightarrow ①$$

For $c=2$

$$\ddot{x}^2 + \Gamma^2_{ab}\,\dot{x}^a\,\dot{x}^b = 0$$

$$\ddot{x}^2 + \Gamma^2_{12}\,\dot{x}^1\,\dot{x}^2 + \Gamma^2_{21}\,\dot{x}^2\,\dot{x}^1 = 0$$ because $\Gamma^2_{11} = \Gamma^2_{22} = 0$

$$\ddot{\phi} + \cot\theta\,\dot{\phi}\,\dot{\theta} + \cot\theta\,\dot{\phi}\,\dot{\theta} = 0$$

$$\ddot{\phi} + 2\cot\theta\,\dot{\phi}\,\dot{\theta} = 0 \longrightarrow ②$$

Multiplying eq. ② by $\sin^2\theta$.

$$\sin^2\theta\,\ddot{\phi} + 2\sin\theta\cos\theta\,\dot{\phi}\,\dot{\theta} = 0$$

$$(\sin^2\theta\cdot\dot{\phi})^{\cdot} = 0$$

Integrating by s.

$$\sin^2\theta\cdot\dot{\phi} = h \text{ (Constant)}$$

$$\dot{\phi} = h\,\mathrm{cosec}^2\theta$$

Again integrating gives,

$$\boxed{\phi = h\,\mathrm{cosec}^2\theta\; s + h_1}$$

As $\dot{\phi} = h\,\mathrm{cosec}^2\theta$

Now $$\frac{d}{ds} = \frac{d\phi}{ds}\cdot\frac{d}{d\phi} = \dot{\phi}\,\frac{d}{d\phi} = (h\,\mathrm{cosec}^2\theta)\frac{d}{d\phi}$$

Now

$$\ddot{\theta} = \frac{d}{ds}\cdot\frac{d\theta}{ds}$$

$$= \left(h\,\text{cosec}^2\theta\,\frac{d}{d\phi}\right)\left(h\,\text{cosec}^2\theta\,\frac{d\theta}{d\phi}\right) \longrightarrow ③$$

eq. ① becomes

$$\left(h\,\text{cosec}^2\theta\,\frac{d}{d\phi}\right)\left(h\,\text{cosec}^2\theta\,\frac{d\theta}{d\phi}\right) - \sin\theta\cos\theta\left(h\,\text{cosec}^2\theta\right)^2 = 0$$

Diving by $h^2\,\text{cosec}^2\theta$.

$$\left(\frac{d}{d\phi}\right)\left(\text{cosec}^2\theta\,\frac{d\theta}{d\phi}\right) - \cot\theta = 0 \longrightarrow ④$$

Put $\cot\theta = u$

$$\frac{du}{d\phi} = -\,\text{cosec}^2\theta\cdot\left(\frac{d\theta}{d\phi}\right)$$

$$\text{cosec}^2\theta\,\frac{d\theta}{d\phi} = -\frac{du}{d\phi}$$

eq. ④ becomes.

$$\frac{d}{d\phi}\left(-\frac{du}{d\phi}\right) - u = 0$$

$$\frac{d^2u}{d\phi^2} + u = 0$$

$D^2+1=0$
$D^2=\pm i$

$$\Rightarrow u = c_1\cos\phi + c_2\sin\phi$$

$$\cot\theta = A\cos(\phi+B)$$

$$\theta = \cot^{-1}[A\cos(\phi+B)]$$

$$\Rightarrow c_1\cos\phi + c_2\sin\phi = A\cos(\phi+B)$$

$$= A\cos\phi\cos B - A\sin\phi\sin B$$

By comparing,

$$C_1 = A\cos B \longrightarrow (i)$$

$$C_2 = -A\sin B \longrightarrow (ii)$$

Dividing eq. (ii) by eq. (i)

$$-\tan B = \frac{C_2}{C_1}$$

$$B = -\tan^{-1}\left(\frac{C_2}{C_1}\right)$$

By squaring and adding (i) & (ii),

$$A^2 = C_1^2 + C_2^2$$

$$A = \sqrt{C_1^2 + C_2^2}$$

So,

$$\boxed{\theta = \cot^{-1}\left[\left(\sqrt{c_1^2+c_2^2}\right)^{-1}\cos\left(\phi - \tan^{-1}\left(\frac{c_2}{c_1}\right)\right)\right.}$$

15-1-15

Exercise:- 1):- Workout Geodesic eq. in a flat space in n-dimensional cartesian space.

2):- Workout Geodesic eq. for

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} , \quad x^a = (r, \theta, \phi)$$

Solution:-(1) We know that

$$\ddot{x}^c + \Gamma^c_{ab}\,\dot{x}^a\,\dot{x}^b = 0$$

For flat space

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Since all $g_{ab,c} = 0$

So, $\Gamma^c_{ab} = 0$

Thus $\ddot{x}^c + (0)(\dot{x}^a)(\dot{x}^b) = 0$

$\Rightarrow \ddot{x}^c = 0$

On integrating, we have

$\dot{x}^c = \alpha^c$ (constant)

Again integrating.

$x^c = \alpha^c s + \beta^c$

For

$c = 1 \Rightarrow x^1 = \alpha^1 s + \beta^1$

$c = 2 \Rightarrow x^2 = \alpha^2 s + \beta^2$

$c = 3 \Rightarrow x^3 = \alpha^3 s + \beta^3$

$\vdots \qquad \vdots \qquad \vdots$

$c = n \Rightarrow x^n = \alpha^n s + \beta^n$

Solution. 2:-

$\Gamma^2_{12}, \Gamma^1_{22} \Leftarrow g_{22,1} = 2r$

$\Gamma^3_{13}, \Gamma^1_{33} \Leftarrow g_{33,1} = 2.r \sin^2\theta$

$\Gamma^3_{23}, \Gamma^2_{33} \Leftarrow g_{33,2} = 2r^2 \sin\theta \cos\theta$

$$\Gamma^c_{ab} = \frac{1}{2} g^{cl}\left[g_{lb,a} + g_{al,b} - g_{ab,l}\right]$$

$\Gamma^1_{22}, \Gamma^2_{12} = \Gamma^2_{21}, \Gamma^1_{33}, \Gamma^3_{13} = \Gamma^3_{31}, \Gamma^2_{33}, \Gamma^3_{23} = \Gamma^3_{32}$

$$\Gamma^1_{22} = \frac{1}{2} g^{1l}\left[g_{l2,2} + g_{2l,2} - g_{22,l}\right].$$

for $l = 2, 3 \Rightarrow g^{13} = g^{12} = 0$

So,

$$\Gamma^1_{22} = \frac{1}{2} g^{11}\left[\underset{0}{\cancel{g_{12,2}}} + \underset{0}{\cancel{g_{21,2}}} - g_{22,1}\right]$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11}\left(-g_{22,1}\right).$$

$$\Gamma^1_{22} = \frac{1}{\cancel{2}}(1)(-\cancel{2}r)$$

$$\boxed{\Gamma^1_{22} = -r}$$

Now

$$\Gamma^2_{12} = \frac{1}{2} g^{2l}\left[g_{l2,1} + g_{1l,2} - g_{12,l}\right]$$

$$\Gamma^2_{12} = \frac{1}{2} g^{22}\left[g_{22,1} + \cancel{g_{12,2}} - \cancel{g_{12,2}}\right]$$

$$\Gamma^2_{12} = \frac{1}{2} g^{22} \quad g_{22,1}$$

$$\Gamma^2_{12} = \frac{1}{\cancel{2}}\left(\frac{1}{r^2}\right)(\cancel{2}r)$$

$$\boxed{\Gamma^2_{12} = \frac{1}{r}}$$

$$\Gamma^2_{33} = \frac{1}{2} g^{2l}\left[g_{l3,3} + g_{3l,3} - g_{33,l}\right]$$

$$\Gamma^2_{33} = \frac{1}{2} g^{22}\left[\underset{0}{\cancel{g_{23,3}}} + \underset{0}{\cancel{g_{32,3}}} - g_{33,2}\right]$$

$$\Gamma^2_{33} = -\frac{1}{2} g^{22} g_{33,2}$$

$$\Gamma^2_{33} = -\frac{1}{\cancel{2}}\left(\frac{1}{\cancel{r^2}}\right)(\cancel{2} \cancel{r^2} \sin\theta \cos\theta)$$

$$\boxed{\Gamma^2_{33} = - \sin\theta \cos\theta}$$

$$\Gamma^3_{23} = \frac{1}{2} g^{3l}[g_{2l,3} + g_{l3,2} - g_{23,l}]$$

$$\Gamma^3_{23} = \frac{1}{2} g^{33}[\cancel{g_{23,3}} + g_{33,2} - \cancel{g_{23,3}}]$$

$$\Gamma^3_{23} = \frac{1}{2} g^{33} g_{33,2}$$

$$\Gamma^3_{23} = \frac{1}{\cancel{2}}\left(\frac{1}{\cancel{r^2} \sin^2\theta}\right)(\cancel{2} \cancel{r^2} \sin\theta \cos\theta)$$

$$\boxed{\Gamma^3_{23} = \cot\theta}$$

$$\Gamma^3_{13} = \frac{1}{2} g^{3l}[g_{1l,3} + g_{l3,1} - g_{13,l}]$$

$$\Gamma^3_{13} = \frac{1}{2} g^{33}[\cancel{g_{13,3}} + g_{33,1} - \cancel{g_{13,3}}]$$

$$\Gamma^3_{13} = \frac{1}{2} g^{33} g_{33,1}$$

$$\Gamma^3_{13} = \frac{1}{\cancel{2}} \frac{1}{r^2 \cancel{\sin^2\theta}} (\cancel{2} r \cancel{\sin^2\theta})$$

$$\boxed{\Gamma^3_{13} = \frac{1}{r}}$$

$$\Gamma^1_{33} = \frac{1}{2} g^{1l}\left[g_{l3,3} + g_{3l,3} - g_{33,l}\right]$$

$$\Gamma^1_{33} = \frac{1}{2} g^{11}\left[\underset{0}{\cancel{g_{13,3}}} + \underset{0}{\cancel{g_{31,3}}} - g_{33,1}\right]$$

$$\Gamma^1_{33} = -\frac{1}{2} g^{11} g_{33,1}$$

$$\Gamma^1_{33} = -\frac{1}{\cancel{2}}(1)(2r\sin^2\theta)$$

$$\boxed{\Gamma^1_{33} = -r\sin^2\theta}$$

For $c=1$

$$\ddot{x}^1 + \Gamma^1_{ab}\dot{x}^a\dot{x}^b = 0$$

$$\ddot{x}^1 + \Gamma^1_{22}\dot{x}^2\dot{x}^2 + \Gamma^1_{33}\dot{x}^3\dot{x}^3 = 0$$

$$\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2 = 0 \longrightarrow ①$$

For $c=2$

$$\ddot{x}^2 + \Gamma^2_{ab}\dot{x}^a\dot{x}^b = 0$$

$$\ddot{x}^2 + 2\Gamma^2_{12}\dot{x}^1\dot{x}^2 + \Gamma^2_{33}\dot{x}^3\dot{x}^3 = 0$$

$$\ddot{\theta} + \frac{2}{r}\dot{\theta}\dot{r} - \sin\theta\cos\phi\,\dot{\phi}^2 = 0 \longrightarrow ②$$

For $c=3$

$$\ddot{x}^3 + \Gamma^3_{ab}\dot{x}^a\dot{x}^b = 0$$

$$\ddot{x}^3 + 2\Gamma^3_{31}\dot{x}^1\dot{x}^3 + 2\Gamma^3_{23}\dot{x}^2\dot{x}^3 = 0$$

$$\ddot{\phi} + \frac{2}{r}\dot{\phi}\dot{r} + 2\cot\theta\,\dot{\phi}\dot{\theta} = 0 \longrightarrow ③$$

Q:-

Work out Geodesic eq. for

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^{1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x^a = (\rho, \theta, z)$$

Solution:- $g_{22,1} = 1$

Γ^1_{22}, $\Gamma^2_{12} = \Gamma^2_{21}$

We know that

$$\ddot{x}^c + \Gamma^c_{ab}\,\dot{x}^a\,\dot{x}^b = 0$$

& $$\Gamma^c_{ab} = \frac{1}{2} g^{cl}\left[g_{al,b} + g_{lb,a} - g_{ab,l}\right]$$

$$\Gamma^1_{22} = \frac{1}{2} g^{1l}\left[g_{2l,2} + g_{l2,2} - g_{22,l}\right]$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11}\left[\overset{\nearrow 0}{g_{21,2}} + \overset{\nearrow 0}{g_{12,2}} - g_{22,1}\right] \qquad \because g^{12} = 0$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11}\left[-g_{22,1}\right]$$

$$\Gamma^1_{22} = \frac{1}{2}(1)(-1)$$

$$\boxed{\Gamma^1_{22} = -\frac{1}{2}}$$

$$\Gamma^2_{12} = \frac{1}{2} g^{2l}\left[g_{1l,2} + g_{l2,1} - g_{12,l}\right]$$

$$\Gamma^2_{12} = \frac{1}{2} g^{22}\left[\cancel{g_{12,2}} + g_{22,1} - \cancel{g_{12,2}}\right] \qquad \because g^{21} = 0$$

$$\Gamma^2_{12} = \frac{1}{2} g^{22} g_{22,1}$$

$$\Gamma^{2}_{12} = \frac{1}{2}\left(\frac{1}{\rho}\right) \quad (1)$$

$$\boxed{\Gamma^{2}_{12} = \frac{2}{\rho}}$$

For $C=1$

$$\ddot{x}^{1} + \Gamma^{1}_{ab}\,\dot{x}^{a}\,\dot{x}^{b} = 0$$

$$\ddot{\rho} + \Gamma^{1}_{22}\,\dot{x}^{2}\,\dot{x}^{2} = 0$$

$$\ddot{\rho} + \left(\frac{-1}{2}\right)\dot{\theta}\,\dot{\theta} = 0$$

$$2\ddot{\rho} - \dot{\theta}^{2} = 0$$

$$2\ddot{\rho} = \dot{\theta}^{2} \longrightarrow ①$$

For $C=2$

$$\ddot{x}^{2} + \Gamma^{2}_{ab}\,\dot{x}^{a}\,\dot{x}^{b} = 0$$

$$\ddot{\theta} + 2\Gamma^{2}_{12}\,\dot{x}^{1}\,\dot{x}^{2} = 0$$

$$\ddot{\theta} + \cancel{2}\left(\frac{1}{\cancel{2}\rho}\right)\dot{\rho}\,\dot{\theta} = 0$$

$$\rho\ddot{\theta} + \dot{\rho}\,\dot{\theta} = 0 \longrightarrow ②$$

For $c=3$

$$\ddot{x}^{3} + \Gamma^{3}_{ab}\,\dot{x}^{a}\,\dot{x}^{b} = 0$$

$$\ddot{x}^{3} = 0 \qquad \because \Gamma^{3}_{ab} = 0$$

$$\ddot{z} = 0 \longrightarrow ③$$

On Integrating,

$$\dot{z} = h \text{(constant)}$$

again integrating

$$\boxed{z = hs + h_1}$$

eq ② ⇒

$$\rho\ddot{\theta} + \dot{\rho}\dot{\theta} = 0$$

$$(\rho\dot{\theta})^{\cdot} = 0$$

On Integrating

$$\rho\dot{\theta} = K \text{ (constant)}$$

$$\dot{\theta} = \frac{K}{\rho}$$

Again integrating,

$$\boxed{\theta = \frac{K}{\rho}s + K_1}$$

eq. ① ⇒ $2\ddot{\rho} = \dot{\theta}^2$

$$2\ddot{\rho} = \left(\frac{K}{\rho}\right)^2$$

$$2\rho^2\ddot{\rho} = K^2$$

Consider $\frac{d}{ds} = \frac{d\theta}{ds}\cdot\frac{d}{d\theta} = \frac{K}{\rho}\frac{d}{d\theta}$

& $\ddot{\rho} = \frac{d}{ds}\cdot\frac{d\rho}{ds} = \left(\frac{K}{\rho}\frac{d}{d\theta}\right)\left(\frac{K}{\rho}\frac{d\rho}{d\theta}\right)$

$$\ddot{\rho} = \frac{K^2}{\rho^2}\frac{d^2\rho}{d\theta^2}$$

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$$2\rho^2\left(\frac{K^2}{\rho^2}\,\frac{d^2\rho}{d\theta^2}\right) = K^2$$

$$\frac{d^2\rho}{d\theta^2} = \frac{1}{2}$$

$$\frac{d\rho}{d\theta} = \frac{1}{2}\theta + M$$

(constant)

$$\rho = \frac{1}{2}\left(\frac{\theta^2}{2}\right) + M\theta + M_1$$

$$\rho = \frac{\theta^2}{4} + M\theta + M_1$$

Put $\theta = \frac{K}{\rho}s + K_1$

$$\rho = \frac{1}{4}\left(\frac{K}{\rho}s + K_1\right)^2 + M\left(\frac{K}{\rho}s + K_1\right) + M_1$$

$$\rho = \frac{1}{4}\frac{K^2}{\rho^2}s^2 + \frac{1}{4}K_1^2 + \frac{1}{2}\frac{KK_1}{\rho}s + \frac{MK}{\rho}s + MK_1 + M_1$$

$$\rho = \frac{K^2}{4\rho^2}s^2 + \left(\frac{KK_1}{2\rho} + \frac{MK}{\rho}\right)s + \frac{1}{4}K_1^2 + MK_1 + M_1$$

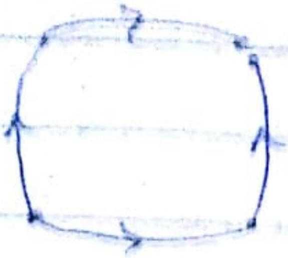
$$\boxed{\rho = As^2 + Bs + C}$$

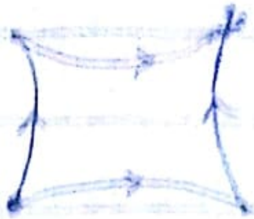
where $A = \frac{K^2}{4\rho^2}$

$$B = \frac{KK_1}{2\rho} + \frac{MK}{\rho}$$

& $C = \frac{1}{4}K_1^2 + MK_1 + M_1$

Curvature on Manifold 19-01-15





π-radian

We know that a local measure of the curvature of manifold may be obtain by the difference b/w the sum of the angles of a triangle drawn in the manifold & π-radian. The sides of triangles are offcourse the geodesic. Another way to measure the curvature is to see the extend to which a parallelogram closes over or stays open.

Mathematically we will calculate

$$\xi^a_{;d;c} - \xi^a_{;c;d}$$

let $\xi^a_{;c} = T^a_c$

and

$$\xi^a_{;c;d} = (\xi^a_{;c})_{;d} = (T^a_c)_{;d} = T^a_{c;d}$$

$$= T^a_{c,d} + \Gamma^a_{db} T^b_c - \Gamma^b_{cd} T^a_b$$

$$= (\xi^a_{;c})_{,d} + \Gamma^a_{db} \xi^b_{;c} - \Gamma^b_{cd} \xi^a_{;b}$$

$$= (\xi^a_{,c} + \Gamma^a_{ce} \xi^e)_{,d} + \Gamma^a_{db}(\xi^b_{,c} + \Gamma^b_{ce} \xi^e)$$

$$- \Gamma^b_{cd}(\xi^a_{,b} + \Gamma^a_{be} \xi^e)$$

$$= \xi^a_{,c,d} + \underline{\Gamma^a_{ce,d}\,\xi^e + \Gamma^a_{ce}\,\xi^e_{,d}}$$

$$+ \Gamma^a_{db}\,\xi^b_{,c} + \Gamma^a_{db}\Gamma^b_{ce}\,\xi^e - \Gamma^b_{cd}\,\xi^a_{,b}$$

$$- \Gamma^b_{cd}\,\Gamma^a_{be}\,\xi^e \longrightarrow ①$$

Similarly we can write.

$$\xi^a_{;d;c} = \xi^a_{,d,c} + \Gamma^a_{de,c}\,\xi^e + \Gamma^a_{de}\,\xi^e_{,c}$$

$$+ \Gamma^a_{cb}\,\xi^b_{,d} + \Gamma^a_{cb}\Gamma^b_{de}\,\xi^e - \Gamma^b_{dc}\,\xi^a_{,b} - \Gamma^b_{dc}\Gamma^a_{be}\,\xi^e \longrightarrow ②$$

By ② − ①

$\{e \Leftrightarrow b$

$$\xi^a_{;d;c} - \xi^a_{;c;d} = \cancel{\xi^a_{,d,c}} + \Gamma^a_{de,c}\,\xi^e + \Gamma^a_{de}\,\xi^e_{,c}$$

$b \Leftrightarrow e$

$$+ \Gamma^a_{cb}\,\xi^b_{,d} + \Gamma^a_{cb}\Gamma^b_{de}\,\xi^e - \cancel{\Gamma^b_{dc}\,\xi^a_{,b}} - \cancel{\Gamma^b_{dc}\Gamma^a_{be}\,\xi^e}$$

$$- \cancel{\xi^a_{,c,d}} - \Gamma^a_{ce,d}\,\xi^e - \Gamma^a_{ce}\,\xi^e_{,d} - \Gamma^a_{db}\,\xi^b_{,c}$$

$$- \Gamma^a_{db}\Gamma^b_{ce}\,\xi^e + \cancel{\Gamma^b_{cd}\,\xi^a_{,b}} + \cancel{\Gamma^b_{cd}\Gamma^a_{be}\,\xi^e}$$

$$= \Gamma^a_{de,c}\,\xi^e + \cancel{\Gamma^a_{db}\,\xi^b_{,c}} + \cancel{\Gamma^a_{ce}\,\xi^e_{,d}} + \Gamma^a_{cb}\Gamma^b_{de}\,\xi^e$$

$$- \Gamma^a_{ce,d}\,\xi^e - \cancel{\Gamma^a_{ce}\,\xi^e_{,d}} - \cancel{\Gamma^a_{db}\,\xi^b_{,c}} - \Gamma^a_{db}\Gamma^b_{ce}\,\xi^e$$

$$= \left(\Gamma^a_{de,c} - \Gamma^a_{ce,d} + \Gamma^a_{cb}\Gamma^b_{de} - \Gamma^a_{db}\Gamma^b_{ce}\right)\xi^e$$

$$= R^a_{ecd}\,\xi^e \qquad e \Leftrightarrow b$$

$$= R^a_{bcd}\,\xi^b$$

Here R^a_{bcd} is called Riemann ~~curvature~~ tensor ~~et~~ or simply curvature tensor.

Example:- Workout non-zero components of R^a_{bcd} for the metric tensor.

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

$$g_{ab} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{pmatrix}$$

Rici Tensor

20-1-15

The rici tensor is obtained by contracting first and third indices of the riemann curvature as

$$R_{bd} = R^a_{bad}$$

$$= \Gamma^a_{bd,a} - \Gamma^a_{ba,d} + \Gamma^a_{ea}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{ab}$$

Rici Scalar is denoted by R and defined as

$$R = g^{bd} R_{bd}$$

$$= g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} + g^{44}R_{44} \cancel{+ g^{mm}R_{mm}}$$

Examples:- Workout Rici Scalar for a sphere of radius "a".

Solution:-

$$g_{ab} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{pmatrix}$$

$x^a = (\theta, \phi)$

$$g_{22,1} = 2a^2\sin\theta\cos\theta \Rightarrow \Gamma^1_{22}, \Gamma^2_{12}$$

$$\Gamma^c_{ab} = \frac{1}{2} g^{cl}[g_{al,b} + g_{lb,a} - g_{ab,l}]$$

$$\Gamma^1_{22} = \frac{1}{2} g^{1l}[g_{2l,2} + g_{12,2} - g_{22,l}]$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11}[\underset{0}{\cancel{g_{21,2}}} + \underset{0}{\cancel{g_{12,2}}} - g_{22,1}]$$

$$\Gamma^1_{22} = \frac{-1}{2} g^{11} g_{22,1}$$

$$\Gamma^1_{22} = \frac{-1}{\cancel{2}}\left(\frac{1}{\cancel{a^2}}\right)(\cancel{2}\cancel{a^2}\sin\theta\cos\theta)$$

$$\boxed{\Gamma^1_{22} = -\sin\theta\cos\theta}$$

$$\Gamma^2_{12} = \frac{1}{2} g^{2l}[g_{1l,2} + g_{l2,1} - g_{12,l}]$$

$$\Gamma^2_{12} = \frac{1}{2} g^{22}[\cancel{g_{12,2}} + g_{22,1} - \cancel{g_{12,2}}]$$

$$\Gamma^2_{12} = \frac{1}{2} g^{22} g_{22,1}$$

$$\Gamma^2_{12} = \frac{1}{\cancel{2}}\left(\frac{1}{\cancel{a^2}\sin^2\theta}\right)(\cancel{2}\cancel{a^2}\sin\theta\cos\theta)$$

$$\boxed{\Gamma^2_{12} = \cot\theta}$$

We know that

$$R = g^{11} R_{11} + g^{22} R_{22}$$

&

$$R_{bd} = \Gamma^a_{bd,a} - \Gamma^a_{ba,d} + \Gamma^a_{ea} \Gamma^e_{bd} - \Gamma^a_{ed} \Gamma^e_{ab}$$

$$R_{11} = \underset{0}{\cancel{\Gamma^a_{11,a}}} - \Gamma^a_{1a,1} + \Gamma^a_{ea} \underset{0}{\cancel{\Gamma^e_{11}}} - \Gamma^a_{e1} \Gamma^e_{a1}$$

$$= -\Gamma^2_{12,1} - \Gamma^2_{e1} \Gamma^e_{21}$$

$$= -\Gamma^2_{12,1} - \Gamma^2_{21} \Gamma^2_{21}$$

$$= -(-\operatorname{cosec}^2\theta) - (\cot\theta)(\cot\theta)$$

$$= \operatorname{cosec}^2\theta - \cot^2\theta = 1$$

$$\Rightarrow \boxed{R_{11} = 1}$$

$$R_{22} = \Gamma^a_{22,a} - \Gamma^a_{2a,2} + \Gamma^a_{ea} \Gamma^e_{22} - \Gamma^a_{e2} \Gamma^e_{a2}$$

$$= \Gamma^1_{22,1} - \underbrace{\cancel{\Gamma^1_{21,2}} - \cancel{\Gamma^2_{22,2}}}_{0} + \underbrace{\Gamma^2_{12} \Gamma^1_{22}} - \Gamma^a_{12} \Gamma^1_{a2} - \Gamma^a_{22} \Gamma^2_{a2}$$

$$= \Gamma^1_{22,1} + \cancel{\Gamma^2_{12} \Gamma^1_{22}} - \cancel{\Gamma^2_{12} \Gamma^1_{22}} - \Gamma^1_{22} \Gamma^2_{12}$$

$$= \Gamma^1_{22,1} - \Gamma^1_{22} \Gamma^2_{12}$$

$$= -[\sin\theta(-\sin\theta) + \cos\theta\cos\theta] - (-\sin\theta\cos\theta)(\cot\theta)$$

$$= \sin^2\theta - \cancel{\cos^2\theta} + \cancel{\cos^2\theta}$$

$$\Rightarrow \boxed{R_{22} = \sin^2\theta}$$

So,

$$R = \frac{1}{a^2}.(1) + \frac{1}{a^2 \cancel{\sin^2\theta}} (\cancel{\sin^2\theta})$$

$$R = \frac{1}{a^2} + \frac{1}{a^2}$$

$$R = \frac{2}{a^2}$$

Answer.

Properties of Riemann Curvature tensor. **22-1-15**

$$R_{abcd} = g_{ae} R^e_{bcd}$$

(i):- $R_{abcd} = -R_{\underline{ba}cd}$

$= -R_{ab\underline{dc}}$

$= R_{\underline{ba}\underline{dc}}$

$= R_{cdab}$

Bianchi First Identity:-

$$R^a_{[bcd]} = 0$$

Bianchi Second Identity:-

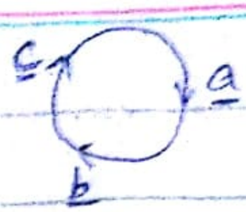
$$R^a_{b[cd;e]} = 0$$

Symetric Bracket:-

$$A_{(ab)} = \frac{A_{ab} + A_{\underline{ba}}}{2!}$$

Skew Symetric Bracket:-

$$A_{[ab]} = \frac{A_{ab} - A_{ba}}{2!}$$



$$A_{[\underline{a}\,\underline{b}\,\underline{c}]} = \frac{1}{3!}\left[A_{\underline{a}\underline{b}\underline{c}} + A_{\underline{b}\underline{c}\underline{a}} + A_{\underline{c}\underline{a}\underline{b}} - A_{\underline{a}\underline{c}\underline{b}} - A_{\underline{c}\underline{b}\underline{a}} - A_{\underline{b}\underline{a}\underline{c}}\right]$$

$A[a_1 \; a_n] = \frac{1}{n!}$[Sum of even permutation − Sum of odd permutation]

Associative Property:-

$$A_{[\underline{a}[\underline{b}\,\underline{c}]]} = A_{[[\underline{a}\,\underline{b}]\underline{c}]} = A_{[\underline{a}\,\underline{b}\,\underline{c}]}$$

Einstein Tensor

Consider

$$R^a{}_{b[cd;e]} = \frac{1}{3!}\left[R^a{}_{bcd;e} + R^a{}_{bde;c} + R^a{}_{bec;d} - R^a{}_{bce;d} - R^a{}_{bdc;e} - R^a{}_{bed;c}\right]$$

$$R^a{}_{b[cd;e]} = \frac{1}{3!}\left[R^a{}_{bcd;e} + R^a{}_{bde;c} + R^a{}_{bec;d} + R^a{}_{bec;d} + R^a{}_{bcd;e} + R^a{}_{bde;c}\right]$$

By Bianchi Second Identity.

$$R^a{}_{b[cd;e]} = 0 \Rightarrow$$

$$\frac{2}{6}\left[R^a{}_{bcd;e} + R^a{}_{bde;c} + R^a{}_{bec;d}\right] = 0$$

$$\Rightarrow R^a{}_{bcd;e} + R^a{}_{bde;c} + R^a{}_{bec;d} = 0$$

Contracting a & c.

$$R^a{}_{bad;e} + R^a{}_{bde;a} + R^a{}_{bea;d} = 0.$$

$$R_{bd;e} + R^a_{bde;a} - R^a_{bae;d} = 0$$

$$R_{bd;e} - R_{be;d} + R^a_{bde;a} = 0$$

Multiplying by g^{bq}

$$g^{bq} R_{bd;e} - g^{bq} . P_{be;d} + g^{bq} R^a_{bde;a} = 0$$

$$R^q_{d;e} - R^q_{e;d} + R^{aq}_{de;a} = 0$$

Contracting q & e.

$$R^e_{d;e} - R^e_{e;d} + R^{ae}_{de;a} = 0$$

$$\underline{R^e_{d;e} - R^e_{e;d}} + R^{ea}_{ed;a} = 0$$

$e \Leftrightarrow a$

$$R^a_{d;a} - R^a_{a;d} + R^a_{d;a} = 0$$

$$2R^a_{d;a} - R_{;d} = 0$$

$$2R^a_{d;a} - \delta^a_d R_{;a} = 0$$

$$R^a_{d;a} - \frac{1}{2}\delta^a_d R_{;a} = 0$$

$$\left(R^a_d - \frac{1}{2}\delta^a_d R\right)_{;a} = 0$$

$$G^a_{d;a} = 0 \quad (A)$$

where $G^a_d = R^a_d - \frac{1}{2}\delta^a_d R$ is called Einstein tensor.

Multiplying g_{ac} with ①-

$$g_{ac}G^{a}_{b} = g_{ac}R^{a}_{b} - \frac{1}{2}g_{ac}\delta^{a}_{b}R$$

$$\boxed{G_{bc} = R_{bc} - \frac{1}{2}g_{bc}R}$$

Now Multiplying by g^{bc} with eq ①.

$$g^{bc}G^{a}_{b} = g^{bc}R^{a}_{b} - \frac{1}{2}g^{bc}\delta^{a}_{b}R$$

$$\boxed{G^{ac} = R^{ac} - \frac{1}{2}g^{ac}R}$$

Curvature Invariants 29-1-15

The Riemann tensor is useful for determining where the singularity is essential or coordinate. If the curvature become infinite (∞) the singularity is called essential.

Since we know that

$$R^{a}_{bcd} = \begin{Bmatrix} a \\ bd \end{Bmatrix}_{,c} - \begin{Bmatrix} a \\ bc \end{Bmatrix}_{,d} + \begin{Bmatrix} a \\ ec \end{Bmatrix}\begin{Bmatrix} e \\ bd \end{Bmatrix} - \begin{Bmatrix} a \\ ed \end{Bmatrix}\begin{Bmatrix} e \\ bc \end{Bmatrix}$$

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

If $\theta = 0, \pi$ then it will be singular.

Where $\left\{ {a \atop bc} \right\} = \Gamma^{a}_{bc}$ is expressed in coordinates terms.

Consequently these will effect riemann tensor components. However scalar quantities are invariant under coordinates transformation, such as we construct Ricci scalar from Riemann curvature tensor.

It is obvious that infinity many scalars can be construct from R^{a}_{bcd}. However symmetry consideration can be used to show that there are only finite number of independent scalars. All the other can be expressed in terms of these scalars.

The simplest scalars can be construct as;

$$R_1 = g^{ab} R_{ab} = R$$

$$R_2 = R^{ab}_{cd} R^{cd}_{ab}$$

$$R_3 = R^{ab}_{cd} R^{cd}_{ef} R^{ef}_{ab} \quad \text{and so on.}$$

These are called Curvature Invariants.

The points where the curvature invariants becomes infinite are called essential singular point.

If curvature invariants are finite then singularity is called coordinate singularity.

Hint // For example:-

$$g_{ab} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \sin^2\theta \end{pmatrix}$$

which is singular at $\theta = 0, \pi$

But $R = \frac{2}{a^2}$

which is finite at $\theta = 0, \pi$.

So, $\theta = 0, \pi$ are the coordinate singularity.

Q:- Find the nature of the singularity of a right cone.

Ans:- We know that for right cone,

$$x(u,v) = (u\cos v, u\sin v, u)$$

$$\Rightarrow x_u = (\cos v, \sin v, 1)$$

$$\& \quad x_v = (-u\sin v, u\cos v, 0)$$

$$E = x_u \cdot x_u = \cos^2 v + \sin^2 v + 1$$

$$\Rightarrow \boxed{E = 2}$$

$$F = x_u \cdot x_v = \cancel{-u \sin v \cos v + u \cos v \sin v}$$

$$\Rightarrow \boxed{F = 0}$$

$$G = x_v \cdot x_v = u^2 \sin^2 v + u^2 \cos^2 v$$

$$\Rightarrow \boxed{G = u^2}$$

So,
$$g_{ab} = \begin{pmatrix} 2 & 0 \\ 0 & u^2 \end{pmatrix}, \quad x^a = (u, v)$$

Which is singular at $u = 0$.

$$g_{22,1} = 2u$$

$$\Gamma^c_{ab} = \frac{1}{2} g^{cl} \left[g_{lb,a} + g_{al,b} - g_{ab,l} \right]$$

$$\Gamma^1_{22} = \frac{1}{2} g^{1l} \left[g_{l2,2} + g_{2l,2} - g_{22,l} \right]$$

$$= \frac{1}{2} g^{11} \left[\underbrace{\cancel{g_{12,2}}}_{0} + \underbrace{\cancel{g_{21,2}}}_{0} - g_{22,1} \right]$$

$$= \frac{1}{2} g^{11} \left[-g_{22,1} \right]$$

$$= \frac{1}{2} \cdot \left(\frac{1}{\cancel{2}} \right) \left(-\cancel{2}u \right)$$

$$\boxed{\Gamma^1_{22} = \frac{-u}{2}}$$

$$\Gamma^2_{12} = \frac{1}{2} g^{2l}\left[g_{l2,1} + g_{1l,2} - g_{12,2}\right]$$

$$= \frac{1}{2} g^{22}\left[g_{22,1} + \cancel{g_{12,2}} - \cancel{g_{12,2}}\right]$$

$$= \frac{1}{2} g^{22}\left[g_{22,1}\right]$$

$$= \frac{1}{\cancel{2}} \cdot \left(\frac{1}{u^2}\right) \cdot (\cancel{2}u)$$

$$\boxed{\Gamma^2_{12} = \frac{1}{u}}$$

We know that

$$R = g^{11}R_{11} + g^{22}R_{22}$$

&

$$R_{bd} = \Gamma^a_{bd,a} - \Gamma^a_{ba,d} + \Gamma^a_{ea}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{ab}$$

$$R_{11} = \underset{0}{\cancel{\Gamma^a_{11,a}}} - \Gamma^a_{1a,1} + \Gamma^1_{ea}\underset{0}{\cancel{\Gamma^e_{11}}} - \Gamma^a_{e1}\Gamma^e_{a1}$$

$$= -\Gamma^2_{12,1} - \left[\Gamma^1_{e1}\underset{0}{\cancel{\Gamma^e_{11}}} + \Gamma^2_{e1}\Gamma^e_{21}\right]$$

$$= -\Gamma^2_{12,1} - \Gamma^2_{21}\Gamma^2_{21}$$

$$= -\left(-u^{-1-1}\right) - \left(\frac{1}{u}\right)\left(\frac{1}{u}\right)$$

$$= \cancel{\frac{1}{u^2}} - \cancel{\frac{1}{u^2}}$$

$$= 0$$

$$R_{22} = \Gamma^a_{22,a} - \Gamma^a_{2a,2} + \Gamma^a_{ea}\Gamma^e_{22} - \Gamma^a_{e2}\Gamma^e_{a2}$$

$$= \Gamma^1_{22,1} + \left[\Gamma^2_{12}\Gamma^1_{22}\right] - \left[\Gamma^a_{12}\Gamma^1_{a2} + \Gamma^a_{22}\Gamma^2_{a2}\right]$$

$$= \Gamma^1_{22,1} + \cancel{\Gamma^2_{12}\Gamma^1_{22}} - \cancel{\Gamma^2_{12}\Gamma^1_{22}} - \Gamma^1_{22}\Gamma^2_{12}$$

$$= \Gamma^1_{22,1} - \Gamma^1_{22}\Gamma^2_{12}$$

$$= \frac{-1}{2} - \left(\frac{-u}{2}\right)\left(\frac{1}{u}\right)$$

$$= \cancel{\frac{-1}{2}} + \cancel{\frac{1}{2}}$$

$$= 0$$

So,

$$R = g^{11}(0) + g^{22}(0)$$

$$R = 0.$$

which is finite at $u=0$

So, $u=0$ is the coordinate singularity.

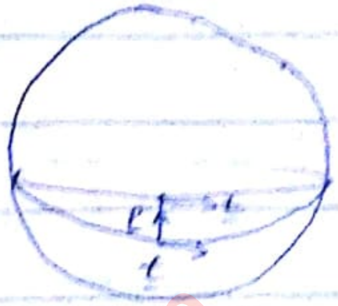
Geodesic Deviation 02-02-15

Let us consider two families of Geodesic with tangent vector $\underline{t}$ i.e

$$\underline{t}\,[\underline{t}] = 0$$

In components form

$$t^b\, t^a_{;b} = 0 \longrightarrow ①$$



P is Separation vector.

Let $\underline{P}$ be the vector field Joining the two Geodesics. Then $\underline{P}$ is Lei transported along $\underline{t}$. i.e $\underline{P}$ remains invariant under Lei transport. Thus

$$\underline{t}\,[\underline{P}] = \underline{P}\,[\underline{t}]$$

In components forms

$$\Rightarrow t^a P^b_{;a} = P^a t^b_{;a} \longrightarrow ②$$

$$\because \mathcal{L}_{\underline{\xi}}\,\eta^b = \xi^a \eta^b_{;a} - \eta^a \xi^b_{;a} = 0$$

However $\underline{P}$ needs not be parallel transported along $\underline{t}$, i.e

$$\underline{t}\,(\underline{P}) \neq 0 \longrightarrow ③$$

If we consider $\underline{P}$ as position vector. Then we can write the acclaration vector as

$$\underline{A} = \ddot{P} = \frac{d^2P}{ds^2} = \underline{t}\left[\underline{t}(P)\right] \longrightarrow ④$$

In componehents form eq ④ implies that

$$A^a = t^c(t^b P^a_{;b})_{;c}$$

Using eq ②.

$$A^a = t^c(P^b t^a_{;b})_{;c}$$

$$A^a = t^c(P^b t^a_{;b;c} + P^b_{;c} t^a_{;b})$$

$$A^a = t^c P^b t^a_{;b;c} + \underbrace{t^c P^b_{;c}}_{\uparrow \text{ using eq ②.}} t^a_{;b}$$

$$A^a = t^c P^b t^a_{;b;c} + P^c t^b_{;c} t^a_{;b}$$

we know

$$(t^b t^a_{;b})_{;c} = t^b_{;c} t^a_{;b} + t^b t^a_{;b;c}$$

$$\Rightarrow t^b_{;c} t^a_{;b} = (t^b t^a_{;b})_{;c} - t^b t^a_{;b;c}$$

$$A^a = t^c P^b t^a_{;b;c} + P^c\left[\underset{0}{\cancel{(t^b t^a_{;b})}}_{;c} - t^b t^a_{;b;c}\right]$$

$\uparrow$ By eq ①

$$A^a = t^c P^b t^a_{;b;c} - \underbrace{P^c t^b t^a_{;b;c}}_{\uparrow\, b \longleftrightarrow c}$$

$$A^a = t^c P^b t^a_{;b;c} - P^b t^c t^a_{;c;b}$$

$$A^a = t^c P^b\left[t^a_{;b;c} - t^a_{;c;b}\right]$$

$$A^a = t^c p^b \left[R^a_{dbc} \, t^d \right]$$

$$A^a = t^d R^a_{dbc} \, p^b t^c$$

This gives Geodesic Deviations.

We can define **Tidal Force** as

$$F_T^a = m \underbrace{A^a}_{\text{Geodesic Deviation}}$$

6-2-2015

Killing Vectors (or Isometries)

(A mathematician)

An isometry is a direction along with the metric tensor is Lie transported. If K is an isometry then

$$\underset{K}{\mathcal{L}} g = 0 \longrightarrow ①$$

$$K^c \nabla_c g_{ab} = 0$$

Using index notation we can write

$$\cancelto{0}{g_{ab;c}} K^c + g_{cb} K^c_{;a} + g_{ac} K^c_{;b} = 0 \longrightarrow ②$$

$$\Rightarrow \quad K_{b;a} + K_{a;b} = 0 \longrightarrow ③$$

$$\Rightarrow \quad K_{(a;b)} = 0 \longrightarrow ④$$

The equation ① → ④ are different forms of Killing Equations. Any vector satisfying these equations is called Killing Vector or Isometry.

Case I: As for the case of flat space all the christoffel symbols becomes zero. So the equation (4) simply becomes

$$K_{(a,b)} = 0$$

$$\Rightarrow \quad K_{a,b} + K_{b,a} = 0 \longrightarrow ⑤$$

when $a = b$.

$$K_{a,a} = 0 \longrightarrow ⑥$$

Now differentiate equation ⑤ w.r.t x^a

$$K_{a,ba} + K_{b,aa} = 0$$

$$K_{a,ab} + K_{b,aa} = 0$$

$$\therefore \underbrace{(K_{a,a})}_{0}{}_{,b} + K_{b,aa} = 0$$

by Eq. (6)

$$\Rightarrow K_{b,aa} = 0$$

On integrating,

$$K_{b,a} = C_{ba}$$

Again Integrating,

$$K_b = C_{ba} x^a + D_b$$

Similarly,

$$K_{a,bb} = 0$$

On integrating,

$$K_{a,b} = C_{ab}$$

Again,

$$K_a = C_{ab} x^b + D_a$$

Eq. ⑤ becomes

$$C_{ab} = -C_{ba}$$

$\Rightarrow$ C_{ab} is skew symmetric in it's indices.

In general, we have $\frac{n(n-1)}{2}$ independent components of C_{ab} in an n-dimensional space. These correspond to rotation metrix. There are also n-independent components of D_a which correspond to translation.

Thus in general, killing vectors depends upon $n+\frac{n(n-1)}{2}=\frac{n(n+1)}{2}$ independent components in a flat n-dimensional space.

Case II: Now we consider the curvilinear coordinates for which all the christoffell symboles are not zero. In this case, it is more convenient to keep the Killing vectors in contravarient form give as in Eq. ③

$$g_{cb}K^c_{;a}+g_{ac}K^c_{;b}=0$$

$$g_{cb}\left[K^c_{,a}+\Gamma^c_{da}K^d\right]+g_{ac}\left[K^c_{,b}+\Gamma^c_{db}K^d\right]=0$$

$$g_{cb}K^c_{,a}+g_{cb}\Gamma^c_{da}K^d+g_{ac}K^c_{,b}+g_{ac}\Gamma^c_{db}K^d=0$$

$$g_{cb}K^c_{,a}+g_{ac}K^c_{,b}+\left[g_{cb}\Gamma^c_{da}+g_{ac}\Gamma^c_{db}\right]K^d=0 \rightarrow ⑦$$

Now consider,

$$g_{cb}\Gamma^{c}_{da} = g_{cb}\left(\tfrac{1}{2}\right)g^{ce}\left[g_{de,a}+g_{ea,d}-g_{da,e}\right]$$

$$= \tfrac{1}{2}\delta^{e}_{b}\left[g_{de,a}+g_{ea,d}-g_{da,e}\right]$$

$$= \tfrac{1}{2}\left[\delta^{e}_{b}g_{de,a}+\delta^{e}_{b}g_{ea,d}-\delta^{e}_{b}g_{da,e}\right]$$

$$= \tfrac{1}{2}\left[g_{db,a}+g_{ba,d}-g_{da,b}\right]$$

Similarly,

$$g_{ac}\Gamma^{c}_{db} = \tfrac{1}{2}\left[g_{ab,d}+g_{da,b}-g_{db,a}\right]$$

Eq. ② becomes

$$g_{cb}K^{c}_{,a}+g_{ac}K^{c}_{,b}+\tfrac{1}{2}\Big[\cancel{g_{db,a}}+\overset{a\leftrightarrow b}{g_{ba,d}}-\cancel{g_{da,b}}$$

$$+g_{ab,d}+\cancel{g_{da,b}}-\cancel{g_{db,a}}\Big]K^{d}=0$$

$$\Rightarrow g_{cb}K^{c}_{,a}+g_{ac}K^{c}_{,b}+\tfrac{1}{2}\left[g_{ab,d}+g_{ab,d}\right]K^{d}=0$$

$$g_{cb}K^{c}_{,a}+g_{ac}K^{c}_{,b}+g_{ab,d}K^{d}=0$$

replace d with c

$$K_{ab}:\ g_{ab,c}K^{c}+g_{cb}K^{c}_{,a}+g_{ac}K^{c}_{,b}=0$$

These are called the Killing Equations in covariant form.

Q:-

Workout the killing Eq for

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad x^i = (r, \theta)$$

and also solve the Killing Equations.

Solution:- We know that

$$K_{ab} : g_{ab,c}K^c + g_{cb}K^c_{,a} + g_{ac}K^c_{,b} = 0$$

$$K_{11} : g_{11,c}K^c + g_{c1}K^c_{,1} + g_{1c}K^c_{,1} = 0$$

$$\underset{0}{\cancel{g_{11,1}}}K^1 + \underset{0}{\cancel{g_{11,2}}}K^2 + g_{11}K^1_{,1} + g_{11}K^1_{,1} = 0$$

$$2g_{11}K^1_{,1} = 0$$

$$2(1)K^1_{,1} = 0$$

$$\Rightarrow K^1_{,1} = 0$$

$$\therefore \Rightarrow K^1 = A(\theta) \longrightarrow ①$$

$$K_{21} = K_{12} : \underset{0}{\cancel{g_{12,c}}}K^c + g_{c2}K^c_{,1} + g_{1c}K^c_{,2} = 0$$

$$g_{22}K^2_{,1} + g_{11}K^1_{,2} = 0$$

$$r^2K^2_{,1} + (1)K^1_{,2} = 0$$

$$K^1_{,2} + r^2K^2_{,1} = 0 \longrightarrow ②$$

$$K_{22} : g_{22,c}K^c + g_{2c}K^c_{,2} + g_{2c}K^c_{,2} = 0$$

$$g_{22,1} K^1 + 2 g_{22} K^2_{,2} = 0$$

$$2r K^1 + 2 r^2 K^2_{,2} = 0$$

Dividing $2r$.

$$K^1 + r K^2_{,2} = 0 \longrightarrow ③$$

Differentiate Eq. ② with respect to θ.

$$K^1_{,22} + r^2 K^2_{,12} = 0$$

OR $$A_{,\theta\theta} + r^2 K^2_{,12} = 0 \longrightarrow ④$$

Eq. ③ ⇒ $$K^2_{,2} = \frac{-1}{r} A(\theta) \longrightarrow ⑤$$

Diff. w.r.t r.

$$K^2_{,21} = \frac{1}{r^2} A(\theta)$$

Eq. ④ ⇒ $$A_{,\theta\theta} + \cancel{r^2}\left[\frac{1}{\cancel{r^2}} A(\theta)\right] = 0$$

$$A_{,\theta\theta} + A(\theta) = 0$$

$D^2 + 1 = 0$

$D = \pm i$

$$\Rightarrow \boxed{K^1 = A(\theta) = c_1 \cos\theta + c_2 \sin\theta} \longrightarrow ⑥$$

Eq. ⑤ ⇒ $$K^2_{,2} = \frac{-1}{r}\left(c_1 \cos\theta + c_2 \sin\theta\right)$$

Integrating w.r.t θ.

$$K^2 = \frac{-1}{r}\left[c_1 \sin\theta - c_2 \cos\theta\right] + B(r) \longrightarrow ⑦$$

Diff. w.r.t r.

$$K^2_{,1} = \frac{1}{r^2}\left[c_1 \sin\theta - c_2 \cos\theta\right] + B'(r)$$

Diff. Eq. ⑥ w.r.t θ.

$$K^1_{,2} = -c_1\sin\theta + c_2\cos\theta$$

So, equation ② becomes.

$$-c_1\sin\theta + c_2\cos\theta + r^2\left[\frac{1}{r^2}\left(c_1\sin\theta - c_2\cos\theta\right) + B'(r)\right] = 0$$

$$\cancel{-c_1\sin\theta} + \cancel{c_2\cos\theta} + \cancel{c_1\sin\theta} - \cancel{c_2\cos\theta} + r^2 B'(r) = 0$$

$$\Rightarrow r \neq 0 \ , \ B'(r) = 0$$

$$\Rightarrow B = c_3$$

So, equation ⑦ $\Rightarrow$

$$\boxed{K^2 = \frac{-1}{r}\left[c_1\sin\theta - c_2\cos\theta\right] + c_3}$$

So

$$K^a = \begin{pmatrix} c_1\cos\theta + c_2\sin\theta \\ \frac{-1}{r}\left[c_1\sin\theta - c_2\cos\theta\right] + c_3 \end{pmatrix}$$

Q:- Work out the Killing equation for the following metrices.

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r\sin^2\theta \end{pmatrix} , \quad g_{ab} = \begin{pmatrix} p & -q \\ -q & p \end{pmatrix}$$

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Q:- Workout the Killing Eq's for

$$g_{ab} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{pmatrix}, \quad x^a = (\theta, \phi)$$

and find

$$K^a = \begin{pmatrix} K^1 \\ K^2 \end{pmatrix}$$

Solution:- We know that the Killing equation;

$$K_{ab} : g_{ab,c}K^c + g_{ac}K^c_{,b} + g_{bc}K^c_{,a} = 0$$

First we find three independent Killing equations

$$K_{11} : \underset{0}{\cancel{g_{11,c}}}K^c + g_{1c}K^c_{,1} + g_{1c}K^c_{,1} = 0$$

$$g_{11}K^1_{,1} + g_{11}K^1_{,1} = 0$$

$$\Rightarrow 2a^2K^1_{,1} = 0$$

$$\Rightarrow K^1_{,1} = 0 \longrightarrow ①$$

Integrating it w.r.t θ.

$$K^1 = A(\phi) \longrightarrow ②$$

Now

$$K_{21} = K_{12} : \underset{0}{\cancel{g_{12,c}}}K^c + g_{1c}K^c_{,2} + g_{2c}K^c_{,1} = 0$$

$$g_{11}K^1_{,2} + g_{22}K^2_{,1} = 0$$

$$a^2 K'_{,2} + a^2\sin^2\theta\, K^2_{,1} = 0$$

$$\Rightarrow K'_{,2} + \sin^2\theta\, K^2_{,1} = 0 \longrightarrow ③$$

Now

$$K_{22} : g_{22,c}K^c + g_{2c}K^c_{,2} + g_{2c}K^c_{,2} = 0$$

$$g_{22,1}K^1 + g_{22}K^2_{,2} + g_{22}K^2_{,2} = 0$$

$$2a^2\sin\theta\cos\theta\, K' + 2a^2\sin^2\theta\, K^2_{,2} = 0$$

Dividing by $2a^2\sin\theta\cos\theta$

~~$\cos\theta\, K' + \sin\theta\, K^2_{,2} = 0$~~

$$\Rightarrow K' + \tan\theta\, K^2_{,2} = 0 \longrightarrow ④$$

$$\tan^{-1}\theta\, K' + K^2_{,2} = 0 \longrightarrow ⑤$$

$$\Rightarrow K^2_{,2} = -\frac{K'}{\tan\theta} \longrightarrow ⑥$$

Integrating Eq. ⑤ w.r.t ϕ we have

$$K^2 = -\frac{1}{\tan\theta}\int A(\phi)d\phi + B(\theta) \longrightarrow \overset{Ⓐ}{⑧}$$

Diff. equation ③ w.r.t ϕ.

$$K'_{,22} + \sin^2\theta\, K^2_{,12} = 0$$

$$A_{,\phi\phi} + \sin^2\theta\, K^2_{,12} = 0 \longrightarrow ⑦$$

Diff. Eq. 5 w.r.t θ.

$$-\tan^{-2}\theta \cdot \sec^2\theta\, A(\phi) + K^2_{,21} = 0$$

$$-\frac{1}{\sin^2\theta}A(\phi) + K^2_{,21} = 0$$

$$\Rightarrow \quad -A(\phi) + \sin^2\theta \, K^2_{,21} = 0 \longrightarrow ⑧$$

Subtracting Eq ⑧ from Eq ⑦.

$$A_{,\phi\phi} + A(\phi) = 0 \qquad m^2+1=0$$

$$\Rightarrow \quad A(\phi) = c_1\cos\phi + c_2\sin\phi$$

$$\Rightarrow \quad \boxed{K^1 = c_1\cos\phi + c_2\sin\phi} \longrightarrow ⑨$$

Putting Eq ⑨ in Eq ⑥

$$K^2_{,2} = \frac{-1}{\tan\theta}\left[c_1\cos\phi + c_2\sin\phi\right]$$

Integrating w.r.t ϕ we get,

$$K^2 = \frac{-1}{\tan\theta}\left[c_1\sin\phi - c_2\cos\phi\right] + B(\theta) \longrightarrow ⑩$$

Diff. Eq ⑩ w.r.t θ.

$$K^2_{,1} = \tan^{-2}\theta \sec^2\theta\left[c_1\sin\phi - c_2\cos\phi\right] + B'(\theta)$$

$$K^2_{,1} = \frac{1}{\sin^2\theta}\left[c_1\sin\phi - c_2\cos\phi\right] + B'(\theta) \longrightarrow ⑪$$

Diff. Eq ⑨ w.r.t ϕ, we have

$$K^1_{,2} = -c_1\sin\phi + c_2\cos\phi \longrightarrow ⑫$$

Putting Eq ⑫ & ⑪ in ③ we have

$$-c_1\sin\phi + c_2\cos\phi + \sin^2\theta\left[\frac{1}{\sin^2\theta}\left(c_1\sin\phi - c_2\cos\phi\right) + B'(\theta)\right] = 0$$

$$\Rightarrow -\cancel{c_1\sin\phi} + \cancel{c_2\cos\phi} + \cancel{c_1\sin\phi} - \cancel{c_2\cos\phi} + \sin^2\theta\, B'(\theta) =$$

$$\Rightarrow \sin^2\theta \, B'(\theta) = 0$$

$$\Rightarrow B'(\theta) = 0$$

Integrating w.r.t θ.

$B\theta = C_3 \longrightarrow$ ⑬

Putt Eq. ⑬ in Eq. ⑩

$$K^2 = -\cot\theta[c_1 \sin\phi - c_2 \cos\phi] + c_3$$

So,

$$K^a = \begin{pmatrix} K^1 \\ K^2 \end{pmatrix} = \begin{pmatrix} c_1 \cos\phi + c_2 \sin\phi \\ -\cot\theta[c_1 \sin\phi - c_2 \cos\phi] + c_3 \end{pmatrix}$$

DEPARTMENT OF MATHEMATICS UNIVERSITY OF SARGODHA

BS-VII(A1 & A2), MID Term Examination, Nov 17, 2014

Course Title: Riemannian Geometry

Time:60 min

ROLL NO: 177

Total Marks: 20

Q 1. Define the surface give an example. **(02)**

Q 2. For the metric tensor given by

$$g_{ab} = \begin{pmatrix} \frac{1}{x} & -y \\ -y & x \end{pmatrix},$$

where "x" and "y" are the Cartesian coordinates,

a). Work out the Christoffel symbols with lower subscript same. **(05)**

b). Transform its components (with different lower subscript) into polar coordinates. **(05)**

Q 3. Derive the extreme values of principal curvature and also find the conditions for extreme values of λ. **(08)**

DEPARTMENT OF MATHEMATICS UNIVERSITY OF SARGODHA

M.Sc-III & BS-VII (R+SS) Final Term Examination

Course Title: Riemannian Geometry Total Time: 2.0 hrs

ROLL NO. 177 Total Marks: 60

Q 1. Define the following terms: $(2 \times 5 = 10)$

i). Affine connection ii). Geodesics.

iii). Ricci Tensor iv). Isometry

v). Write down the expression of Bianchi First Identity

Q 2. Derive the expression of Einstein's tensor using Bianchi second identity. Also, convert it in covariant and contravariant forms. (10)

Q 3. After obtaining the expression for the function of several variables $f(x^a)$ by using generalized Taylor's theorem, explain Parallel and Lei Transports. (10)

Q 4. Show that Christoffel symbol is not a tensor. Find the condition under which it becomes a tensor. (10)

Q 5. Find the Ricci scalar R for the metric tensor, given by:

$$g_{ab} = \begin{pmatrix} v^2 & 0 \\ 0 & u^2 \end{pmatrix} \quad x^a = (u, v)$$ (10)

✓**Q 6.** Obtain and solve geodesic equations for a sphere of radius 5. (10)

Best of luck