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⇒ Difference Equation:-

A difference equation is a relation between the differences of an unknown function at one or more general values of the arguments. For Example:

$$y_{n+2} + 2y_{n+1} + y_n = n^2$$
$$\Delta^2 y_n + \Delta 2 y_n + y_n = n^2$$
$$(E^2 + 2E + 1) y_n = n^2$$

★ **Order of difference Equation:-** The order of difference equation is the difference between the largest and smallest arguments (suffix) in equation. For example:-

$$y_{k+2} + 2y_{k+1} + y_k = k^2$$

order is $k+2-k = 2$

★ **Formation of Difference Equation:-** The following examples illustrates the way in which difference eqution is formed.

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Example:- From $y_n = A2^n + B(-3)^n$. Derive difference equation not containing A and B.

Solution:-

$$y_n = A2^n + B(-3)^n \quad \text{—— (i)}$$

$$y_{n+1} = A(2)^{n+1} + B(-3)^{n+1} \quad \text{—— (ii)}$$

$$y_{n+2} = A(2)^{n+2} + B(-3)^{n+2} \quad \text{——— (iii)}$$

From (i), (ii) & (iii)

$$y_n = A2^n + B(-3^n)$$
$$y_{n+1} = 2A2^n + (-3)B(-3)^n$$
$$y_{n+2} = 4A2^n + 9B(-3)^n$$

Eliminate A & B from above equati

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 2 & -3 \\ y_{n+2} & 4 & 9 \end{vmatrix} = 0$$

$$\Rightarrow y_n[18+12] - 1[9y_{n+1} + 3y_{n+2}] + 1[4y_{n+1} - 2y_{n+2}] = 0$$

$$\Rightarrow 30y_n - 9y_{n+1} - 3y_{n+2} + 4y_{n+1} - 2y_{n+2} = 0$$

$$\Rightarrow -5y_{n+2} - 5y_{n+1} + 30y_n = 0$$

$$\Rightarrow -5[y_{n+2} + y_{n+1} - 6y_n] = 0$$

$$\Rightarrow y_{n+2} + y_{n+1} - 6y_n = 0$$

which is required difference equation.

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Example:- Eliminate Constant from the equation $y_n = A_1 + (-1)^n A_2 + A_3 n$ and derive corresponding difference equation.

Solution:- $y_n = A_1 + (-1)^n A_2 + A_3 n$ ——— (i)

$$y_{n+1} = A_1 + (-1)^{n+1} A_2 + A_3 n+1 \quad \text{——— (ii)}$$

$$y_{n+2} = A_1 + (-1)^{n+2} A_2 + A_3 n+2 \quad \text{——— (iii)}$$

$$y_{n+3} = A_1 + (-1)^{n+3} A_2 + A_3\, n+3 \quad \text{——— (iv)}$$

From (i) to (iv)

$$y_n = A_1 + (-1)^n A_2 + A_3\, n \quad \text{——— (1)}$$

$$y_{n+1} = A_1 + (-1)(-1)^n A_2 + A_3 n + A_3 \quad \text{——— (2)}$$

$$y_{n+2} = A_1 + (-1)^n A_2 + A_3 n + 2A_3 \quad \text{——— (3)}$$

$$y_{n+3} = A_1 + (-1)^3(-1)^n A_2 + A_3 n + 3A_3 \quad \text{——— (4)}$$

Subtract (1) from (3)

$$\Rightarrow y_{n+2} - y_n = 2A_3 \quad \text{——— (5)}$$

Subtract (2) from (4)

$$y_{n+3} - y_{n+1} = 2A_3 \quad \text{——— (6)}$$

From (5) & (6)

$$y_{n+3} - y_{n+1} = y_{n+2} - y_n$$

$$\Rightarrow y_{n+3} - y_{n+2} - y_{n+1} + y_n = 0$$

is required difference equation.

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Example:- Also for $y_n = (A+Bn)3^n$

The difference equation is

$$y_{n+2} - 6y_{n+1} + 9y_n = 0 \quad \text{As.}$$

★ **Linear Difference Equations:-** Linear difference equation is that in which y_n, y_{n+1}, y_{n+2} etc occure to first degree only and are not multiplied.

A linear difference equation with constant co-efficient is of form

$$a_0 y_{n+r} + a_1 y_{n+r-1} + a_2 y_{n+r-2} + \cdots + a_n y_n = \phi(n)$$

The equation can be written symbolically as

$$f(E)\, y_n = \phi(n)$$

If $\phi(n) = 0$ Homogeneous $\longrightarrow$ (i)
If $\phi(n) \neq 0$ Non-Homogeneous $\longrightarrow$ (ii)

1st we solve associated Homogeneous

$$f(E)\, y_n = 0$$

The solution of that equation is called complementry function (C.F). It contains orbitrary constant equal to order of that equation.

2nd solve associated Non-Homogeneous.

$$f(E)\, y_n = \phi(n)$$

The solution is called perticular solution which does not contains arbitrary constants.

General Solution of $f(E)\, y_n = \phi(n)$ is

$$y_n = C.F + P.S$$

1) Rule of Finding Complementry Function [C.F] of Homogeneous equations

$f(E)y_n = 0$

Solve $f(E) = 0$ which is called Auxiliary equation (characteristic equation)

Let $\alpha_1, \alpha_2, \alpha_3, \ldots$ be roots of this equation

I:- If $\alpha_1, \alpha_2, \alpha_3, \ldots$ are all real and distinct Then C.F is

$$y_n = C_1\alpha_1^n + C_2\alpha_2^n + C_3\alpha_3^n + \ldots$$

II:- If two roots are equal $\alpha_1 = \alpha_2 = \alpha_1$ then C.F is

$$y_n = C_1\alpha_1^n + C_2 n\alpha_1^n + C_3\alpha_3^n + \ldots$$

$$= (C_1 + C_2 n)\alpha_1^n + C_3\alpha_3^n + \ldots$$

similarly if three roots are equal i.e $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ Then

$$y_n = (C_1 + C_2 n + C_3 n^2)\alpha^n + C_4\alpha_4^n + \ldots$$

III:- If two roots are imaginary say $\alpha \pm i\beta$ then C.F is

$$y_n = R^n(C_1\cos n\theta + C_2\sin n\theta) + C_3\alpha_3^n + \ldots$$

where $R = \sqrt{\alpha^2 + \beta^2}$ &

$$\theta = \tan^{-1}(\beta/\alpha)$$

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Question:- Solve $y_{n+3} - 2y_{n+2} - 5y_{n+1} + 6y_n = 0$

Solution:-

$$y_{n+3} - 2y_{n+2} - 5y_{n+1} + 6y_n = 0$$

$$\Rightarrow E^3 y_n - 2E^2 y_n - 5E y_n + 6y_n = 0$$

$$\Rightarrow [E^3 - 2E^2 - 5E + 6] y_n = 0$$

Characteristic equation is

$$E^3 - 2E^2 - 5E + 6 = 0$$

$$\Rightarrow E = 1, -2, 3$$

$$\Rightarrow y_n = A(1)^n + B(-2)^n + C(3)^n$$

Question Solve $y_{n+2} + 6y_{n+1} + 9y_n = 0$

Solution:-

$$y_{n+2} + 6y_{n+1} + 9y_n = 0$$

Given equation can be written as

$$(E^2 + 6E + 9) y_n = 0$$

Characteristic equation is

$$E^2 + 6E + 9 = 0$$

$$\Rightarrow E = -3, -3$$

$$\Rightarrow y_n = A(-3)^n + Bn(-3)^n$$

$$= (A + Bn)(-3)^n$$

Question:- Solve $y_{k+2} - 3y_{k+1} + 2y_k = 0$

Solution:- $y_{k+2} - 3y_{k+1} + 2y_k = 0$

Equation in E-operator form is

$$E^2 y_K - 3E y_K + 2 y_K = 0$$

$$\Rightarrow [E^2 - 3E + 2] y_K = 0$$

Charactrestic equation is

$$E^2 - 3E + 2 = 0$$

$$\Rightarrow E = 1, 2$$

$$\Rightarrow y_K = c_1 (1)^n + c_2 (2)^n$$

Question:- Find C.F of $(E^2+2E+4) y_K = 0$

Solution:- Given $(E^2+2E+4) y_K = 0$

Characteristic equation is

$$E^2 + 2E + 4 = 0$$

$$\Rightarrow E = \frac{-2 \pm \sqrt{4 - 4(1)(4)}}{2}$$

$$= \frac{-2 \pm \sqrt{12}\, i}{2} = -1 \pm \sqrt{3}\, i$$

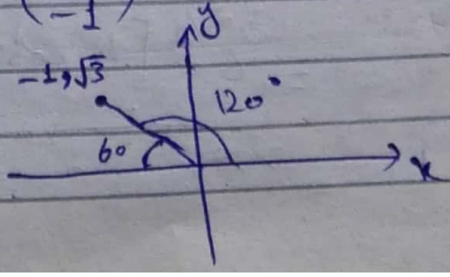
$$\Rightarrow a = -1, \quad b = \pm\sqrt{3}$$

$$R = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4}$$

$$R = 2$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$$

$$\theta = 120^\circ = \frac{2\pi}{3}$$



Solution is

$$y_k = R^k (C_1 \cos k\theta + C_2 \sin k\theta)$$

$$= 2^k \left(C_1 \cos \frac{2\pi}{3} k + C_2 \sin \frac{2\pi}{3} k\right)$$

Question:- Solve $\Delta^2 y_n + 2\Delta y_n + 7 y_n = 0$

Solution:

$$\Delta^2 y_n + 2\Delta y_n + 7 y_n = 0$$

As $\Delta = E - 1$

So given equation becomes

$$(E-1)^2 y_n + 2(E-1) y_n + 7 y_n = 0$$

$$\Rightarrow (E^2 - 2E + 1) y_n + (2E - 2) y_n + 7 y_n = 0$$

$$\Rightarrow (E^2 - \cancel{2E} + 1 + \cancel{2E} - 2 + 7) y_n = 0$$

$$\Rightarrow E^2 + 6 = 0 \Rightarrow E^2 = -6$$

$$E = \pm\sqrt{6}\, i \Rightarrow E = 0 \pm \sqrt{6}\, i$$

Now $R = \sqrt{(0)^2 + (\sqrt{6})^2} = \sqrt{6}$

$$\theta = \tan^{-1}\left(\frac{\sqrt{6}}{0}\right) = \tan^{-1}(\infty) = \pi/2$$

So $y_n = R^n [A \cos n\theta + B \sin n\theta]$

$$= (\sqrt{6})^n \left[A \cos n\frac{\pi}{2} + B \sin n\frac{\pi}{2}\right]$$

$$= 6^{n/2} \left[A \cos\left(\frac{n\pi}{2}\right) + B \sin\left(\frac{n\pi}{2}\right)\right]$$

Assignment

Solve following difference equations.

(i) $u_{n+2} - 2u_{n+1} + u_n = 0$

(ii) $y_{n+3} - 3y_{n+2} + 4y_n = 0$

(iii) $[E^2 + 2E + 2] f_n = 0$

(iv) $y_{k+4} - 2y_{k+3} + 2y_{k+2} - 2y_{k+1} + y_k = 0$

(v) $y_{n+2} - y_{n+1} + y_n = 0$; $y_0 = 1$ & $y_1 = 1 + \frac{\sqrt{3}}{2}$

(vi) $y_{k+3} + 6y_{k+2} + 11y_{k+1} + 6y_k = 0$

(vii) $y_{t+2} + \frac{1}{4} y_t = 0$; $y_0 = 1$, $y_1 = 2$

(viii) $\Delta^3 y_n - 5\Delta y_n + 4y_n = 0$

(ix) $y_{m+2} + 16 y_{m-1} = 0$

(x) Let u_k and v_k be solution of
$\rightarrow \star$ $y_{k+2} + a_1 y_{k+1} + a_2 y_k = 0$ then show that
$c_1 u_k + c_2 v_k$ is also solution.

Sol u_k & v_k be solution then

$u_{k+2} + a_1 u_{k+1} + a_2 u_k = 0$ ——— (i)

$v_{k+2} + a_1 v_{k+1} + a_2 v_k = 0$ ——— (ii)

We show that $c_1 u_k + c_2 v_k$ is also solution
$\times$ (i) by c_1 & (ii) by c_2 then add And then comparing with $\star$ it follows that $c_1 u_k + c_2 v_k$ is also solution.

Solutions

(i) $U_{n+2} - 2U_{n+1} + U_n = 0$

Given equation can be written as

$E^2 U_n - 2E U_n + U_n = 0$

$\Rightarrow (E^2 - 2E + 1) U_n = 0$

Characteristic equation is

$E^2 - 2E + 1 = 0 \Rightarrow E^2 - E - E + 1 = 0$

$\Rightarrow E(E-1) - 1(E-1) = 0 \Rightarrow (E-1)(E-1) = 0$

$\Rightarrow E = 1, 1$

$\Rightarrow U_n = C_1 (1)^n + C_2 n (1)^n$

$= C_1 + C_2 n$

(ii) Given $y_{n+3} - 3y_{n+2} + 4y_n = 0$

$\Rightarrow E^3 y_n - 3E^2 y_n + 4y_n = 0$

$\Rightarrow (E^3 - 3E^2 + 4) y_n = 0$

Ch. equation is

$E^3 - 3E^2 + 4 = 0$

$E = -1$ is root. So by synthetic division

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 0 & 4 \\ & & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$\Rightarrow (E+1)(E^2 - 4E + 4) = 0 \Rightarrow (E+1)(E-2)^2 = 0$

$\Rightarrow (E+1)(E-2)(E-2) = 0 \Rightarrow E = -1, 2, 2$

So complementary solution is

$y_n = C_1 (-1)^n + C_2 (2)^n + C_3 n (2)^n$

(iii) $[E^2 + 2E + 2] f_n = 0$

Ch. equation is

$E^2 + 2E + 2 = 0$

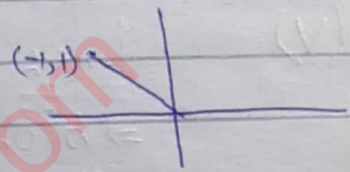
$$\Rightarrow E = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(2)}}{2(1)} \Rightarrow E = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$\Rightarrow E = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} \Rightarrow E = -1 \pm i$$

So complementry solution is

$$y_h = R^n [A \cos n\theta + B \sin n\theta]$$

where $R = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$



& $\theta = \tan^{-1}\left(\frac{1}{-1}\right) \Rightarrow \theta = \frac{3\pi}{4}$

$$\Rightarrow y_h = 2^{n/2}\left[A \cos \frac{3\pi n}{4} + B \sin \frac{3\pi n}{4}\right]$$

(iv) $y_{k+4} - 2y_{k+3} + 2y_{k+2} - 2y_{k+1} + y_k = 0$

$$\Rightarrow E^4 y_k - 2E^3 y_k + 2E^2 y_k - 2E y_k + y_k = 0$$

$$\Rightarrow (E^4 - 2E^3 + 2E^2 - 2E + 1) y_k = 0$$

ch. equation is

$$E^4 - 2E^3 + 2E^2 - 2E + 1 = 0$$

$E = 1$ is root. So by synthetic division

1	1	-2	2	-2	1
		1	-1	1	-1
	1	-1	1	-1	$\boxed{0}$

$$\Rightarrow (E-1)(E^3 - E^2 + E - 1) = 0$$

$E = 1$ is root of eqn $E^3 - E^2 + E - 1 = 0$ so

1	1	-1	1	-1
	1	0	1	$\boxed{0}$

$$\Rightarrow (E-1)(E-1)(E^2+1) = 0$$

$$\Rightarrow E = 1, 1, \pm i \text{ are roots.}$$

So $y_k = (C_1 + C_2 k) + R^k [C_3 \cos k\theta + C_4 \sin k\theta]$

where $R = \sqrt{(0)^2 + (1)^2} \Rightarrow R = 1$

$\theta = \tan^{-1}(\infty) = \pi/2$

$\Rightarrow y_k = (C_1 + C_2 k) + [C_3 \cos \frac{k\pi}{2} + C_4 \sin \frac{k\pi}{2}]$

(V) $y_{n+2} - y_{n+1} + y_n = 0$; $y_0 = 1$ & $y_1 = \cancel{1+}\frac{\sqrt{3}}{2}$

$\Rightarrow E^2 y_n - E y_n + 1 = 0 \Rightarrow (E^2 - E + 1) = 0$

Characteristic equation is

$E^2 - E + 1 = 0$

$\Rightarrow E = \frac{1 \pm \sqrt{1-4}}{2} \Rightarrow E = \frac{1 \pm \sqrt{-3}}{2}$

$\Rightarrow E = \frac{1 \pm \sqrt{3}i}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$\Rightarrow y_n = R^n [A_1 \cos n\theta + A_2 \sin n\theta]$

$R = \sqrt{(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1}$

$\Rightarrow R = 1$

& $\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right) = \tan^{-1}(\sqrt{3}) = \pi/3$

$\Rightarrow y_n = [A_1 \cos \frac{n\pi}{3} + A_2 \sin \frac{n\pi}{3}]$

$y_0 = [A_1 \cos(0) + A_2 \sin(0)] \Rightarrow A_1 = 1$

& $y_1 = [A_1 \cos \frac{\pi}{3} + A_2 \sin \frac{\pi}{3}]$

$\Rightarrow \frac{1}{2} + \frac{\sqrt{3}}{2} = [A_1 \frac{1}{2} + A_2 \sqrt{3}/2]$

$\Rightarrow \frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{1}{2} + \frac{\sqrt{3}}{2} A_2 \Rightarrow \frac{\sqrt{3}}{2} A_2 = \frac{\sqrt{3}}{2} \Rightarrow A_2 = 1$

$\Rightarrow y_n = \cos \frac{n\pi}{3} + \sin \frac{n\pi}{3}$

(vi) $y_{k+3} + 6y_{k+2} + 11y_{k+1} + 6y_k = 0$

$\Rightarrow E^3 y_k + 6E^2 y_k + 11E y_k + 6y_k = 0$

$\Rightarrow (E^3 + 6E^2 + 11E + 6) y_k = 0$

Ch. equ is $E^3 + 6E^2 + 11E + 6 = 0$

$E = -1$ is root of equ so by synthetic division

$$\begin{array}{r|rrrr} -1 & 1 & 6 & 11 & 6 \\ & & -1 & -5 & -6 \\ \hline & 1 & 5 & 6 & \boxed{0} \end{array}$$

$\Rightarrow (E+1)(E^2 + 5E + 6) = 0 \Rightarrow (E+1)(E^2 + 2E + 3E + 6) = 0$

$\Rightarrow (E+1)[E(E+2) + 3(E+2)] = 0$

$\Rightarrow (E+1)(E+2)(E+3) = 0$

$\Rightarrow E = -1, -2, -3$

So complementry solution is

$y_k = C_1(-1)^k + C_2(-2)^k + C_3(-3)^k$

(vii) $y_{t+2} + \frac{1}{4} y_t = 0$

$\Rightarrow E^2 y_t + \frac{1}{4} y_t = 0 \Rightarrow (E^2 + \frac{1}{4}) y_t = 0$

characteristic equation is

$E^2 + \frac{1}{4} = 0 \Rightarrow E^2 = -1/4$

$\Rightarrow E = \sqrt{\frac{-1}{4}} \Rightarrow E = \pm \frac{1}{2} i$

CF is

$y_t = R^t [A_1 \cos t\theta + A_2 \sin t\theta]$

where $R = \sqrt{\left(\frac{1}{2}\right)^2} \Rightarrow R = \frac{1}{2}$

& $\theta = \tan^{-1}\left(\frac{1/2}{0}\right) \Rightarrow \theta = \frac{\pi}{2}$

$$y_t = \left(\frac{1}{2}\right)^t \left[A_1 \cos\frac{t\pi}{2} + A_2 \sin\frac{t\pi}{2}\right] \longrightarrow \star$$

Put $t = 0$

$$\Rightarrow y_0 = \left(\frac{1}{2}\right)^0 \left[A_1 \cos(0) + A_2 \sin(0)\right]$$

$$\Rightarrow 1 = A_1(1) + A_2(0) \Rightarrow A_1 = 1$$

Put $t = 1$

$$\Rightarrow y_1 = \left(\frac{1}{2}\right)^1 \left[A_1 \cos\frac{\pi}{2} + A_2 \sin\frac{\pi}{2}\right]$$

$$\Rightarrow 2 = \frac{1}{2}(0 + A_2) \Rightarrow \frac{A_2}{2} = 2$$

$$\Rightarrow A_2 = 4$$

$$\Rightarrow y_t = \left(\frac{1}{2}\right)^t \left[\cos\frac{t\pi}{2} + 4 \sin\frac{t\pi}{2}\right]$$

(viii)

$$\Delta^3 y_n - 5\Delta y_n + 4y_n = 0$$

As $\Delta = E - 1$

$$\Rightarrow (E-1)^3 y_n - 5(E-1)y_n + 4y_n = 0$$

$$\Rightarrow (E-1)(E-1)^2 y_n - 5(E-1)y_n + 4y_n = 0$$

$$\Rightarrow (E-1)(E^2 + 1 - 2E)y_n - (5E - 5)y_n + 4y_n = 0$$

$$\Rightarrow (E^3 + E - 2E^2 - E^2 - 1 + 2E)y_n - 5Ey_n + 5y_n + 4y_n = 0$$

$$\Rightarrow (E^3 - 3E^2 + 3E - 1)y_n - 5Ey_n + 9y_n = 0$$

$$\Rightarrow (E^3 - 3E^2 + 3E - 1 - 5E + 9)y_n = 0$$

$$\Rightarrow (E^3 - 3E^2 - 2E + 8)y_n = 0$$

Characteristic equation is

$E^3 - 3E^2 - 2E + 8 = 0$

2 is root of equ. So by synthetic division

$$\begin{array}{c|cccc} 2 & 1 & -3 & -2 & 8 \\ & & 2 & -2 & -8 \\ \hline & 1 & -1 & -4 & \boxed{0} \end{array}$$

$\Rightarrow (E-2)(E^2 - E - 4) = 0 \quad \Rightarrow E = 2$ &

$$\Rightarrow E = \frac{1 \pm \sqrt{1 - 4(1)(-4)}}{2} \Rightarrow E = \frac{1 \pm \sqrt{17}}{2}$$

$$\Rightarrow E = 2, \frac{1 \pm \sqrt{17}}{2}$$

So complementry solution is

$$y_n = C_1(2)^n + C_2\left(\frac{1+\sqrt{17}}{2}\right)^n + C_3\left(\frac{1-\sqrt{17}}{2}\right)^n$$

(ix) $y_{m+2} + 16\, y_{m-1} = 0$

(x) $y_{k+2} + a_1 y_{k+1} + a_2 y_k = 0 \longrightarrow (*)$

As u_k and v_k be solution then

$u_{k+2} + a_1 u_{k+1} + a_2 u_k = 0$ ——— ①

$v_{k+2} + a_1 v_{k+1} + a_2 v_k = 0$ ——— ②

X equ ① by c_1 and equ ② by c_2

$\Rightarrow c_1 u_{k+2} + c_1 a_1 u_{k+1} + c_1 a_2 u_k = 0$ ——— ③

$c_2 v_{k+2} + c_2 a_1 v_{k+1} + c_2 a_2 v_k = 0$ ——— ④

Add equ ③ and ④

$(c_1 u_{k+2} + c_2 v_{k+2}) + (c_1 a_1 u_{k+1} + c_2 a_1 v_{k+1})$
$+ (c_1 a_2 u_k + c_2 a_2 v_k) = 0$

Comparing with (*) is followed that $c_1 u_k + c_2 v_k$ is also solution

2) Solution of Non-Homogenuous Linear Equation $f(E)y_n = \phi(n)$

Type I :- When $\phi(n)$ = constant then 1st calculate $y_n^{(c)}$. The trail substitution for perticular solution is

$$y_n = C$$

(i) If sum of co-efficient of characteristic equation is zero then $y_n = C$ fails. In this case $y_n = C$ (some power of n) i.e $y_n = Cn^k$

(ii) If sum of co-efficients of characteristic equation is zero then "1" be root of equation

(iii) Now $y_n^{(p)} = Cn^k$, where

$k=0$ if 1 is not root of $f(E)=0$ i.e $y_n^{(p)} = C$

$k=1$ " 1 is single " " " $=0$ i.e $y_n = Cn$

$k=2$ " " " double " " " $=0$ i.e $y_n = Cn^2$

$k=3$ " " " triple " " " $=0$ i.e $y_n = Cn^3$

& so on.

★★

Example:- Solve $y_{n+2} + 7y_{n+1} + 12y_n = 5$

Solution

$$y_{n+2} + 7y_{n+1} + 12y_n = 5$$

$$\Rightarrow E^2y_2 + 7Ey_n + 12y_n = 5$$

$$\Rightarrow (E^2 + 7E + 12)y_n = 5$$

Characteristic equation is

$$E^2 + 7E + 12 = 0$$

$$\Rightarrow E = -3, -4$$

$$\Rightarrow y_n^{(c)} = A(-3)^n + B(-4)^n$$

for Perticular solution put $y_n^{(P)} = \alpha \cdot n^K$

where $K=0$ as 1 is not root of $f(E)=0$

$$\Rightarrow y_n^{(P)} = \alpha \quad \text{then } y_{n+1} = \alpha$$

$$\& \; y_{n+2}^{(P)} = \alpha$$

Then from given equation

$$\alpha + 7\alpha + 12\alpha = 5$$

$$\Rightarrow 20\alpha = 5 \Rightarrow \alpha = 1/4$$

$$\Rightarrow y_n^{(P)} = 1/4$$

So general solution is

$$y_n = y_n^{(c)} + y_n^{(P)}$$

$$y_n = A(-3)^n + B(-4)^n + 1/4$$

**

Question:- Solve $y_{K+3} + 3y_{K+2} - 2y_K = 5$

Solution:- $E^3 y_K + 3E^2 y_K - 2 y_5 = 5$

$$\Rightarrow (E^3 + 3E^2 - 2E) y_K = 5$$

Characteristic equation is

$$E^3 + 3E^2 - 2E = 0$$

$E = -1 \Rightarrow (-1)^3 + 3(-1)^2 - 2(-1) = 0$

$$-1 + 3 - 2 = 0$$

$$0 = 0$$

So $E=-1$ is root

then

-1	1	3	0	-2
		-1	-2	2
	1	2	-2	0

$\Rightarrow E^2+2E-2=0$

$\Rightarrow E=-1\pm\sqrt{3}$

$\Rightarrow$ Roots are $-1, -1\pm\sqrt{3}$

Complement function is

$$y_k^{(c)} = A(-1)^k + B(-1+\sqrt{3})^k + C(-1-\sqrt{3})^k$$

& for Perticular Solution Put $y_k^{(P)} = \alpha.n^k$

where $k=0$ as 1 is not root of eq

$\Rightarrow y_k^{(P)}=\alpha, \; y_{k+1}^{(P)}=\alpha,$

$y_{k+2}^{(P)}=\alpha, \; y_{k+3}^{(P)}=\alpha$

Then from given equation

$\alpha+3\alpha-2\alpha=5 \Rightarrow 2\alpha=5$

$\Rightarrow \alpha=5/2$

$\Rightarrow y_k^{(P)}=5/2$

So G.S is $y_k = y_k^{(c)} + y_k^{(P)}$

$$y_k = A(-1)^k + B(-1+\sqrt{3})^k + C(-1-\sqrt{3})^k + 5/2$$

Question:- $y_{t+2} + \frac{1}{4}y_t = 2$; $y_0 = y_1 = 1$

Solution:- $y_{t+2} + \frac{1}{4}y_t = 2$

$\Rightarrow E^2 y_t + \frac{1}{4}y_t = 2 \Rightarrow (E^2 + 1/4)y_t = 2$

Ch. equation is $E^2 + 1/4 = 0$

$\Rightarrow E = \pm \frac{1}{2} i$

$C.F = y_t^{(c)} = R^t(A\cos t\theta + B\sin t\theta)$

where $R = \sqrt{a^2+b^2} = \sqrt{0 + (\frac{1}{2})^2}$

$= \frac{1}{2}$

$\theta = Tan^{-1}(\frac{b}{a}) = Tan^{-1}(\frac{1/2}{0}) = Tan^{-1}(\infty) = \frac{\pi}{2}$

$\Rightarrow y_t = (\frac{1}{2})^t [A\cos\frac{\pi}{2}t + B\sin\frac{\pi}{2}t]$

For Perticular function put

$y_t^{(P)} = C$

$\Rightarrow y_{t+1}^{(P)} = C$, $y_{t+2}^{(P)} = C$

Then $C + \frac{1}{4}C = 2 \Rightarrow \frac{5}{4}C = 2$

$\Rightarrow C = 8/5$

$y_t^{(P)} = 8/5$

$G.S = C.F + P.S$

$$\Rightarrow y_t = \left(\frac{1}{2}\right)^t \left[A\cos\frac{\pi}{2}t + B\sin\frac{\pi}{2}t\right] + \frac{8}{5}$$

Also Find A and B by Given $y_0 = y_1 = 1$

When $t=0$:-

$$y_0 = \left(\frac{1}{2}\right)^0 \left[A\cos(0) + B\sin(0)\right] + \frac{8}{5}.$$

$$1 = A + \frac{8}{5} \qquad \because y_0 = 1$$

$$\Rightarrow A = -3/5$$

When $t=1$:-

$$y_1 = \left(\frac{1}{2}\right)^1 \left[A\cos\frac{\pi}{2} + B\sin\frac{\pi}{2}\right] + \frac{8}{5}$$

$$\Rightarrow 1 = \frac{1}{2}\left[0 + B\right] + \frac{8}{5} \Rightarrow 1 = \frac{B}{2} + \frac{8}{5}$$

$$\Rightarrow B = 2\left(1 - \frac{8}{5}\right) = -6/5$$

Then $y_t = \left(\frac{1}{2}\right)^t \left[-\frac{3}{5}\cos\frac{\pi}{2}t - \frac{6}{5}\sin\frac{\pi}{2}t\right] + 8/5$

*** ——— ***

Assignment

Solve

Q1:- $y_{n+2} - 4y_{n+1} + y_n = 1$

Q2:- $y_{n+2} - 3y_{n+1} - 4y_n = 6$

Q3:- $U_{n+2} - 4U_{n+1} + U_n = 3$

Q4:- $y_{n+3} + 6y_{n+2} + 11y_{n+1} = 5$

Q5:- $\Delta^2 y_k + 3\Delta y_k = 9$; $y_0 = 1$, $y_1 = 0$

Q6:- $(E^3 - 4E^2 + 2E + 1)y_n = 14$

Type II :- If $\phi(n) = \alpha \cdot a^n$, where α constant

Then put $y_n^{(P)} = c \cdot a^n \cdot n^k$, where

$K = 0$ if a is not root of Ch. equ.

$K = 1$ if " " single " " " "

$K = 2$ if " " double " " " "

& so on

★ ——— ★★★ ——— ★

Example:- Solve $y_{k+2} - 4y_{k+1} + 4y_k = 3 \cdot 2^{k+1}$

Solution:- $y_{k+2} - 4y_{k+1} + 4y_k = 3 \cdot 2^{k+1}$

we can write given equation as

$$E^2 y_k - 4E y_k + 4y_k = 6 \cdot 2^k$$

$$\Rightarrow \quad (E^2 - 4E + 4) y_k = 6 \cdot 2^k$$

Ch. equation is

$$E^2 - 4E + 4 = 0$$

$$\Rightarrow E = 2, 2$$

$$\Rightarrow y_k^{(c)} = A(2)^k + BK(2)^k$$

$$= (A + BK)\, 2^k$$

Now as 2 is double root of ch. eq

so put

$$y_k^{(P)} = C \cdot 2^k \cdot k^P$$

Put $P = 2$

$$y_k^{(P)} = C \cdot 2^k \cdot k^2$$

Then $y_{K+1}^{(P)} = C \cdot 2^{K+1} (K+1)^2$

$y_{K+2}^{(P)} = C \cdot 2^{K+2} (K+2)^2$

Then from given equation

$C \cdot 2^{K+2} (K+2)^2 - 4C \cdot 2^{K+1} (K+1)^2 + 4C\, 2^K K^2 = 6 \cdot 2^K$

$\Rightarrow [C \cdot 4 (K^2+4K+4) - 4C \cdot 2 (K^2+2K+1) + 4CK^2] \cancel{2^K} = 6 \cdot \cancel{2^K}$

$C[\cancel{4K^2} + \cancel{16K} + 16 - 8K^2 - \cancel{16K} - 8 + 4K^2] = 6$

$\Rightarrow 8C = 6 \quad \Rightarrow \quad C = 3/4$

$\Rightarrow y_K^{(P)} = \frac{3}{4} 2^K \cdot K^2$

Hence $y_K = A(2)^K + BK(2)^K + \frac{3}{4} K^2 \cdot 2^K$

**

Question:- Solve $y_{K+2} - 4y_{K+1} + 4y_K = 2^{K+1}$

Solution:- $y_{K+2} - 4y_{K+1} + 4y_K = 2 \cdot 2^K$ —— ①

$\Rightarrow E^2 y_K - 4E y_K + 4y_K = 2 \cdot 2^K$

$(E^2 - 4E + 4) y_K = 2 \cdot 2^K$

For C.F

Ch. equ $E^2 - 4E + 4 = 0$

$\Rightarrow E^2 - 2E - 2E + 4 = 0 \Rightarrow E(E-2) - 2(E-2) = 0$

$\Rightarrow (E-2)(E-2) = 0$

$\Rightarrow E = 2, 2$

& so on solve same step as in previous question.

Question:- Solve $U_{n+2} - 7U_{n+1} + 10U_n = 12\cdot 5^n$

Solution:- $U_{n+2} - 7U_{n+1} + 10U_n = 12\cdot 5^n$ ——— (1)

$\Rightarrow E^2 U_n - 7E U_n + 10 U_n = 12\cdot 5^n$

$\Rightarrow (E^2 - 7E + 10) U_n = 12\cdot 5^n$

For C.F

Ch. equation is $E^2 - 7E + 10 = 0$

$\Rightarrow E = 2, 5$

$\Rightarrow U_n = C_1 2^n + C_2 5^n$

Now for P.S Put $U_n^{(P)} = Cn5^n$

($\because$ 5 is single root of ch. equ)

$\Rightarrow U_{n+1}^{(P)} = C(n+1)\cdot 5^{n+1}$

$U_{n+2}^{(P)} = C(n+2)\cdot 5^{n+2}$

Put in equ (1)

$C(n+2)\cdot 5^{n+2} - 7[C(n+1)\cdot 5^{n+1}] + 10Cn5^n = 12\cdot 5^n$

$\Rightarrow \cancel{5^n}[C(n+2)\cdot 5^2 - 7C(n+1)5 + 10Cn] = 12\cdot \cancel{5^n}$

$\Rightarrow 25C(n+2) - 35C(n+1) + 10Cn = 12$

$\Rightarrow C[\cancel{25n} + 50 - \cancel{35n} - 35 + \cancel{10n}] = 12$

$\rightarrow C[15] = 12 \Rightarrow C = \frac{12}{15}$

$\Rightarrow C = 3/4$

$$\Rightarrow U_n^{(P)} = \frac{4}{5} n 5^n$$

So G.S is ~~$y_n^{(c)}$~~ $U_n^{(c)} + U_n^{(P)} = U_n$

$$\Rightarrow U_n = c_1 2^n + c_2 5^n + \frac{4}{5} n 5^n$$

****

Assignment

Solve

Q 1:- $y_{n+4} - A^4 y_n = 2^n$.

Q2:- $y_{n+2} + 4y_{n+1} + 4y_n = 2^{n+2}$; $y_0 = 1, y_1 = 2$

Q3:- $y_{k+2} - 6y_{k+1} + 8y_k = 2 \cdot 3^k$

Q4:- $y_{k+3} - y_{k+2} + y_{k+1} - y_k = 3^k$

Q5:- $y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = 3^n$

Q6:- $y_{n+2} - 3y_{n+1} + 2y_n = 7 \cdot 2^n$

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Type III:- If $\phi(n)$ = Polynomial of degree m
Then choose

$$y_n^{(P)} = [A_0 + A_1 n + A_2 n^2 + \dots + A_m n^m] \cdot n^K$$

where

$K=0$ if 1 is not root of ch. eqn

$K=1$ " " " single " " " "

$K=2$ " " " double " " " "

$K=3$ " " " triple " " " "

& so on

* * *

Examples- Solve $y_{n+2} - 2y_{n+1} + y_n = n^2$

Solution:-

$$y_{n+2} - 2y_{n+1} + y_n = n^2 \quad \text{——} \ (1)$$

$$\Rightarrow E^2 y_n - 2E y_n + y_n = n^2$$

$$\Rightarrow (E^2 - 2E + 1) y_n = n^2$$

Ch. equation is

$$E^2 - 2E + 1 = 0$$

$$\Rightarrow (E-1)^2 = 0 \quad \Rightarrow E = 1, 1.$$

C.F $\Rightarrow$ $y_n^{(c)} = (C_1 + C_2 n)(1)^n$

$$\Rightarrow y_n^{(c)} = C_1 + C_2 n$$

As 1 is double root so put for P.S

$$y_n^{(P)} = n^2 (B_0 + B_1 n + B_2 n^2)$$

$$y_n = n^2 B_0 + B_1 n^3 + B_2 n^4$$

$$y_{n+1} = (n+1)^2 B_0 + (n+1)^3 B_1 + (n+1)^4 B_2$$

$$y_{n+2} = (n+2)^2 B_0 + (n+2)^3 B_1 + (n+2)^4 B_2$$

Then $y_{n+2} - 2y_{n+1} + y_n = n^2$ becomes

$$B_0(n+2)^2 + B_1(n+2)^3 + B_2(n+2)^4 - 2[B_0(n+1)^2 + B_1(n+1)^3 + B_2(n+1)^4] + B_0 n^2 + B_1 n^3 + B_2 n^4 = n^2$$

$$\Rightarrow B_0(n^2+4+4n) + B_1(n^3+6n^2+12n+8) + B_2(n^4+8n^3 + 24n^2+32n+16) - 2B_0(n^2+2n+1) - 2B_1(n^3+3n^2+3n+1) - 2B_2(n^4+4n^3+6n^2+4n+1) + B_0 n^2 + B_1 n^3 + B_2 n^4 = n^2$$

★ Equating co-efficient of n^4:-

$$B_2 - 2B_2 + B_2 = 0$$

Not gives information

★ Equating Co-efficient of n^3:-

$$B_1 + 8B_2 - 2B_1 - 8B_2 + B_1 = 0$$

$$\Rightarrow 0 = 0$$

★ Equating Co-efficient of n^2:-

$$\cancel{B_0} + \cancel{6B_1} + 24B_2 - \cancel{2B_0} - \cancel{6B_1} - 12B_2 + \cancel{B_0} = 1$$

$$\Rightarrow 12B_2 = 1$$

$$\Rightarrow B_2 = 1/12$$

★ Equating Co-efficient of n:-

$$4B_0 + 12B_1 + 32B_2 - 4B_0 - 6B_1 - 8B_2 = 0$$

$$6B_1 + 24B_2 = 0$$

$$\Rightarrow B_1 = -4B_2 \quad \Rightarrow B_1 = -4\left(\frac{1}{12}\right) = -1/3$$

★ Equating constant terms:-

$$4B_0 + 8B_1 + 16B_2 - 2B_0 - 2B_1 - 2B_2 = 0$$

$$B_0 + 3B_1 + 7B_2 = 0$$

$$\Rightarrow B_0 + 3(-1/3) + 7(1/12) = 0$$

$$B_0 - 1 + 7/12 = 0$$

$$\Rightarrow B_0 = 5/12$$

So $y_n^{(P)} = n^2\left[\frac{5}{12} - \frac{1}{3}n + \frac{1}{12}n^2\right]$

$$= \frac{5}{12}n^2 - \frac{1}{3}n^3 + \frac{1}{12}n^4$$

General solution is

$$y_n = y_n^{(C)} + y_n^{(P)}$$

$$= C_1 + C_2 n + \frac{5}{12}n^2 - \frac{1}{3}n^3 + \frac{1}{12}n^4$$

★★★

Example:- Solve $\Delta^2 y_K + 3\Delta y_K + 3y_K = K^2 + 1$

Solution:- $\Delta^2 y_K + 3\Delta y_K + 3y_K = K^2 + 1$ ——— ①

As $\Delta y_K = y_{K+1} - y_K$

So $\Delta(\Delta y_K) + 3\Delta y_K + 3y_K = K^2 + 1$

$$\Delta[y_{K+1} - y_K] + 3[y_{K+1} - y_K] + 3y_K = K^2 + 1$$

$$\Delta y_{K+1} - \Delta y_K + 3y_{K+1} - \cancel{3y_K} + \cancel{3y_K} = K^2 + 1$$

$$\Rightarrow y_{k+2} - y_{k+1} - y_{k+1} + y_k + 3y_{k+1} = k^2+1$$

$$\Rightarrow y_{k+2} + y_{k+1} + y_k = k^2+1 \quad \text{———} (*)$$

For Complementry function

ch. equation is $y_{k+2} + y_{k+1} + y_k = 0$

$$E^2 y_k + E y_k + y_k = 0$$

$$\Rightarrow E^2 + E + 1 = 0$$

$$\Rightarrow E = \frac{-1 \pm \sqrt{1-4(1)(1)}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\,i}{2}$$

As

$$y_k^{(c)} = R^k\left[A_1 \cos k\theta + A_2 \sin k\theta\right]$$

$$R = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{\frac{4}{4}}$$

$$\Rightarrow R = 1$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = \frac{-\pi}{3}$$

$$y_k^{(c)} = (1)^k\left[A_1 \cos\left(\frac{-\pi}{3}\right)k + A_2 \sin\left(\frac{-\pi}{3}\right)k\right]$$

$$= (1)^k\left[A_1 \cos\left(\frac{k\pi}{3}\right) - A_2 \sin\left(\frac{k\pi}{3}\right)\right]$$

For Perticular Solution

$$y_k^{(P)} = \left[B_0 + B_1 k + B_2 k^2\right]$$

$$y_{k+1}^{(P)} = \left[B_0 + B_1(k+1) + B_2(k+1)^2\right]$$

$$y_{K+2}^{(P)} = B_0 + B_1(K+2) + B_2(K+2)^2$$

Then ⊛ becomes

$$B_0 + B_1(K+2) + B_2(K+2)^2 - B_0 - B_1(K+1) - B_2(K+1)^2 + B_0$$
$$+ B_1 K + B_2 K^2 = K^2 + 1$$

★ Equating coefficient of K^2:-

$$B_2 - \cancel{B_2} + \cancel{B_2} = 1$$
$$\Rightarrow B_2 = 1$$

★ Equating Co-efficient of K:-

$$B_1 + 4B_2 - \cancel{B_1} - 2B_2 + \cancel{B_1} = 0$$
$$B_1 + 2B_2 = 0$$
$$\Rightarrow B_1 + 2(1) = 0 \quad \Rightarrow B_1 = -2$$

★ Equating constant terms:-

$$B_0 + 2B_1 + 4B_2 - \cancel{B_0} - B_1 - B_2 + \cancel{B_0} = 1$$
$$\Rightarrow B_0 + B_1 + 3B_2 = 1$$
$$\Rightarrow B_0 - 2 + 3(1) = 1 \quad \Rightarrow B_0 + 1 = 1$$
$$B_0 = 0$$

So $y_K^{(P)} = B_0 + B_1 K + B_2 K^2$

$$= -2K + K^2$$

General Solution $= y_K = y_K^{(c)} + y_K^{(P)}$

$$y_K = (1)^K \left[A_1 \cos\left(\frac{K\pi}{3}\right) - A_2 \sin\left(\frac{K\pi}{3}\right)\right] - 2K + K^2$$

★ ——— ★ ——— ★ ———

Example:- Solve $y_{k+4} - 2y_{k+3} + 2y_{k+2} - 2y_{k+1} + y_k = k^2$

Solution,:- $y_{k+4} - 2y_{k+3} + 2y_{k+2} - 2y_{k+1} + y_k = k^2$ ——— ①

Given equation can be written as

$$E^4 y_k - 2E^3 y_k + 2E^2 y_k - 2E y_k + y_k = k^2$$

$$\Rightarrow (E^4 - 2E^3 + 2E^2 - 2E + 1) y_k = k^2$$

For C.F Ch. equation is

$$E^4 - 2E^3 + 2E^2 - 2E + 1 = 0$$

$$\Rightarrow E = 1 \Rightarrow (1)^4 - 2(1)^3 + 2(1)^2 - 2(1) + 1 = 0$$

$$\Rightarrow 0 = 0$$

So $E = 1$ is root of ch. eq.

So by Synthetic division

1	1	-2	2	-2	1
		1	-1	1	-1
1	1	-1	1	-1	$\boxed{0}$
		1	0	1	
	1	0	1	$\boxed{0}$	

$$\Rightarrow (E-1)(E-1)(E^2+1) = 0$$

$$\Rightarrow E = 1, 1,$$

$$E = \sqrt{-1} = \pm i$$

So roots ar

$$E = 1, 1, \pm i$$

$$y_k^{(c)} = (A + Bk)(1)^k + R^k [A_3 \cos k\theta + A_4 \sin k\theta]$$

Now $R = \sqrt{0 + (1)^2} = 1$

$$\theta = \tan^{-1}\left(\frac{1}{0}\right) = \frac{\pi}{2}$$

So $y_k = (A + Bk)(1)^k + (1)^k \left[A_3 \cos \frac{\pi}{2} k + A_4 \sin \frac{\pi}{2} k\right]$

For Perticular Solution Put
$$y_K^{(P)} = K^2(B_0 + B_1 K + B_2 K^2)$$ Then equ ① becom

$$(K+4)^2[B_0 + B_1(K+4) + B_2(K+4)^2] - 2(K+3)^2[B_0 + B_1(K+3) + B_2(K+3)^2] + 2(K+2)^2[B_0 + B_1(K+2) + B_2(K+2)^2] - 2(K+1)^2 [B_0 + B_1(K+1) + B_2(K+1)^2] + K^2[B_0 + B_1 K + B_2 K^2] = K^2$$

Comparing Coefficient of like powers of K on both sides

$$B_0 + 12B_1 + 9B_2 - 2B_1 - 18B_1 - 108B_2 + 2B_0 + 12B_1 + 48B_2 - 2B_0 - 6B_1 - 12B_2 + B_0 = 1 \longrightarrow \star$$

$$8B_0 + 96B_1 + 256B_2 - 12B_0 - 54B_1 - 216B_2 + 8B_0 + 24B_1 + 16B_2 - 4B_0 - 6B_1 - 8B_2 = 0 \longrightarrow \star'$$

$$16B_0 + 64B_1 + 256B_2 - 18B_0 - 54B_1 - 162B_2 + 8B_0 + 16B_1 + 32B_2 - 2B_0 - 2B_1 - 2B_2 = 0 \longrightarrow \star''$$

From $\star, \star', \star''$ we have
$$B_1 = 4/315 , \quad B_0 = 131/315$$
$$\& \quad B_2 = -1/63$$

So Perticular Solution becomes
$$y_K^{(P)} = K^2\left[\frac{131}{315} + \frac{4}{315}K - \frac{1}{63}K^2\right]$$

General Solution is
$$y_K = y_K^{(c)} + y_K^{(P)}$$

$$y_K = [A_1 + BK](1)^K + [A_3 \cos\frac{\pi}{2}K + A_4 \sin\frac{\pi}{2}K](1)^K$$
$$+ K^2\left[\frac{131}{315} + \frac{4}{315}K - \frac{1}{63}K^2\right]$$

** ——————————————— **

Assignment

Q1:- $\Delta^2 y_n - 7\Delta y_n - 60 y_n = 2^n n^2 + 1$

Q2:- $y_{n+2} - 7y_{n+1} + 12 y_n = 12n + 8$

Q3:- $\Delta^4 y_n = n$

Q4:- $y_{K+2} - 2y_{K+1} + y_K = K + 1$

Q5:- $y_{n+2} - 5y_{n+1} + 6y_n = 2n + 1 + 2^n$

Q6:- $\Delta^2 y_K + 3\Delta y_K + 3y_K = K^2 + 1$

——————— *** ———————

Type IV :- If $\phi(n) = a^n \cdot$ Polynomial of degree m

Then Put
$$y_n^{(P)} = a^n\left[A_0 + A_1 n + A_2 n^2 + \cdots + A_m n^m\right] \cdot n^K$$
where $K = 0$ if a is not root of Ch. equ
$K = 1$ if " " Single " " " "
$K = 2$ if " " double " " " "
$K = 3$ if " " triple " " " "
& so on.

Example:- Solve
$$y_{n+2} - 2y_{n+1} + y_n = 2^n \cdot n^2$$

Solution
$$y_{n+2} - 2y_{n+1} + y_n = 2^n \cdot n^2 \quad \text{——— (1)}$$
$$\Rightarrow E^2 y_n - 2E y_n + y_n = 2^n \cdot n^2$$
$$(E^2 - 2E + 1)\, y_n = 2^n \cdot n^2$$
Ch. equation is
$$E^2 - 2E + 1 = 0$$
$$\Rightarrow (E-1)^2 = 0 \quad \Rightarrow E = 1, 1$$
So Complementry function is
$$y_n^{(c)} = (c_1 + c_2 n)(1)^n$$
For Perticular Solution Put
$$y_n^{(P)} = 2^n\left[B_0 + B_1 n + B_2 n^2\right] \text{ in (1) we have}$$
$$2^{(n+2)}\left[B_0 + B_1(n+2) + B_2(n+2)^2\right] - 2\left[2^{n+1}\left[B_0 + B_1(n+1)\right.\right.$$

$$+B_2(n+1)^2]] + 2^n[B_0+B_1 n+B_2 n^2] = 2^n \cdot n^2$$

$$\Rightarrow 2^n[4B_0+4nB_1+8B_1+4B_2n^2+16B_2n+16B_2-4B_0-4B_1n$$
$$-4B_1-4B_2n^2+8B_2n-4B_2+B_0+B_1n+B_2n^2] = 2^n \cdot n^2$$

Dividing both sides by 2^n

$$\Rightarrow 4B_0+4nB_1+8B_1+4B_2n^2+16B_2n+16B_2n^2-4B_0-4B_1n$$
$$-4B_1-4B_2n^2+8B_2n-4B_2+B_0+B_1n+B_2n^2 = n^2$$

★ Equating Coefficients of n^2:-

$$4B_2+B_2-4B_2 = 1 \quad \Rightarrow B_2 = 1$$

★ Equating Coefficients of n:-

$$4B_1+16B_2-4B_1-8B_2+B_1 = 0$$
$$\Rightarrow 8B_2+B_1 = 0 \quad \Rightarrow 8(1)+B_1 = 0$$
$$\Rightarrow B_1 = -8$$

★ Equating constant terms:-

$$8B_0+16B_2-4B_0-4B_2+B_0-4B_1 = 0$$
$$\Rightarrow B_0+12B_2-4B_1 = 0$$
$$\Rightarrow B_0+12(1)-4(-8) = 0 \quad \Rightarrow B_0-20 = 0$$
$$\Rightarrow B_0 = 20$$

$$\Rightarrow y_n^{(P)} = 2^n[20-8n+n^2]$$

So $G.S = y_n^{(C)} + y_n^{(P)}$

$$\Rightarrow y_n = (C_1+C_2 n)(1)^n + 2^n(20-8n+n^2)$$

★ ★ ★

Example:- Solve $y_{K+3} - 5y_{K+2} + 8y_{K+1} - 4y_K = 2^K \cdot K$

Solution:- $y_{K+3} - 5y_{K+2} + 8y_{K+1} - 4y_K = 2^K \cdot K$ ——— ①

$\Rightarrow E^3 y_K - 5E^2 y_K + 8E y_K - 4y_K = 2^K \cdot K$

$\Rightarrow (E^3 - 5E^2 + 8E - 4) y_K = 2^K \cdot K$

For Complementry function Ch. equ is

$$E^3 - 5E^2 + 8E - 4 = 0$$

$$(E-1)(E^2 - 4E + 4) = 0$$

$$\Rightarrow (E-1)(E-2)(E-2) = 0$$

$$\Rightarrow E = 1, 2, 2$$

So $y_K^{(c)} = C_1 (1)^K + (C_2 + C_3 K)\, 2^{(K)}$ ———→ ⊛

Now For Perticular solution

$y_K^{(P)} = K^2 \cdot 2^K [B_0 + B_1 K]$ $\because K^2$ bcz 2 is double root

Put in equ ①

$(K+3)^2\, 2^{(K+3)} [B_0 + B_1 (K+3)] - 5[(K+2)^2\, 2^{(K+2)} (B_0 + B_1 (K+2))]$
$+ 8[(K+1)^2\, 2^{(K+1)} (B_0 + B_1 (K+1))] - 4K^2 2^K (B_0 + B_1 K) = 2^K \cdot K$

$= 2^K [8(K+3)^2 \cdot (B_0 + B_1 K + 3B_1) - 5(4)(K+2)^2 (B_0 + B_1 K + 2B_1)$
$+ 8(2)(K+1)^2 (B_0 + B_1 K + B_1) - 4K^2 (B_0 + B_1 K)] = 2^K \cdot K$

Dividing by 2^K

$\Rightarrow 8(K+3)^2 \cdot (B_0 + B_1 K + 3B_1) - 5(4)(K+2)^2 (B_0 + B_1 K + 2B_1) + 8(2)$
$(K+1)^2 (B_0 + B_1 K + B_1) - 4K^2 (B_0 + B_1 K) = K$

Equating co-efficients of like powers of K

$$\Rightarrow 48B_0 + 216B_1 - 80B_0 - 240B_1 + 32B_1 + 48B_1 = 1 \quad \text{---} ②$$

$$72B_0 + 216B_1 - 80B_0 - 160B_1 + 16B_1 + 16B_1 = 0 \quad \text{---} ③$$

$$② \Rightarrow 24B_1 = 1 \quad \Rightarrow B_1 = 1/24$$

$$③ \Rightarrow 8B_0 + 72B_1 = 0 \quad \Rightarrow B_0 = -3/8$$

Perticular Solution is

$$y_K^{(P)} = K^2 \cdot 2^K \left(-\frac{3}{8} + \frac{1}{24}K\right)$$

General solution of given difference eqn is

$$y_K = C_1(1)^K + (C_2 + C_3K)2^K + K^2 \cdot 2^K\left(-\frac{3}{8} + \frac{1}{24}K\right)$$

** **

Assignment

Q1:- $y_{n+2} - 9y_{n+1} + 20y_n = 3^n(n^2+1)$

Q2:- $y_{K+2} - 2y_{K+1} + y_K = K2^K - 2^K$

Q3:- $U_{n+3} + 8U_n = (2n+3)2^n$

Q4:- $y_{K+2} - 13y_{K+1} + 36y_K = 2^K(K^2+1)$

Q5:- $y_{K+2} + 6y_{K+1} + 25y_K = 2^K + K + 4$

Q6:- $(E^2-6E+9)y_K = 3^K \cdot K^2$

Type V :- If $f(n) = \alpha \sin a_n$ or $\beta \cos a_n$ or $\alpha \sin a_n + \beta \cos a_n$

Then in either case choose

$$y_n^{(P)} = (A \cos a_n + B \sin a_n)\, n^K$$

where K is least non-negative integer, so that $y_n^{(c)}$ & $y_n^{(P)}$ are not coincident.

Example:- Solve $y_{n+2} - 4y_{n+1} + 4y_n = \sin n$

Solution

$$y_{n+2} - 4y_{n+1} + 4y_n = \sin n \quad \text{--- (1)}$$

$$\Rightarrow E^2 y_n - 4E y_n + 4y_n = \sin n$$

$$\Rightarrow (E^2 - 4E + 4) y_n = \sin n$$

Ch. equation is

$$E^2 - 4E + 4 = 0$$

$$\Rightarrow (E-2)^2 = 0 \quad \Rightarrow E = 2, 2$$

C.F is $y_n^{(c)} = (C_1 + C_2 n) 2^n$

For P.S Put $y_n^{(P)} = A_1 \sin n + A_2 \cos n$

in (1) we have

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$$A_1\sin(n+2)+A_2\cos(n+2)-4[A_1\sin(n+1)+A_2\cos(n+1)] + 4[A_1\sin n + A_2\cos n] = \sin n$$

$$\Rightarrow A_1[\sin(n)\cos(2)+\cos(n)\sin(2)]+A_2[\cos(n)\cos(2)-\sin(n)\sin(2)]-4A_1[\sin(n)\cos(1)+\cos(n)\sin(1)]-4A_2[\cos(n)\cos(1)-\sin(n)\sin(1)]+4A_1\sin(n)+4A_2\cos n=\sin(n)$$

$$\Rightarrow [A_1\cos(2)-A_2\sin(2)-4A_1\cos(1)-4A_2\sin(1)+4A_1]\sin(n) + [A_1\sin(2)+A_2\cos(2)-4A_1\sin(1)-4A_2\cos 1+4A_2]\cos n = \sin(n)$$

★ Equating coefficients of $\sin(n)$:-

$$(\cos 2-4\cos 1+4)A_1-(\sin 2+4\sin 1)A_2=1 \longrightarrow (A)$$

★ Equating Coefficient of $\cos(n)$:-

$$(\sin 2-4\sin 1)A_1+(\cos 2-4\cos 1+4)A_2=0 \longrightarrow (B)$$

Let $K_1 = \cos 2 - 4\cos 1 + 4$

$K_2 = \sin 2 + 4\sin 1$

Then A & B becomes

$$K_1A_1-K_2A_2=1 \Rightarrow K_1A_1-K_2A_2-1=0$$

$$K_2A_1-K_1A_2=0 \Rightarrow K_2A_1-K_1A_2-0=0$$

$$\frac{A_1}{0+K_1}=\frac{-A_2}{0+K_2}=\frac{1}{K_1^2+K_2^2}$$

$$\Rightarrow A_1=\frac{K_1}{K_1^2+K_2^2}, \quad A_2=\frac{-K_2}{K_1^2+K_2^2}$$

P.S is $y_n^{(P)} = A_1 \sin n + A_2 \cos n$

$= \frac{K_1}{K_1^2+K_2^2} \sin(n) - \frac{K_2}{K_1^2+K_2^2} \cos(n)$

General Solution is

$y_n = C.F + P.S$

$= (C_1 + C_2 n) 2^n + \frac{K_1}{K_1^2+K_2^2} \sin(n) - \frac{K_2}{K_1^2+K_2^2} \cos(n)$

where $K_1 = \cos 2 - 4\cos 1 + 4$

$K_2 = \sin 2 - 4 \sin 1$

Example:- $y_{k+2} - 2y_{k+1} + y_k = \sin 5k + \cos 5k + 6$

Solution:- $y_{k+2} - 2y_{k+1} + y_k = \sin 5k + \cos 5k + 6$ ——①

$\Rightarrow E^2 y_k - 2E y_k + y_k = \sin 5k + \cos 5k + 6$

$\Rightarrow (E^2 - 2E + 1) y_k = \sin 5k + \cos 5k + 6$

For C.F ch equ is

$E^2 - 2E + 1 = 0$

$\Rightarrow (E-1)^2 = 0 \quad \Rightarrow E = 1, 1$

$y_k^{(c)} = (C_1 + C_2 k)(1)^k$

For P.S Put $y_k^{(P)} = A \sin 5k + B \cos 5k + Ck^2$

Put in equ ① we have

$A\sin 5(K+2) + B\cos 5(K+2) + C(K+2)^2 - 2[A\sin 5(K+1) + B\cos 5 (K+1) + C(K+1)^2] + A\sin 5K + B\cos 5K + CK^2$
$= \sin 5K + \cos 5K + 6$

$\Rightarrow A[\sin(5K)\cos 10 + \cos 5K \sin 10] + B[\cos 5K \cos 10 - \sin 5K \sin 10] + C[K^2 + 4K + 4] - 2A[\sin 5K \cos 5 + \cos 5K \sin 5] - 2B[\cos 5K \cos 5 - \sin 5K \sin 5] - 2C[K^2 + 2K + 1] + A\sin 5K + B\cos 5K + CK^2 = \sin 5K + \cos 5K + 6$

Comparing co-efficient of $\sin 5K, \cos 5K$ & constant we have

$(\cos 10 - 2\cos 5 + 1)A - (\sin 10 - 2\sin 5)B = 1$ ——(i)

$(\sin 10 - 2\sin 5)A + (\cos 10 - 2\cos 5 + 1)B = 1$ ——(ii)

$C(\cancel{K^2} + \cancel{4K} + 4 - \cancel{2K^2} - \cancel{4K} - 2 + \cancel{K^2}) = 6$

$\Rightarrow C(2) = 6 \quad \Rightarrow C = 3$

Now (i) & (ii) written as

$K_1 A - K_2 B = 1 \Rightarrow K_1 A - K_2 B - 1 = 0$

$K_2 A + K_1 B = 1 \Rightarrow K_2 A + K_1 B - 1 = 0$

where $K_1 = \cos 10 - 2\cos 5 + 1$

$K_2 = \sin 10 - 2\sin 5$

Solving these equations by Cross Multiplicat

$$\frac{A}{K_2 + K_1} = \frac{-B}{-K_1 + K_2} = \frac{1}{K_1^2 + K_2^2}$$

$$\Rightarrow A = \frac{K_1 + K_2}{K_1^2 + K_2^2}, \quad B = \frac{K_1 - K_2}{K_1^2 + K_2^2}$$

So P.S is

$$y_k^{(p)} = \left(\frac{K_1+K_2}{K_1^2+K_2^2}\right)\sin 5K + \left(\frac{K_1-K_2}{K_1^2+K_2^2}\right)\cos 5K + 3K^2$$

So The General solution is

$$y_n = y_n^{(c)} + y_n^{(p)}$$

* ** ** ** *

Assignment

Solve the following difference equs.

Q1:- $y_{n+2} - 3y_{n+1} - 4y_n = \sin 2n$

Q2:- $y_{n+2} - 2y_{n+1} + y_n = \sin 5n + \cos 5n + 9$

Q3:- $y_{n+2} + y_n = \sin \frac{n\pi}{2}$

Q4:- $y_{n+2} - 8y_{n+1} + y_n = 2^n + \sin n$

Q5:- $y_{n+4} - 6y_n = \sin 3n$

Q6:- $y_{n+2} + y_n = \sin x$

Q7:- $y_{n+2} + y_n = \cos \frac{n}{2}$

Q8:- $y_{k+2} - 7y_{k+1} + 12y_k = \cos K$

Type VI :- When $\phi(n) = a^n[\cos An$ or $\sin An]$ where A is constant

In order to find particular solution we shall make substitution

$$y_n = a^n[c_1 \sin An + c_2 \cos An]$$

And find value of c_1 and c_2

★★

Example :- $y_{t+2} - 7y_{t+1} - 8y_t = 7^t[\cos 3t + \sin 3t]$

Solution $y_{t+2} - 7y_{t+1} - 8y_t = 7^t(\cos 3t + \sin 3t)$ ——— ①

$\Rightarrow E^2 y_t - 7E y_t - 8y_t = 7^t(\cos 3t + \sin 3t)$

$\Rightarrow (E^2 - 7E - 8)y_t = 7^t(\cos 3t + \sin 3t)$

characteristic equation is

$E^2 - 7E - 8 = 0$

$\Rightarrow E^2 - 8E + E - 8 = 0 \Rightarrow E(E-8) + 1(E-8) = 0$

$\Rightarrow (E-8)(E+1) = 0$

$\Rightarrow E = -1, 8$

C.F is $y_t^{(c)} = c_1(-1)^t + c_2(8)^t \longrightarrow$ ★

For particular solution

Put $y_t^{(P)} = 7^t[A_1 \cos 3t + A_2 \sin 3t]$

Put in eq ①

$7^{(t+2)}[A_1 \cos 3(t+2) + A_2 \sin 3(t+2)] - 7.7^{t+1}[A_1 \cos 3(t+1)$
$+ A_2 \sin 3(t+1)] - 8[7^t(A_1 \cos 3t + A_2 \sin 3t)] = 7^t(\cos 3t$
$+ \sin 3t)$

Dividing both sides by 7^t

$\Rightarrow 49A_1[\cos 3t \cos 6 - \sin 3t \sin 6] + 49A_2[\sin 3t \cos 6 + \cos 3t \sin 6] - 49A_1[\cos 3t \cos 3 - \sin 3t \sin 3] - 49A_2[\sin 3t \cos 3 - \cos 3t \sin 3] - 8A_1 \cos 3t - 8A_2 \sin 3t = \cos 3t + \sin 3t$

Comparing co-efficients of $\cos 3t$ & $\sin 3t$

$(49\cos 6 - 49\cos 3 - 8)A_1 + (49\sin 6 - 49\sin 3)A_2 = 1$

$-(49\sin 6 - 49\sin 3)A_1 + (49\cos 6 - 49\cos 3 - 8)A_2 = 1$

Let $K_1 = 49\cos 6 - 49\cos 3 - 8$

$K_2 = 49\sin 6 - 49\sin 3$

$\Rightarrow K_1A_1 + K_2A_2 - 1 = 0$

$-K_2A_1 + K_1A_2 - 1 = 0$

$$\Rightarrow \frac{A_1}{-K_2 + K_1} = \frac{-A_2}{-K_1 - K_2} = \frac{1}{K_1^2 + K_2^2}$$

$$\Rightarrow A_1 = \frac{K_1 - K_2}{K_1^2 + K_2^2}, \quad A_2 = \frac{K_1 + K_2}{K_1^2 + K_2^2}$$

So P.S is

$$y_t^{(p)} = 7^t\left[\left(\frac{K_1 - K_2}{K_1^2 + K_2^2}\right)\cos 3t + \left(\frac{K_1 + K_2}{K_1^2 + K_2^2}\right)\sin 3t\right] \longrightarrow \star'$$

by $\star$ & $\star'$

$$y_t = y_t^{(c)} + y_t^{(P)}$$

$$y_t = C_2 8^t + C_1(-1)^t + 7^t\left[\frac{(k_1-k_2)\cos 3t + (k_1+k_2)\sin 3t}{k_1^2+k_2^2}\right]$$

* * *

Example: Solve $y_{k+2} + 13y_{k+1} + 3y_k = 3^k \cos 4k$

Solution Given $y_{k+2} + 13y_{k+1} + 3y_k = 3^k \cos 4k$ —— ①

$$E^2 y_k + 13E y_k + 3y_k = 3^k \cos 4k$$

$$\Rightarrow (E^2 + 13E + 3)y_k = 3^k \cos 4k$$

Auxiliarly equation is

$$E^2 + 13E + 3 = 0$$

$$\Rightarrow E = \frac{-13 \pm \sqrt{169 - 4(1)(3)}}{2}$$

$$= \frac{-13 \pm \sqrt{157}}{2}$$

So Complementry Solution is

$$y_k^{(c)} = A_1\left(\frac{-13+\sqrt{157}}{2}\right)^k + A_2\left(\frac{-13-\sqrt{157}}{2}\right)^k \longrightarrow *$$

For Perticular Solution Put

$$y_k = 3^k[C_1 \sin 4k + C_2 \cos 4k]$$

Put in ① we have

$$\Rightarrow 3^{k+2}[C_1 \sin 4(k+2) + C_2 \cos 4(k+2)] + 13 \cdot 3^{k+1}[C_1 \sin 4(k+1)$$
$$+ C_2 \cos 4(k+1)] + 3 \cdot 3^k[C_1 \sin 4k + C_2 \cos 4k]$$
$$= 3^k \cos 4k$$

Dividing both sides by 3^K we get

$9C_1[\sin 4K \cos 8 + \cos 4K \sin 8] + 9C_2[\cos 4K \cos 8 - \sin 4K \sin 8] + 39C_1[\sin 4K \cos 4 + \cos 4K \sin 4] + 39C_2[\cos 4K \cos 4 - \sin 4K \sin 4] + 3C_1 \sin 4K + 3C_2 \cos 4K = \cos 4K$

Comparing Coefficients of $\cos 4K$ & $\sin 4K$

$C_1[9\cos 8 + 39\cos 4 + 3] - C_2[9\sin 8 + 39\sin] = 0$

$C_1[9\sin 8 + 39\sin 4] + C_2[9\cos 8 + 39\cos 4 + 3] = 1$

Let

$9\cos 8 + 39\cos 4 + 3 = K_1$

$9\sin 8 + 39\sin 4 = K_2$

Then

$K_1C_1 - K_2C_2 - 0 = 0$

$K_2C_1 - K_1C_2 - 1 = 0$

$$\Rightarrow \frac{C_1}{K_2} = \frac{-C_2}{-K_1} = \frac{1}{K_1^2 + K_2^2}$$

So $C_1 = \dfrac{K_2}{K_1^2 + K_2^2}$, $C_2 = \dfrac{K_1}{K_1^2 + K_2^2}$

So P.S is

$$y_K^{(P)} = 3^K\left[\frac{K_2 \sin 4K + K_1 \cos 4K}{K_1^2 + K_2^2}\right] \longrightarrow \star'$$

by $\star$ and $\star'$ General Solution is

$$y_K = y_K^{(c)} + y_K^{(P)}$$

$$y_k = A_1\left[\frac{-13+\sqrt{157}}{2}\right]^k + A_2\left[\frac{-13-\sqrt{157}}{2}\right]^k +$$

$$3^k\left[\frac{k_2 \sin 4k + k_1 \cos 4k}{k_1^2 + k_2^2}\right]$$

★ ———— ★ ★ ★ ———— ★

Example:- Solve $y_{k+2}+y_{k+1}+y_k = 2^k \cdot \sin k \cos 3k$

Solution Given $y_{k+2}+y_{k+1}+y_k = 2^k \cdot \sin k \cos 3k$

$$\Rightarrow y_{k+2}+y_{k+1}+y_k = \frac{2^k}{2}\left[2 \sin k \cos 3k\right]$$

$$\Rightarrow y_{k+2}+y_{k+1}+y_k = \frac{2^k}{2}\left[\sin 2k - \sin k\right] \longrightarrow ①$$

For Complementry function

$$y_{k+2} + y_{k+1} + y_k = 0$$

$$E^2 y_k + E y_k + y_k = 0$$

$$\Rightarrow (E^2+E+1)\, y_k = 0$$

$$\Rightarrow E^2+E+1 = 0$$

$$\Rightarrow E = \frac{-1 \pm \sqrt{1-4}}{2} \Rightarrow E = \frac{-1 \pm \sqrt{3}\, i}{2}$$

So Complementry solution is

$$y_k = R^k\left[A_1 \cos k\theta + A_2 \sin k\theta\right]$$

where $R = \sqrt{\frac{1}{4}+\frac{3}{4}} \Rightarrow R = 1$

$$\theta = \tan^{-1}\frac{\sqrt{3}}{-1} \Rightarrow \theta = 120^\circ$$

$$\Rightarrow y_k^{(c)} = (1)^k\left[A_1 \cos 120k + A_2 \sin 120k\right] \longrightarrow ★$$

For Perticular Solution we dive equ ① into two equations.

$$y_{k+2} + y_{k+1} + y_k = 2^k \cdot \frac{1}{2} \sin 2k \longrightarrow (A)$$

& $y_{k+2} + y_{k+1} + y_k = 2^k \cdot \frac{1}{2} \sin k \longrightarrow (B)$

First we solve (A)

Put $y_k = 2^k (c_1 \sin 2k + c_2 \cos 2k)$

(A) $\Rightarrow$ $2^{k+2}[c_1 \sin 2(k+2) + c_2 \cos(k+2)] + 2^{k+1}[c_1 \sin 2(k+1)$
$+ c_2 \cos(k+1)] + 2^k [c_1 \sin 2k + c_2 \cos 2k]$
$= 2^k \cdot \frac{1}{2} \sin 2k$

Dividing by 2^k and comparing coeff. of $\sin 2k$ and $\cos 2k$ we have

$(4\cos 4 + 2\cos 2 + 1)c_1 - (4\sin 4 + 2\sin 2)c_2 = 1/2$

$(4\sin 4 + 2\sin 2)c_1 + (4\cos 4 + 2\cos 2 + 1)c_2 = 0$

Let $k_1 = 4\cos 4 + 2\cos 2 + 1$

$k_2 = 4\sin 4 + 2\sin 2$

Then above write as

$k_1 c_1 - k_2 c_2 = 1/2$

$k_2 c_1 + k_1 c_2 = 0$

$\Rightarrow$ $k_1 c_1 - k_2 c_2 - 1/2 = 0$

$k_2 c_1 + k_1 c_2 - 0 = 0$

by Solving $\dfrac{c_1}{k_1/2} = \dfrac{-c_2}{-k_2/2} = \dfrac{1}{k_1^2 + k_2^2}$

$$\Rightarrow c_1 = \frac{k_1}{2(k_1^2+k_2^2)} \quad , \quad c_2 = \frac{-k_2}{2(k_1^2+k_2^2)}$$

So Perticular Solution is

$$y_k = 2^k\left[\frac{k_1 \sin 2k - k_2 \cos 2k}{2(k_1^2+k_2^2)}\right]$$

Similarly for Ⓑ Perticular Solution

$$y_k = 2^k\left[\frac{k_1 \sin k - k_2 \cos k}{2(k_1^2+k_2^2)}\right]$$

Particular Solution of Ⓐ and Ⓑ is actually P.S of (i)

So P.S of ① = P.S of Ⓐ − P.S of Ⓑ

$$y_k = 2^k\left[\frac{k_1(\sin 2k - \sin k) - k_2(\cos 2k - \cos k)}{2(k_1^2+k_2^2)}\right] \longrightarrow \star'$$

General solution of (i) is

$$y_k = (1)^k\left[A_1 \cos 120^\circ k + A_2 \sin 120^\circ k\right] + 2^k\left[\frac{k_1(\sin 2k - \sin k) - k_2(\cos 2k - \cos k)}{2(k_1^2+k_2^2)}\right]$$

★★★ ★

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Solutions of Assignments

Type 1 → 6

Type I:-

Q1:- $y_{n+2} - 4y_{n+1} + y_n = 1$ ——— ①

$\Rightarrow E^2 y_n - 4E y_n + y_n = 1$

$\Rightarrow (E^2 - 4E + 1) y_n = 1$ $\Rightarrow$ Ch. equ $E^2 - 4E + 1 = 0$

$\Rightarrow E = \dfrac{4 \pm \sqrt{(4)^2 - 4(1)(1)}}{2} = \dfrac{4 \pm \sqrt{12}}{2}$

$\Rightarrow E = \dfrac{4 \pm 2\sqrt{3}}{2}$ $\Rightarrow E = 2 \pm \sqrt{3}$

So complementary function is

$y_n^{(c)} = c_1 (2+\sqrt{3})^n + c_2 (2-\sqrt{3})^n$

Now for particular solution

$y_n^{(p)} = c \cdot n^k$, $k=0$ $\because 1$ is not root

$\Rightarrow y_n^{(p)} = c$, $y_{n+1}^{(p)} = c$, $y_{n+2}^{(p)} = c$

Put all in equ ①

$\Rightarrow c - 4c + c = 1$ $\Rightarrow$ $-2c = 1$

$\Rightarrow c = -\frac{1}{2}$ $\Rightarrow y_n^{(p)} = -\frac{1}{2}$

So General solution is $y_n = y_n^{(c)} + y_n^{(p)}$

$\Rightarrow y_n = c_1 (2+\sqrt{3})^n + c_2 (2-\sqrt{3})^n - \frac{1}{2}$

Q2:- $y_{n+2} - 3y_{n+1} - 4y_n = 6$ ⟶ ①

Given equ can be written as

$E^2 y_n - 3E y_n - 4 y_n = 6$

$$\Rightarrow (E^2 - 3E - 4) y_n = 0$$

Characteristic equation is

$E^2 - 3E - 4 = 0 \quad \Rightarrow \quad E^2 - 4E + E - 4 = 0$

$\Rightarrow E(E-4) + 1(E-4) = 0 \quad \Rightarrow (E+1)(E-4) = 0$

$\Rightarrow E = 4, -1$ are roots

So Complementry function is

$$y_n^{(c)} = c_1(-1)^n + c_2(4)^n$$

Now for perticular Solution

$$y_n^{(P)} = C \cdot n^k$$

As 1 is not root so $k=0$

$\Rightarrow y_n^{(P)} = C, \quad y_{n+1}^{(P)} = C, \quad y_{n+2}^{(P)} = C$

Put in equ ①

$C - 3C - 4C = 6 \quad \Rightarrow \quad -6C = 6 \quad \Rightarrow C = -1$

So $y_n^{(P)} = -1$

So general solution is $y_n = y_n^{(c)} + y_n^{(P)}$

$$\Rightarrow y_n = c_1(-1)^n + c_2(4)^n - 1$$

Q3:- $U_{n+2} - 4U_{n+1} + U_n = 3 \longrightarrow$ ①

$\Rightarrow E^2 U_n - 4E U_n + U_n = 3$

$\Rightarrow (E^2 - 4E + 1) U_n = 3$

Characteristic equation is

$E^2 - 4E + 1 = 0$

$$\Rightarrow E = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

So C.F is

$$y_n^{(c)} = c_1(2+\sqrt{3})^n + c_2(2-\sqrt{3})^n$$

For Particular Solution

$y_n^{(P)} = C \cdot n^k$, where $k=0$

$\Rightarrow y_n^{(P)} = C$, $y_{n+1}^{(P)} = C$, $y_{n+2}^{(P)} = C$

Put all in equ (1)

$\Rightarrow C - 4C + C = 3 \Rightarrow -2C = 3$

$\Rightarrow C = -3/2$

So general solution is

$y_n = C_1(2+\sqrt{3})^n + C_2(2-\sqrt{3})^n - 3/2$

★ ★ ——————— ★ ★

Q4:- $y_{n+3} + 6y_{n+2} + 11y_{n+1} = 5$

$\Rightarrow E^3 y_n + 6E^2 y_n + 11E y_n = 5$

$\Rightarrow (E^3 + 6E^2 + 11E) y_n = 5$

So Characteristic equation is

$E^3 + 6E^2 + 11E = 0 \Rightarrow E(E^2 + 6E + 11) = 0$

$\Rightarrow E = 0$, $E^2 + 6E + 11 = 0$

$\Rightarrow E = \dfrac{-6 \pm \sqrt{36-44}}{2} = \dfrac{-6 \pm \sqrt{-8}}{2}$

$\Rightarrow E = \dfrac{-6 \pm 2\sqrt{2}\,i}{2} \Rightarrow E = -3 \pm \sqrt{2}\,i$

$\Rightarrow y_n^{(c)} = R^n [A_1 \cos n\theta + A_2 \sin n\theta]$

where $R = \sqrt{(-3)^2 + (\sqrt{2})^2} = \sqrt{11}$

$\theta = \tan^{-1}(\sqrt{2}/3)$

$$\Rightarrow y_n^{(c)} = (\sqrt{13})^n \left[A_1 \cos n\left(\tan^{-1}\frac{\sqrt{2}}{-3}\right) + A_2 \sin n\left(\tan^{-1}\frac{\sqrt{2}}{-3}\right)\right]$$

For Particular solution $y_n^{(P)} = C$,
$y_{n+1}^{(P)} = C$, $y_{n+2}^{(P)} = C$, $y_{n+3}^{(P)} = C$

$$\Rightarrow C + 6C + 11C = 5 \quad \Rightarrow 18C = 5$$

$$\Rightarrow C = 5/18$$

So $y_n = 11^{n/2}\left[A_1 \cos n \tan^{-1}\left(\frac{\sqrt{2}}{-3}\right) + A_2 \sin n \tan^{-1}\left(\frac{\sqrt{2}}{-3}\right)\right] + \frac{5}{18}$

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Q5:- $\Delta^2 y_K + 3\Delta y_K = 9$

As $\Delta = E - 1$ so equ becomes

$$(E-1)^2 y_K + 3(E-1)y_K = 9$$

$$\Rightarrow (E^2 + 1 - 2E)y_K + (3E - 3)y_K = 9$$

$$\Rightarrow (E^2 + 1 - 2E + 3E - 3)y_K = 9$$

$$\Rightarrow (E^2 + E - 2)y_K = 9$$

& $y_{K+2} + y_{K+1} - 2y_K = 9 \longrightarrow (1)$

So Ch. equation is $E^2 + E - 2 = 0$

$\Rightarrow E^2 + 2E - E - 2 = 0 \Rightarrow E(E+2) - 1(E+2) = 0$

$\Rightarrow (E-1)(E+2) = 0 \Rightarrow E = 1, -2$ are roots

So complementry function is

$$y_K^{(c)} = C_1(1)^K + C_2(-2)^K$$

And for Perticular solution

$y_K^{(P)} = \alpha K^n$, $n = 1$ ∵ 1 is single root

$$\Rightarrow y_K^{(P)} = \alpha K$$

$$y_{k+1}^{(p)} = \alpha(k+1) \quad , \quad y_{k+2}^{(p)} = \alpha(k+2)$$

Put in equ ①

$$\Rightarrow \alpha k + 2\alpha + \alpha k + \alpha \cancel{-2}\alpha k = 9$$

$$\Rightarrow 3\alpha = 9 \quad \Rightarrow \quad \alpha = 3$$

So general solution is

$$y_n = c_1(1)^k + c_2(-2)^k + 3$$

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Q6:- $(E^3 - 4E^2 + 2E + 1)y_n = 14$,

Given equation can be written as

$$y_{n+3} - 4y_{n+2} + 2y_{n+1} + y_n = 14 \longrightarrow ①$$

Characteristic equation is

$$E^3 - 4E^2 + 2E + 1 = 0$$

$E = 1$ is root so by Synthetic division

1	1	-4	2	1
		1	-3	-1
	1	-3	-1	0

$$\Rightarrow (E-1)(E^2 - 3E - 1) = 0 \quad \Rightarrow E = 1 \text{ \&}$$

$$E = \frac{3 \pm \sqrt{9+4}}{2} \quad \Rightarrow E = \frac{3 \pm \sqrt{13}}{2}$$

$$\Rightarrow y_n^{(c)} = c_1(1)^n + c_2\left(\frac{3+\sqrt{13}}{2}\right)^n + c_3\left(\frac{3-\sqrt{13}}{2}\right)^n$$

& for Perticular Solution Put

$y_n^{(p)} = c\,n^k$, $k=1$ as 1 is root.

$$\Rightarrow y_{n+1}^{(p)} = c(n+1) \quad , \quad y_{n+2}^{(p)} = c(n+2) \quad , \quad y_{n+3}^{(p)} = c(n+3)$$

So equ ① becomes

$$Cn + 3C - 4Cn - 8C + 2Cn + 2C + Cn = 14$$

$$\Rightarrow -3C = 14 \qquad \Rightarrow C = \frac{-14}{3}$$

$$\Rightarrow y_n = C_1(1)^n + C_2\left(\frac{3+\sqrt{13}}{2}\right)^n + C_3\left(\frac{3-\sqrt{13}}{2}\right)^n - \frac{14}{3}$$

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Type II :-

Q1:- $y_{n+4} - A^4 y_n = 2^n \longrightarrow$ ①

$$\Rightarrow E^4 y_n - A^4 y_n = 2^n \Rightarrow (E^4 - A^4) y_n = 2^n$$

Characteristic equation is

$$E^4 - A^4 = 0 \qquad \Rightarrow E^4 = A^4 \qquad \Rightarrow E = \pm A$$

$$\Rightarrow y_n^{(c)} = C_1(A)^n + C_2(-A)^n$$

Now for Particular Solution Put

$y_n^{(p)} = \alpha \cdot 2^n \cdot n^K$, where $K = 0$ $\because 2$ is not root

$$\Rightarrow y_n^{(p)} = \alpha \cdot 2^n , \quad y_{n+4}^{(p)} = \alpha \cdot 2^{n+4} \quad \text{Put in ①}$$

$$\Rightarrow \alpha \cdot 2^{n+4} - A^4 \cdot \alpha \cdot 2^n = 2^n$$

$$\Rightarrow 2^n [\alpha \cdot 2^4 - A^4 \alpha] = 2^n \quad \Rightarrow 16\alpha - A^4 \alpha = 1$$

$$\Rightarrow \alpha(16 - A^4) = 1 \qquad \Rightarrow \alpha = \frac{1}{16 - A^4}$$

$$\Rightarrow y_n^{(p)} = \frac{1}{16 - A^4} \cdot 2^n$$

$\Rightarrow$ General solution is $y_n = y_n^{(c)} + y_n^{(p)}$

$$\Rightarrow y_n = C_1(A)^n + C_2(-A)^n + \frac{1}{16 - A^4} \cdot 2^n$$

Q2:- Given $y_{n+2} + 4y_{n+1} + 4y_n = 2^{n+2}$; $y_0 = 1, y_1 = 2$

$$y_{n+2} + 4y_{n+1} + 4y_n = 2^{n+2} \longrightarrow (1)$$

$\Rightarrow E^2 y_n + 4E y_n + 4y_n = 2^2 \cdot 2^n \Rightarrow (E^2 + 4E + 4) y_n = 4 \cdot 2^n$

Characteristic equation is

$E^2 + 4E + 4 = 0 \quad \Rightarrow E^2 + 2E + 2E + 4 = 0$

$\Rightarrow E(E+2) + 2(E+2) = 0 \quad \Rightarrow (E+2)(E+2) = 0$

$\Rightarrow E = -2, -2$ are roots

$\Rightarrow$ Complementry function is

$y_n^{(c)} = C_1(-2)^n + C_2 n(-2)^n$

Now by given condition

$y_0 = C_1(-2)^0 + C_2 n(-2)^0 \quad \Rightarrow 1 = C_1 + C_2 n$

$y_1 = C_1(-2)^1 + C_2 n(-2)^1 \Rightarrow 2 = -2C_1 - 2C_2 n$

equations are not satisfied

Now for perticular solution

$y_n^{(p)} = \alpha \cdot 2^n \cdot n^K$, where $K = 0 \quad \because 2$ is not root

$\Rightarrow y_n^{(p)} = \alpha \cdot 2^n \quad , y_{n+1}^{(p)} = \alpha \cdot 2^{n+1} \quad , y_{n+2}^{(p)} = \alpha \cdot 2^{n+2}$

Put all in equ (1)

$\alpha \cdot 2^{n+2} + 4(\alpha \cdot 2^{n+1}) + 4\alpha 2^n = 4 \cdot 2^n$

$\Rightarrow \alpha \cdot 2^{n+2} + 4\alpha 2^{n+1} + 4\alpha 2^n = 4 \cdot 2^n$

$\Rightarrow 2^n[4\alpha + 8\alpha + 4\alpha] = 4 \cdot 2^n \quad \Rightarrow \quad 16\alpha = 4$

$\Rightarrow \alpha = 1/4 \quad \Rightarrow y_n^{(p)} = \frac{1}{4} 2^n$

So General solution is

$y_n = C_1(-2)^n + C_2 n(-2)^n + 2^{n-2}$

Q3:- $y_{k+2} - 6y_{k+1} + 8y_k = 2\cdot 3^k \longrightarrow ①$

$\Rightarrow E^2 y_k - 6E y_k + 8y_k = 2\cdot 3^k$

$\Rightarrow (E^2 - 6E + 8) y_k = 2\cdot 3^k$

Characteristic equation is

$E^2 - 6E + 8 = 0 \quad \Rightarrow E^2 - 4E - 2E + 8 = 0$

$\Rightarrow E(E-4) - 2(E-4) = 0 \quad \Rightarrow (E-4)(E-2) = 0$

$\Rightarrow E = 4, 2$ are roots.

$\Rightarrow$ C.F is $y_k^{(c)} = C_1(2)^k + C_2(4)^k$

Now for Particular solution

$y_k^{(p)} = \alpha \cdot 3^k \cdot k^n \quad , n = 0 \because 3$ is not root

$\Rightarrow y_k^{(p)} = \alpha \cdot 3^k \quad , y_{k+1}^{(p)} = \alpha \cdot 3^{k+1} \quad , y_{k+2}^{(p)} = \alpha \cdot 3^{k+2}$

Put all in equ ①

$\alpha \cdot 3^{k+2} - 6(\alpha 3^{k+1}) + 8(\alpha 3^k) = 2\cdot 3^k$

$\alpha 3^{k+2} - 6\alpha 3^{k+1} + 8\alpha 3^k = 2\cdot 3^k$

$\Rightarrow 3^k[9\alpha - 18\alpha + 8\alpha] = 2\cdot 3^k \quad \Rightarrow -\alpha = 2$

$\Rightarrow \alpha = -2$

$\Rightarrow y_k^{(p)} = -2\cdot 3^k$

So General solution is

$y_n = C_1(2)^k + C_2(4)^k - 2\cdot 3^k$

★★

Q4:- $y_{k+3} - y_{k+2} + y_{k+1} - y_k = 3^k \longrightarrow ①$

$\Rightarrow E^3 y_k - E^2 y_k + E y_k - y_k = 3^k$

$\Rightarrow (E^3 - E^2 + E - 1) y_k = 3^k$

Ch. equ is

$E^3 - E^2 + E - 1 = 0$, 1 is root so

1	1	-1	1	-1
		1	0	1
	1	0	1	$\boxed{0}$

$\Rightarrow (E-1)(E^2+1) = 0$

$\Rightarrow E = 1$ & $E^2 = -1 \Rightarrow E = \pm i$

$\Rightarrow E = 1, \pm i$ are roots

So Complementary function is

$y_K^{(c)} = C_1 + R^n [C_2 \cos n\theta + C_3 \sin n\theta]$

where $R = \sqrt{(1)^2} = 1$,

$\theta = \tan^{-1}(1/0) \Rightarrow \theta = \pi/2$

$\Rightarrow y_K^{(c)} = C_1 + \left[C_2 \cos \frac{n\pi}{2} + C_3 \sin \frac{n\pi}{2}\right]$

Now for Particular Solution

$y_K^{(P)} = \alpha . 3^K . K^n$, $n = 0$ $\because$ 3 is not root

$\Rightarrow y_K^{(P)} = \alpha . 3^K$, $y_{K+1}^{(P)} = \alpha . 3^{K+1}$, $y_{K+2}^{(P)} = \alpha . 3^{K+2}$

$y_{K+3}^{(P)} = \alpha . 3^{K+3}$ Put in equ①

$\alpha 3^{K+3} - \alpha 3^{K+2} + \alpha 3^{K+1} - \alpha 3^K = 3^K$

$\Rightarrow 3^K [27\alpha - 9\alpha - 3\alpha - \alpha] = 3^K$

$\Rightarrow 20\alpha = 1 \Rightarrow \alpha = 1/20$

$\Rightarrow y_K^{(P)} = \frac{1}{20} . 3^K$

$\Rightarrow y_n = C_1 + C_2 \cos \frac{n\pi}{2} + C_3 \sin \frac{n\pi}{2} + \frac{1}{20} . 3^K$

Q5:- $y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = 3^n \longrightarrow ①$

$\Rightarrow E^3 y_n - 3E^2 y_n + 3E y_n - y_n = 3^n$

$\Rightarrow (E^3 - 3E^2 + 3E - 1) y_n = 3^n$

Characteristic equation is

$E^3 - 3E^2 + 3E - 1 = 0$, $E = 1$ is root so

$$\begin{array}{c|cccc} 1 & 1 & -3 & 3 & -1 \\ & & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & \boxed{0} \end{array}$$

$\Rightarrow (E-1)(E^2 - 2E + 1) = 0$

$\Rightarrow (E-1)(E^2 - E - E + 1) = 0$

$\Rightarrow (E-1)[E(E-1) - 1(E-1)] = 0$

$\Rightarrow (E-1)(E-1)(E-1) = 0$

$\Rightarrow E = 1, 1, 1$ are roots

So Complementry Solution is

$y_n^{(c)} = c_1(1)^n + c_2 n(1)^n + c_3 n^2 (1)^n$

$= c_1 + c_2 n + c_3 n^2$

Now for Perticular solution

$y_n^{(P)} = \alpha \cdot 3^n$, $y_{n+1}^{(P)} = \alpha \cdot 3^{n+1}$, $y_{n+2}^{(P)} = \alpha \cdot 3^{n+2}$

$y_{n+3}^{(P)} = \alpha \cdot 3^{n+3}$ So equ ① becomes

$\alpha \cdot 3^{n+3} - 3\alpha \cdot 3^{n+2} + 3\alpha \cdot 3^{n+1} - \alpha \cdot 3^n = 3^n$

$\Rightarrow 3^n [27\alpha - 27\alpha + 9\alpha - \alpha] = 3^n$

$\Rightarrow 8\alpha = 1 \Rightarrow \alpha = 1/8$

$\Rightarrow y_n^{(P)} = 1/8 \cdot 3^n$

So General solution is

$$y_n = C_1 + nC_2 + n^2C_3 + \frac{1}{8}3^n$$

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Q6:- $y_{n+2} - 3y_{n+1} + 2y_n = 7\cdot 2^n \longrightarrow (1)$

$\Rightarrow E^2y_n - 3Ey_n + 2y_n = 7\cdot 2^n$

$\Rightarrow (E^2 - 3E + 2)y_n = 7\cdot 2^n$

Characteristic equation is

$E^2 - 3E + 2 = 0 \quad \Rightarrow E^2 - 2E - E + 2 = 0$

$\Rightarrow E(E-2) - 1(E-2) = 0 \quad \Rightarrow (E-2)(E-1) = 0$

$\Rightarrow E = 1, 2$ are roots

$\Rightarrow$ C.F is

$$y_n^{(c)} = C_1(1)^n + C_2(2)^n$$

Now for particular solution

$y_n^{(P)} = \alpha\cdot 2^n\cdot n^K$, where $K=1$ $\because$ 2 is not root

$\Rightarrow y_n^{(P)} = \alpha\cdot 2^n\cdot n$, $y_{n+1}^{(P)} = \alpha\cdot 2^{n+1}(n+1)$

$y_{n+2}^{(P)} = \alpha\cdot 2^{n+2}(n+2)$

So equ (1) becomes

$(n+2)\alpha\cdot 2^{n+2} - 3(n+1)\cdot\alpha\cdot 2^{n+1} + 2\alpha\cdot 2^n\cdot n = 7\cdot 2^n$

$\Rightarrow n\alpha\cdot 2^{n+2} + 2\alpha\cdot 2^{n+2} - 3n\alpha\cdot 2^{n+1} + 2^n\alpha\cdot 2^n = 7\cdot 2^n$

$\Rightarrow \cancel{2^n}[4n\alpha + 8\alpha - 6n\alpha - 6\alpha + 2n\alpha] = 7\cdot\cancel{2^n}$

$\Rightarrow 2\alpha = 7 \quad \Rightarrow \alpha = 7/2$ So

$$y_n^{(P)} = \frac{7}{2}\cdot 2^n\cdot n \Rightarrow y_n^{(P)} = 7n\cdot 2^{n-1}$$

So General Solution is

$$y_n = y_n^{(c)} + y_n^{(P)}$$

Type III :-

Q1:- $\Delta^2 y_n - 7\Delta y_n - 60 y_n = 2^n + n^2 + 1$

As $\Delta = E - 1$ So

$(E-1)^2 y_n - 7(E-1)y_n - 60 y_n = 2^n + n^2 + 1$

$\Rightarrow (E^2 + 1 - 2E + 7 - 60) y_n = 2^n + n^2 + 1$

$\Rightarrow (E^2 - 9E - 52) y_n = 2^n + n^2 + 1$

characteristic equation is

$E^2 - 9E - 52 = 0 \quad \Rightarrow E^2 - 13E + 4E - 52 = 0$

$\Rightarrow E(E-13) + 4(E-13) = 0 \quad \Rightarrow (E+4)(E-13) = 0$

$\Rightarrow E = -4, 13$ are roots

So complementry function is

$y_n^{(c)} = c_1(-4)^n + c_2(13)^n$

For perticular solution we divide equ ① into two parts

$E^2 y_n - 9E y_n - 52 y_n = 2^n \longrightarrow (A)$

$E^2 y_n - 9E y_n - 52 y_n = n^2 + 1 \longrightarrow (B)$

First we solve equ (A)

Put $y_n^{(P)} = \alpha \cdot 2^n$, $y_{n+1}^{(P)} = \alpha \cdot 2^{n+1}$, $y_{n+2}^{(P)} = \alpha \cdot 2^{n+2}$

Put all in equation (A)

$\Rightarrow \alpha \cdot 2^{n+2} - 9\alpha 2^{n+1} - 52\alpha 2^n = 2^n$

$\Rightarrow \cancel{2^n}[4\alpha - 18\alpha - 52\alpha] = \cancel{2^n} \quad \Rightarrow -66\alpha = 1$

$\Rightarrow \alpha = -1/66$

So Perticular solution of equ (A) is

$y_n^{(P)} = \frac{-1}{66} \cdot 2^n$

Now we solve equ B. For Perticular sol

$y_n^{(P)} = [B_1 + B_2 n + B_3 n^2] \cdot n^K \quad , K=0 \quad \because 1$ is not root

$\Rightarrow y_n^{(P)} = B_1 + B_2 n + B_3 n^2 , \quad y_{n+1}^{P} = B_1 + B_2(n+1) + B_3(n+1)^2$

$y_{n+2}^{(P)} = B_1 + B_2(n+2) + B_3(n+2)^2$ Put in (B)

$\Rightarrow B_1 + B_2(n+2) + B_3(n^2+4+4n) - 9[B_1 + B_2 n + B_2 + B_3(n^2+1+2n)] - 52[B_1 + B_2 n + B_3 n^2] = n^2+1$

$\Rightarrow B_1 + B_2 n + 2B_2 + B_3 n^2 + 4B_3 + 4B_3 n - 9B_1 - 9B_2 n - 9B_2 - 9B_3 n^2 - 9B_3 - 18B_3 n - 52B_1 - 52B_2 n - 52B_3 n^2 = n^2+1$

$\Rightarrow [-52B_3 - 9B_3 + B_3]n^2 + [B_2 + 4B_3 - 9B_2 - 18B_3 - 52B_2]n + [B_1 + 2B_2 + 4B_3 - 9B_1 - 9B_2 - 9B_3 - 52B_1] = n^2+1$

Equating co-efficients of same terms

$-60B_3 = 1 \quad \Rightarrow B_3 = -1/60$

$-14B_3 - 60B_2 = 0 \quad \Rightarrow \left(-\frac{1}{60}\right)(-14) - 60B_2 = 0$

$\Rightarrow \frac{14}{60} = 60B_2 \quad \Rightarrow B_2 = \frac{7}{1800}$

$-60B_1 - 7B_2 - 5B_3 = 1$

$\Rightarrow -60[B_1] - 7\left[\frac{7}{1800}\right] - 5\left[\frac{-1}{60}\right] = 1$

$\Rightarrow -60B_1 - \frac{21}{1800} + 5/60 = 1$

$\Rightarrow -60B_1 = 1 + \frac{21}{1800} - 5/60 = \frac{1800+21-150}{1800}$

$\Rightarrow -60B_1 = \cancel{1 + \frac{21}{1800} - \frac{5}{60}} \; \frac{1671}{1800} \Rightarrow B_1 = \frac{-1671}{60 \times 1800}$

So $y_n^{(P)} = \frac{1}{60}\left[\frac{-1671}{1800} + \frac{7n}{30} - n^2\right]$

So General solution is $y_n = y_n^{(C)} + y_n^{(P)}$

$$y_n = C_1(-4)^n + C_2(13)^n - \frac{1}{66}2^n - \frac{1}{60}\left[n^2 - \frac{7n}{30} + \frac{1671}{1800}\right]$$

★★

Q2:- $y_{n+2} - 7y_{n+1} + 12y_n = 12n + 8$

$\Rightarrow E^2y_n - 7Ey_n + 12y_n = 12n + 8$

$\Rightarrow [E^2 - 7E + 12]y_n = 12n + 8$

Characteristic equation is

$E^2 - 7E + 12 = 0 \quad \Rightarrow E^2 - 3E - 4E + 12 = 0$

$\Rightarrow E(E-3) - 4(E-3) = 0 \quad \Rightarrow (E-3)(E-4) = 0$

$\Rightarrow E = 3, 4$ are roots

So Complementry function is

$y_n^{(C)} = C_1(3)^n + C_2(4)^n$

Now for Particular solution Put

$y_n^{(P)} = (B_0 + B_1 n)\cdot n^K$, where $K = 0$ as 1 is not root

$\Rightarrow y_n^{(P)} = B_0 + B_1 n$, $y_{n+1}^{(P)} = B_0 + B_1(n+1)$

$y_{n+2}^{(P)} = B_0 + B_1(n+2)$ Put in given eqn.

$\Rightarrow B_0 + B_1 n + 2B_1 - 7B_0 - 7B_1 n - 7B_1 + 12B_0 + 12B_1 n = 12n + 8$

$\Rightarrow (B_1 - 7B_1 + 12B_1)n + B_0 + 2B_1 - 7B_0 - 7B_1 + 12B_0 = 12n + 8$

Equating coefficients of same terms

$\Rightarrow 6B_1 = 12 \quad \Rightarrow B_1 = 2$

& $6B_0 - 5B_1 = 8 \quad \Rightarrow 6B_0 - 5(2) = 8$

$\Rightarrow 6B_0 = 18 \quad \Rightarrow B_0 = 3$

So $y_n^{(P)} = 3+2n$

And general solution is $y_n = y_n^{(c)} + y_n^{(P)}$

$\Rightarrow y_n = c_1(3)^n + c_2(4)^n + 2n + 3$

★————————★

Q3:- $\Delta^4 y_n = n$

As $\Delta = E - 1$

$\Rightarrow (E-1)^4 y_n = n$ $\Rightarrow (E-1)^2(E-1)^2 y_n = n$

$\Rightarrow (E^2+1-2E)(E^2+1-2E)y_n = n$

$\Rightarrow (E^4+E^2-2E^3+E^2+1-2E-2E^3-2E+4E^2)y_n = n$

$\Rightarrow (E^4-4E^3+6E^2-4E+1)y_n = n$

$\Rightarrow y_{n+4} - 4y_{n+3} + 6y_{n+2} - 4y_{n+1} + y_n = n \longrightarrow (1)$

Characteristic equation is

$(E-1)^4 = 0 \Rightarrow (E-1)(E-1)(E-1)(E-1) = 0$

$\Rightarrow E = 1, 1, 1, 1$ are roots

So complementry function is

$y_n^{(c)} = c_1(1)^n + c_2 n(1)^n + c_3 n^2(1)^n + c_4 n^3(1)^n$

$= c_1 + c_2 n + c_3 n^2 + c_4 n^3$

Now for perticular solution

$y_n^{(P)} = (B_0 + B_1 n)\cdot n^k$, where $k=4$ $\because 1$ is four time root

$\Rightarrow y_n^{(P)} = (B_0 + B_1 n)n^4$, $y_{n+1}^{(P)} = [B_0 + B_1(n+1)](n+1)^4$

$y_{n+2}^{(P)} = [B_0 + B_1(n+2)](n+2)^4$, $y_{n+3}^{(P)} = [B_0 + B_1(n+3)](n+3)^4$

$y_{n+4}^{(P)} = [B_0 + B_1(n+4)](n+4)^4$

Put all in equ (1)

$\Rightarrow (B_0+B_1 n+4B_1)(n+4)^4 - (4B_0+4B_1 n+12B_1)(n+3)^4 +$
$(6B_0+6B_1 n+12B_1)(n+2)^4 - (4B_0+4B_1 n+4B_1)(n+1)^4$
$+(B_0+B_1 n) n^4 = n$

$\Rightarrow (B_0+B_1+4B_1)(n^4+16n^3+96n^2+256n+256) - (4B_0+4B_1 n+$
$12B_1)(n^4+12n^3+54n^2+108n+81) + (16B_0+6B_1 n+12B_1)$
$(n^4+24n^2+32n-16) - (4B_0+4B_1 n+4B_1)(n^4+4n^3+6n^2$
$+4n+1) + (B_0+B_1 n) n^4 = n$

After multiplying & equating co-effi-

$$y_n^{(P)} = \frac{-1}{12} n^4 + \frac{1}{120} n^5$$

So General Solution $= y_n = y_n^{(c)} + y_n^{(P)}$

$$\Rightarrow y_n = C_1 + C_2 n + C_3 n^2 + C_4 n^3 - \frac{1}{12} n^4 + \frac{1}{120} n^5$$

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Q4:- $y_{K+2} - 2y_{K+1} + y_K = K+1 \longrightarrow (1)$

$\Rightarrow E^2 y_K - 2E y_K + y_K = K+1$

$\Rightarrow (E^2 - 2E + 1) y_K = K+1$

Characteristic equation is

$(E^2 - 2E + 1) = 0 \quad \Rightarrow E^2 - E - E + 1 = 0$

$\Rightarrow E(E-1) - 1(E-1) = 0 \quad \Rightarrow (E-1)(E-1) = 0$

$\Rightarrow E = 1, 1$ are roots

So complementry function is

$$y_K^{(c)} = C_1(1)^K + C_2 K(1)^K$$

$$= C_1 + C_2 K$$

Now for perticular solution Put

$y_K^{(P)} = (B_0 + B_1 K) \cdot K^n, \quad n = 2 \because 1$ is double root

$\Rightarrow y_k^{(p)} = (B_0 + B_1 k)k^2$, $y_{k+1}^{(p)} = [B_0 + B_1(k+1)](k+1)^2$

$y_{k+2}^{(p)} = [B_0 + B_1(k+2)](k+2)^2$ Put in equ ①

$\Rightarrow (B_0 + B_1 k + 2B_1)(k^2+4+4k) - 2[(B_0+B_1k+B_1)(k^2+1+2k)]$
$+ B_0k^2 + B_1k^3 = k+1$

$\Rightarrow B_0k^2+4B_0+4B_0k+B_1k^3+4B_1k+4B_1k^2+2B_1k^2+8B_1+8B_1k$
$-2B_0k^2-2B_0-4B_0k-2B_1k^3-2B_1k-4B_1k^2-2B_1k^2$
$-2B_1-4B_1k+B_0k^2+B_1k^3 = k+1$

$\Rightarrow (B_1-2B_1+B_1)k^3+(B_0+4B_1+2B_1-2B_0-4B_1-2B_1+B_0)k^2$
$+(4B_0+4B_1+8B_1-4B_0-2B_1-4B_1)k+(4B_0+8B_1-2B_0-$
$2B_1) = k+1$

Equating same terms

$6B_1 = 1 \Rightarrow B_1 = 1/6$

$2B_0 - 6B_1 = 1 \Rightarrow 2B_0 - 6(1/6) = 1$

$2B_0 = 1-1 \Rightarrow B_0 = 0$

So $y_k^{(p)} = (\frac{1}{6}k)k^2 = \frac{1}{6}k^3$

So G. S is

$y_n = c_1 + c_2 k + \frac{1}{6}k^3$

★★ ——— ★★

Q5:- $y_{n+2} - 5y_{n+1} + 6y_n = 2n+1+2^n \longrightarrow ①$

$\Rightarrow E^2 y_n - 5E y_n + 6 y_n = 2n+1+2^n$

$\Rightarrow (E^2 - 5E + 6) y_n = 2n+1+2^n$

Characteristic equ is

$E^2 - 5E + 6 = 0 \Rightarrow E^2 - 2E - 3E - 6 = 0$

$\Rightarrow E(E-2) - 3(E-2) = 0$

$\Rightarrow (E-2)(E-3) = 0 \qquad \Rightarrow E = 2, 3$

So complementry function is

$$y_n^{(c)} = C_1(2)^n + C_2(3)^n$$

Now for perticular solution divide equ ① into two parts

$$\Rightarrow y_{n+2} - 5y_{n+1} + 6y_n = 2n+1 \longrightarrow Ⓐ$$

$$y_{n+2} - 5y_{n+1} + 6y_n = 2^n \longrightarrow Ⓑ$$

First we solve equ Ⓐ

For P.S of A

Put $y_n^{(P)} = (B_0 + B_1 n)n^K, \; K = 0$

$\Rightarrow y_n^{(P)} = B_0 + B_1 n \;, \; y_{n+1}^{(P)} = B_0 + B_1(n+1)$

$y_{n+2}^{(P)} = B_0 + B_1(n+2)$ Put in Ⓐ

$$B_0 + B_1 n + 2B_1 - 5B_0 - 5B_1 n - 5B_1 + 6B_0 + 6B_1 n = 2n+1$$

$$\Rightarrow (B_1 - 5B_1 + 6B_1)n + B_0 + 2B_1 - 5B_0 - 5B_1 + 6B_0 = 2n+1$$

Equating coefficients of n & constant

$\Rightarrow 2B_1 = 2 \qquad \Rightarrow B_1 = 1$

& $2B_0 - 3B_1 = 1 \qquad \Rightarrow 2B_0 - 3 = 1 \qquad \Rightarrow 2B_0 = 4$

$\Rightarrow B_0 = 2$

So Perticular solution of equ Ⓐ is

$$y_n^{(P)} = 2 + n$$

Now for perticular solution of equ Ⓑ

Put $y_n^{(P)} = \alpha \cdot 2^n \cdot n^K$, where $K = 1$

$\Rightarrow y_n^{(P)} = \alpha \cdot 2^n \cdot n \;, \; y_{n+1}^{(P)} = \alpha \cdot 2^{n+1}(n+1)$

$y_{n+2}^{(P)} = \alpha \cdot 2^{n+2}(n+2)$

Put all in B

$$\Rightarrow n\alpha 2^{n+2} + 2\alpha \cdot 2^{n+2} - 5n\alpha \cdot 2^{n+1} - 5\alpha 2^{n+1} + 6n\alpha 2^n = 2^n$$

$$\Rightarrow 2^n[4n\alpha + 8\alpha - 10n\alpha - 10\alpha + 6n\alpha] = 2^n$$

$$\Rightarrow -2\alpha = 1 \qquad \Rightarrow \alpha = -1/2$$

$$\Rightarrow y_n^{(P)} = -\frac{1}{2} \cdot 2^n \cdot n \qquad \Rightarrow y_n^{(P)} = -n2^{n-1}$$

So for Perticular solution of equ ①

$= P.S$ of Ⓐ $+ P.S$ of Ⓑ

$$= -n2^{n-1} + 2 + n$$

& General solution is $y_n = y_n^{(c)} + y_n^{(P)}$

$$\Rightarrow y_n = c_1(2)^n + c_2(3)^n - n2^{n-1} + 2 + n$$

★———★★★———★

Q6:- $\Delta^2 y_K + 3\Delta y_K + 3y_K = K^2 + 1$

$$\Rightarrow (E-1)^2 y_K + 3(E-1)y_K + 3y_K = K^2 + 1$$

$$\Rightarrow (E^2 + 1 - 2E)y_K + (3E-3)y_K + 3y_K = K^2 + 1$$

$$\Rightarrow (E^2 + 1 - 2E + 3E - \cancel{3} + \cancel{3})y_K = K^2 + 1$$

$$\Rightarrow (E^2 + E + 1)y_K = K^2 + 1$$

$$\Rightarrow y_{K+2} + y_{K+1} + y_K = K^2 + 1 \longrightarrow ①$$

Characteristic equation is

$$E^2 + E + 1 = 0$$

$$\Rightarrow E = \frac{-1 \pm \sqrt{1-4}}{2} \qquad \Rightarrow E = \frac{-1 \pm \sqrt{3}\, i}{2}$$

So Complementry function is

$$y_K^{(c)} = R^K[A_1 \cos K\theta + A_2 \sin K\theta]$$

where $R = \sqrt{(-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{4}{4}}$

$\Rightarrow R = 1$

& $\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right) \Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\Rightarrow \theta = \pi/3$

$\Rightarrow y_K^{(c)} = A_1 \cos\frac{K\pi}{3} + A_2 \sin\frac{K\pi}{3}$

Now for Perticular Solution

$y_K^{(P)} = (B_0 + B_1K + B_2K^2)K^n$ where $n=0$

$\Rightarrow y_K^{(P)} = B_0 + B_1K + B_2K^2$, $y_{K+1}^{(P)} = B_0 + B_1(K+1) + B_2(K+1)^2$

$y_{K+2}^{(P)} = B_0 + B_1(K+2) + B_2(K+2)^2$ Put in ①

$\Rightarrow B_0 + B_1K + 2B_1 + B_2K^2 + 4B_2 + 4B_2K + B_0 + B_1K + B_1 + B_2K^2 + B_2 + 2B_2K + B_0 + B_1K + B_2K^2 = K^2 + 1$

$\Rightarrow (B_2 + B_2 + B_2)K^2 + (B_1 + 4B_2 + B_1 + 2B_2 + B_1)K + B_0 + 2B_1 + 4B_2 + B_0 + B_1 + B_2 + B_0 = K^2 + 1$

Equating same terms.

$3B_2 = 1 \Rightarrow B_2 = 1/3$

& $3B_1 + 6B_2 = 0 \Rightarrow 3B_1 + 6(1/3) = 0$

$\Rightarrow 3B_1 = -2 \Rightarrow B_1 = -2/3$

& $3B_0 + 3B_1 + 5B_2 = 1 \Rightarrow 3B_0 + 3(-2/3) + 5(1/3) = 1$

$\Rightarrow 3B_0 - 2 + 5/3 = 1 \Rightarrow B_0 = 4/9$

So $y_n^{(P)} = \frac{4}{9} - \frac{2}{3}K + \frac{1}{3}K^2$

And $G.S = y_n^{(c)} + y_n^{(P)}$

Type IV :-

Q1:- $y_{n+2} - 9y_{n+1} + 20y_n = 3^n(n^2+1) \longrightarrow (1)$

$\Rightarrow E^2 y_n - 9E y_n + 20y_n = 3^n(n^2+1)$

$\Rightarrow (E^2 - 9E + 20) y_n = 3^n(n^2+1)$

Characteristic equation is

$E^2 - 9E + 20 = 0 \quad \Rightarrow E^2 - 5E - 4E + 20 = 0$

$\Rightarrow E(E-5) - 4(E-5) = 0 \quad \Rightarrow (E-4)(E-5) = 0$

$\Rightarrow E = 4, 5$

So Complementry function is

$y_n^{(c)} = C_1(4)^n + C_2(5)^n$

Now for perticular solution put

$y_n^{(P)} = 3^n(B_0 + B_1 n + B_2 n^2) \cdot n^k$, $K = 0 \because 3$ is not root

$\Rightarrow y_n^{(P)} = 3^n(B_0 + B_1 n + B_2 n^2)$

$y_{n+1}^{(P)} = 3^{n+1}(B_0 + B_1(n+1) + B_2(n+1)^2)$

$y_{n+2}^{(P)} = 3^{n+2}(B_0 + B_1(n+2) + B_2(n+2)^2)$

So equation (1) becomes

$3^{n+2}[B_0 + B_1 n + 2B_1 + B_2 n^2 + 4B_2 + 4B_2 n] - 3^{n+1}[B_0 + B_1 n + B_1 + B_2 n^2 + B_2 + 2B_2 n] + 20 \cdot 3^n[B_0 + B_1 n + B_2 n^2] = 3^n(n^2+1)$

$\Rightarrow 3^n[9B_0 + 9B_1 n + 18B_1 + 9B_2 n^2 + 36B_2 + 36B_2 n - 27B_0 - 27B_1 n - 27B_1 - 27B_2 n^2 - 27B_2 - 54B_2 n + 20B_0 + 20B_1 n + 20B_2 n^2] = (n^2+1)3^n$

$\Rightarrow (9B_2 - 27B_2 + 20B_2)n^2 + (9B_1 + 36B_2 - 27B_1 - 54B_2 + 20B_1)n + (9B_0 + 18B_1 - 27B_0 - 27B_1 + 36B_2 - 27B_2 + 20B_0) = n^2 + 1$

Equating co-efficients of n^2 and 1

$\Rightarrow 2B_2 = 1 \quad \Rightarrow \quad B_2 = 1/2$

& $2B_1 - 18B_2 = 0 \quad \Rightarrow 2B_1 - 18(1/2) = 0$

$\Rightarrow 2B_1 = 9 \quad \Rightarrow \quad B_1 = 9/2$

& $2B_0 - 9B_1 + 9B_2 = 1$

$\Rightarrow 2B_0 - 9(9/2) + 9(1/2) = 1$

$\Rightarrow 2B_0 = 1 - 9/2 + \frac{81}{2} \quad \Rightarrow \quad B_0 = 37/2$

So $y_n^{(P)} = \left(\frac{37}{2} + \frac{9}{2}n + \frac{1}{2}n^2\right)3^n$

And $y_n = C_1(4)^n + C_2(5)^n + \left[\frac{37}{2} + \frac{9}{2}n + \frac{1}{2}n^2\right] \cdot 3^n$

Q2:- $y_{k+2} - 2y_{k+1} + y_k = k2^k - 2^k$

$\Rightarrow y_{k+2} - 2y_{k+1} + y_k = 2^k[k-1]$

$\Rightarrow E^2 y_k - 2Ey_k + y_k = 2^k[k-1]$

Characteristic equation is

$(E^2 - 2E + 1) = 0 \quad \Rightarrow E^2 - E - E + 1 = 0$

$\Rightarrow E(E-1) - 1(E-1) = 0 \quad \Rightarrow (E-1)(E-1) = 0$

$\Rightarrow E = 1, 1$ are roots

$\Rightarrow y_k^{(c)} = C_1 + C_2 k$

Now for Particular solution Put

$y_k^{(P)} = 2^k(B_0 + B_1 k) \cdot k^n$, where $n = 2$

$\Rightarrow y_k^{(P)} = 2^k(B_0 + B_1 k)k^2$

$$y^{(P)}_{k+1} = 2^{k+1}[B_0 + B_1(k+1)](k+1)^2$$

$$y^{(P)}_{k+2} = 2^{k+2}[B_0 + B_1(k+1)](k+2)^2$$

Put all in equ (1)

$$2^{k+2}[B_0 + B_1k + B_1](k^2+4+4k) - 2\cdot 2^{k+1}[B_0 + B_1k + B_1](k^2+1+2k) + 2^k(B_0 + B_1k)k^2 = 2^k(k-1)$$

$$\Rightarrow 2^{k+2}[B_0k^2 + 4B_0 + 4B_0k + B_1k^3 + 4B_1k + 4B_1k^2 + B_1k^2 + 4B_1 + 4B_1k] - 2^{k+2}[B_0k^2 + B_0 + 2B_0k + B_1k^3 + B_1k + B_1k^2 + B_1k^2 + B_1 + 2B_1k] + 2^k[B_0k + B_1k^3] = 2^k(k-1)$$

$$\Rightarrow 2^k[4B_0k^2 + 16B_0 + 16B_0k + 4B_1k^3 + 16B_1k + 16B_1k^2 + 4B_1k^2 + 16B_1 + 16B_1k - 4B_0k^2 - 4B_0 - 8B_0k - 4B_1k^3 - 4B_1k - 4B_1k^2 - 4B_1k^2 - 4B_1 - 8B_1k + B_0k + B_1k^3] = 2^k(k-1)$$

$$\Rightarrow (4B_1 - 4B_1 + B_1)k^3 + (4B_0 + 16B_1 + 4B_1 - 4B_0 - 4B_1 - 4B_1)k^2 + (16B_0 + 16B_1 + 16B_1 - 8B_0 - 4B_1 - 8B_1 + B_0)k + 16B_0 + 16B_1 - 4B_0 - 4B_1 = k-1$$

By equating co-efficients

$$B_0 = -1 \quad \& \quad B_1 = 1/6$$

So $y^{(P)}_k = (-1 + 1/6\, k)k^2$

So General Solution is

$$y_n = C_1 + C_2k + \left[\frac{1}{6}k^3 - k^2\right]$$

***

Q3:- $U_{n+3} + 8U_n = (2n+3)2^n \longrightarrow (1)$

$\Rightarrow E^3 U_n + 8U_n = (2n+3)2^n$

$\Rightarrow (E^3+8)U_n = (2n+3)2^n$

Characteristic equation is

$E^3 + 8 = 0$

$E = -2$ is root. So by Synthetic division

$$\begin{array}{c|cccc} -2 & 1 & 0 & 0 & 8 \\ & & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & \boxed{0} \end{array}$$

$\Rightarrow (E+2)(E^2 - 2E + 4) = 0$

$\Rightarrow E = -2 \, , \quad E = \dfrac{2 \pm \sqrt{4-16}}{2}$

$= \dfrac{2 \pm 2\sqrt{3}\,i}{2}$

$\Rightarrow E = -2, \; 1 \pm \sqrt{3}\,i$

So complementry function is

$U_n^{(c)} = c_1(-2)^n + R^n[A_1 \cos n\theta + A_2 \sin n\theta]$

where $R = \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$

& $\theta = \tan^{-1} \sqrt{3}/1 \quad \Rightarrow \theta = \pi/3$

So C.F is

$U_n^{(c)} = c_1(-2)^n + 2^n\left[A_1 \cos \dfrac{n\pi}{3} + A_2 \sin \dfrac{n\pi}{3}\right]$

Now for Perticular Solution.

$$U_n^{(P)} = (B_0 + B_1 n) 2^n$$

$$U_{n+3}^{(P)} = [B_0 + B_1(n+3)] 2^{n+3} \quad \text{Put in } ①$$

$$B_0 2^{n+3} + B_1 n \cdot 2^{n+3} + 3B_1 2^{n+3} + 8B_0 2^n + 8B_1 n 2^n = 2^n[2n+3]$$

$$\Rightarrow \cancel{2^n}[8B_0 + 8B_1 n + 24B_1 + 8B_0 + 8B_1 n] = \cancel{2^n}[2n+3]$$

$$\Rightarrow (8B_1 + 8B_1)n + 8B_0 + 24B_1 + 8B_0 = 2n + 3$$

$$\Rightarrow 16B_1 = 2n \qquad \Rightarrow B_1 = 1/8$$

$$\& \ 16B_0 + 24B_1 = 3 \qquad \Rightarrow 16B_0 + 24(1/8) = 3$$

$$16B_0 + \cancel{3} = \cancel{3} \qquad \Rightarrow B_0 = 0$$

$$y_n^{(P)} = \left(\frac{1}{8}n\right) 2^n$$

$$\Rightarrow y_n = y_n^{(C)} + y_n^{(P)}$$

**

Q4:- $y_{k+2} - 13y_{k+1} + 36y_k = 2^k(k^2+1) \longrightarrow ①$

$$\Rightarrow E^2 y_k - 13E y_k + 36y_k = 2^k(k^2+1)$$

$$\Rightarrow (E^2 - 13E + 36) y_k = 2^k(k^2+1)$$

Characteristic equ is

$$E^2 - 13E + 36 = 0 \qquad \Rightarrow E^2 - 9E - 4E + 36 = 0$$

$$\Rightarrow E(E-9) - 4(E-9) = 0$$

$$\Rightarrow (E-9)(E-4) = 0$$

$$\Rightarrow E = 4, 9 \text{ are roots}$$

So complementary function is

$$y_k^{(c)} = c_1(4)^k + c_2(9)^k$$

Now for perticular solution

$$y_k^{(p)} = [B_0 + B_1 k + B_2 k^2]\, 2^k$$

$$y_{k+1}^{(p)} = [B_0 + B_1(k+1) + B_2(k+1)^2]\, 2^{k+1}$$

$$y_{k+2}^{(p)} = [B_0 + B_1(k+2) + B_2(k+2)^2]\, 2^{k+2}$$

Put all in eqn ①

$$[B_0 + B_1 k + 2B_1 + B_2(k^2+4+4k)]\,2^{k+2} - 13[B_0 + B_1 k + B_1 + B_2(k^2+1+2k)]\,2^{k+1} + 36[B_0 + B_1 k + B_2 k^2]\cdot 2^k = 2^k(k^2+1)$$

$$\Rightarrow 2^{k+2}[B_0 + B_1 k + 2B_1 + B_2 k^2 + 4B_2 + 4B_2 k] - 13\cdot 2^{k+1}[B_0 + B_1 k + B_1 + B_2 k^2 + B_2 + 2B_2 k] + 36\, 2^k[B_0 + B_1 k + B_2 k^2] = 2^k(k^2+1)$$

$$\Rightarrow \cancel{2^k}[4B_0 + 4B_1 k + 8B_1 + 4B_2 k^2 + 16B_2 + 16B_2 k - 26B_0 - 26B_1 k - 26B_1 - 26B_2 k^2 - 26B_2 - 52B_2 k + 36B_0 + 36B_1 k + 36B_2 k^2 = \cancel{2^k}(k^2+1)$$

$$\Rightarrow [4B_2 - 26B_2 + 36B_2]k^2 + [4B_1 + 16B_2 - 26B_1 - 52B_2 + 36B_1]k + [4B_0 + 8B_1 + 16B_2 - 26B_0 - 26B_1 - 26B_2 + 36B_0] = k^2 + 1$$

Equating coefficients of same terms.

$14B_2 = 1 \Rightarrow B_2 = 1/14$

& $14B_1 - 36B_2 = 0 \Rightarrow 14B_1 - 36\left(\frac{1}{14}\right) = 0$

$\Rightarrow 14B_1 = \frac{36}{14} \Rightarrow B_1 = 9/49$

& $14B_0 - 18B_1 - 10B_2 = 1$

$\Rightarrow 14B_0 - 18\left[\frac{9}{49}\right] - 10\left[\frac{1}{14}\right] = 0$

$\Rightarrow 14B_0 = \frac{197}{49} \Rightarrow B_0 = \frac{197}{686}$

So $y_k^{(P)} = \left[\frac{197}{686} + \frac{9}{49}k + \frac{1}{14}k^2\right] \cdot 2^k$

So General solution is

$y_n = C_1(4)^k + C_2(9)^k + \left[\frac{197}{686} + \frac{9}{49}k + \frac{1}{14}k^2\right] \cdot 2^k$

Q5:- $y_{k+2} + 6y_{k+1} + 25y_k = 2^k + k + 4$

$\Rightarrow E^2 y_k + 6E y_k + 25 y_k = 2^k + k + 4$

$\Rightarrow (E^2 + 6E + 25) y_k = 2^k + k + 4$

Ch. equation is

$E^2 + 6E + 25 = 0$

$\Rightarrow E = \frac{-6 \pm \sqrt{36 - 100}}{2} = \frac{-6 \pm \sqrt{-64}}{2}$

$\Rightarrow E = \frac{-6 \pm 8i}{2} \Rightarrow E = -3 \pm 4i$

So $y_k^{(c)} = R^k \left[A_1 \cos k\theta + A_2 \sin k\theta\right]$

where $R = \sqrt{(3)^2 + (4)^2} = \sqrt{9+16} = 5$

& $\theta = \tan^{-1}\left(\frac{4}{3}\right)$

So $y_k^{(c)} = 5^n\left[A_1 \cos k \tan^{-1}\left(\frac{4}{3}\right) + A_2 \sin k \tan^{-1}\left(\frac{4}{3}\right)\right]$

Now for Perticular Solution

Divide given equation into ② parts

$y_{k+2} + 6y_{k+1} + 2y_k = 2^k \longrightarrow$ Ⓐ

$y_{k+2} + 6y_{k+1} + 2y_k = k+4 \longrightarrow$ Ⓑ

First we find perticular solution of Ⓐ

$\Rightarrow y_k^{(p)} = \alpha 2^k, \; y_{k+1} = \alpha 2^{k+1}, \; y_{k+2} = \alpha \cdot 2^{k+2}$

Put all in Ⓐ

$\Rightarrow \alpha 2^{k+2} + 6\alpha 2^{k+1} + 2\alpha 2^k = 2^k$

$\Rightarrow 2^k[4\alpha + 12\alpha + 2\alpha] = 2^k \Rightarrow 18\alpha = 1$

$\Rightarrow \alpha = 1/18$

So $y_k^{(p)} = \frac{1}{18} 2^k$

Now for B put

$y_k^{(p)} = B_0 + B_1 k, \; y_{k+1}^{(p)} = B_0 + B_1(k+1)$

$y_{k+2}^{(p)} = B_0 + B_1(k+2)$ Put in Ⓑ

$B_0 + B_1 k + 2B_1 + 6B_0 + 6B_1 k + 6B_1 + 2B_0 + 2B_1 k = k+4$

$\Rightarrow (B_1 + 6B_1 + 2B_1)k + B_0 + 2B_1 + 6B_1 + 2B_0 = k+4$

Equating coefficients

$\Rightarrow 9B_1 = 1 \quad \Rightarrow B_1 = 1/9$

$9B_0 + 8B_1 = 4 \quad \Rightarrow 9B_0 + 8\left(\frac{1}{9}\right) = 4$

$\Rightarrow 9B_0 = 4 - \frac{8}{9} \quad \Rightarrow B_0 = \frac{28}{81}$

So $y_k^{(P)} = \frac{28}{81} + \frac{1}{9}k$

So Perticular Solution of given eqn is

$y_n^{(P)} = \frac{28}{81} + \frac{1}{9}k + \frac{1}{8}2^k$

And General Solution is

$y_k = y_k^{(c)} + y_k^{(P)}$

***

Q6:- $(E^2 - 6E + 9)y_k = 3^k \cdot k^2$

$\Rightarrow E^2 y_k - 6E y_k + 9y_k = 3^k \cdot k^2$

$\Rightarrow y_{k+2} - 6y_{k+1} + 9y_k = 3^k \cdot k^2 \longrightarrow (1)$

Characteristic equation is

$E^2 - 6E + 9 = 0 \quad \Rightarrow E^2 - 3E - 3E + 9 = 0$

$\Rightarrow E(E-3) - 3(E-3) = 0 \Rightarrow (E-3)(E-3) = 0$

$\Rightarrow E = 3, 3$ are roots

So Complementary Solution is

$y_k^{(c)} = c_1(3)^k + c_2 k(3)^k$

Now for Perticular Solution

$$y_K^{(P)} = (B_0 + B_1 K + B_2 K^2)\, 3^K \cdot K^n \quad , n = 2$$

So $$y_K^{(P)} = (B_0 + B_1 K + B_2 K^2)\, 3^K \cdot K^2$$

$$y_{K+1}^{(P)} = [B_0 + B_1(K+1) + B_2(K+1)^2]\, 3^{K+1} (K+1)^2$$

$$y_{K+2}^{(P)} = [B_0 + B_1(K+2) + B_2(K+2)^2]\, 3^{K+2} (K+2)^2$$

Put all in equ (1)

$$\Rightarrow [B_0 + B_1 K + 2B_1 + B_2 K^2 + 4B_2 + 4B_2 K](K^2 + 4 + 4K) \cdot 3^{K+2}$$
$$- 6\, 3^{K+1} [B_0 + B_1 K + B_1 + B_2 K^2 + B_2 + 2B_2 K](K^2 + 1 + 2K)$$
$$+ 9\, 3^K [B_0 + B_1 K + B_2 K^2] K^2 = 3^K \cdot K^2$$

After Simplyfing & equating

$$y_K^{(P)} = 3^K \cdot K^2 \left[\frac{1}{108} - \frac{1}{27} K + \frac{1}{108} K^2\right]$$

So General Solution is $y_K = y_K^{(c)} + y_K^{(P)}$

$$\Rightarrow y_K = (C_1 + C_2 K) 3^K + 3^K \cdot K^2 \left[\frac{1}{108} - \frac{1}{27} K + \frac{1}{108} K^2\right]$$

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⇒ Simultaneous Linear Difference Equations.

If two or more difference equations are given with same number of unknown functions, then we can solve such equations simultaneously by using a procedure which eliminates all but one of the unknown.

★★★

Example:- Solve the system

$$U_{n+1} + U_n + 3V_n = 7$$

$$3V_{n+1} + V_n - 2U_n = 6$$

Solution Given $U_{n+1} - U_n + 3V_n = 7$

$$3V_{n+1} - V_n - 2U_n = 6$$

In operator notation, the above equations can be written as

$$EU_n - U_n + 3V_n = 7$$

$$\Rightarrow (E-1)U_n + 3V_n = 7 \longrightarrow \text{(i)}$$

&

$$3V_{n+1} + V_n - 2U_n = 6$$

$$3EV_n + V_n - 2U_n = 6$$

$$\Rightarrow (3E+1)V_n - 2U_n = 6 \longrightarrow \text{(ii)}$$

Multiplying equ (i) by $(3E+1)$ and (ii) by 3 then subtract them we get

$$(3E+1)(E-1)U_n + 3(3E+1)V_n = 7(3E+1)$$

$$\underset{+}{-}6U_n \underset{-}{+} 3(3E+1)V_n = \underset{-}{+}18$$

$$(3E^2-2E-1)U_n+6U_n=21E+7-18$$

$$(3E^2-2E+5)U_n = 21+7-18$$

$$\Rightarrow (3E^2-2E+5)U_n = 10$$

$$\Rightarrow 3U_{n+1}-2U_{n+1}+5U_n=10 \longrightarrow ③$$

which is a difference equation (Non-homogeneous) to solve this we find C.S & P.S of ③

For complementry solution Ch. equ is

$$3E^2-2E+5=0$$

$$\Rightarrow E = \frac{2\pm\sqrt{4-4(3)(5)}}{6}$$

$$= \frac{2\pm\sqrt{56}}{6} = \frac{1\pm\sqrt{14}\,i}{3}$$

So C.F is

$$U_n = R^n[A_1\cos n\theta + A_2\sin n\theta]$$

$$R = \sqrt{\frac{1}{9}+\frac{14}{9}} = \sqrt{\frac{15}{9}} = \frac{\sqrt{15}}{3}$$

$$\theta = \tan^{-1}\sqrt{14}$$

$$\Rightarrow U_n = \left(\frac{\sqrt{15}}{3}\right)^n[A_1\cos n\theta + A_2\sin n\theta] \quad \because \theta\tan^{-1}(\sqrt{14})$$

To find Perticular solution of ③ Put

$$U_n^* = c$$

③ $3c-2c+5c=10 \Rightarrow 6c=10$

$$\Rightarrow c=5/3 \Rightarrow U_n^* = 5/3$$

General Solution $= U_n + U_n^*$

$$= \left(\frac{\sqrt{15}}{3}\right)^n \left[A_1 \cos n\theta + A_2 \sin n\theta\right] + 5/3$$

where $\theta = \tan^{-1}\sqrt{14}$

which is solution of U_n

To find V_n Put value of U_n in ①

$$3V_n = 7 - (E-1)U_n$$

$$= 7 - U_{n+1} + U_n$$

$$= \frac{7}{3} - \frac{1}{3}\left(\frac{\sqrt{15}}{3}\right)^{n+1}\left[A_1 \cos(n+1)\theta + A_2 \sin(n+1)\theta\right.$$

$$\left. + \left(\frac{\sqrt{15}}{3}\right)^n \left[A_1 \cos(n)\theta + A_2 \sin(n)\theta\right] + 5/3\right.$$

which is required solution of V_n.

★ ——————————— ★

Example:- Solve the system of equation

$$U_{n+1} + V_n - 3U_n = n$$
$$3U_n + V_{n+1} - 5V_n = 4^n$$

Solution Given $U_{n+1} + V_n - 3U_n = n \longrightarrow$ (i)

$V_{n+1} - 5V_n + 3U_n = 4^n \longrightarrow$ (ii)

In operator notation. above equation (i) and (ii) becomes

$$EU_n + V_n - 3U_n = n$$
$$EV_n - 5V_n + 3U_n = 4^n$$

$\Rightarrow$ $(E-3)U_n + V_n = n \longrightarrow$ (iii)

$(E-5)V_n + 3U_n = 4^n \longrightarrow$ (iv)

Multiplying (iii) by (E−5) then subtract from (iv)

$$(E-5)(E-3)u_n+\cancel{(E-5)v_n}=(E-5)n$$

$$\pm 3u_n \pm \cancel{(E-5)v_n} = \pm 4^n$$

$$(E^2-8E+15)u_n-3u_n=-5n+En-4^n$$

$$(E^2-8E+12)u_n = n+1-5n-4^n \quad (\because En=n+1)$$

$$(E^2-8E+12)u_n = 1-4n-4^n \longrightarrow (V)$$

Characteristic equation is

$$E^2-8E+12=0$$

$$E^2-2E-6E+12=0$$

$$E(E-2)-6(E-2)=0$$

$$(E-2)(E-6)=0$$

$$\Rightarrow E=2,6$$

So GF is

$$u_n = C_1(2)^n + C_2(6)^n \longrightarrow (\star)$$

Equ (V) be write it as

$$u_{n+2}-8u_{n+1}+12u_n = 1-4n-4^n$$

For perticular solution consider (1) equ.

$$u_{n+2}-8u_{n+1}+12u_n = 1-4n \longrightarrow (A)$$

$$u_{n+2}-8u_{n+1}+12u_n = -4^n \longrightarrow (B)$$

Now Consider (A) for Perticular solution

Put $u_n = B_0+B_1n$

Put in (A)

$$\Rightarrow B_0+B_1(n+2)-8B_0-8B_1(n+1)+12B_0+12B_1n = 1-4n$$

$$\Rightarrow B_0+B_1n+2B_1-8B_0-8B_1n-8B_1+12B_0+12B_1n=1-4n$$

Equating co-efficients of n and constant terms

$$B_1 - 8B_1 + 12B_1 = -4$$

$$\Rightarrow 5B_1 = -4 \quad \Rightarrow B_1 = -4/5$$

&

$$B_0 + 2B_1 - 8B_0 - 8B_1 + 12B_0 = 1$$

$$5B_0 - 6B_1 = 1 \Rightarrow 5B_0 - 6(-4/5) = 1$$

$$5B_0 = 1 - 24/5$$

$$\Rightarrow B_0 = -19/25$$

So P.S for (A) is

$$U_n = \frac{-19}{25} - \frac{4}{5}n$$

For equ (B) let $U_n = C4^n$

$$U_{n+1} = C4^{n+1}, U_{n+2} = C4^{n+2}$$

Put in (B)

$$C4^{n+2} - 8C4^{n+1} + 12C4^n = -4^n$$

$$4^n[16C - 32C + 12C] = 4^n(-1)$$

$$16C - 32C + 12C = -1$$

$$-4C = -1 \quad \Rightarrow \quad C = 1/4$$

P.S is $U_n = \frac{1}{4}4^n$

So Perticular solution for (V)

Perticular Soln of (A) + P.S of (B)

$$\Rightarrow U_n^{(P)} = \frac{-19}{25} - \frac{4}{5}n + \frac{1}{4}4^n \longrightarrow (*')$$

General solution of U_n is

(by $*$ & $*'$)

$$U_n = C_1(2)^n + C_2(6)^n - \frac{19}{25} - \frac{4}{5}n + \frac{1}{4}4^n$$

which is required solution for U_n

Now by equ (iii)

$$V_n = n - U_{n+1} - 3U_n$$

$$= n - \left[C_1(2)^{n+1} + C_2(6)^{n+1} - \frac{19}{25} - \frac{4}{5}(n+1) + \frac{1}{4}4^{(n+1)}\right]$$

$$-3\left[C_1(2)^n + C_2(6)^n - \frac{19}{25} - \frac{4}{5}(n) + \frac{1}{4}4^n\right]$$

$$= n + (-2C_1 + 3C_1)2^n + (-6C_2 + 3C_2)6^n +$$

$$\left(\frac{19}{25} - \frac{57}{25} + \frac{4}{5}\right) + \left(\frac{4}{5} - \frac{12}{5}\right)n + \left(-1 + \frac{3}{4}\right)4^n$$

$$\Rightarrow V_n = n + C_1 2^n - 3C_2 6^n - \frac{18}{25} - \frac{8}{5}n - \frac{4^n}{4}$$

$$\Rightarrow V_n = C_1 2^n - 3C_2 6^n - 3/5\, n - 18/25 - \frac{4^n}{4}$$

which is required solution for V_n

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Example:- Solve system of equations.

$$U_{n+1} - U_n + V_n = 7$$

$$3V_{n+1} - 2V_n + U_n = 2$$

Solution Given

$$U_{n+1} - U_n + V_n = 7$$

$$3V_{n+1} - 2V_n + U_n = 2$$

In operator notation

$$EU_n - U_n + V_n = 7$$

$$\Rightarrow (E-1)U_n + V_n = 7 \quad \text{——— (1)}$$

$$(3E-2)V_n + U_n = 2 \quad \text{——— (2)}$$

Multiplying ② by $(E-1)$ and subtract from ①

$$(3E-2)(E-1)V_n + \cancel{(E-1)U_n} = 2(E-1)$$
$$\pm V_n \pm \cancel{(E-1)U_n} = \pm 7$$

$$(3E^2-5E+2)V_n - V_n = 2-2-7 \quad (\because 2E = 2$$

$$\Rightarrow (3E^2-5E+1)V_n = -7$$

$$3V_{n+2} - 5V_{n+1} + V_n = -7 \quad \text{---} ③$$

Ch. equation is

$$3E^2 - 5E + 1 = 0$$

$$\Rightarrow E = \frac{5 \pm \sqrt{13}}{6}$$

Then C.F

$$V_n^{(c)} = C_1\left(\frac{5+\sqrt{13}}{6}\right)^n + C_2\left(\frac{5-\sqrt{13}}{6}\right)^n$$

To find P.S Put $V_n = C$ in (iii)

$$\Rightarrow 3C - 5C + C = -7$$

$$\Rightarrow -C = -7 \Rightarrow C = 7$$

P.S of $V_n = 7$

General Solution = C.F + P.S

$$\Rightarrow V_n = C_1\left(\frac{5+\sqrt{13}}{6}\right)^n + C_2\left(\frac{5-\sqrt{13}}{6}\right)^n + 7$$

Now $U_n = 2 - 3V_{n+1} + 2V_n$

$$= 2 - 3\left[C_1\left(\frac{5+\sqrt{13}}{6}\right)^{n+1} + C_2\left(\frac{5-\sqrt{13}}{6}\right)^n + 7\right]$$

$$+ 2\left[C_1\left(\frac{5+\sqrt{13}}{6}\right)^n + C_2\left(\frac{5-\sqrt{13}}{6}\right)^n + 7\right]$$

$$U_n = 2 - 3\left[C_1\left(\frac{5+\sqrt{13}}{6}\right)^{n+1} + C_2\left(\frac{5-\sqrt{13}}{6}\right)^{n+1}\right]$$
$$+ 2\left[C_1\left(\frac{5+\sqrt{13}}{6}\right)^{n} + C_2\left(\frac{5-\sqrt{13}}{6}\right)^{n}\right] - 7$$

which is required solution for U_n

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ASSIGNMENT

Solve the followings

Q1:- $3U_n + 4V_{n+1} - 5V_n = 2^n$
$2U_{n+1} + 3U_n + 4V_n = 7$

Q2:- $U_{n+1} - U_n + V_n = 2^n$
$3V_{n+1} + 2V_n + U_n = 7$

Q3:- $3V_{n+1} - 2V_n + U_n = n^2$
$U_{n+1} - 3U_n + V_n = 3^n$

Q4:- $U_{n+1} - U_n + V_n = 1$
$3V_{n+1} - 2V_n + U_n = 2$

⇒ Numerical Methods for Ordinary Difference Equations:-

An ordinary difference equation is a relation b/w a function, its derivatives and variable upon which they depend. The most general form of an O.D.E is

$$\phi(y, y', y'', \ldots, y^n) = 0$$

where y and its derivatives are all function of x.

In this chapter we shall derive and analyse numerical methods for solving the main form of problem that we shall study is initial value problem.

$$y' = f(x, y) \quad ; \quad y(x_0) = y_0$$

The important methods of solving O.D.E's of first order are as follows.

(i) Euler's Method.
(ii) Heun's Method (Improved Euler Method)
(iii) Modified Euler's Method.
(iv) Taylor's Series Method.
(v) Runge-Kutta Method.
(vi) Adams-Bashforth Method.

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1) **Euler's Method:-**

Consider the first order differential equation together with an initial condition is

$$\frac{dy}{dx} = f(x,y) \longrightarrow ① \quad ; \; y(x_0) = y_0$$

Note that

$$\frac{dy}{dx} = f(x,y) \quad \text{means}$$

$$y'(x) = f(x,y)$$

$$\Rightarrow y'(x_0) = f(x_0, y_0)$$

Now $\quad y(x_1) = y(x_0+h)$

$$\Rightarrow y(x_1) = y(x_0+h) = y(x_0) + h y'(x_0) + \frac{h^2}{2!} y''(x_0) + \cdots$$

$$\Rightarrow y_1 = y(x_1) = y(x_0) + h y'(x_0) \quad \text{neglecting}$$

$$\Rightarrow y_1 = y(x_0) + h f(x_0, y_0)$$

Similarly

$$y_2 = y_1 + h f(x_1, y_1)$$

$$y_3 = y_2 + h f(x_2, y_2)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y_{n+1} = y_n + h f(x_n, y_n)$$

This formula is called Euler's Method.

Remark:- Drawback of Euler's method, if h is small (h is unit step size), Euler's Method is too small.

However if h is large this method is inaccurate.

Example:- Solve $\frac{dy}{dx} = x+y$ where $y(0) = 1$, $h = 0.2$

find $y(0.4)$ by Euler's Method and compare with exact value.

Solution The Given initial value problem is

$$\frac{dy}{dx} = x+y \longrightarrow ①$$

$y(0) = 1$, $h = 0.2$

The Euler's formula is

$$y_{n+1} = y_n + h f(x_n, y_n)$$

As $f(x,y) = x+y$

$$f(x_n, y_n) = x_n + y_n$$

So $$y_{n+1} = y_n + h(x_n + y_n) \longrightarrow ②$$

1st Iteration:- Put $n=0$ in ②

$$\Rightarrow y_1 = y_0 + h(x_0 + y_0)$$
$$= 1 + (0.2)[0+1]$$
$$= 1 + 0.2$$

$y_1 = y(x_1)$
$x_1 = x_0 + h$
$= 0 + 0.2$
$x_1 = 0.2$

$y_1 = 1.2 \Rightarrow y(0.2) = 1.2$

2nd Iteration:- Put $n=1$ in ②

$$y_2 = y_1 + h(x_1 + y_1)$$
$$= 1.2 + 0.2[0.2 + 1.2] = 1.2 + 0.2[1.4]$$

$y_2 = 1.48 \Rightarrow y(0.4) = 1.48$

Exact Solution:- Given O.D.E is

$$\frac{dy}{dx} = x + y \longrightarrow ③$$

$$\frac{dy}{dx} - y = x \longrightarrow ③' \text{ (Linear Diff eqn.)}$$

I.F is $\int -1\,dx = e^{-x}$

Then ③' becomes

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}x$$

$$\frac{d}{dx}[ye^{-x}] = xe^{-x}$$

Integrating both sides

$$ye^{-x} = -xe^{-x} - e^{-x} + c$$

$$\Rightarrow ye^{-x} = -e^{-x}(x+1) + c$$

$$\Rightarrow y = -(x+1) + ce^{x}$$

$$\Rightarrow y = ce^{x} - (x+1) \longrightarrow ④$$

Applying condition $y(0) = 1$

$$1 = c - 1 \Rightarrow c = 2$$

Then ④ becomes

$$y = 2e^{x} - (x+1)$$

At $x = 0.4$, $y = 2e^{0.4} - (0.4+1)$

$$y = 1.5836$$

So,

$$\text{Error} = \text{Exact value} - \text{Approx Value}$$
$$= 1.5836 - 1.48$$
$$E = 0.1036$$

Example:- Solve $\frac{dy}{dx} = 3x^2 + 2xy$, $y(0) = 1$ find $y(1)$

Solution

Put $h = \frac{1-0}{5} \Rightarrow h = 0.2$

Then $x_0 = 0$, $x_1 = 0 + 0.2 = 0.2$, $x_2 = 0.4$

$x_3 = 0.6$, $x_4 = 0.8$, $x_5 = 1$

To find $y(1)$ i.e $y(x_5)$ i.e y_5

Euler's formula is given by

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

$$\Rightarrow y_{n+1} = y_n + h(3x_n^2 + 2x_n y_n) \longrightarrow (*)$$

1st Iteration:- Put $n = 0$ in $(*)$

$$y_1 = y_0 + h(3x_0^2 + 2x_0 y_0)$$
$$= 1 + 0.2[3(0)^2 + 2(0)(1)]$$
$$= 1$$

2nd Iteration:- Put $n = 1$ in $(*)$

$$y_2 = y_1 + h\{3x_1^2 + 2(x_1)(y_1)\}$$
$$= 1 + (0.2)\{3(0.2)^2 + 2(0.2)(1)\}$$
$$= 1.0104$$

3rd Iteration:- Put $n = 2$ in $(*)$

$$y_3 = y_2 + h(3x_2^2 + 2x_2 y_2)$$

$$y_3 = 1.0104 + 0.2\left[3(0.4)^2 + 2(0.4)(1.0104)\right]$$
$$= 1.2681$$

4th Iteration:- Put $n=3$ in ⊛

$$y_4 = y_3 + h(3x_3^2 + 2x_3 y_3)$$
$$= 1.2681 + (0.2)\left[3(0.6)^2 + 2(0.6)(1.2681)\right]$$
$$= 1.7884$$

5th Iteration:- Put $n=4$ in ⊛

$$y_5 = y_4 + h\left[3x_4^2 + 2x_4 y_4\right]$$
$$= 1.7884 + (0.2)\left[3(0.8)^2 + 2(0.8)(1.7884)\right]$$
$$= 2.7447$$

So $y(1) = 2.7447$

★ ★

Example:- Find solution of $\frac{dy}{dx} = 2xy^2$, $y(0) = 1$ & $h = 0.05$. Find $y(0.3)$ using Euler's method.

Solution

$x_0 = 0, \quad h = 0.05$

$x_1 = 0 + 0.05 = 0.05, \quad x_2 = 0 + 2h = 0.1$

$x_3 = 0.15, \quad x_4 = 0.2, \quad x_5 = 0.25$

$x_6 = 0.3$

To find $y(0.3)$ i.e $y(x_6)$ i.e y_6

Euler's formula is given by

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$\Rightarrow y_{n+1} = y_n + h(2x_n y_n^2) \longrightarrow (*)$

1st Iteration:- Put $n=0$ in $(*)$

$y_1 = y_0 + h(2x_0 y_0^2)$

$= 1 + 0.05[2(0)(1)^2] = 1 + 0.05(0)$

$y_1 = 1$ i.e $y(0.05) = 1$

2nd Iteration:- Put $n=1$ in $(*)$

$y_2 = y_1 + h(2x_1 y_1^2)$

$= 1 + 0.05[2(0.05)(1)^2]$

$= 1.0050 \Rightarrow y(0.1) = 1.0050$

3rd Iteration:- Put $n=2$ in $(*)$

$y_3 = y_2 + h(2x_2 y_2^2)$

$= 1.0050 + 0.05[2(0.1)(1.005)^2]$

$= 1.0151$

4th Iteration:- Put $n=3$ in $(*)$

$y_4 = y_3 + h(2x_3 y_3^2)$

$= 1.0151 + 0.05[2(0.15)(1.0151)^2]$

$= 1.0306$

5th Iteration:- Put $n=4$ in $(*)$

$y_5 = y_4 + h(2x_4 y_4^2)$

$= 1.0306 + 0.05[2(0.2)(1.0306)^2]$

$= 1.0518$

6th Iteration:- Put $n=5$ in $(*)$

$$y_6 = y_5 + h(2x_5 y_5^2)$$

$$= 1.0518 + 0.05[2(0.25)(1.0518)^2]$$

$$= 1.0795$$

$$\Rightarrow \quad y(0.3) = 1.0795$$

* * * *

Example:- Find an approximation value for the solution of initial value problems.

$$y' = x + y^2 \quad , \quad y(1) = 0$$

at $x = 1, 1.1, 1.2, 1.3, 1.4, 1.5$

Solution:- The initial value problem is

$$y' = x + y^2 \quad ; \quad y(1) = 0$$

The Euler's formula is

$$y_{n+1} = y_n + h f(x_n, y_n) \longrightarrow ①$$

As $f(x, y) = x + y^2 \Rightarrow f(x_n, y_n) = x_n + y_n^2$

$\Rightarrow$ ① becomes

$$y_{n+1} = y_n + h(x_n + y_n^2) \longrightarrow ②$$

1st Iteration:- when $n = 0$

$$y_1 = y_0 + h(x_0 + y_0^2)$$

$$= 0 + 0.1[1 + 0^2]$$

$$= 0.1$$

2nd Iteration:- When $n = 1$

$$y_2 = y_1 + h(x_1 + y_1^2)$$

$$= 0.1 + 0.1[1.1 + (0.1)^2]$$

$$= 0.2110$$

3rd Iteration:- when $n=2$

$$y_3 = y_2 + h(x_2 + y_2^2)$$
$$= 0.2110 + 0.1[1.2 + (0.2110)^2]$$
$$= 0.33$$

4th Iteration:- when $n=3$

$$y_4 = y_3 + h(x_3 + y_3^2)$$
$$= 0.33 + 0.1[1.3 + (0.33)^2]$$
$$= 0.4709$$

5th Iteration:- when $n=4$

$$y_5 = y_4 + h(x_4 + y_4^2)$$
$$= 0.4709 + 0.1[1.4 + (0.4709)^2]$$
$$= 0.6321$$

6th Iteration:- when $n=5$

$$y_6 = y_5 + h(x_5 + y_5^2)$$
$$= 0.6321 + 0.1[1.5 + (0.6321)^2]$$
$$= 0.8221$$

$$\Rightarrow y(1.5) = 0.8221$$

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2) Improved Euler's Method (OR) Heun's method :-

Given $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$

where h is a step size. Integrate from x_0 to x_{0+1}

$$\int_{x_0}^{x_1} \frac{dy}{dx} \cdot dx = \int_{x_0}^{x_1} f(x,y)\, dx$$

$$y \Big|_{x_0}^{x_1} = \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

$\because$ using Trapezoidal Rule

$$y(x_1) - y(x_0) = \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

$$y_1 - y_0 = \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

Similarly

$$y_2 = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2)\right]$$

$$y_3 = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3)\right]$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

Here we write it as

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f\left(x_n + h, y_n + h f(x_n, y_n)\right)\right]$$

using Euler's formula.

Example:- Solve $\frac{dy}{dx} = 3x - 5y$, $y(1) = 3$
find $y(1.6)$ here $h = 0.2$

Solution

Here $f(x,y) = 3x - 5y$, $x_0 = 1$

$y_0 = 3$, $h = 0.2$

Now $x_0 = 1$, $x_1 = x_0 + h = 1 + 0.2 \Rightarrow x_1 = 1.2$

$x_2 = 1.4$, $x_3 = 1.6$

To find $y(1.6)$ i.e $y(x_3)$ i.e y_3

Now by Improved Euler's method

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))]$$

$$= y_n + \frac{h}{2}[3x_n - 5y_n) + f(x_n + h , y_n + h(3x_n - 5y_n))]$$

$$= y_n + \frac{h}{2}[(3x_n - 5y_n) + 3(x_n + h) - 5(y_n + 3hx_n - 5hy_n)]$$

$$= y_n + \frac{h}{2}[3x_n - 5y_n + 3x_n + 3h - 5y_n - 15hx_n + 25hy_n]$$

$$= y_n + \frac{h}{2}[6x_n - 10y_n \overset{+3h}{-} 15hx_n + 25hy_n] \longrightarrow (*)$$

1st Iteration:- Put $n = 0$ in (*)

$$y_1 = y_0 + \frac{h}{2}[6x_0 - 10y_0 \overset{+3h}{-} 15hx_0 + 25hy_0]$$

$$= 3 + \frac{0.2}{2}[6(1) - 10(3) \overset{+3(0.2)}{-} 15(0.2)(1) + 25(0.2)(3)]$$

$$= 1.86$$

2nd Iteration:- Put $n = 1$ in (*)

$$y_2 = y_1 + \frac{h}{2}[6x_1 - 10y_1 + 3h - 15hx_1 + 25hy_1]$$

$$y_2 = 1.86 + \frac{0.2}{2}\left[6(1.2) - 10(1.86) + 3(0.2) - 15(0.2)(1.2) + 25(0.2)(1.86)\right]$$

$$= 1.35$$

3rd Iteration:- Put $n = 2$ in (A)

$$y_3 = y_2 + \frac{h}{2}\left[6x_2 - 10y_2 + 3h - 15hx_2 + 25hy_2\right]$$

$$= 1.35 + \frac{0.2}{2}\left[6(1.4) - 10(1.35) + 3(0.2) - 15(0.2)(1.4) + 25(0.2)(1.35)\right]$$

$$= 1.16$$

So $y(1.6) = 1.16$

Example:- Solve $\frac{dy}{dx} = x^2 + y^2$; $y(0) = 1$

$h = 0.1$

find $y(0.1)$ using Euler's Improved formula

Solution:- Given

$$\frac{dy}{dx} = x^2 + y^2, \quad x_0 = 0, \; y_0 = 1 \quad h = 0.1$$

$$x_1 = 0 + h = 0.1$$

By Euler's improved formula

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))\right]$$

$$= y_n + \frac{h}{2}\left[(x_n^2 + y_n^2) + f(x_n + h, y_n + h(x_n^2 + y_n^2))\right]$$

$$= y_n + \frac{h}{2}\left[(x_n^2 + y_n^2) + (x_n + h)^2 + (y_n + h(x_n^2 + y_n^2))^2\right]$$

$$y_{n+1} = y_n + \frac{h}{2}\left[x_n^2 + y_n^2 + x_n^2 + h^2 + 2hx_n + y_n^2 + 2hy_n(x_n^2+y_n^2) + \{h(x_n^2+y_n^2)\}^2\right]$$

$$= y_n + \frac{h}{2}\left[2x_n^2 + 2y_n^2 + h^2 + 2hx_n + 2hy_nx_n^2 + 2hy_n^3 + h^2\{x_n^4 + y_n^4 + 2x_n^2y_n^2\}\right]$$

$$= y_n + \frac{h}{2}\left[2x_n^2 + 2y_n^2 + h^2 + 2hx_n + 2hy_nx_n^2 + 2hy_n^3 + h^2\{x_n^4 + y_n^4 + 2x_n^2y_n^2\}\right] \longrightarrow (*)$$

At $x = 0.1$ i.e $y(0.1)$

i.e y_1

$$\Rightarrow y_1 = y_0 + \frac{h}{2}\left[2x_0^2 + 2y_0^2 + h^2 + 2hx_0 + 2hy_0x_0^2 + 2hy_0^3 + h^2\{x_0^4 + y_0^4 + 2x_0^2y_0^2\}\right]$$

$$= 1 + \frac{0.1}{2}\left[2(0) + 2(1)^2 + (0.1)^2 + 2(0.1)(0) + 2hy_0(0) + 2(0.1)(1)^3 + (0.1)^2\{0 + (0.1)^4 + 2(0)y_0^2\}\right]$$

$$\Rightarrow y_1 = 1.111 \quad \text{Ans.}$$

***

Example:- $\frac{dy}{dx} = x+y$; $y(0) = 1$, $h = 0.1$

find $y(0.2)$ using Improved Euler's method.

Solution Given $\frac{dy}{dx} = x+y$

$\therefore f(x,y) = x+y$

$x_0 = 0, \quad y_0 = 1, \quad h = 0.1$

Improved Euler's formula.

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

$$= y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_n+h, y_n + hf(x_n, y_n))\right]$$

$$\Rightarrow y_{n+1} = y_n + \frac{h}{2}\left[(x_n+y_n) + f(x_n+h, y_n + h(x_n+y_n))\right]$$

$$= y_n + \frac{h}{2}\left[x_n + y_n + x_n + h + y_n + hx_n + hy_n\right]$$

$$y_{n+1} = y_n + \frac{h}{2}\left[2x_n + 2y_n + h + hx_n + hy_n\right] \longrightarrow \circledast$$

To find $y(0.2)$ i.e $y(x_2)$ $\because x_2 = x_0 + 2h$

$\Rightarrow$ find y_2 $\quad = 0 + 2(0.1)$

$\quad = 0.2$

1st Iteration:- Put $n=0$ in $\circledast$

$$y_1 = y_0 + \frac{h}{2}\left[2x_0 + 2y_0 + h + hx_0 + hy_0\right]$$

$$= 1 + \frac{0.1}{2}\left[2(0) + 2(1) + 0.1 + 0.2(0) + 0.1(1)\right]$$

$$= 1.11$$

2nd Iteration:- Put $n=1$ in $\circledast$

$$y_2 = y_1 + \frac{h}{2}\left[2x_1 + 2y_1 + h + hx_1 + hy_1\right]$$

$$= 1.11 + \frac{0.1}{2}\left[2(0.1) + 2(1.11) + 0.1 + 0.1(0.1) + 0.1(1.11)\right]$$

$$= 1.2421$$

$$\Rightarrow y(0.2) = 1.2421$$

3) Modified Euler's Method:-

$$y_{n+1} = y_n + h f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)$$

Example:- Apply modified Euler's formula to find $y(0.3)$ given that

$$\frac{dy}{dx} = x + y \; ; \quad y(0) = 1 \;, \; h = 0.1$$

Solution:- Given $\frac{dy}{dx} = x + y$, $y_0 = 1$, $x_0 = 0$
$h = 0.1$

$x_1 = x_0 + h \Rightarrow x_1 = 0.1$, $x_2 = 0.2$ $x_3 = 0.3$

To find $y(0.3)$ i.e $y(x_3)$ i.e y_3

Euler's Modified formula is

$$y_{n+1} = y_n + h f\left(x_n + h/2, y_n + h/2 f(x_n, y_n)\right)$$

As $f(x, y) = x + y$

$$\Rightarrow y_{n+1} = y_n + h\left[x_n + h/2 + y_n + h/2(x_n + y_n)\right]$$

$$= y_n + h\left[\frac{h}{2}\{1 + x_n + y_n\} + x_n + y_n\right] \longrightarrow \circledast$$

1st Iteration:- Put $n = 0$ in $\circledast$

$$\Rightarrow y_1 = y_0 + h\left[\frac{h}{2}\{1 + x_0 + y_0\} + x_0 + y_0\right]$$

$$= 1 + 0.1\left[\frac{0.1}{2}\{1 + 0 + 1\} + 0 + 1\right]$$

$$y_1 = 1.11$$

$$\Rightarrow y(0.1) = 1.11$$

<u>2nd Iteration</u>:- Put $n=1$ in (★)

$$\Rightarrow y_2 = y_1 + h\left[\frac{h}{2}\{1+x_1+y_1\} + y_1 + x_1\right]$$

$$= 1.11 + 0.1\left[\frac{0.1}{2}\{1+0.1+1.11\} + 1.11 + 0.1\right]$$

$$= 1.2421$$

<u>3rd Iteration</u>:- Put $n=2$ in (★)

$$\Rightarrow y_3 = y_2 + h\left[\frac{h}{2}\{1+x_2+y_2\} + y_2 + x_2\right]$$

$$= 1.2421 + 0.1\left[\frac{0.1}{2}\{1+0.2+1.2421\} + 1.2421 + 0.2\right]$$

$$= 1.3985$$

$$\Rightarrow y(0.3) = 1.3985 \quad \text{Ans.}$$

★★

Example:- Solve $\frac{dy}{dx} = x+2y$; $0 \leq x \leq 0.2$

$h = 0.1$; $y(0) = 1$

using modified Euler's Method.

Solution Given $\frac{dy}{dx} = x+2y$

$x_0 = 0$, $y_0 = 1$, $h = 0.1$

So $x_1 = x_0 + h = 0+0.1 \Rightarrow x_1 = 0.1$

$x_2 = x_1 + h = 0.1 + 0.1 \Rightarrow x_2 = 0.2$

we find $y(0.2)$ i.e $y(x_2)$ i.e y_2

Euler's Modified Method is

$$y_{n+1} = y_n + h\, f\left(x_n + \frac{h}{2},\, y_n + \frac{h}{2} f(x_n, y_n)\right)$$

$$= y_n + h\, f\left(x_n + \frac{h}{2},\, y_n + \frac{h}{2}(x_n + 2y_n)\right)$$

$$= y_n + h\left[x_n + \frac{h}{2} + 2\left\{y_n + \frac{h}{2}(x_n + 2y_n)\right\}\right]$$

$$\Rightarrow y_{n+1} = y_n + h\left[x_n + \frac{h}{2} + 2y_n + hx_n + 2hy_n\right]$$

$$y_{n+1} = y_n + \frac{h}{2}\left[2x_n + h + 4y_n + 2hx_n + 4hy_n\right] \longrightarrow (*)$$

<u>1st Iteration:-</u> Put $n=0$ in $(*)$

$$y_1 = y_0 + \frac{h}{2}\left[2x_0 + h + 4y_0 + 2hx_0 + 4hy_0\right]$$

$$= 1 + \frac{0.1}{2}\left[2(0) + 0.1 + 4(1) + 2(0.1)(0) + 4(0.1)(1)\right]$$

$$= 1.2250 \qquad \Rightarrow y(0.1) = 1.2250$$

<u>2nd Iteration:-</u> Put $n=1$ in $(*)$

$$\Rightarrow y_2 = y_1 + \frac{h}{2}\left[2x_1 + h + 4y_1 + 2hx_1 + 4hy_1\right]$$

$$= 1.225 + \frac{0.1}{2}\left[2(0.1) + 0.1 + 4(1.225) + 2(0.1)(0.1) + 4(0.1)(1.225)\right]$$

$$= 1.5105$$

$$\Rightarrow y(0.2) = 1.5105 \quad \text{Ans.}$$

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<u>Assignment</u>

Q1:- Solve the following questions by Euler's Method.

(i) $y' = x + y$; $y(0) = 0$, $h = 0.2$ carry out six steps.

(ii) $\frac{dy}{dx} = x + y^2$ and $y = 1$ at $x = 0$ find $y(0.5)$: $h = 0.1$

(iii) If $xy' = x+y$, when y has at value $x=1$ is 2
find y when $x=2$ (step size is 0.1)

(iv) Solve $\frac{dy}{dx} = -xy^2$, $y=2$ at $x=0$
obtain y at $x=0.2$ where $h=0.05$

(vi) Solve $\frac{dy}{dx} = 1 + x\sin(xy)$. $0 \le x \le 1$
with $h=0.1$ & $y(0)=0$

(v) $y' = \frac{-y^2}{1+x^2}$ estimate $y(0.2)$
with $y(0)=1$, $h=0.05$

Q2:- Solve the following by Improved Euler's Method.

(i) Derive Improved Euler's method. Apply this to find $y(0.2)$ from $y' = -2xy^2$
with $y(0)=1$, $h=0.1$.
Compare your result with exact value.

(ii) Find approximate value of y when $x=0.06$ given $\frac{dy}{dx} = x^2+y$; $y(0)=1$
taking interval 0.02

(iii) Tabulate $y(x)$ for $x = 0.1, 0.2, 0.3$
from IVP $y' = 1+xy$ where $y(0)=1$

(iv) Find $y(0.2)$ given $\frac{dy}{dx} = x + 3y$; $y(0) = 1$
$h = 0.1$

Q3:- Solve the following questions by Modified Euler's Method:-

(i) Tabulate $y(0.3)$ for $y' = 1 + xy$ where $y(0) = 1$

(ii) Solve $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$, $h = 0.5$ at $x = 1$

(iii) Solve following differential equation $dy/dx = x - y$ $\quad 0 \leq x \leq 0.2$, $y(0) = 1$ taking $h = 0.1$ and working to four decimal places by Modified Euler's method.

(iv) Show that modified Euler's method for solution of differential equation $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ is given by

$$y_{n+1} = y_n + \frac{h}{2}\{f(x_n, y_n) + f(x_n + h, y_n + h f(x_n, y_n))\}$$

Hint:- Use Euler's & improved Euler's Method.

SOLUTIONS

Q1:-(i) Given $\frac{dy}{dx} = x+y$; $y(0) = 0$
$h = 0.2$

So $x_0 = 0$, $y_0 = 0$, $x_1 = 0+0.2 = 0.2$
$x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, $x_5 = 1.0$
$x_6 = 1.2$

$f(x,y) = x+y \Rightarrow f(x_n, y_n) = x_n + y_n$

Euler's formula is

$$y_{n+1} = y_n + h\, f(x_n, y_n) \longrightarrow (*)$$

<u>1st Iteration:-</u> Put $n=0$ in $(*)$

$\Rightarrow y_1 = y_0 + h f(x_0, y_0)$
$= 0 + 0.2(0+0) = 0$

<u>2nd Iteration:-</u> Put $n=1$ in $(*)$

$\Rightarrow y_2 = y_1 + h f(x_1, y_1)$
$= y_1 + h(x_1 + y_1) = 0 + 0.2(0.2+0)$

$\Rightarrow y(0.4) = 0.04$

<u>3rd Iteration:-</u> Put $n=2$ in $(*)$

$\Rightarrow y_3 = y_2 + h f(x_2, y_2)$
$= y_2 + h(x_2 + y_2) = 0.04 + 0.2(0.4 + 0.04)$
$= 0.128$

<u>4th Iteration:-</u> Put $n=3$ in $(*)$

$\Rightarrow y_4 = y_3 + h f(x_3, y_3)$
$= y_3 + h(x_3 + y_3) = 0.128 + 0.2(0.6 + 0.128)$
$= 0.2736$

<u>5th Iteration:-</u> Put $n=4$ in $(*)$

$\Rightarrow y_5 = y_4 + h f(x_4, y_4)$
$= y_4 + h(x_4 + y_4) = 0.2736 + 0.2(0.8 + 0.2736)$

$= 0.48832$

6 th Iteration:- Put $n=5$ in ⊛

$\Rightarrow y_6 = y_5 + h f(x_5, y_5)$

$= y_5 + h(x_5 + y_5) = 0.48832 + 0.2(1 + 0.48832)$

$\Rightarrow y(1.2) = 0.786$ Ans.

(ii) Given $y' = x + y^2$, $y_0 = 1$, $x_0 = 0$

$h = 0.1$

So $x_1 = x_0 + h = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$

$x_4 = 0.4$, $x_5 = 0.5$

To find $y(0.5)$ i.e $y(x_5)$ i.e y_5

$f(x,y) = x + y^2 \Rightarrow f(x_n, y_n) = x_n + y_n^2$

Euler's formula is

$y_{n+1} = y_n + h f(x_n, y_n) \longrightarrow$ ⊛

1st Iteration:- Put $n=0$ in ⊛

$\Rightarrow y_1 = y_0 + h f(x_0, y_0)$

$= y_0 + h(x_0 + y_0^2) = 1 + 0.1(0 + 1^2)$

$= 1.1$

2 nd Iteration:- Put $n=1$ in ⊛

$\Rightarrow y_2 = y_1 + h f(x_1, y_1)$

$= y_1 + h(x_1 + (y_1)^2) = 1.1 + 0.1[0.1 + (1.1)^2]$

$= 1.231$

3rd Iteration:- Put $n=2$ in ⊛

$\Rightarrow y_3 = y_2 + h f(x_2, y_2)$

$= y_2 + h(x_2 + y_2^2) = 1.231 + 0.1[0.2 + (1.231)^2]$

$= 1.4025$

4th Iteration:- Put $n=3$ in ⊛

$\Rightarrow y_4 = h f(x_3, y_3) + y_3$

$= y_3 + h(x_3 + y_3^2) = 1.4025 + 0.1[0.3 + (1.4025)^2]$

$= 1.6292$

5th Iteration:- Put $n = 4$ in ⊛

$\Rightarrow y_5 = h f(x_4, y_4) + y_4$

$= y_4 + h[x_4 + (y_4)^2]$

$= 1.6292 + 0.1[0.4 + (1.6292)^2]$

$= 1.9346$ Ans.

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(iii)

Given $xy' = x + y$; $x_0 = 1$, $y_0 = 2$, $h = 0.1$

$\Rightarrow y' = \dfrac{x+y}{x}$

Now $x_0 = 1$, $x_1 = x_0 + h = 1.1$, $x_2 = 1.2$

$x_3 = 1.3$, $x_4 = 1.4$, $x_5 = 1.5$, $x_6 = 1.6$

$x_7 = 1.7$, $x_8 = 1.8$, $x_9 = 1.9$, $x_{10} = 2$

To find $y(2)$ i.e $y(x_{10})$ i.e y_{10}

$f(x_n, y_n) = \dfrac{x_n + y_n}{x_n}$

Euler's formula is

$y_{n+1} = y_n + h f(x_n, y_n)$

$\Rightarrow y_{n+1} = y_n + h\left[\dfrac{x_n + y_n}{x_n}\right] \longrightarrow$ ⊛

1st Iteration:- Put $n = 0$ in ⊛

$\Rightarrow y_1 = y_0 + h\left[\dfrac{x_0 + y_0}{x_0}\right]$

$$\Rightarrow y_1 = 2 + 0.1\left[\frac{1+2}{1}\right] = 2.3$$

2nd Iteration:- Put $n = 1$ in ⊛

$$\Rightarrow y_2 = y_1 + h\left[\frac{x_1 + y_1}{x_1}\right]$$

$$= 2.3 + 0.1\left[\frac{1.1 + 2.3}{1.1}\right] = 2.6091$$

3rd Iteration:- Put $n=2$ in ⊛

$$\Rightarrow y_3 = y_2 + h\left[\frac{x_2 + y_2}{x_2}\right]$$

$$= 2.6091 + 0.1\left[\frac{1.2 + 2.6091}{1.2}\right] = 2.9265$$

4th Iteration:- Put $n=3$ in ⊛

$$\Rightarrow y_4 = y_3 + h\left[\frac{x_3 + y_3}{x_3}\right]$$

$$= 2.9265 + 0.1\left[\frac{1.3 + 2.9265}{1.3}\right] = 3.2516$$

5th Iteration:- Put $n=4$ in ⊛

$$\Rightarrow y_5 = y_4 + h\left[\frac{x_4 + y_4}{x_4}\right]$$

$$= 3.2516 + 0.1\left[\frac{1.4 + 3.2516}{1.4}\right] = 3.5839$$

6th Iteration:- Put $n=5$ in ⊛

$$\Rightarrow y_6 = y_5 + h\left[\frac{x_5 + y_5}{x_5}\right]$$

$$= 3.5839 + 0.1\left[\frac{1.5 + 3.5839}{1.5}\right] = 3.9228$$

7th Iteration:- Put $n=6$ in ⊛

$$\Rightarrow y_7 = y_6 + h\left[\frac{x_6 + y_6}{x_6}\right] = 4.268$$

8th Iteration:- Put $n=7$ in ⊛

$$\Rightarrow y_8 = y_7 + h\left[\frac{x_7+y_7}{x_7}\right]$$

$$= 4.268 + 0.1\left[\frac{1.7+4.268}{1.7}\right] = 4.6188$$

9th Iteration:- Put $n=8$ in ⊛

$$\Rightarrow y_9 = y_8 + h\left[\frac{x_8+y_8}{x_8}\right]$$

$$= 4.6188 + 0.1\left[\frac{1.8+4.6188}{1.8}\right] = 4.9754$$

10th Iteration:- Put $n=9$ in ⊛

$$\Rightarrow y_{10} = y_9 + h\left[\frac{x_9+y_9}{x_9}\right]$$

$$= 4.9754 + 0.1\left[\frac{1.9+4.9754}{1.9}\right] = 5.3373 \text{ Ans}$$

(iv)

$$\frac{dy}{dx} = -xy^2 \; ; \; x_0 = 0 \; , \; y_0 = 2 \; , \; h = 0.05$$

So $x_1 = x_0 + h = 0 + 0.05 = 0.05$

$x_2 = 0.1$, $x_3 = 0.15$, $x_4 = 0.2$

To find $y(0.2)$ i.e $y(x_4)$ i.e y_4

$$f(x_n, y_n) = -x_n y_n^2$$

Euler's formula is

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$\Rightarrow y_{n+1} = y_n + h\left[-x_n y_n^2\right] \longrightarrow ⊛$$

1st Iteration:- Put $n=0$ in ⊛

$$\Rightarrow y_1 = y_0 + h\left[-x_0 y_0^2\right]$$

$$= 2 + 0.05\left[-(0)(2)^2\right] = 2$$

2nd Iteration:- Put $n=1$ in ⊛

$$\Rightarrow y_2 = y_1 + h[-x_1 y_1^2]$$

$$= 2 + 0.05[-(0.05)(2)^2] = 1.99$$

3rd Iteration:- Put $n=2$ in ⊛

$$\Rightarrow y_3 = y_2 + h[-x_2 y_2^2]$$

$$= 1.99 + h[-(0.1)(1.99)^2] = 1.9702$$

4th Iteration:- Put $n=3$ in ⊛

$$\Rightarrow y_4 = y_3 + h[-x_3 y_3^2]$$

$$= 1.9702 + 0.05[-(0.15)(1.9702)^2] = 1.9411$$

$$\Rightarrow y(0.2) = 1.9411 \quad \text{Ans.}$$

****

(V)

$$y' = \frac{-y^2}{1+x^2}; \quad y_0 = 1, \quad x_0 = 0, \quad h = 0.05$$

So

$$x_1 = 0.05, \quad x_2 = 0.1, \quad x_3 = 0.15, \quad x_4 = 0.2$$

To find $y(0.2)$ i.e $y(x_4)$ i.e y_4

$$f(x_n, y_n) = \frac{-y_n^2}{1+x_n^2}$$

Euler's formula is

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$\Rightarrow y_{n+1} = y_n + h\left[\frac{-y_n^2}{1+x_n^2}\right] \longrightarrow ⊛$$

1st Iteration:- Put $n=0$ in ⊛

$$\Rightarrow y_1 = y_0 + h\left[\frac{-y_0^2}{1+x_0^2}\right]$$

$$\Rightarrow y_1 = 1 + 0.05\left[\frac{-(1)^2}{1+(0)^2}\right] = 0.95$$

<u>2nd Iteration:-</u> Put $n = 1$ in ⊛

$$\Rightarrow y_2 = y_1 + h\left[\frac{-y_1^2}{1+x_1^2}\right] =$$

$$= 0.95 + 0.05\left[\frac{-(0.95)^2}{1+(0.05)^2}\right] = 0.905$$

<u>3rd Iteration:-</u> Put $n = 2$ in ⊛

$$\Rightarrow y_3 = y_2 + h\left[\frac{-y_2^2}{1+x_2^2}\right]$$

$$= 0.905 + 0.05\left[\frac{-(0.905)^2}{1+(0.10)^2}\right] = 0.8645$$

<u>4th Iteration:-</u> Put $n = 3$ in ⊛

$$\Rightarrow y_4 = y_3 + h\left[\frac{-y_3^2}{1+x_3^2}\right]$$

$$= 0.8645 + 0.05\left[\frac{-(0.8645)^2}{1+(0.15)^2}\right] = 0.828$$

$$\Rightarrow y(0.2) = 0.828$$

(vi) Given,

$$y' = 1 + x\sin(xy) \qquad x_0 = 0,\ y_0 = 0$$
$$h = 0.1$$

$x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, $x_4 = 0.4$, $x_5 = 0.5$
$x_6 = 0.6$, $x_7 = 0.7$, $x_8 = 0.8$, $x_9 = 0.9$, $x_{10} = 1$

$$f(x_n, y_n) = 1 + x_n \sin(x_n y_n)$$

Euler's formula is

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$\Rightarrow y_{n+1} = y_n + h\left[1 + x_n \sin(x_n y_n)\right] \longrightarrow ⊛$$

1st Iteration:- Put $n=0$ in ⊛

$\Rightarrow y_1 = y_0 + h[1 + x_0 \sin(x_0 y_0)]$

$= 0 + 0.1[1 + 0 \sin(0)(0)] = 0.1$

2nd Iteration:- Put $n=1$ in ⊛

$\Rightarrow y_2 = y_1 + h[1 + x_1 \sin(x_1 y_1)]$

$= 0.1 + 0.1[1 + 0.1 \sin((0.1)(0.1))] = 0.2001$

3rd Iteration:- Put $n=2$ in ⊛

$\Rightarrow y_3 = y_2 + h[1 + x_2 \sin(x_2 y_2)]$

$= 0.2001 + 0.1[1 + 0.2 \sin((0.2)(0.2001))] = 0.3009$

4th Iteration:- Put $n=3$ in ⊛

$\Rightarrow y_4 = y_3 + h[1 + x_3 \sin(x_3 y_3)]$

$= 0.3009 + 0.1[1 + 0.3 \sin((0.3)(0.3009))] = 0.4036$

5th Iteration:- Put $n=4$ in ⊛

$\Rightarrow y_5 = y_4 + h[1 + x_4 \sin(x_4 y_4)]$

$= 0.4036 + 0.1[1 + 0.4 \sin((0.4)(0.4036))] = 0.51$

6th Iteration:- Put $n=5$ in ⊛

$\Rightarrow y_6 = y_5 + h[1 + x_5 \sin(x_5 y_5)]$

$= 0.51 + 0.1[1 + 0.5 \sin((0.5)(0.51))] = 0.6226$

7th Iteration:- Put $n=6$ in ⊛

$\Rightarrow y_7 = y_6 + h[1 + x_6 \sin(x_6 y_6)]$

$= 0.6226 + 0.1[1 + 0.6 \sin((0.6)(0.6226))]$

$= 0.7445$

8th Iteration:- Put $n=7$ in ⊛

$\Rightarrow y_8 = y_7 + h\{1 + x_7 \sin(x_7 y_7)\}$

$= 0.7445 + 0.1[1 + 0.7\sin((0.7)(0.7445)] = 0.8794$

9th Iteration:- Put $n=8$ in ⊛

$\Rightarrow y_9 = y_8 + h[1 + x_8 \sin(x_8 y_8)]$

$= 0.8794 + 0.1[1 + 0.8\sin((0.8)(0.8794)] = 1.0312$

10th Iteration:- Put $n=9$ in ⊛

$\Rightarrow y_{10} = y_9 + h[1 + x_9 \sin(x_9 y_9)]$

$= 1.0312 + 0.1[1 + 0.9\sin((0.9)(1.0312))] = 1.2032$

$\Rightarrow y(1) = 1.2032$

★★★ ——————————— ★★★

Q2:-

(i) Given $y' = -2xy^2$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$

To find $y(0.2)$ i.e $y(x_2)$ i.e y_2

$f(x_n, y_n) = -2x_n y_n^2$

Euler's Improved formula is

$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n)]$

$\Rightarrow y_{n+1} = y_n + \frac{h}{2}[-2x_n y_n^2 + f(x_n + h, y_n + h(-2x_n y_n^2))]$

$= y_n + \frac{h}{2}[-2x_n y_n^2 + f(x_n + h, y_n - 2hx_n y_n^2)]$

$= y_n + \frac{h}{2}[-2x_n y_n^2 + \{-2(x_n + h)(y_n - 2hx_n y_n^2)^2\}]$

$$\Rightarrow y_{n+1} = y_n + \frac{h}{2}\left[-2x_n y_n^2 + \{(-2x_n - 2h)(y_n^2 + 4h^2 x_n^2 y_n^2 - 4y_n^2 h x_n)\}\right]$$

$$= y_n + \frac{h}{2}\left[-2x_n y_n^2 + [-2x_n y_n^2 - 8h^2 x_n^3 y_n^2 + 8h x_n^2 y_n^2 - 2h y_n^2 - 8h^3 x_n^2 y_n^2 + 8h^2 x_n y_n^2\right]$$

$$= y_n + \frac{h}{2}\left[-4x_n y_n^2 - 8h^2 x_n^3 y_n^2 + 8h x_n^2 y_n^2 - 2h y_n^2 - 8h^3 x_n^2 y_n^2 + 8h^2 x_n y_n^2\right]$$

$$\Rightarrow y_{n+1} = y_n - h\left[2x_n y_n^2 + 4h^2 x_n^3 y_n^2 - 4h x_n^2 y_n^2 + h y_n^2 + 4h^3 x_n^2 y_n^2 - 4h^2 x_n y_n^2\right] \longrightarrow (*)$$

<u>1st Iteration:-</u> Put $n=0$ in $(*)$

$$y_1 = y_0 - h\left[2x_0 y_0^2 + 4h^2 x_0^3 y_0^2 - 4h x_0^2 y_0^2 + h y_0^2 + 4h^3 x_0^2 y_0^2 - 4h^2 x_0 y_0^2\right]$$

$$= 1 - 0.1\left[2(0)(1)^2 + 4(0.1)^2(0)^3(1)^2 - 4(0.1)(0)^2(1)^2 + (0.1)(1)^2 + 4(0.1)^3(0)^2(1)^2 - 4(0.1)^2(0)(1)^2\right]$$

$$= 0.99$$

<u>2nd Iteration:-</u> Put $n=1$ in $(*)$

$$\Rightarrow y_2 = y_1 - h\left[2x_1 y_1^2 + 4h^2 x_1^3 y_1^2 - 4h x_1^2 y_1^2 + h y_1^2 + 4h^3 x_1^2 y_1^2 - 4h^2 x_1 y_1^2\right]$$

$$= 0.99 - 0.1\left[2(0.1)(0.99)^2 + 4(0.1)^2(0.1)^3(0.99)^2 - 4(0.1)(0.1)^2(0.99)^2 + 0.1(0.99)^2 + 4(0.1)^3(0.1)^2(0.99)^2 - 4(0.1)^2(0.1)(0.99)^2\right]$$

$$= 0.9614$$

Exact Value:-

(ii)

Given $y' = x^2 + y$; $x_0 = 0$, $y_0 = 1$, $h = .02$

$\Rightarrow x_0 = 0$, $x_1 = .02$, $x_2 = .04$, $x_3 = .06$

$$f(x_n, y_n) = x_n^2 + y_n$$

To find $y(.06)$ i.e $y(x_3)$ i.e y_3

Euler's Improved formula is

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_n + h/2 , y_n + h f(x_n, y_n))\right]$$

$$\Rightarrow y_{n+1} = y_n + \frac{h}{2}\left[x_n^2 + y_n + f\left(x_n + \frac{h}{2}, y_n + h(x_n^2 + y_n)\right)\right]$$

$$= y_n + \frac{h}{2}\left[x_n^2 + y_n + \left(x_n + \frac{h}{2}\right)^2 + y_n + h(x_n^2 + y_n)\right]$$

$$= y_n + \frac{h}{2}\left[x_n^2 + y_n + x_n^2 + \frac{h^2}{4} + hx_n + y_n + hx_n^2 + hy_n\right]$$

$$= y_n + \frac{h}{2}\left[2x_n^2 + 2y_n + hx_n + hx_n^2 + hy_n + \frac{h^2}{2}\right] \longrightarrow ⊛$$

<u>1st Iteration:-</u> Put $n = 0$ in ⊛

$$\Rightarrow y_1 = y_0 + \frac{h}{2}\left[2x_0^2 + 2y_0 + hx_0 + hx_0^2 + hy_0 + \frac{h^2}{2}\right]$$

$$= 1 + \frac{.02}{2}\left[2(0)^2 + 2(1) + .02(0) + .02(0)^2 + .02(1) + \frac{(.02)^2}{2}\right]$$

$$= 1.0202$$

<u>2nd Iteration:-</u> Put $n = 1$ in ⊛

$$\Rightarrow y_2 = y_1 + \frac{h}{2}\left[2x_1^2 + 2y_1 + hx_1 + hx_1^2 + hy_1 + \frac{h^2}{2}\right]$$

$$= 1.0202 + \frac{.02}{2}\left[2(.02)^2 + 2(1.0202) + .02(.02) + .02(.02)^2 + .02(1.0202) + \frac{(.02)^2}{2}\right]$$

$$= 1.0408$$

<u>3rd Iteration:-</u> Put $n = 2$ in ⊛

$$\Rightarrow y_3 = y_2 + \frac{h}{2}\left[2x_2^2 + 2y_2 + hx_2 + hx_2^2 + hy_2 + \frac{h^2}{2}\right]$$

$$= 1.0408 + \frac{.02}{2}\left[2(.04)^2 + 2(1.0408) + .02(.04) + .02(.04)^2 + .02(1.0408) + \frac{(.02)^2}{2}\right]$$

$$= 1.0619$$

$$\Rightarrow y(.06) = 1.0619$$

(iii) Given $\frac{dy}{dx} = 1 + xy$; $x_0 = 0$, $y_0 = 1$
$h = 0.1$

Now $x_0 = 0$, $x_1 = x_0 + h$
$\Rightarrow x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$

$$f(x_n, y_n) = 1 + x_n y_n$$

Euler's Improved formula is

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_n + h, y_n + h f(x_n, y_n))]$$

$$\Rightarrow y_{n+1} = y_n + \frac{h}{2}[1 + x_n y_n + f(x_n + h, y_n + h(1 + x_n y_n))]$$

$$= y_n + \frac{h}{2}[1 + x_n y_n + f(x_n + h, y_n + h + h x_n y_n)]$$

$$= y_n + \frac{h}{2}[1 + x_n y_n + \{1 + (x_n + h)(y_n + h + h x_n y_n)\}]$$

$$= y_n + \frac{h}{2}[1 + x_n y_n + 1 + x_n y_n + h x_n + h x_n^2 y_n + h y_n + h^2 + h^2 x_n y_n]$$

$$y_{n+1} = y_n + \frac{h}{2}[2 + 2x_n y_n + h x_n + h y_n + h x_n^2 y_n + h^2 x_n y_n + h^2] \longrightarrow (*)$$

1st Iteration: Put $n = 0$ in $(*)$

$$\Rightarrow y_1 = y_0 + \frac{h}{2}[2 + 2x_0 y_0 + h x_0 + h y_0 + h x_0^2 y_0 + h^2 x_0 y_0 + h^2]$$

$$= 1 + \frac{0.1}{2}[2 + 2(0)(1) + 0.1(0) + 0.1(1) + 0.1(0)^2(1) + (0.1)^2(0)(1) + (0.1)^2]$$

$$= 1.1055$$

2nd Iteration: Put $n = 1$ in $(*)$

$$\Rightarrow y_2 = y_1 + \frac{h}{2}[2 + 2x_1 y_1 + h x_1 + h y_1 + h x_1^2 y_1 + h^2 x_1 y_1 + h^2]$$

$$= 1.1055 + \frac{0.1}{2}[2 + 2(0.1)(1.1055) + 0.1(0.1) + 0.1(1.1055) + 0.1(0.1)^2(1.1055) + (0.1)^2(0.1)(1.1055) + (0.1)^2]$$

$$\Rightarrow y_2 = 1.2232$$

3rd Iteration Put $n=2$ in ⊛

$$\Rightarrow y_3 = y_2 + \frac{h}{2}\left[2+2x_2y_2+hx_2+hy_2+hx_2^2y_2+h^2x_2y_2+h^2\right]$$

$$= 1.2232 + \frac{0.1}{2}\left[2+2(0.2)(1.2232)+(0.1)(0.2)+(0.1)(1.2232)+(0.1)(0.2)^2(1.2232)+(0.1)^2(0.2)(1.2232)+(0.1)^2\right]$$

$$= 1.3556$$

(iv) Given,

$$y' = x+3y \quad ; \quad y_0 = 1,\ x_0 = 0 \quad , h = 0.1$$

Now $x_0 = 0$, $x_1 = x_0 + h = 0 + 0.1 \Rightarrow x_1 = 0.1$

$$x_2 = 0.2$$

To find $y(0.2)$ i.e $y(x_2)$ i.e y_2

Euler's Improved formula is

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n,y_n) + f(x_n+h,\ y_n + hf(x_n,y_n))\right]$$

$$f(x_n,y_n) = x_n + 3y_n$$

$$\Rightarrow y_{n+1} = y_n + \frac{h}{2}\left[x_n+3y_n+f(x_n+h,\ y_n+h(x_n+3y_n))\right]$$

$$= y_n + \frac{h}{2}\left[x_n+3y_n+f(x_n+h,\ y_n+hx_n+3hy_n)\right]$$

$$= y_n + \frac{h}{2}\left[x_n+3y_n+x_n+h+3(y_n+hx_n+3hy_n)\right]$$

$$= y_n + \frac{h}{2}\left[x_n+3y_n+x_n+h+3y_n+3hx_n+9hy_n\right]$$

$$= y_n + \frac{h}{2}\left[2x_n+6y_n+3hx_n+9hy_n+h\right] \longrightarrow ⊛$$

1st Iteration Put $n=0$ in ⊛

$$\Rightarrow y_1 = y_0 + \frac{h}{2}\left[2x_0+6y_0+3hx_0+9hy_0+h\right]$$

$$= 1 + \frac{0.1}{2}\left[2(0)+6(1)+3(0.1)(0)+9(0.1)(1)+0.1\right]$$

$$1.35$$

2nd Iteration:- Put $n=1$ in ⊛

$\Rightarrow y_2 = y_1 + \frac{h}{2}[2x_1 + 6y_1 + 3hx_1 + 9hy_1 + h]$

$= 1.35 + \frac{0.1}{2}[2(0.1) + 6(1.35) + 3(0.1)(0.1) + 9(0.1)(1.35) + 0.1]$

$= 1.8323$

3rd Iteration:- Put $n=2$ in ⊛

$\Rightarrow y_3 = 2.4924$

Q3:- (i) Given $y' = 1 + xy$; $y(0) = 1$. Let $h = 0.1$

$\Rightarrow x_1 = x_0 + h = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$

To find $y(0.3)$ i.e y_3.

Here $f(x_n, y_n) = 1 + x_n y_n$

Euler's Modified formulas is

$y_{n+1} = y_n + h f[x_n + h/2, y_n + h/2 f(x_n, y_n)]$

$= y_n + h f[x_n + h/2, y_n + h/2(1 + x_n y_n)]$

$= y_n + h f[x_n + h/2, y_n + \frac{h}{2} + \frac{h x_n y_n}{2}]$

$= y_n + h[1 + (x_n + h/2)(y_n + \frac{h}{2} + \frac{h x_n y_n}{2})]$

$= y_n + h[1 + x_n y_n + \frac{h x_n}{2} + \frac{h x_n^2}{2} + \frac{h y_n}{2} + \frac{h^2}{4} + \frac{h^2 x_n y_n}{4}] \longrightarrow$ ⊛

1st Iteration:- Put $n=0$ in ⊛

$\Rightarrow y_1 = y_0 + h[1 + x_0 y_0 + \frac{h x_0}{2} + \frac{h x_0^2}{2} + \frac{h y_0}{2} + \frac{h^2}{4} + \frac{h^2 x_0 y_0}{4}]$

$= 1 + 0.1[1 + (0)(1) + \frac{0.1(0)}{2} + \frac{0.1(0)^2}{2} + \frac{0.1(1)}{2} + \frac{(0.1)^2}{4} + \frac{(0.1)^2(0)(1)}{4}]$

$\Rightarrow y_1 = 1.1053$

<u>2nd Iteration:-</u> Put $n=1$ in ⊛

$$\Rightarrow y_2 = y_1 + h\left[1 + x_1 y_1 + \frac{h x_1}{2} + \frac{h x_1^2}{2} + \frac{h y_1}{2} + \frac{h^2}{4} + \frac{h^2 x_1 y_1}{4}\right]$$

$$= 1.1053 + 0.1\left[1 + (0.1)(1.1053) + \frac{(0.1)(0.1)}{2} + \frac{(0.1)(0.1)^2}{2} + \frac{(0.1)(1.1053)}{2} + \frac{(0.1)^2}{4} + \frac{(0.1)^2(0.1)(1.1053)}{4}\right]$$

$= 1.2227$

<u>3rd Iteration:-</u> Put $n=2$ in ⊛

$$\Rightarrow y_3 = y_2 + h\left[1 + x_2 y_2 + \frac{h x_2}{2} + \frac{h x_2^2}{2} + \frac{h y_2}{2} + \frac{h^2}{4} + \frac{h^2 x_2 y_2}{4}\right]$$

$$= 1.2227 + 0.1\left[1 + (0.2)(1.2227) + \frac{0.1(0.2)}{2} + \frac{0.1(0.2)^2}{2} + \frac{(0.1)(1.2227)}{2} + \frac{(0.1)^2}{4} + \frac{(0.1)^2(0.2)(1.2227)}{4}\right]$$

$= 1.3548$ Ans.

(ii) Given $\frac{dy}{dx} = x^2 + y^2$; $y_0 = 1$, $x_0 = 0$, $h = 0.5$

So $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$

To find $f(1)$ i.e $y(x_2)$ i.e y_2

$$f(x_n, y_n) = x_n^2 + y_n^2$$

Euler's modified formula is

$$y_{n+1} = y_n + h f\left[x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right]$$

$$= y_n + h f\left[x_n + \frac{h}{2}, y_n + \frac{h}{2}(x_n^2 + y_n^2)\right]$$

$$= y_n + h f\left[x_n + \frac{h}{2}, y_n + \frac{h x_n^2}{2} + \frac{h y_n^2}{2}\right]$$

$$y_{n+1} = y_n + h\left[\left(x_n + \frac{h}{2}\right)^2 + \left(y_n + \frac{h x_n^2}{2} + \frac{h y_n^2}{2}\right)^2\right] \longrightarrow ⊛$$

1st Iteration:- Put $n=0$ in (*)

$$\Rightarrow y_1 = y_0 + h\left[(x_0 + h/2)^2 + \left(y_0 + \frac{hx_0^2}{2} + \frac{hy_0^2}{2}\right)^2\right]$$

$$= 1 + 0.5\left[\left\{0 + \frac{0.5}{2}\right\}^2 + \left\{1 + \frac{0.5(0)^2}{2} + \frac{0.5(1)^2}{2}\right\}^2\right]$$

$$= 1 + 0.5[0.0625 + 1.5625]$$

$$= 1.8125$$

2nd Iteration:- Put $n=1$ in (*)

$$\Rightarrow y_2 = y_1 + h\left[(x_1 + h/2)^2 + \left(y_1 + \frac{hx_1^2}{2} + \frac{hy_1^2}{2}\right)^2\right]$$

$$= 1.8125 + 0.5\left[\left\{0.5 + \frac{0.5}{2}\right\}^2 + \left\{1.8125 + \frac{0.5(0.5)^2}{2} + \frac{0.5(1.8125)^2}{2}\right\}^2\right]$$

$$= 1.8125 + 0.5[0.5625 + 7.27]$$

$$= 5.7287$$

***

(iii) $\frac{dy}{dx} = x - y$; $x_0 = 0$, $y = 1$, $h = 0.1$

$\Rightarrow x_1 = 0.1$, $x_2 = 0.2$

$f(x_n, y_n) = x - y$

Euler's Modified formula is

$$y_{n+1} = y_n + hf\left[x_n + h/2,\ y_n + h/2\, f(x_n, y_n)\right]$$

$$= y_n + hf\left[x_n + h/2,\ y_n + h/2(x_n - y_n)\right]$$

$$= y_n + hf\left[\left(x_n + h/2\right), \left(y_n + \frac{hx_n}{2} - \frac{hy_n}{2}\right)\right]$$

$$= y_n + h\left[x_n + \frac{h}{2} - y_n - \frac{hx_n}{2} + \frac{hy_n}{2}\right] \longrightarrow (*)$$

1st Iteration:- Put $n=0$ in ⊛

$$\Rightarrow y_1 = y_0 + h\left[x_0 + \frac{h}{2} - y_0 - \frac{hx_0}{2} + \frac{hy_0}{2}\right]$$

$$= 1 + 0.1\left[0 + \frac{0.1}{2} - 1 - \frac{0.1(0)}{2} + \frac{(0.1)(1)}{2}\right]$$

$$= 0.9055$$

2nd Iteration:- Put $n=1$ in ⊛

$$\Rightarrow y_2 = y_1 + h\left[x_1 + \frac{h}{2} - y_1 - \frac{hx_1}{2} + \frac{hy_1}{2}\right]$$

$$= 0.9055 + 0.1\left[0.1 + \frac{0.1}{2} - 0.9055 - \frac{0.1(0.1)}{2} + \frac{0.1(0.9055)}{2}\right]$$

$$= 0.834$$

Exact Value:-

$$\frac{dy}{dx} = x - y$$

$$\Rightarrow \frac{dy}{dx} + y = x \qquad \text{Inte. F} = e^{\int 1\,dx}$$

$$\text{I.F} = e^{x}$$

$$\Rightarrow e^{x}y = \int e^{x}x\,dx \Rightarrow e^{x}y = e^{x}x - \int e^{x}$$

$$\Rightarrow e^{x}y = e^{x}x - e^{x} + c \qquad \Rightarrow y = x - 1 + ce^{-x}$$

Now $y(0) = 1$

$$\Rightarrow 0 - 1 + c = 1 \Rightarrow c = 2$$

$$\Rightarrow y = x - 1 + 2e^{-x}$$

$$y(0.2) = 0.8375$$

iv) Taylor's Series Algorithm:-

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \; ; \; y(x_0) = y_0 \;, \; h \text{ being step size.}$$

$$y_1 = y(x_1) = y(x_0 + h)$$
$$= y(x_0) + h\, y'(x_0) + \frac{h^2}{2!} y''(x_0) + \cdots$$

$$\Rightarrow y_1 = y_0 + h\, y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

Similarly

$$y_2 = y_1 + h\, y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$$

$$y_3 = y_2 + h\, y_2' + \frac{h^2}{2!} y_2'' + \frac{h^3}{3!} y_2''' + \cdots$$

$$\vdots$$

$$y_{n+1} = y_n + h\, y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \cdots$$

This is called Taylor's Series Algorithm

further

$$y_{n+1} = y_n + h\, y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \cdots + \frac{h^k}{k!} y_n^{(k)}$$

is called Taylor's series formula of order K.

***

Example:- Given $\frac{dy}{dx} = 2xy + x^2$, $y(1) = 3$, $h = 0.2$ find $y(2)$ by using Taylor's Series formula of order 3

Solution Given $y' = 2xy + x^2$

$x_0 = 1$, $y_0 = 3$, $h = 0.2$

Now $x_0 = 1$, $x_1 = x_0 + h = 1 + 0.2 = 1.2$

$x_2 = 1.4$, $x_3 = 1.6$, $x_4 = 1.8$, $x_5 = 2$

To find $y(2)$ i.e $y(x_5)$ i.e y_5

Now $y' = 2xy + x^2$

$y'' = 2xy' + 2y + 2x$

$y''' = 2xy'' + 2y' + 2y' + 2$

$= 2xy'' + 4y' + 2$

Now by Taylor series formula of order 3

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n'''$$

$$= y_n + hy_n' + \frac{h^2}{2}y_n'' + \frac{h^3}{6}y_n''' \longrightarrow (*)$$

where $y_n' = 2x_n y_n + x_n^2$

$y_n'' = 2x_n y_n' + 2y_n + 2x_n$

$y_n''' = 2x_n y_n'' + 4y_n' + 2$

1st Iteration:- when $n = 0$

$$y_1 = y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0'''$$

$y_0' = 2x_0 y_0 + x_0^2$
$= 2(1)(3) + (1)^2 = 7$

$y_0'' = 2x_0 y_0' + 2y_0 + 2x_0$
$= 2(1)(7) + 2(3) + 2(1) = 22$

$y_0''' = 2x_0 y_0'' + 4y_0' + 2$
$= 2(1)(22) + 4(7) + 2 = 74$

$\Rightarrow y_1 = 3 + 0.2(7) + \frac{(0.2)^2}{2}(22) + \frac{(0.2)^3}{6}(74)$

$= \cancel{5.037}\ 4.9387$

2nd Iteration:- Put $n=1$ in ⓐ

$y_2 = y_1 + h y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1'''$

Now $y_1' = 2x_1 y_1 + x_1^2$
$= 2(1.2)(5.037) + (1.2)^2 = 13.5288$

$y_1'' = 2x_1 y_1' + 2y_1 + 2x_1$
$= 2(1.2)(13.5288) + 2(5.037) + 2(1.2)$
$= 44.9431$

$y_1''' = 2x_1 y_1'' + 4y_1' + 2$
$= 2(1.2)(44.9431) + 4(13.5288) + 2$
$= 163.9787$

So $y_2 = 5.037 + 0.2(13.5288) + \frac{(0.2)^2}{2}(44.9431) + \frac{(0.2)^3}{6}(163.9787)$

$= 9.2975$

3rd Iteration:- Put $n=2$ in ⊛

$$y_3 = y_2 + h y_2' + \frac{h^2}{2} y_2'' + \frac{h^3}{6} y_2'''$$

Now $y_2' = 2x_2 y_2 + x_2^2$

$= 2(1.4)(9.2975) + (1.4)^2 = 27.993$

$y_2'' = 2x_2 y_2' + 2y_2 + 2x_2$

$= 2(1.4)(27.993) + 2(9.2975) + 2(1.4)$

$= 99.7754$

$y_2''' = 2x_2 y_2'' + 4y_2' + 2$

$= 2(1.4)(99.7754) + 4(27.993) + 2$

$= 393.3431$

$\Rightarrow y_3 = 9.2975 + 0.2(27.993) + \frac{(0.2)^2}{2}(99.7754) + \frac{(0.2)^3}{6}(393.3431)$

$= 17.4161$

4th Iteration:- Put $n=3$ in ⊛

$\Rightarrow y_4 = y_3 + h y_3' + \frac{h^2}{2} y_3'' + \frac{h^3}{6} y_3'''$

Now $y_3' = 2x_3 y_3 + x_3^2$

$y_3'' = 2x_3 y_3' + 2y_3 + 2x_3$

$y_3''' = 2x_3 y_3'' + 4y_3' + 2$

& so on

upto y_6

★★★

Example:- Find value of $y(0.3)$ for I.V.P (initial value problem)

$y' = -2xy^2$; $y(0) = 1$, $h = 0.1$

using Taylor series Algorithm of order 2.

Solution Given $y' = -2xy^2$

$x_0 = 0$, $y_0 = 1$, $h = 0.1$

Now $x_0 = 0$, $x_1 = x_0 + h \Rightarrow x_1 = 0.1$

$x_2 = 0.2$, $x_3 = 0.3$

To find $y(0.3)$ i.e $y(x_3)$ i.e y_3

Now $y' = -2xy^2$

$y'' = -2x2yy' - 2y^2 = -4xyy' - 2y^2$

$= -4xy(-2xy^2) - 2y^2$

$y'' = 8x^2y^3 - 2y^2$

Talyor series Algorithm of order 2 is

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!}y_n''$$

Putting value

$$y_{n+1} = y_n + h(-2x_ny_n^2) + \frac{h^2}{2}(8x_n^2y_n^3 - 2y_n^2)$$

$$y_{n+1} = y_n - 2hx_ny_n^2 + h^2(4x_n^2y_n^3 - y_n^2) \longrightarrow *$$

Since $y_n' = -2x_ny_n^2$ 4

$y_n'' = 8x_n^2y_n^3 - 2y_n^2$

1st Iteration:- Put $n=0$ in ⊛

$$\Rightarrow y_1 = y_0 - 2hx_0y_0^2 + h^2(4x_0^2y_0^3 - y_0^2)$$

$$= 1 - 2(0.1)(0)(1)^2 + (0.1)^2[4(0.0)(1)^2 - 1^2]$$

$$= 0.990$$

2nd Iteration:- Put $n=1$ in ⊛

$$y_2 = y_1 - 2hx_1y_1^2 + h^2(4x_1^2y_1^3 - y_1^2)$$

$$= 0.99 - 2(0.1)(0.1)(0.99)^2 + (0.1)^2[4(0.1)^2(0.99)^3 - (0.99)^2]$$

$$y_2 = 0.961$$

3rd Iteration:- Put $n=2$ in ⊛

$$y_3 = y_2 - 2hx_2y_2^2 + h^2[4x_2^2y_2^3 - y_2^2]$$

$$= 0.961 - 2(0.1)(0.2)(0.961)^2 + (0.1)^2[4(0.2)^2(0.961)^3 - (0.961)^2]$$

$$= 0.916244$$

$$\Rightarrow y(0.3) = 0.916244 \text{ Ans.}$$

Exact Value:- $\frac{dy}{dx} = -2xy^2$

$$\int \frac{dy}{y^2} = \int -2x\,dx$$

$$\frac{y^{-1}}{-1} = -2\frac{x^2}{2} + c \Rightarrow \frac{-1}{y} = -x^2 + c$$

Given $y(0) = 1$

$$-\frac{1}{y} = 0 + c \Rightarrow -1 = c$$

$$\Rightarrow C = -1$$

$$\Rightarrow -\frac{1}{y} = -x^2 - 1 \Rightarrow \frac{1}{y} = 1 + x^2$$

$$y = \frac{1}{1+x^2}$$

$$y(0.3) = \frac{1}{1+(0.3^2)} = 0.917431$$

$$\text{Error} = 0.917431 - 0.916244$$

$$= 0.001187$$

***

Example If y satisfies equation $dy/dx = x^2 y - 1$, $y(0) = 1$ using Taylor's expension obtain y as series in power of x. Also find $y(0.3)$ to 5 d.p.

Solution Taylor's series

$$y = y_0 + x y_0' + \frac{x^2}{2!} y_0'' + \frac{x^3}{3!} y_0''' + \frac{x^4}{4!} y_0^{iv} + \cdots \quad (1)$$

when $x_0 = 0$, $y_0 = 1$

Given $y' = x^2 y - 1 \Rightarrow y_0' = x_0^2 y_0 - 1$

$$\Rightarrow y_0' = (0)^2(1) - 1 = -1$$

$$y'' = 2xy + x^2 y' \Rightarrow y_0'' = 2x_0 y_0 + x_0^2 y_0'$$

$$y_0'' = 2(0)(1) + (0)^2(-1)^2 = 0$$

$$y_0''' = 2y_0 + 2x_0 y_0' + x_0^2 y_0'' + 2x_0 y_0'$$

$$= 2(1) + 2(0)(-1)^2 + (0)^2(0)^2 + 2(0)(-1)$$

$$= 2$$

$$y_0^{IV} = 2y_0' + 2x_0 y_0'' + 2y_0' + 2x_0 y_0'' + x_0^2 y_0''' + 2x_0 y_0'' + 2y_0'$$

$$= 2(-1) + 2(0)(0) + 2(-1) + 2(0)(0) + (0)^2(2) + 2(0)(0) + 2(-1)$$

$$= -6$$

Put all value in ①

$$y = 1 + x(-1) + 0 + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \cdots$$

$$y = 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

which is required Taylor series expension of y in power of x.

Put $x = 0.3$

$$y(0.3) = 1 - (0.3) + \frac{(0.3)^3}{3} - \frac{(0.3)^4}{4} + \cdots$$

$$= 1 - 0.3 + 0.0090 - 0.0020$$

$$= 0.7070 \text{ Ans.}$$

Example:- Find $y(0.3)$ for I.V.P
$y' = x + y$ where $y(0) = 1$.
Apply Taylor series Algorithm of order 3 with $h = 0.1$ comparing with exact result.

Solution As Taylor's series Algorithm of order 3 is

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' \longrightarrow (1)$$

Given $y' = x + y$

$$y'' = 1 + y' \quad , \quad y''' = y''$$

So

$$y_n' = x_n + y_n$$

$$y_n'' = 1 + y_n' = 1 + x_n + y_n$$

$$y_n''' = y_n'' = 1 + x_n + y_n$$

Put all in (1)

$$y_{n+1} = y_n + h(x_n + y_n) + \frac{h^2}{2}(1 + x_n + y_n) + \frac{h^3}{6}(1 + x_n + y_n) \longrightarrow (*)$$

To find $y(0.3)$

Now $x_0 = 0$, $h = 0.1$

$\therefore x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$

we have to find

$y(x_3)$ i.e y_3

1st Iteration:- Put $n=0$ in ⊛

$$y_1 = y_0 + h(x_0+y_0) + \frac{h^2}{2}(1+x_0+y_0) + \frac{h^3}{6}(1+x_0+y_0)$$

$$= 1 + 0.1(0+1) + \frac{(0.1)^2}{2}(1+0+01) + \frac{(0.1)^3}{6}(1+0+01)$$

$$= 1.11033$$

2nd Iteration:- Put $n=1$ in ⊛

$$y_2 = y_1 + h(x_1+y_1) + \frac{h^2}{2}(1+x_1+y_1) + \frac{h^3}{6}(1+x_1+y_1)$$

$$= 1.11033 + 0.1(0.1+1.11033) + \frac{(0.1)^2}{2}(1+0.1+1.11033) + \frac{(0.1)^3}{6}(1+0.1+1.11033)$$

$$= 1.2428$$

3rd Iteration:- Put $n=2$ in ⊛

$$y_3 = y_2 + h(x_2+y_2) + \frac{h^2}{2}(1+0.2+y_2) + \frac{h^3}{6}(1+x_2+y_2)$$

$$= 1.2428 + 0.1(0.2+1.2428) + \frac{(0.1)^2}{2}(1+0.2+1.2428) + \frac{(0.1)^3}{6}(1+0.2+1.2428)$$

$$= 1.39986$$

Exact Solution:- $\frac{dy}{dx} = x+y$

$$\frac{dy}{dx} - y = x$$

$$\text{I.F} \quad e^{-\int dx} = e^{-x}$$

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}x$$

$$\int \frac{d}{dx}(e^{-x}y) = \int e^{-x}x$$

$$e^{-x}y = -xe^{-x} - e^{-x} + c$$

$$y = -x + ce^{x} - 1$$

using condition $y(0) = 1$

$$1 = -(0) + ce^{0} - 1$$

$$\Rightarrow 2 = c$$

$$\Rightarrow y = -x - 1 + 2e^{x\cancel{(0.3)}}$$

Put $x = 0.3$

$$\Rightarrow y = -x - 1 + 2e^{(0.3)}$$

$$\Rightarrow y = 1.399718$$

Error = Exact value − Approx. Value

$$= 1.399718 - 1.399686$$

$$= 0.000032$$

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ASSIGNMENT

Q1:- Using Taylor's series Algorithm of order 2 find $y(0.2)$, $y(0.4)$ and $y(0.6)$ of I.V.P is given that

$$y' = -xy^2 \,, \quad y(0) = 2$$

Compare with exact value.

Q2:- Find approximate value of $y(0.1)$, $y(0.2)$ of $y' = (x+1)y \; ; \; y(0) = 1$ using T.S algorithm of order 3 $(h = 0.1)$

Q3:- Using Taylor's series find solution of

$$y' = -2xy^2 \,, \quad y(0) = 1$$

at $x = 0.3$ correct upto 5 d.p

Q4:- Using Taylor's series find solution of $xy' = x - y \; ; \; y(2) = 2$ at $x = 2.1$ correct to 5 d.p (Hint : $y' = 1 - y/x$)

Q5:- Given $y' = y^2 + 1 \; ; \; y(0) = 0$ obtain y as a series of power of x. Also find $y(0.2)$. check your answer with exact solution.

Ans: $y(0.2) = 0.202709$

& $y = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots$

SOLUTIONS

Q1:- Given,

$y' = -xy^2$; $x_0 = 0$, $y_0 = 2$, $h = 0.2$

So $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$

Taylor series of order 2 is

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' \longrightarrow ①$$

Here $y_n' = -x_n y_n^2$

$$y_n'' = -x_n \cdot 2y_n \cdot y_n' - y_n^2 = -x_n \cdot 2y_n(-x_n y_n^2) - y_n^2$$

$$= 2x_n^2 y_n^3 - y_n^2 \quad \text{put in eqn ①}$$

$$\Rightarrow y_{n+1} = y_n + h(-x_n y_n^2) + \frac{h^2}{2!}(2x_n^2 y_n^3 - y_n^2)$$

$$= y_n - h(x_n y_n^2) + \frac{h^2}{2}(2x_n^2 y_n^3 - y_n^2) \longrightarrow ⊛$$

<u>1st Iteration</u>- Put $n = 0$ in ⊛

$$\Rightarrow y_1 = y_0 - h(x_0 y_0^2) + \frac{h^2}{2}(2x_0^2 y_0^3 - y_0^2)$$

$$= 2 - 0.2((0)(2)^2) + \frac{(0.2)^2}{2}[2(0)^2(2)^3 - (2)^2]$$

$$= 1.98$$

<u>2nd Iteration</u>- Put $n = 1$ in ⊛

$$\Rightarrow y_2 = y_1 - h(x_1 y_1^2) + \frac{h^2}{2}(2x_1^2 y_1^3 - y_1^2)$$

$$= 1.98 - 0.2[(0.2)(1.98)^2] + \frac{(0.2)^2}{2}[2(0.2)^2(1.98)^3 - (0.2)^2]$$

$$= 1.98 - 0.156816 - 0.065988$$

$$= 1.7572$$

<u>Exact Value:</u>-

$$\frac{dy}{dx} = -xy^2$$

$$\Rightarrow \frac{dy}{y^2} = -x\,dx$$

$$\Rightarrow \int \frac{dy}{y^2} = -\int x\,dx$$

$$\Rightarrow \int y^{-2}\,dy = -\frac{x^2}{2} + c$$

$$\Rightarrow \frac{y^{-1}}{-1} = \frac{-x^2}{2} + c \quad \Rightarrow \frac{-1}{y} = \frac{-x^2}{2} + c$$

$$\Rightarrow \frac{1}{y} = \frac{x^2}{2} - c \quad \Rightarrow \frac{1}{y} = \frac{x^2-2c}{2}$$

$$\Rightarrow y = \frac{2}{x^2-2c} \longrightarrow ②$$

Now

$$2 = \frac{2}{0-2c} \Rightarrow 2 = -\frac{1}{c} \Rightarrow 2c = -1$$

$$\Rightarrow c = -0.5 \quad \text{Put in ②}$$

$$y = \frac{2}{x^2-2(-0.5)} \Rightarrow \text{At } x = 0.4$$

$$y = \frac{2}{(0.4)^2-2(-0.5)} \Rightarrow y = 1.7241$$

* * *

Q2:- Given $y' = (x+1)y$; $y_0 = 1$, $x_0 = 0$, $h = 0.1$

$\Rightarrow x_1 = 0.1$, $x_2 = 0.2$

$y' = xy + y$, $y'' = xy' + y + y'$

$y''' = xy'' + y' + y' + y''$

$= xy'' + 2y' + y''$

Taylor's series algorithm of order 3 is

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' \longrightarrow ①$$

$y_n' = x_n y_n + y_n$, $y_n'' = x_n y_n' + y_n + y_n'$

$y_n''' = x_n y_n'' + 2y_n' + y_n''$

1st Iteration:- When $n=0$

$$\Rightarrow y_1 = y_0 + h y_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{6} y_0'''$$

Now

$$y_0' = x_0 y_0 + y_0 = 0(1)+1 = 1$$

$$y_0'' = x_0 y_0' + y_0 + y_0' = 0(1)+1+1 = 2$$

$$y_0''' = x_0 y_0'' + 2y_0' + y_0'' = 0(2)+2(1)+2 = 4$$

$$\Rightarrow y_1 = 1 + 0.1(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(4)$$

$$= 1.1107$$

2nd Iteration:- When $n=1$

$$\Rightarrow y_2 = y_1 + h y_1' + \frac{h^2}{2} y_1'' + \frac{h^3}{6} y_0'''$$

Now

$$y_1' = x_1 y_1 + y_1 = 0.1(1.1107) + 1.1107$$

$$= 1.12177$$

$$y_1'' = x_1 y_1' + y_1 + y_1' = 0.1(1.12177) + 1.1107 + 1.12177$$

$$= 2.34465$$

$$y_1''' = x_1 y_1'' + 2y_1' + y_1'' = 0.1(2.34465) + 2(1.12177) + 2.34465$$

$$= 4.82266$$

$$\Rightarrow y_2 = 1.1107 + 0.1(1.12177) + \frac{(0.1)^2}{2}[2.34465] + \frac{(0.1)^3}{6}(4.82266)$$

$$= 1.2347$$

Q3:- Given

$$y' = -2xy^2 \quad ; \; y_0 = 1, \; x_0 = 0$$

Let $h = 0.1$

$$\Rightarrow x_1 = 0.1, \quad x_2 = 0.2, \quad x_3 = 0.3$$

$$y' = -2xy^2$$

$$y'' = -2x \cdot 2yy' - 2y^2 = -4xyy' - 2y^2$$

$$\Rightarrow y'' = -4xy(-2xy^2) - 2y^2$$
$$= 8x^2y^3 - 2y^3$$

Taylor's Series Algorithm of order 2 is

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' \longrightarrow (1)$$

Here
$$y_n' = -2x_n y_n^2$$
$$y_n'' = 8x_n^2 y_n^3 - 2y_n^3 \quad \text{Put in (1)}$$

$$\Rightarrow y_{n+1} = y_n + h(-2x_n y_n^2) + \frac{h^2}{2}(8x_n^2 y_n^3 - 2y_n^3)$$
$$= y_n - 2h[x_n y_n^2] + h^2[4x_n^2 y_n^3 - y_n^2]$$

<u>1st Iteration:-</u> When $n=0$

$$\Rightarrow y_1 = y_0 - 2h[x_0 y_0^2] + h^2[4x_0^2 y_0^3 - y_0^2]$$
$$= 1 - 2(0.1)[(0)(1)^2] + (0.1)^2[4(0)^2(1)^3 - (1)^2]$$
$$= 0.99$$

<u>2nd Iteration:-</u> When $n=1$

$$\Rightarrow y_2 = y_1 - 2h[x_1 y_1^2] + h^2[4x_1^2 y_1^3 - y_1^2]$$
$$= 0.99 - 2(0.1)[(0.1)(0.99)^2] + (0.1)^2[4(0.1)^2(0.99)^3 - (0.99)^2]$$
$$= 0.99 - 0.019602 - 0.0094128804$$
$$= 0.961427$$

3rd Iteration:- when $n=2$

$$\Rightarrow y_3 = y_2 - 2h[x_2 y_2^2] + h^2[4x_2^2 y_2^3 - y_2^2]$$

$$= 0.961427 - 2(0.1)[(0.2)(0.961427)^2] + (0.1)^2[4(0.2)^2 (0.961427)^3 - (0.961427)^2]$$

$$= 0.961427 - 0.0369736 7505 - 0.00782151918$$

$$= 0.9166 3181$$

★★★★

Q4:- Given

$$y' = 1 - y/x \quad ; y_0 = 2, \; x_0 = 2 \;, h = 0.1$$

$$\Rightarrow x_0 = 2 \quad , x_1 = x_0 + h = 2.1$$

To find $y(2.1)$ i.e $y(x_1)$ i.e y_1

$$y' = 1 - y/x$$

$$y'' = -\frac{xy' - y}{x^2} = \frac{y - xy'}{x^2}$$

$$y''' = \frac{x^2 \frac{d}{dx}(y - xy') - (y - xy)\, d/dx(x^2)}{x^4}$$

$$= \frac{x^2[y' - xy'' - y'] - (y - xy)(2x)}{x^4}$$

$$= \frac{x^2y' - x^3y'' - x^2y' - [2xy - 2x^2y]}{x^4}$$

$$\Rightarrow y''' = \frac{2x^2y - 2xy - x^3y''}{x^4}$$

So $y_n' = 1 - y_n/x_n$

$$y_n'' = \frac{y_n - x_n y_n'}{x_n^2}$$

$$y_n''' = \frac{2x_n^2 y_n - 2x_n y_n - x_n^3 y_n''}{x_n^4}$$

Taylor Series Algorithm of order 3 is

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' \longrightarrow (1)$$

Put $n=0$

$$\Rightarrow y_1 = y_0 + h y_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{6} y_0'''$$

Now $y_0' = 1 - y_0/x_0 = 1 - 2/2 = 0$

$$y_0'' = \frac{y_0 - x_0 y_0'}{x_0^2} = \frac{2 - 2(0)}{(2)^4} = 0.5$$

$$y_0''' = \frac{2x_0^2 y_0 - 2x_0 y_0 - x_0^3 y_0''}{x_0^4} = \frac{2(2)^2(2) - 2(2)(2) - (2)^3(0.5)}{(2)^4}$$

$$\Rightarrow y_0''' = 0.25$$

$$\Rightarrow y_1 = 2 + 0.1(0) + \frac{(0.1)^2}{2}[0.5] + \frac{(0.1)^3}{6}[0.25]$$

$$= 2.0025417$$

& $x_1 = x_0 + h = 2 + 0.1 \Rightarrow x_1 = 2.1$

$$\Rightarrow y(2.1) = 2.0025417$$

Exact Value:-

$$\Rightarrow \frac{dy}{dx} = 1 - \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = 1 \longrightarrow (1)$$

$$I.F \quad e^{\int \frac{1}{x} dx} = e^{\int x^{-1} dx} = e^{\ln x}$$

$$I.F = x$$

$$\Rightarrow xy = \int x\,dx$$

$$\Rightarrow xy = \frac{x^2}{2} + c \qquad \Rightarrow y = \frac{x}{2} + \frac{c}{x}$$

$$\Rightarrow y = \frac{x^2 + 2c}{2x}$$

Now $2 = \frac{4 + 2c}{4} \Rightarrow 2 = \frac{\cancel{2}(2+c)}{\cancel{4}2}$

$$\Rightarrow 2 + c = 4 \Rightarrow c = 2$$

So $y = \frac{x^2 + 4}{2x}$

At $x = 2.1$

$$y = \frac{(2.1)^2 + 4}{2(2.1)} \Rightarrow y = 2.0024$$

Q5:- Given $y' = y^2 + 1$

$y_0 = 0, \quad x_0 = 0, \quad h = 0.1$

Series in power of x

$$y = y_0 + x y_0' + \frac{x^2}{2!} y_0'' + \frac{x^3}{3!} y_0''' + \cdots$$

Now

$$y_0' = y_0^2 + 1 = 0 + 1 = 1$$

$$y_0'' = 2y_0 y_0' + 0 = 2(0)(1) + 0 = 0$$

$$y_0''' = 2y_0' y_0' + 2y_0 y_0'' = 2(1)(1) + 2(0)(0) = 2$$

$$y_0^{iv} = 2y_0'' y_0' + 2y_0' y_0'' + 2y_0' y_0'' + 2y_0 y_0'''$$

$$= 6y_0' y_0'' + 2y_0 y_0''' = 6(0)(0) + 2(0)(0)$$

$$= 0$$

$$y_0^{(v)} = 6y_0'' y_0'' + 6y_0' y_0''' + 2y_0' y_0''' + 2y_0 y_0^{iv}$$

$$= 6(0)(0) + 6(1)(2) + 2(1)(0) + 2(0)(2)$$

$$= 12$$

$$\Rightarrow y = 0 + x(1) + \frac{x^2}{2}(0) + \frac{x^3}{6}\{2\} + \frac{x^4}{24}\{0\} + \frac{x^5}{120}\{12\} + \cdots$$

$$= x + \frac{x^3}{3} + \frac{x^5}{10} + \cdots$$

$$\Rightarrow y(0.2) = 0.2 + \frac{(0.2)^3}{3} + \frac{(0.2)^5}{10} + \cdots$$

$$= 0.202699$$

**

5) Runge-Kutta Method:-
(R-K Method)

Consider the differential equation

$$\frac{dy}{dx} = f(x,y) \quad ; \quad y(x_0) = y_0$$

Integrate from x_0 to x_1

$$\int_{x_0}^{x_1} \frac{dy}{dx}\, dx = \int_{x_0}^{x_1} f(x,y)\, dx$$

$$y\Big|_{x_0}^{x_1} = \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1)\right] \quad \because \text{By Trapezoidal Rule}$$

$$y(x_1) - y(x_0) = \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1)\right]$$

$$y_1 - y_0 = \frac{h}{2}\left[f(x_0,y_0) + f(x_0+h, y_0+hf(x_0,y_0))\right]$$

$$\Rightarrow \quad y_1 = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_0+h, y_0+hf(x_0,y_0))\right]$$

Similarly

$$y_2 = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_1+h, y_1+hf(x_1,y_1))\right]$$

$$y_3 = y_2 + \frac{h}{2}\left[f(x_2,y_2) + f(x_2+h, y_2+hf(x_2,y_2))\right]$$

$$\vdots$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n,y_n) + f(x_n+h, y_n+hf(x_n,y_n))\right]$$

Put $K_1 = hf(x_n,y_n)$

$K_2 = hf(x_n+h, y_n+k_1)$

This is called R-K method of order 2.

i.e $y_{n+1} = y_n + \frac{1}{2}(K_1+K_2)$

⇒ R-K Method of order 2:-

$$y_{n+1} = y_n + \frac{1}{2}[K_1 + K_2]$$

where $K_1 = h f(x_n, y_n)$

$K_2 = h f(x_n + h, y_n + K_1)$

⇒ R-K Method of order 4:-

$$y_{n+1} = y_n + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

where $K_1 = h f(x_n, y_n)$

$K_2 = h f(x_n + h/2, y_n + \frac{K_1}{2})$

$K_3 = h f(x_n + h/2, y_n + \frac{K_2}{2})$

$K_4 = h f(x_n + h, y_n + K_3)$

⇒ R-K Method of Order 3:-

$$y_{n+1} = y_n + \frac{1}{6}[K_1 + 4K_2 + K_3]$$

where

$K_1 = h f(x_n, y_n)$

$K_2 = h f(x_n + h/2, y_n + K_1/2)$

$K_3 = h f(x_n + h, y_n - K_1 + 2K_2)$

★★ ———————————— ★★

Example:- Apply R-K formula of order 2 to approximate value of y when $x = 1.2$ given
$\frac{dy}{dx} = 3x + y^2$ and $y = 1.2$ when $x = 1$

Solution:- Given $\frac{dy}{dx} = 3x + y^2$

$x_0 = 1$, $y_0 = 1.2$ Let $h = 0.1$

$\therefore$ $x_1 = 1.1$, $x_2 = 1.2$

To find y at $x = 1.2$ we have to find $y(1.2)$ i.e $y(x_2)$ i.e y_2

R-K method of order 2 is

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2) \longrightarrow (*)$$

where $k_1 = h f(x_n, y_n)$ &

$k_2 = h f(x_n + h, y_n + k_1)$

1st Iteration:- Put $n = 0$ then

$$y_1 = y_0 + \frac{1}{2}[k_1 + k_2] \longrightarrow (1)$$

where $k_1 = h f(x_0, y_0)$

$= h(3x_0 + y_0^2)$ $\because f(x,y) = 3x + y^2$

$= h(3(1) + (1.2)^2)$

$= 0.444$

$k_2 = h f(x_0 + h, y_0 + k_1)$

$= h[3(x_0 + h) + (y_0 + k_1)^2]$

$\Rightarrow \quad K_2 = 0.1[3(1+0.1) + (1.2+0.444)^2]$

$= 0.600$ Put values in equ ①

$y_1 = 1.2 + \frac{1}{2}[0.444 + 0.600]$

$y_1 = 1.722$

<u>2nd Iteration:-</u> Put $n=1$ then

$y_2 = y_1 + \frac{1}{2}[K_1 + K_2]$

where $K_1 = h f(x_1, y_1)$

$= h[3x_1 + y_1^2] = 0.1[3(1.1) + (1.722)^2]$

$= 0.6265$

$K_2 = h f[x_1 + h, y_1 + k_1]$

$= h[3(x_1+h) + (y_1+k_1)^2]$

$= 0.1[3(1.1+0.1) + (1.722+0.6265)^2]$

$= 0.91155$

So $y_2 = 1.722 + \frac{1}{2}[0.6265 + 0.91155]$

$= 2.4910$

$\therefore \quad y(1.2) = 2.4910$ Ans.

***

Example:- Solve following differential equation using R-K method of order 4 $\frac{dy}{dx} = 1+y^2$; $y(0) = 0$
$h = 0.2$

find $y(0.2)$ and $y(0.4)$. compare with exact value.

Solution:- Given $\frac{dy}{dx} = 1+y^2$

$$f(x,y) = 1+y^2 \Rightarrow f(x_n, y_n) = 1+y_n^2$$

$$x_0 = 0 \quad , \quad y_0 = 0 \quad , \quad h = 0.2$$

R-K method of order 4 is

$$y_{n+1} = y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

where $k_1 = h f(x_n, y_n)$, $k_2 = h f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$

$k_3 = h f(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$, $k_4 = h f(x_n + h, y_n + k_3)$

Step I :- When $n=0$

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4],$$

$$k_1 = h f(x_0, y_0) = h(1 + y_0^2)$$

$$= 0.2[1 + 0^2] = 0.2$$

$$k_2 = h f[x_0 + h/2, y_0 + \frac{k_1}{2}] = h[1 + (y_0 + \frac{k_1}{2})^2]$$

$$= 0.2[1 + (0 + \frac{0.2}{2})^2] \Rightarrow k_2 = 0.202$$

$$k_3 = h f[x_0 + h/2, y_0 + \frac{k_2}{2}] = h[1 + (y_0 + \frac{k_2}{2})^2]$$

$$= 0.2[1 + (0 + \frac{0.202}{2})^2] \Rightarrow k_3 = 0.2020402$$

$$k_4 = h f[x_0 + h, y_0 + k_3] \Rightarrow k_4 = h[1 + (y_0 + k_3)^2]$$

$$= 0.2[1 + (0 + 0.20204)^2] = 0.208164048$$

So $y_1 = 0 + \frac{1}{6}[0.2 + 2(0.202) + 2(0.20204) + 0.208164]$

$= 0.202707$

$x_1 = x_0 + h = 0 + 0.2 = 0.2$

$\Rightarrow y(0.2) = 0.202707$

<u>Step II</u> :- When $n = 1$

$y_2 = y_1 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

where $K_1 = hf(x_1, y_1) = h(1 + y_1^2)$

$\Rightarrow K_1 = 0.2(1 + (0.202707)^2) = 0.2082181$

$K_2 = hf[x_1 + h/2, y_1 + K_1/2]$

$= h\left[1 + \left(y_1 + \frac{K_1}{2}\right)^2\right] = 0.2\left[1 + \left(0.202707 + \frac{0.208218}{2}\right)^2\right]$

$= 0.2188272177$

$K_3 = hf\left[x_1 + h/2, y_1 + \frac{K_2}{2}\right]$

$= h\left[1 + \left(y_1 + \frac{K_2}{2}\right)^2\right]$

$= 0.2\left[1 + \left(0.202707 + \frac{0.2188272177}{2}\right)^2\right]$

$= 0.2194838549$

$K_4 = hf(x_1 + h, y_1 + K_3) = h[1 + (y_1 + K_3)^2]$

$= 0.2[1 + (0.202707 + 0.2194838549)^2]$

$= 0.2356490236$

$\Rightarrow y_2 = 0.202707 + \frac{1}{6}[0.2082181 + 2(0.2188272177)$

$+ 2(0.219483549) + 0.2356490236]$

$y_2 = 0.4227884428$

$x_2 = x_1 + h = 0.2 + 0.2$

$= 0.4$

$y(0.4) = 0.4227884428$

Exact value :- $\frac{dy}{dx} = 1 + y^2$

$\frac{dy}{1+y^2} = dx \Rightarrow \int \frac{dy}{1+y^2} = \int dx$

$\tan^{-1}(y) = x + c$

$y(0) = 0$ so $\tan^{-1}(0) = 0 + c$

$\Rightarrow c = 0$

$\Rightarrow \tan^{-1}(y) = x$

$y = \tan x$

$y(0.2) = \tan(0.2)$

$= 0.2027$

& $y(0.4) = \tan(0.4)$

$= 0.4227932$

Example:- Show that most Popular R-K method of order 4 reduces to simpson's Rule in dy/dx is a function of x alone.

Solution $\frac{dy}{dx} = f(x)$

$$\therefore K_1 = h f(x_n) = h f_n$$

$$K_2 = h f(x_n + h/2) = h f_{n+h/2}$$

$$K_3 = h f(x_n + h/2) = h f_{n+\frac{h}{2}}$$

$$K_4 = h f(x_n + h) = h f_{n+h}$$

Thus

$$y_{n+1} = y_n + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$= y_n + \frac{1}{6}[h f_n + 2h f_{n+\frac{h}{2}} + 2h f_{n+\frac{h}{2}} + h f_{n+h}]$$

$$= y_n + \frac{h}{6}[f_n + 4 f_{n+\frac{h}{2}} + f_{n+h}]$$

which is simpson's Rule with size $\frac{h}{2}$

Replace h by $2h$

$$y_{n+1} = y_n + \frac{h}{3}[f_n + 4 f_{n+h} + f_{n+2h}]$$

$$= y_n + \frac{h}{3}\{f(x_n) + 4 f(x_n + h) + f(x_n + 2h)]$$

which is simpson's $\frac{1}{3}$ rule.

★★★

Example- Use Runge-Kutta 4^{th} order method to find an approximate value of y for $x = 0.2$ in step of 0.1 if $\frac{dy}{dx} = x + y^2$ given that $y(0) = 1$

Solution $\frac{dy}{dx} = x + y^2 \quad , y(0) = 1$

$$f(x, y) = x + y^2 \quad ; x_0 = 0, y_0 = 1 \quad , h = 0.1$$

Step I :- $k_1 = h f(x_0, y_0) = h(x_0 + y_0^2)$

$= 0.1(0 + 1^2)$

$= 0.1$

$K_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = h\left[x_0 + \frac{h}{2} + \left(y_0 + \frac{0.1}{2}\right)^2\right]$

$= 0.1\left[0 + \frac{0.1}{2} + \left(1 + \frac{0.1}{2}\right)^2\right]$

$= 0.11525$

$K_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = h\left[x_0 + \frac{h}{2} + \left(y_0 + \frac{K_2}{2}\right)^2\right]$

$= 0.1\left[0 + \frac{0.1}{2} + \left(1 + \frac{0.11525}{2}\right)^2\right]$

$= 0.11686$

$K_4 = h f(x_0 + h, y_0 + k_3) = h[x_0 + h + (y_0 + k_3)^2]$

$= 0.1[0 + 0.1 + (1 + 0.11686)^2]$

$= 0.1347$

$y_1 = y_0 + \frac{1}{6}[k_1 + 2K_2 + 2K_3 + k_4]$

$= 1 + \frac{1}{6}[0.1 + 2(0.11525) + 2(0.11686) + 0.1347]$

$= 1.1165$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$y(0.1) = 1.1165$

Step II :- $x_1 = 0.1$, $y_1 = 1.1165$, $h = 0.1$

$$k_1 = h f(x_1, y_1) = h(x_1 + y_1^2)$$

$$= 0.1[0.1 + (1.1165)^2]$$

$$= 0.1347$$

$$k_2 = h f\left(x_1 + h/2, y_1 + \frac{k_1}{2}\right) = h\left[x_1 + \frac{h}{2} + \left(y_1 + \frac{k_1}{2}\right)^2\right]$$

$$= 0.1\left[0.1 + \frac{0.1}{2} + \left(1.1165 + \frac{0.1347}{2}\right)^2\right]$$

$$= 0.1551$$

$$k_3 = h f\left(x_1 + h/2, y_1 + \frac{k_2}{2}\right) = h\left[x_1 + \frac{h}{2} + \left(y_1 + \frac{k_2}{2}\right)^2\right]$$

$$= 0.1\left[0.1 + \frac{0.1}{2} + \left(1.1165 + \frac{0.1551}{2}\right)^2\right]$$

$$= 0.1576$$

$$k_4 = h f(x_1 + h, y_1 + k_3) = h[x_1 + h + (y_1 + k_3)^2]$$

$$= 0.1[0.1 + 0.1 + (1.1165 + 0.1576)^2]$$

$$= 0.1823$$

$$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1.1165 + \frac{1}{6}[0.1347 + 2(0.1551) + 2(0.1576) + 0.1823]$$

$$= 1.2736$$

$$x_2 = x_0 + 2h = 0 + 2(0.1) = 0.2$$

$$\Rightarrow y(0.2) = 1.2736$$

***

Example: Solve $\frac{dy}{dx} = 2x^2 + 3y$, $y(1) = 2$, $h = 0.2$ find $y(1.6)$ using R-K method of order 4.

Solution

Given $\frac{dy}{dx} = 2x^2 + 3y$, $y_0 = 2$, $x_0 = 1$, $h = 0.2$

Now $x_1 = x_0 + h = 1 + 0.2 = 1.2$

$x_2 = x_0 + 2h = 1 + 2(0.2) = 1.4$

$x_3 = x_0 + 3h = 1 + 3(0.2) = 1.6$

To find $y(1.6)$ ~~at~~ we have to find $y(x_3)$ i.e y_3

R.K method of order 4 is

$$y_{n+1} = y_n + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

where $K_1 = h f(x_n, y_n)$

$K_2 = h f(x_n + \frac{h}{2}, y_n + \frac{K_1}{2})$, $K_3 = h f(x_n + \frac{h}{2}, y_n + \frac{K_2}{2})$

$K_4 = h f(x_n + h, y_n + K_3)$

$f(x_n, y_n) = 2x_n^2 + 3y_n$

1st Iteration — when $n = 0$

$K_1 = h f(x_0, y_0) = h[2x_0^2 + 3y_0]$

$= 0.2[2(1)^2 + 3(2)]$

$= 1.6$

$K_2 = h f(x_0 + h/2, y_0 + \frac{K_1}{2}) = h[2(x_0 + h/2)^2 + 3(y_0 + \frac{K_1}{2})]$

$= 0.2[2(1 + \frac{0.2}{2})^2 + 3(2 + \frac{1.6}{2})]$

$K_2 = \cancel{1.564}\ 2.164$

$K_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}\right) = h\left[2\left(x_0+\frac{h}{2}\right)^2+3\left(y_0+\frac{K_2}{2}\right)\right]$

$= 0.2\left[2\left(1+\frac{0.2}{2}\right)^2+3\left(2+\frac{2.164}{2}\right)\right]$

$= 2.3332$

$K_4 = hf(x_0+h, y_0+K_3) = h\left[2(x_0+h)^2+3(y_0+K_3)\right]$

$= 0.2\left[2(1+0.2)^2+3(2+2.3332)\right]$

$= 3.1759$

$y_1 = y_0 + \frac{1}{6}\left[K_1+2K_2+2K_3+K_4\right]$

$= 2+\frac{1}{6}\left[1.6+2(2.164)+2(2.3332)+3.1759\right]$

$= 4.2951$

<u>2nd Iteration:-</u> when $n=1$

$K_1 = hf(x_1, y_1) = h(2x_1^2+3y_1)$

$= 0.2(2(1.2)^2+3(4.2951))$

$= 3.1531$

$K_2 = hf\left(x_1+\frac{h}{2}, y_1+\frac{K_1}{2}\right) = h\left[2\left(x_1+\frac{h}{2}\right)^2+3\left(y_1+\frac{K_1}{2}\right)\right]$

$= 0.2\left[2\left(1.2+\frac{0.2}{2}\right)^2+3\left(4.2951+\frac{3.1531}{2}\right)\right]$

$= 4.199$

$K_3 = hf\left(x_1+\frac{h}{2}, y_1+\frac{K_2}{2}\right) = h\left[2\left(x_1+\frac{h}{2}\right)^2+3\left(y_1+\frac{K_2}{2}\right)\right]$

$= 0.2\left[2\left(1.2+\frac{0.2}{2}\right)^2+3\left(4.2951+\frac{4.199}{2}\right)\right]$

$K_3 = 4.5128$

$K_4 = h f(x_1+h, y_1+K_3) = h\left(2(x_1+h)^2+3(y_1+K_3)\right)$

$= 0.2\left[2(1.2+0.2)^2+3(4.2951+4.5128)\right]$

$= 9.3807$

$y_2 = y_1 + \frac{1}{6}\left[K_1+2K_2+2K_3+K_4\right]$

$= 4.2951 + \frac{1}{6}\left[3.1531+2(4.199)+2(4.5128)+9.3807\right]$

$= 9.288$

3rd Iteration:- when $n=2$

$K_1 = h f(x_2, y_2) = h\left[2x_2^2+3y_2\right]$

$= 0.2\left[2(1.4)^2+3(9.288)\right]$

$= 6.3568$

$K_2 = h f\left(x_2+h/2, y_2+\frac{K_1}{2}\right) = h\left[2\left(x_2+\frac{h}{2}\right)^2+3\left(y_2+\frac{K_1}{2}\right)\right]$

$= 0.2\left[2\left(1.4+\frac{0.2}{2}\right)^2+3\left(9.288+\frac{6.3568}{2}\right)\right]$

$= 8.3798$

$K_3 = h f\left(x_2+h/2, y_2+\frac{K_2}{2}\right) = h\left[2\left(x_2+\frac{h}{2}\right)^2+3\left(y_2+\frac{K_2}{2}\right)\right]$

$= 0.2\left[2\left(1.4+\frac{0.2}{2}\right)^2+3\left(9.288+\frac{8.3798}{2}\right)\right]$

$= 8.9867$

$K_4 = h f(x_2+h, y_2+K_3) = h\left[2(x_2+h)^2+3(y_2+K_3)\right]$

$$= 0.2\left[2(1.4+0.2)^2 + 3(8.9867+9.288)\right]$$

$$K_4 = 11.9888$$

& $y_3 = y_2 + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$

$$= 9.288 + \frac{1}{6}\left[6.3568 + 2(8.3798) + 2(8.9867) + 11.9888\right]$$

$$= 18.1344$$

$\Rightarrow$ $y(1.6) = 18.1344$ Ans.

★———★★★———★

ASSIGNMENT

Q1:- Derive R-K method of order 2 and use it to find $y(0.02)$ and $y(0.03)$ with $h = 0.01$ of problem

$$y' = 2y + x \quad ; \quad y(0) = 1$$

Q2:- Apply Ruge's formula (2^{nd} order) to find an approximate value of y when $x = 1.1$ given that

$$dy/dx = x - y \quad \& \quad y = 1 \text{ when } x = 1$$

Q3:- Calculate $y(0.1)$ and $y(0.2)$ from I.V.P $y' = x + 2y$, $y(0) = 0.75$ taking $h = 0.1$ comparing with exact value.

Q4:- Solve equation $y' = x + y$, $y(0) = 0$ at $x = 0.5$ taking $h = 0.1$ compare your result with exact value.

Q5:- Apply Rung's formula of order 2 to find value of y when $x = 0.02$ given that $dy/dx = x^2 + y$ & $y(0) = 1$

Q6:- Obtain y when $x = 1.1$ given that $y = 1.2$ when $x = 1$ and y satisfies equation $dy/dx = 3x + y^2$ using R-K method of order 4.

Q7:- Find $y(0.2)$ for equ $dy/dx = \frac{y-x}{y+x}$, $y(0) = 1$ take $h = 0.2$ (use R-K order 4)

Q8:- At $x = 0.8$ using R-K method of order 4 find approximate value of y given that $y = 0.41$ when $x = 0.4$ and $y' = \sqrt{x+y}$

Q9:- Using R-K method of order 4 find $y(0.2)$ given $y' = 3x + \frac{1}{2}y$, $y(0) = 1$ $h = 0.1$

Q10:- Use R-K method of order 3 find y when $x = 1.2$ in step size 0.1 given that $y' = y/x$ & $y(1) = 1$

Q11:- Using R-K method of order 3 find y when $x = 1.2$ in step size 0.1 given that $y' = x^2 + y^2$, $y(1) = 1.5$

** —— ** —— **

SOLUTIONS

Q1:- $y' = 2y + x$; $y_0 = 1$, $x_0 = 0$, $h = 0.01$

So $x_1 = x_0 + h = 0 + 0.01 = 0.01$

$x_2 = 0.02$, $x_3 = 0.03$

To find $y(0.02)$, $y(0.03)$ i.e $y(x_2)$ and $y(x_3)$ i.e y_2 and y_3

R-K method of order 2 is

$$\left.\begin{aligned} y_{n+1} &= y_n + \frac{1}{2}[K_1 + K_2] \\ \text{where } K_1 &= h f(x_n, y_n) \quad \& \\ K_2 &= h f[x_n + h, y_n + K_1] \end{aligned}\right\} \longrightarrow (*)$$

$$f(x_n, y_n) = 2y_n + x_n$$

<u>1st Iteration:-</u> Put $n = 0$ in (*)

$$\Rightarrow y_1 = y_0 + \frac{1}{2}[K_1 + K_2]$$

Now $K_1 = h f(x_0, y_0) = h[2y_0 + x_0]$

$= 0.01[2(1) + 0] = 0.02$

$K_2 = h f(x_0 + h, y_0 + K_1) = h[2(y_0 + K_1) + x_0 + h]$

$= 0.01[2(1 + 0.02) + 0 + 0.01] = 0.0205$

$$\Rightarrow y_1 = 1 + \frac{1}{2}[.02 + .0205] = 1.02025$$

$$\Rightarrow y(0.01) = 1.02025$$

2nd Iteration:- Put $n=1$ in ⊛

$\Rightarrow y_2 = y_1 + \frac{1}{2}[K_1 + K_2]$

Now $K_1 = hf(x_1, y_1) = h(2y_1 + x_1)$

$= 0.01[2(1.02025) + .01] = .020505$

$K_2 = hf[x_1 + h, y_1 + k_1] = h[2(y_1 + k_1) + x_1 + h]$

$= 0.01[2(1.02025 + .020505) + (.01 + .01)]$

$= 0.0210151$

$\Rightarrow y_2 = y_1 + \frac{1}{2}[0.020505 + .0210151] = 1.04101$

$\Rightarrow y(.02) = 1.04101$

3rd Iteration:- Put $n=2$ in ⊛

$\Rightarrow y_3 = y_2 + \frac{1}{2}[K_1 + K_2]$

$K_1 = hf(x_2, y_2) = h[2y_2 + x_2]$

$= 0.01[2(1.04101) + .02] = 0.0210202$

$K_2 = hf(x_2 + h, y_2 + K_1) = h[2(y_2 + K_1) + (x_2 + h)]$

$= 0.01[2(1.04101 + 0.0210202) + .03] = 0.21540604$

$\Rightarrow y_3 = 1.04101 + \frac{1}{2}[.0210202 + 0.21540604] = 1.0622904$

$\Rightarrow y(0.03) = 1.0622904$

****

Q2:- Given $\frac{dy}{dx} = x - y$; $y_0 = 1, x_0 = 1$

Let $h = 0.05$

$\Rightarrow x_1 = 1.05$, $x_2 = 1.1$

To find $y(1.1)$ i.e $y(x_2)$ i.e y_2

$f(x_n, y_n) = x_n - y_n$

R-K method of order 2 is

$$y_{n+1} = y_n + \frac{1}{2}[k_1 + k_2]$$
where $k_1 = h\,f(x_n, y_n)$
$k_2 = h\,f[x_n + h, y_n + k_1]$ $\longrightarrow$ ⊛

1st Iteration:- Put $n=0$ in ⊛

$\Rightarrow y_1 = y_0 + \frac{1}{2}[k_1 + k_2]$

$k_1 = h\,f(x_0, y_0) = h(x_0 - y_0) = .05[1-1] = 0$

$k_2 = h\,f[x_0 + h, y_0 + k_1] = h[x_0 + h - y_0 - k_1]$

$= [1 + .05 - 1 - 0](.05) = .0025$

$\Rightarrow y_1 = 1 + \frac{1}{2}[0 + .0025] = 1.00125$

2nd Iteration:- Put $n=1$ in ⊛

$\Rightarrow y_2 = y_1 + \frac{1}{2}[k_1 + k_2]$

$k_1 = h\,f(x_1, y_1) = h[x_1 - y_1] = 0.05[1.05 - 1.00125]$

$= 0.0024375$

$k_2 = h\,f(x_1 + h, y_1 + k_1) = h[x_1 + h - (y_1 + k_1)]$

$= .05[1.05 + .05 - (1.00125 + .0024375)]$

$= 0.004815625$

$\Rightarrow y_2 = 1.00125 + \frac{1}{2}[0.0024375 + 0.004815625]$

$= 1.004877$

Q3:-

$$y' = x + 2y \; ; \quad y(0) = 0.75$$

$\Rightarrow y_0 = 0.75, \; x_0 = 0$

To find $y(0.1)$ & $y(0.2)$ Let $h = 0.1$

$\Rightarrow$ To find $y(x_1)$ & $y(x_2)$ i.e y_1 & y_2

$$f(x_n, y_n) = x_n + 2y_n$$

R-K method of order 2 is

$$y_{n+1} = y_n + \frac{1}{2}[k_1 + k_2],$$
$$k_1 = h f(x_n, y_n) \text{ \& } k_2 = h f(x_n + h, y_n + k_1)$$ → ⊛

<u>1st Iteration:-</u> Put $n=0$ in ⊛

$\Rightarrow y_1 = y_0 + \frac{1}{2}[k_1 + k_2]$

$k_1 = h f(x_0, y_0) = h[x_0 + 2y_0]$

$= 0.1[0 + 2(0.75)] = 0.15$

$k_2 = h f[x_0 + h, y_0 + k_1] = h[x_0 + h + 2(y_0 + k_1)]$

$= 0.1[0 + 0.1 + 2(0.75 + 0.15)] = 0.19$

$\Rightarrow y_1 = 0.75 + \frac{1}{2}[0.19 + 0.15] = 0.92$

$\Rightarrow y(0.1) = 0.92$

<u>2nd Iteration:-</u> Put $n=1$ in ⊛

$\Rightarrow y_2 = y_1 + \frac{1}{2}[k_1 + k_2]$

$k_1 = h f(x_1, y_1) = h[x_1 + 2y_1]$

$= 0.1[0.1 + 2(0.92)] = 0.194$

$k_2 = h f(x_1 + h, y_1 + k_1) = h[x_1 + h + 2(y_1 + k_1)]$

$= 0.1[0.2 + 2(0.92 + 0.194)] = 0.2428$

$\Rightarrow y_2 = 0.92 + \frac{1}{2}[0.194 + 0.2428]$

$= 1.1384$

$\Rightarrow y(0.2) = 1.1384$

*——***——*

Q4:- Given $y' = x+y$; $y(0) = 0$

$\Rightarrow y_0 = 0$, $x_0 = 0$, $h = 0.1$

$\Rightarrow x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, $x_4 = 0.4$

$x_5 = 0.5$

To find $y(0.5)$ i.e $y(x_5)$ i.e y_5

$f(x_n, y_n) = x_n + y_n$

R-K method of order 2 is

$$\left.\begin{aligned} y_{n+1} &= y_n + \frac{1}{2}[K_1 + K_2], \text{ where} \\ K_1 &= h\, f(x_n, y_n) \ \ \& \ \ K_2 = h f[x_n + h, y_n + K_1] \end{aligned}\right\} \longrightarrow (*)$$

<u>1st Iteration</u>- Put $n=0$ in $(*)$

$\Rightarrow y_1 = y_0 + \frac{1}{2}[K_1 + K_2]$

$K_1 = h f(x_0, y_0) = h[x_0 + y_0]$

$= 0.1[0+0] = 0$

$K_2 = h f[x_0 + h, y_0 + K_1] = h[(x_0 + h) + (y_0 + K_1)]$

$= 0.1[0.1 + 0 + 0] = 0.01$

$\Rightarrow y_1 = 0 + \frac{1}{2}[0.01] = 0.005$

<u>2nd Iteration</u>:- Put $n=1$ in $(*)$

$\Rightarrow y_2 = y_1 + \frac{1}{2}[K_1 + K_2]$

$K_1 = h f(x_1, y_1) = h(x_1 + y_1)$

$= 0.1[0.1 + 0.005] = 0.0105$

$K_2 = h f(x_1 + h, y_1 + K_1) = h[x_1 + h + y_1 + K_1]$

$= 0.1[0.1 + 0.1 + 0.005 + 0.0105] = 0.021$

$\Rightarrow y_2 = 0.005 + \frac{1}{2}[0.0105 + 0.021]$

$= 0.021025$

3rd Iteration:- Put $n=2$ in ⊛

$\Rightarrow y_3 = y_2 + \frac{1}{2}[K_1 + K_2]$

$K_1 = h f(x_2, y_2) = h(x_2 + y_2)$

$= 0.1[0.2 + 0.021025] = 0.0221025$

$K_2 = h f(x_2 + h, y_2 + K_1) = h[x_2 + h + y_2 + K_1]$

$= 0.1[0.2 + 0.1 + 0.021025 + 0.0221025] = 0.03431275$

$\Rightarrow y_3 = 0.021025 + \frac{1}{2}[0.0221025 + 0.03431275]$

$= 0.049232625$

4th Iteration:- Put $n = 3$ in ⊛

$\Rightarrow y_4 = y_3 + \frac{1}{2}[K_1 + K_2]$

$K_1 = h f(x_3, y_3) = h(x_3 + y_3)$

$= 0.1(0.3 + 0.04923262 5) = 0.0349233$

$K_2 = h f(x_2 + h, y_2 + K_1) = h[x_2 + h + y_2 + K_1]$

$= 0.1[0.3 + 0.04923262 5 + 0.0349233] = 0.0484156$

$\Rightarrow y_4 = 0.04923262 5 + \frac{1}{2}[0.0349233 + 0.0484156]$

$= 0.0909021$

5th Iteration:- Put $n = 4$ in ⊛

$\Rightarrow y_5 = y_4 + \frac{1}{2}[K_1 + K_2]$

$K_1 = h f(x_4, y_4) = h(x_4 + y_4)$

$= 0.1(0.4 + 0.0909021) = 0.0490 9021$

$K_2 = h f(x_4 + h, y_4 + K_1) = h[x_4 + h + y_4 + K_1]$

$= 0.1[0.5 + 0.0909021 + 0.0490 9021] = 0.063999$

$\Rightarrow y_5 = y_4 + \frac{1}{2}[K_1 + K_2] = 0.147447$

Q5:- $\frac{dy}{dx} = x^2 + y$; $y_0 = 1$, $x_0 = 0$

$f(x_n, y_n) = x_n^2 + y_n$

R-K method of order 2 is

$y_{n+1} = y_n + \frac{1}{2}[K_1 + K_2]$

$\Rightarrow K_1 = h f(x_n, y_n)$

$= h[x_n^2 + y_n] = 0.02[0+1] = 0.02$

$K_2 = h f[x_n + h, y_n + k_1]$

$= h[(x_n+h)^2 + (y_n + k_1)]$

$= 0.1[(0+0.02)^2 + (1+0.02)] = 0.020408$

$\Rightarrow y = y_0 + \frac{1}{2}[0.02 + 0.20408] = 1 + 0.020204$

$= 1.020204$

* * *

Q6:- $\frac{dy}{dx} = 3x + y^2$; $x_0 = 1$, $y_0 = 1.2$

To find $y(1.1)$ Let $h = 0.1$

$\Rightarrow x_1 = 1.1$

$\Rightarrow$ To find $y(x_1)$ i.e y_1

$f(x_n, y_n) = 3x_n + y_n^2$

R-K method of order 4 is

$y_{n+1} = y_n + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$K_1 = h f(x_n, y_n)$, $K_2 = h f(x_n + h/2, y_n + k_1/2)$

$K_3 = h f[x_n + h/2, y_n + k_2/2]$ → (*)

$K_4 = h f[x_n + h, y_n + k_3]$

Put $n=0$ in (*)

$$\Rightarrow y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$K_1 = h\,f(x_0, y_0) = h[3x_0 + y_0^2]$$

$$= 0.1[3(1) + (1.2)^2] = 0.444$$

$$K_2 = h\,f[(x_0 + h/2), (y_0 + K_1/2)]$$

$$= h\left[3(x_0 + h/2) + \left(y_0 + \frac{K_1}{2}\right)^2\right]$$

$$= 0.1\left[3\left(1 + \frac{0.1}{2}\right) + \left(1.2 + \frac{0.444}{2}\right)^2\right] = 0.51721$$

$$K_3 = h\,f\left[(x_0 + h/2), \left(y_0 + \frac{K_2}{2}\right)\right]$$

$$= h\left[3(x_0 + h/2) + \left(y_0 + \frac{K_2}{2}\right)^2\right]$$

$$= 0.1\left[3\left(1 + \frac{0.1}{2}\right) + \left(1.2 + \frac{0.51721}{2}\right)^2\right]$$

$$= 0.527753$$

$$K_4 = h\,f(x_0 + h, y_0 + K_3)$$

$$= h[3(x_0 + h) + (y_0 + K_3)^2]$$

$$= 0.1[3(1 + 0.1) + (1.2 + 0.527753)^2] = 0.628513$$

$$\Rightarrow y_1 = 1.2 + \frac{1}{6}[0.444 + 2(0.51721) + 2(0.527753) + 0.628513]$$

$$= 1.7271$$

Q7:- Given $\frac{dy}{dx} = \frac{y-x}{y+x}$; $y(0) = 1$

$\Rightarrow y_0 = 1$, $x_0 = 0$, $h = 0.2$

So $x_1 = x_0 + h = 0 + 0.2 \Rightarrow x_1 = 0.2$

To find $y(0.2)$ i.e $y(x_1)$ i.e y_1

$$f(x_n, y_n) = \frac{y_n - x_n}{y_n + x_n}$$

R-K method of order 4 is

$$\left.\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4], \\ \text{where } k_1 &= h f(x_n, y_n) \\ k_2 &= h f(x_n + h/2, y_n + k_1/2) \\ k_3 &= h f(x_n + h/2, y_n + k_2/2) \\ k_4 &= h f(x_n + h, y_n + k_3) \end{aligned}\right\} \longrightarrow (*)$$

Put $n = 0$ in $(*)$

$$\Rightarrow y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = h f(x_0, y_0) = h\left[\frac{y_0 - x_0}{y_0 + x_0}\right]$$

$$= 0.2\left[\frac{1 - 0}{1 + 0}\right] = 0.2$$

$$k_2 = h f\left(x_0 + h/2, y_0 + \frac{k_1}{2}\right) = h\left[\frac{\left(y_0 + \frac{k_1}{2}\right) - \left(x_0 + \frac{h}{2}\right)}{\left(y_0 + \frac{k_1}{2}\right) + \left(x_0 + \frac{h}{2}\right)}\right]$$

$$= 0.2\left[\frac{\left(1 + \frac{0.2}{2}\right) - \left(0 + \frac{0.2}{2}\right)}{\left(1 + \frac{0.2}{2}\right) + \left(0 + \frac{0.2}{2}\right)}\right] = 0.1666667$$

$$K_3 = h\,f\left(x_0+\tfrac{h}{2},\, y_0+\tfrac{K_2}{2}\right) = h\left[\frac{\left(y_0+\frac{K_2}{2}\right)-\left(x_0+\frac{h}{2}\right)}{\left(y_0+\frac{K_2}{2}\right)+\left(x_0+\frac{h}{2}\right)}\right]$$

$$= 0.2\left[\frac{\left(1+\frac{0.1666667}{2}\right)-\left(0+\frac{0.2}{2}\right)}{\left(1+\frac{0.1666667}{2}\right)+\left(0+\frac{0.2}{2}\right)}\right] = 0.2\left[\frac{0.983333}{1.183333}\right]$$

$$= 0.166197$$

$$K_4 = h\,f(x_0+h,\, y_0+K_3) = h\left[\frac{(y_0+K_3)-(x_0+h)}{(y_0+K_3)+(x_0+h)}\right]$$

$$= 0.2\left[\frac{(1+0.166197)-(0+0.2)}{(1+0.166197)+(0+0.2)}\right] = 0.2\left[\frac{0.966197}{1.366197}\right]$$

$$= 0.7072165$$

$$\Rightarrow y_1 = 1 + \frac{1}{6}\left[0.2 + 2(0.1666667 + 0.166197) + 0.7072165\right]$$

$$= 1.2622 \quad \text{Ans.}$$

****

Q8:-

$$\frac{dy}{dx} = \sqrt{x+y} \quad ; \; y_0 = 0.41 \;,\; x_0 = 0.4$$

Let $h = 0.2$

So $x_0 = 0.4$, $x_1 = 0.6$, $x_2 = 0.8$

To find $y(0.8)$

i.e $y(x_2)$ i.e y_2

$$f(x_n, y_n) = \sqrt{x_n + y_n}$$

R-K method of order 4

$$y_{n+1} = y_n + \frac{1}{6}[k_1 + 2(k_2 + k_3) + k_4]$$

where $k_1 = h f(x_n, y_n)$

$k_2 = h f(x_n + h/2, y_n + \frac{k_1}{2})$, $k_3 = h f[x_n + h/2, y_n + \frac{k_2}{2}]$

$k_4 = h f[x_n + h, y_n + k_3] \longrightarrow$ ⊛

<u>1st Iteration:-</u> Put $n=0$ in ⊛

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$k_1 = h f(x_0, y_0) = h\sqrt{x_0 + y_0}$

$= 0.2(\sqrt{0.4 + 0.41}) = 0.18$

$k_2 = h f(x_0 + h/2, y_0 + \frac{k_1}{2}) = h\left[\sqrt{(x_0 + h/2) + (y_0 + \frac{k_1}{2})}\right]$

$= 0.2\left[\sqrt{(0.4 + \frac{0.2}{2}) + (0.41 + \frac{0.18}{2})}\right] = 0.2$

$k_3 = h f\left[x_0 + h/2, y_0 + \frac{k_2}{2}\right] = h\left[\sqrt{(x_0 + h/2) + (y_0 + \frac{k_2}{2})}\right]$

$= 0.2\left[\sqrt{(0.4 + \frac{0.2}{2}) + (0.41 + \frac{0.2}{2})}\right] = 0.201$

$k_4 = h f(x_0 + h, y_0 + k_3) = h\left[\sqrt{(x_0 + h) + (y_0 + k_3)}\right]$

$= 0.2\left[\sqrt{(0.4 + 0.2) + (0.41 + 0.201)}\right] = 0.2201$

$\Rightarrow y_1 = 0.41 + \frac{1}{6}[0.18 + 2(0.2 + 0.201) + 0.2201]$

$= 0.6104$

$\Rightarrow y(0.6) = 0.6104$

2nd Iteration :- Put $n=1$ in ⊛

$\Rightarrow y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

where $k_1 = h f(x_1, y_1) = h\sqrt{(x_1 + y_1)}$

$= 0.2(\sqrt{0.6 + 0.6104}) = 0.22004$

$k_2 = h f[x_1 + h/2, y_1 + \frac{k_1}{2}] = h\left[\sqrt{(x_1 + h/2) + (y_1 + \frac{k_1}{2})}\right]$

$= 0.2\left[\sqrt{\left(0.6 + \frac{0.2}{2}\right) + \left(0.6104 + \frac{0.22004}{2}\right)}\right] = 0.2384$

$k_3 = h f[x_1 + h/2, y_1 + \frac{k_2}{2}] = h\left[\sqrt{(x_1 + h/2) + (y_1 + \frac{k_2}{2})}\right]$

$= 0.2\left[\sqrt{\left(0.6 + \frac{0.2}{2}\right) + \left(0.6104 + \frac{0.2384}{2}\right)}\right] = 0.2391$

$k_4 = h f(x_1 + h, y_1 + k_3) = h\sqrt{(x_1 + h) + (y_1 + k_3)}$

$= 0.2\sqrt{(0.6 + 0.2) + (0.6104 + 0.2391)} = 0.2569$

$\Rightarrow y_2 = y_1 + \frac{1}{6}[k_1 + 2(k_2 + k_3) + k_4] = 0.8491$

$\Rightarrow y(0.8) = 0.8491$

Q9:- Given $\frac{dy}{dx} = 3x + \frac{1}{2}y$; $y(0) = 1$, $h = 0.1$

$x_0 = 0$, $y_0 = 1$, $x_1 = x_0 + h = 0.1$, $x_2 = 0.2$

To find $y(0.2)$ i.e $y(x_2)$ i.e y_2

$f(x_n, y_n) = 3x_n + \frac{1}{2}y_n$

R-K method of order 4 is

$$y_{n+1} = y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

where $k_1 = hf(x_n, y_n)$

$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$

$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$

$k_4 = hf(x_n + h, y_n + k_3)$ $\longrightarrow$ ★

1st Iteration:- Put $n=0$ in ★

$$\Rightarrow y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$k_1 = hf(x_0, y_0) = h\left[3x_0 + \frac{1}{2}y_0\right]$

$= 0.1\left[3(0) + \frac{1}{2}(1)\right] = 0.05$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = h\left[3\left(x_0 + \frac{h}{2}\right) + \frac{1}{2}\left(y_0 + \frac{k_1}{2}\right)\right]$

$= 0.1\left[3\left(0 + \frac{0.1}{2}\right) + \frac{1}{2}\left(1 + \frac{0.05}{2}\right)\right] = 0.06625$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = h\left[3\left(x_0 + \frac{h}{2}\right) + \frac{1}{2}\left(y_0 + \frac{k_2}{2}\right)\right]$

$= 0.1\left[3\left(0 + \frac{0.1}{2}\right) + \frac{1}{2}\left(1 + \frac{0.6625}{2}\right)\right] = 0.06666$

$k_4 = hf(x_0 + h, y_0 + k_3) = h\left[3(x_0 + h) + \frac{1}{2}(y_0 + k_3)\right]$

$= 0.1\left[3(0 + 0.1) + \frac{1}{2}(1 + 0.06666)\right] = 0.083333$

$\Rightarrow y_1 = 1 + \frac{1}{6}[0.05 + 2(0.06625 + 0.06666) + 0.083333]$

$= 1.06653$

2nd Iteration:- Put $n=1$ in ★

$$\Rightarrow y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$K_1 = h\,f(x_1, y_1) \quad = h\left[3x_1 + \frac{1}{2}y_1\right]$

$= 0.1\left[3(0.1) + \frac{1}{2}(1.06653)\right] = 0.08333$

$K_2 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = h\left[3\left(x_1 + \frac{h}{2}\right) + \frac{1}{2}\left(y_1 + \frac{K_1}{2}\right)\right]$

$= 0.1\left[3\left(0.1 + \frac{0.1}{2}\right) + \frac{1}{2}\left(1.06653 + \frac{0.08333}{2}\right)\right] = 0.10041$

$K_3 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = h\left[3\left(x_1 + \frac{h}{2}\right) + \frac{1}{2}\left(y_1 + \frac{K_2}{2}\right)\right]$

$= 0.1\left[3\left(0.1 + \frac{0.1}{2}\right) + \frac{1}{2}\left(1.06653 + \frac{0.10041}{2}\right)\right] = 0.10084$

$K_4 = h\,f(x_1 + h, y_1 + K_3) \quad = h\left[3(x_1 + h) + \frac{1}{2}(y_1 + K_3)\right]$

$= 0.1\left[3(0.1 + 0.1) + \frac{1}{2}(1.06653 + 0.10084)\right] = 0.11837$

$\Rightarrow y_2 = 1.06653 + \frac{1}{6}\left[0.08333 + 2(0.10041 + 0.10084)\right.$
$\left. + 0.11837\right]$

$= 1.10723$

★———————————★★★

Q 10:- $\frac{dy}{dx} = y/x \;;\; y(1) = 1, \; h = 0.1$

$\Rightarrow y_0 = 1, \quad x_0 = 1, \quad x_1 = 1.1, \quad x_2 = 1.2$

$\Rightarrow$ To find $y(x_2)$ i.e y_2

$f(x_n, y_n) = y_n/x_n$

R-K method of order 3 is

$y_{n+1} = y_n + \frac{1}{6}\left[K_1 + 4K_2 + K_3\right]$

where

$$\left.\begin{aligned} k_1 &= h\, f(x_n, y_n) \\ k_2 &= h\, f\left[x_n + \tfrac{h}{2}, y_n + \tfrac{k_1}{2}\right] \\ k_3 &= h\, f[x_n + h, y_n - k_1 + 2k_2] \end{aligned}\right\} \longrightarrow (*)$$

<u>1st Iteration:-</u> Put $n = 0$ in $(*)$

$$\Rightarrow y_1 = y_0 + \frac{1}{6}[k_1 + 4k_2 + k_3]$$

$$k_1 = h\, f(x_0, y_0) = h\left[\frac{y_0}{x_0}\right] = 0.1\left[\frac{1}{1}\right]$$

$$= 0.1$$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = h\left(\frac{y_0 + \frac{k_1}{2}}{x_0 + \frac{h}{2}}\right)$$

$$= 0.1\left(\frac{1 + \frac{0.1}{2}}{1 + \frac{0.1}{2}}\right) = 0.1$$

$$k_3 = \boxed{\cancel{h\, f\left(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_2}{2}\right)}}\ h f(x_0 + h, y_0 - k_1 + 2k_2) = \cancel{h\left(\frac{y_0 + \frac{k_2}{2}}{x_0 + \frac{h}{2}}\right)}$$

$$= 0.1\left(\frac{1 + \frac{0.1}{2}}{1 + \frac{0.1}{2}}\right) = 0.1$$

$$\Rightarrow y_1 = 1 + \frac{1}{6}[0.1 + 4(0.1) + 0.1]$$

$$= 1.1$$

<u>2nd Iteration:-</u> Put $n = 1$ in $(*)$

$$\Rightarrow y_2 = y_1 + \frac{1}{6}[k_1 + 4k_2 + k_3]$$

$$k_1 = h\, f(x_1, y_1) = h\left[\frac{y_1}{x_1}\right] = 0.1\left[\frac{1.1}{1.1}\right]$$

$$= 0.1$$

$$k_2 = h f\left(x_1 + \tfrac{h}{2}, y_1 + \tfrac{k_1}{2}\right) = h\left[\frac{y_1 + \frac{k_1}{2}}{x_1 + \frac{h}{2}}\right]$$

$$= 0.1\left[\frac{1.1 + \frac{0.1}{2}}{1.1 + \frac{0.1}{2}}\right] = 0.1$$

$$k_3 = h f(x_1 + h, y_1 - k_1 + 2k_2) = h\left[\frac{y_1 - k_1 + 2k_2}{x_1 + h}\right]$$

$$= 0.1\left[\frac{1.1 - 0.1 + 2(0.1)}{1.1 + 0.1}\right] = 0.1$$

$$\Rightarrow y_2 = 1.1 + \frac{1}{6}\left[0.1 + 4(0.1) + 0.1\right]$$

$$= 1.2 \quad \text{Ans}$$

Q11:- Given $\frac{dy}{dx} = x^2 + y^2$; $y(1) = 1.5$, $h = 0.1$

$y_0 = 1.5$, $x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.2$

$$f(x_n, y_n) = x_n^2 + y_n^2$$

R-K method of order 3 is

$$y_{n+1} = y_n + \frac{1}{6}\left[k_1 + 4k_2 + k_3\right]$$

where $k_1 = h f(x_n, y_n)$

$$k_2 = h f\left(x_n + \tfrac{h}{2}, y_n + \tfrac{k_1}{2}\right)$$

$$k_3 = h f(x_n + h, y_n - k_1 + 2k_2)$$

→ (A)

1st Iteration:- Put $n=0$ in (*)

$$\Rightarrow y_1 = y_0 + \frac{1}{6}[k_1 + 4k_2 + k_3]$$

$$k_1 = h f(x_0, y_0) = h(x_0^2 + y_0^2)$$

$$= 0.1[(1)^2 + (1.5)^2] = 0.325$$

$$k_2 = h f(x_0 + h/2, y_0 + \frac{k_1}{2}) = h\left[\left(x_0 + h/2\right)^2 + \left(y_0 + \frac{k_1}{2}\right)^2\right]$$

$$= 0.1\left[\left(1 + \frac{0.1}{2}\right)^2 + \left(1.5 + \frac{0.325}{2}\right)^2\right] = 0.38664$$

$$k_3 = h f(x_0 + h, y_0 - k_1 + 2k_2) = h\left[(x_0 + h)^2 + (y_0 - k_1 + 2k_2)^2\right]$$

$$= 0.1\left[(1 + 0.1)^2 + (1.5 - 0.325 + 2(0.38664))^2\right]$$

$$= 0.5006$$

$$\Rightarrow y_1 = 1.5 + \frac{1}{6}[0.325 + 4(0.38664) + 0.5006]$$

$$= 1.8954$$

2nd Iteration:- Put $n=1$ in (*)

$$\Rightarrow y_2 = y_1 + \frac{1}{6}[k_1 + 4k_2 + k_3]$$

$$k_1 = h f(x_1, y_1) = h(x_1^2 + y_1^2)$$

$$= 0.1[(1.1)^2 + (1.8954)^2] = 0.48025$$

$$k_2 = h f(x_1 + h/2, y_1 + \frac{k_1}{2}) = h\left[\left(x_1 + h/2\right)^2 + \left(y_1 + \frac{k_1}{2}\right)^2\right]$$

$$= 0.1\left[\left(1.1 + \frac{0.1}{2}\right)^2 + \left(1.8954 + \frac{0.48025}{2}\right)^2\right]$$

$$= 0.5883$$

$$K_3 = h f(x_1+h, y_1 - k_1 + 2k_2) = h\left[(x_1+h)^2 + (y_1 - k_1 + 2k_2)^2\right]$$

$$= 0.1\left[(1.1+0.1)^2 + (1.8954 - 0.48025 + 2(0.5883))^2\right]$$

$$= 0.81572$$

$$\Rightarrow y_2 = 1.8954 + \frac{1}{6}\left[0.48025 + 4(0.5883) + 0.81572\right]$$

$$= 2.5036$$

★———★★★———★

⇒ Predictor - Corrector Methods:-

The methods discussed so far, to solve differential equations. Numarically were self-starting one step method.

To apply these methods, we were required information only at the beginning of interval. But now in present section we shall discuss predictor-corrector methods which require function value at point $x_0, x_{n-1}, x_{n-2}, \ldots$ for the computation of function at x_{n+1}. And these function value can be obtained using Euler's method, Taylor's method, R-K method of order 4 etc.

A predictor formula is used to predict the value of y at x_{n+1} and then a corrector formula is used to improve the value of y.

Here we discuss two methods as following.

(i) Milne's Method

(ii) Adam - Bashforth Method.

1) Milne's Method:-

To solve IVP

$\frac{dy}{dx} = f(x,y)$ where $y(x_0) = y_0$, h is step size

By Milne's method, we first find the approximate value of y_{n+1} by predictor formula and then improve it by corrector formula.

In terms of f, Milne's predictor and corrector formula are.

Predictor Formula:-

$$\tilde{y}_{n+1} = y_{n-3} + \frac{4h}{3}\left[2f_{n-2} - f_{n-1} + 2f_n\right] \longrightarrow (A)$$

Corrector Formula:-

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left\{f_{n-1} + 4f_n + \tilde{f}_{n+1}\right\}$$

where h is step size

and $\tilde{f}_{n+1} = f(x_{n+1}, \tilde{y}_{n+1})$

Also $(\tilde{y}_{n+1} = y^{p}_{n+1})$ p stands for predictor.

*****

Example Find $y(0.4)$ and $y(0.5)$ by Milne's method from

$dy/dx = xy$; $y(0)=1$, $h = 0.1$

Solution The given IVP is

$\frac{dy}{dx} = xy$; $y(0)=1$, $h=0.1$

By R-K method of order 4 is

$$y_{n+1} = y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \longrightarrow (i)$$

where

$K_1 = h f(x_n, y_n)$

$K_2 = h f(x_n + h/2, y_n + k_1/2)$

$K_3 = h f(x_n + h/2, y_n + k_2/2)$

$K_4 = h f(x_n + h, y_n + k_3)$

For $n=0$:-

$K_1 = h f(x_0, y_0) = h(x_0 y_0)$

$= 0.1((0)(1)) = 0$

$K_2 = h f(x_0 + h/2, y_0 + k_1/2)$

$= h(x_0 + h/2)(y_0 + k_1/2) = 0.1[(0+0.5)(1+0)]$

$= 0.005$

$K_3 = h f(x_0 + h/2, y_0 + k_2/2)$

$= h[x_0 + h/2][y_0 + k_1/2] = 0.1[(0+0.5)(1+0.0025)]$

$= 0.0050125$

$K_4 = h f(x_0 + h, y_0 + k_3)$

$= h(x_0 + h)(y_0 + k_3) = 0.1[(0+0.1)(1+0.0050125)]$

$= 0.1005013$

So by (i) we have

$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 1 + \frac{1}{6}[0 + 2(0.005) + 2(0.0050125) + 0.1005013]$

$y(0.1) = 1.0050012512$

For $n=1$:- $K_1 = h f(x_1, y_1) = h x_1 y_1$

$= 0.5(0.1)(1.005013)$

$= 0.010050$

$K_2 = hf(x_1 + h/2, y_1 + K_1/2)$

$= h[x_1 + h/2][y_1 + K_1/2] = 0.1[(0.1 + 0.05)(1.005013 + \frac{0.01005}{2})]$

$= (0.1)(0.15)(0.90038) = 0.015151$

$K_3 = hf(x_1 + h/2, y_1 + K_2/2)$

$= h[x_1 + h/2][y_1 + K_2/2] = 0.1[0.1 + 0.05][1.005013 + \frac{0.015151}{2}]$

$= 0.015189$

$K_4 = hf(x_1 + h, y_1 + K_3)$

$= h(x_1 + h)(y_1 + K_3) = 0.1[0.1 + 0.1][1.005013 + 0.015189]$

$= 0.020404$

So (i) gives

$y_2 = y_1 + \frac{1}{6}[K_1 + 2(K_2 + K_3) + K_4]$

$= 1.005013 + \frac{1}{6}[0.01005 + 2(0.015151 + 0.015189) + 0.020404]$

$= 1.005013 + \frac{1}{6}[0.099134]$

$y(0.2) = 1.020202$

For $n=2$:- $h f(x_2, y_2) = h(x_2 y_2)$

$= 0.1(0.2)(1.020202) = 0.020404$

$K_2 = hf(x_2 + h/2, y_2 + K_1/2)$

$= h[x_2 + h/2][y_2 + K_1/2]$

$= 0.1[0.2 + 0.05][1.020202 + \frac{1.020404}{2}]$

$= 0.025760$

$K_3 = hf(x_2 + h/2, y_2 + \frac{K_2}{2})$

$= h[x_2 + \frac{h}{2}][y_2 + \frac{K_2}{2}] = 0.1[0.2 + 0.05][1.020202 + \frac{0.025760}{2}]$

$= 0.025827$

$k_4 = hf(x_2+h, y_2+k_3)$
$= h[x_2+h][y_2+k_3] = 0.1[0.3][1.020202+0.025827]$
$= 0.031381$

So (i) gives

$y_3 \doteq y_2 + \frac{1}{6}[k_1+2k_2+2k_3+k_4]$
$=1.020202 + \frac{1}{6}[0.020404+2(0.02576+0.025827)+$
$0.031381]$
$= 1.020202 + \frac{1}{6}[0.154959]$
$\Rightarrow y(0.3) = 1.046029$

For $n=3$:- $k_1 = hf(x_3, y_3) = h(x_3 y_3)$
$= 0.1[(0.3)(1.046029)]$
$= 0.031381$

$k_2 = hf(x_3+h/2, y_3+k_1/2)$
$= h[x_3+h/2][y_3+k_1/2]$
$= 0.1[0.3+0.05][1.046029 + \frac{0.031381}{2}]$
$= 0.037160$

$k_3 = hf(x_3+h/2, y_3+k_2/2)$
$= h[x_3+h/2][y_3+k_2/2]$
$= 0.1[0.3+0.05][1.046029+\frac{0.037160}{2}]$
$= 0.037261$

$k_4 = hf(x_3+h, y_3+k_3) = h[x_3+h][y_3+k_3]$
$= 0.1[0.4][1.046029+0.037261]$
$= 0.043332$

So (i) gives

$$y_4 = y_3 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1.046029 + \frac{1}{6}[0.031381 + 2(0.037160 + 0.037261) + 0.043332]$$

$$= 1.046029 + \frac{1}{6}[0.223555]$$

$$y(0.4) = 1.083288$$

For $n=4$:-

$$K_1 = hf(x_4, y_4) = h(x_4 y_4)$$

$$= 0.1(0.4)(1.083288) = 0.043332$$

$$K_2 = hf(x_4 + h/2, y_4 + k_1/2)$$

$$= h[x_4 + h/2][y_4 + k_1/2]$$

$$= 0.1[0.4 + 0.05][1.083288 + \frac{0.043332}{2}]$$

$$= 0.49723$$

$$K_3 = hf(x_4 + h/2, y_4 + k_2/2)$$

$$= h[x_4 + h/2][y_4 + k_2/2]$$

$$= 0.1[0.4 + 0.05][1.083288 + \frac{0.49723}{2}]$$

$$= 0.049867$$

$$K_4 = hf(x_4 + h, y_4 + k_3) = h[x_4 + h][y_4 + k_3]$$

$$= 0.1[0.5][1.083288 + 0.049867]$$

$$= 0.056658$$

So (i) gives

$$y_5 = y_4 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1.083288 + \frac{1}{6}[0.043332 + 2(0.049723 + 0.049867) + 0.056658]$$

$$y(0.5) = 1.133150$$

Now $f(x_n, y_n) = x_n y_n$

$f(x_1, y_1) = x_1 y_1 = 0.1(1.005013)$
$= 0.100501$

$f_2 = f(x_2, y_2) = x_2 y_2 = (0.2)(1.020202)$
$= 0.20404$

$f_3 = f(x_3, y_3) = x_3 y_3 = (0.3)(1.046029)$
$= 0.313809$

$f_4 = f(x_4, y_4) = x_4 y_4 = (0.4)(1.083288)$
$= 0.433315$

$f_5 = f(x_5, y_5) = x_5 y_5 = (0.5)(1.133150)$
$= 0.566575$

So by Milne's predictor and corrector formula, we have

For $n=4$: By Predictor formula

$$\tilde{y}_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 1 + \frac{4(0.1)}{3}[2(0.100501) - 0.20404 + 2(0.313809)]$$

$$\tilde{y}_4 = 1.083277$$

$$\tilde{f}_4 = x_4 \tilde{y}_4 = (0.4)(1.083277)$$
$$= 0.433311$$

And by corrector formula

$$y_4 = y_2 + \frac{h}{3}[f_2 + 4f_3 + \tilde{f}_4]$$

$$= 1.020202 + \frac{0.1}{3}[0.20404 + 4(0.313809) + 0.433311]$$

$$= 1.020202 + \frac{0.1}{3}[1.892587]$$

$$y(0.4) = 1.083288$$

For $n=4$:- $\tilde{y}_5 = y_1 + \frac{4h}{3}[2f_2 - f_3 + 2f_4]$

$$\tilde{y}_5 = 1.005013 + \frac{4(0.1)}{3}[2(0.20404) - 0.313809 + 2(0.433315)]$$

$$= 1.133133$$

$$\Rightarrow \tilde{f}_5 = x_5 \tilde{y}_5 = 0.5(1.133133)$$

$$= 0.566567$$

By corrector formula

$$y_5 = y_3 + \frac{h}{3}[f_3 + 4f_4 + \tilde{f}_5]$$

$$1.046029 + \frac{0.1}{3}[0.313809 + 4(0.433315) + 0.566567]$$

$$= 1.046029 + \frac{0.1}{3}[2.613636]$$

$$\Rightarrow y(0.5) = 1.13315$$

***

Example:- Find $y(0.4)$ by Milne's Method from $y' = x+y$, $y(0) = 1$, $h = 0.1$

Solution:-

The given IVP is

$$\frac{dy}{dx} = x + y, \quad y(0) = 1, \quad h = 0.1$$

Then by Taylor's method of order 4 is

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \frac{h^4}{4!} y_n^{iv} \longrightarrow (1)$$

For $n=0$:-

$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv}$$

given $y' = x + y \Rightarrow y'_0 = x_0 + y_0 = 0 + 1 = 1$

$y'' = 1 + y' \Rightarrow y''_0 = 1 + 1 = 2$

$y''' = y'' \Rightarrow y'''_0 = y''_0 = 2$

$y^{iv} = y''' \Rightarrow y^{iv}_0 = 2$

So by ①

$$y_1 = 1 + (0.1)(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(2) + \frac{(0.1)^4}{24}(2)$$

$$= 1 + 0.1 + 0.01 + 0.000333 + 0.000008$$

$$\Rightarrow y(0.1) = 1.110341$$

<u>For $n = 1$:-</u>

$$y_2 = y_1 + hy'_1 + \frac{h^2}{2!}y''_1 + \frac{h^3}{3!}y'''_1 + \frac{h^4}{4!}y^{iv}_1 \quad \text{——— ②}$$

$y'_1 = x_1 + y_1$

$= 0.1 + 1.110341 = 1.210341$

$y''_1 = 1 + y'_1 = 1 + 1.210341 = 2.210341$

$y'''_1 = y''_1 = 2.210341$

$y^{iv}_1 = y'''_1 = 2.210341$

So ② gives

$$y_2 = 1.110341 + (0.1)(1.210341) + \frac{(0.1)^2}{2}(2.210341) + \frac{(0.1)^3}{6}(2.210341) + \frac{(0.1)^4}{24}(2.210341)$$

$$\Rightarrow y(0.2) = 1.242804$$

<u>For $n = 2$:-</u>

$$y_3 = y_2 + hy'_2 + \frac{h^2}{2!}y''_2 + \frac{h^3}{3!}y'''_2 + \frac{h^4}{4!}y^{iv}_2 \quad \text{——— ③}$$

$y'_2 = x_2 + y_2 = 0.2 + 1.242804$

$= 1.442804$

$y_2'' = 1 + y_2' = 1 + 1.442804 = 2.442804$

$y_2''' = y_2'' \Rightarrow y_2''' = 2.442804$

$y_2^{iv} = y_2''' \Rightarrow y_2^{iv} = 2.442804$

So ③ gives

$$y_3 = 1.242804 + (0.1)(1.442804) + \frac{(0.1)^2}{2}(2.442804) + \frac{(0.1)^3}{6}(2.442804) + \frac{(0.1)^4}{24}(2.442804)$$

$$= 1.242804 + 0.14428 + 0.012214 + 0.000407 + 0.000010$$

$$\Rightarrow y(0.3) = 1.399715$$

For $n=3$:-

$$y_4 = y_3 + h y_3' + \frac{h^2}{2} y_3'' + \frac{h^3}{6} y_3''' + \frac{h^4}{24} y_3^{iv} \quad \text{—— ④}$$

$y_3' = x_3 + y_3 = 0.3 + 1.399715 = 1.699715$

$y_3'' = 1 + y_3' = 1 + 1.699715 = 2.699715$

$y_3''' = y_3'' \Rightarrow y_3''' = 2.699715$

$y_3^{iv} = y_3''' \Rightarrow y_3^{iv} = 2.699715$

So ④ gives

$$y_4 = 1.399715 + (0.1)(1.699715) + \frac{(0.1)^2}{2}(2.699715) + \frac{(0.1)^3}{6}(2.699715) + \frac{(0.1)^4}{24}(2.699715)$$

$$\Rightarrow y(0.4) = 1.582096$$

Now by Milne's predictor and corrector formula we have.

For $n=3$:-

By Predictor formula

$$\tilde{y}_4 = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

Now $f(x_1, y_1) = f_1 = x_1 + y_1 = 0.1 + 1.110341$

$= 1.210341$

$f_2 = x_2 + y_2 = 0.2 + 1.242804$

$= 1.442804$

$f_3 = x_3 + y_3 = 0.3 + 1.399715 = 1.699715$

So $\tilde{y}_4 = 1 + \frac{4(0.1)}{3}[2(1.210341) - 1.442804 + 2(1.699715)]$

$= 1 + \frac{0.4}{3}[4.377308] = 1.583641$

Now $\tilde{f}_4 = x_4 + \tilde{y}_4 = 0.4 + 1.583641$

$= 1.983641$

By Corrector formula

$y_4 = y_2 + \frac{h}{3}[f_2 + 4f_3 + \tilde{f}_4]$

$= 1.242804 + \frac{0.1}{3}[1.442804 + 4(1.699715)$

$+ 1.983641]$

$= 1.242804 + \frac{0.1}{3}[10.225305]$

$\Rightarrow y(0.4) = 1.583648$

2) Adam-Bashforth Method:-

Predictor formula

$$y_{n+1}^{p} = y_n + \frac{h}{24}\left[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right]$$

Corrector formula

$$y_{n+1} = y_n + \frac{h}{24}\left[9f_{n+1}^{p} + 19f_n - 5f_{n-1} + f_{n-2}\right]$$

where $f_{n+1}^{p} = f(x_{n+1}, y_{n+1}^{p})$

Example:- Solve $xy' = x - y$ to find $y(2.2)$ and $y(2.25)$.

given $y(2) = 2$, $y(2.05) = 2.00061$

$y(2.1) = 2.002385$ & $y(2.15) = 2.005242$

Solution

$$xy' = x - y \Rightarrow y' = \frac{x-y}{x}$$

$$\Rightarrow y' = 1 - y/x$$

$$f(x,y) = 1 - y/x$$

$x_0 = 2$, $x_1 = 2.05$, $x_2 = 2.1$

$y_0 = 2$, $y_1 = 2.00061$, $y_2 = 2.002385$

$x_3 = 2.15$ & $y_3 = 2.005242$

$$f_0 = f(x_0, y_0) = f(2,2) = 1 - 2/2 = 0$$

$$f_1 = f(x_1, y_1) = f(2.05, 2.00061)$$

$$= 1 - \frac{2.00061}{2.05} = 0.0240927$$

$$f_2 = f(x_2, y_2) = f(2.1, 2.002385)$$

$$\Rightarrow f_2 = 1 - \frac{2.002385}{2.1} = 0.0464834$$

$$f_3 = f(x_3, y_3) = f(2.15, 2.005242)$$

$$= 1 - \frac{2.005242}{2.15} = 0.0673294$$

By Adam Bashforth Predictor formula

$$y^P_{n+1} = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}] \quad \text{--- (1)}$$

& Corrector formula is

$$y_{n+1} = y_n + \frac{h}{24}[9f^P_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}] \quad \text{--- (2)}$$

Put $n = 1$ in (1)

$$y^P_4 = y_3 + \frac{h}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0]$$

$$= 2.005242 + \frac{0.05}{24}[55(0.0673294) - 59(0.0464834)$$

$$+ 37(0.0240927) - 9(0)]$$

$$\Rightarrow y^P_4 = 2.009\,1003$$

$$f^P_4 = f(x_4, y^P_4) = f(2.2, 2.0091003)$$

$$= 1 - \frac{2.0091003}{2.2} = 0.0867726$$

Again put $n = 3$ in corrector formula

$$y_4 = y_3 + \frac{h}{24}[9f^P_4 + 19f_3 - 5f_2 + f_1]$$

$$y_4 = 2.00524 + \frac{0.05}{24}[9(0.867726) + 19(0.67329) -$$

$$5(0.0464834 + 0.0240927]$$

$$= 2.0091 \Rightarrow y(2.2) = 2.0091$$

Put $n=4$ in predictor formula

$$y_5^P = y_4 + \frac{h}{24}[55f_4 - 59f_3 + 37f_2 - 9f_1]$$

$$y_5^P = 2.00913 + \frac{0.05}{24}[55(0.0867728) - 59(0.0673294) + 37(0.0464834) - 9(0.0240927)]$$

$$y_5^P = 2.01392$$

$$f_5^P = f(x_5, y_5^P) = f(2.25, 2.01392)$$

$$= 1 - \frac{2.01392}{2.25} = 0.1049245$$

Again Put $n=4$ in corrector formula

$$y_5 = y_4 + \frac{h}{24}[9f_5^P + 19f_4 - 5f_3 + f_2]$$

$$y_5 = 2.0091 + \frac{0.05}{24}[9(0.1049245) + 19(0.867728) - 5(0.0673294) + 0.0464834]$$

$$\Rightarrow y_5 = 2.01389$$

$$\Rightarrow y(2.25) = 2.01389$$

ASSIGINMENT

Q1:- Solve $y' = -xy^2$ at $x=0.8$ & $x=1.0$ using starting values $y(0)=2$, $y(0.2)=1.92308$, $y(0.4)=1.72414$, $y(0.6)=1.47059$ using Milne's & Adam Bashforth methods.

Q2:- $2\frac{dy}{dx} = (1+x^2)y^2$ and $y(0) = 1$
$y(0.1) = 1.06$, $y(0.2) = 1.12$, $y(0.3) = 1.21$
Evaluate $y(0.4)$ By both methods

Q3:- Solve $y' = y^2$ where $y(1) = 1$ and $h = 0.1$
find value of IVP at $x = 1.5$

* * * *

SOLUTIONS

Q1:- $y' = -xy^2$, $h = 0.2$
to find $y(0.8)$ and $y(1.0)$
Given initial value $y(0) = 2$
$y(0.2) = 1.92308$, $y(0.4) = 1.72414$, $y(0.6) = 1.47059$
Here $f(x,y) = -xy^2$
As $h = 0.2$ & $x_0 = 0 \Rightarrow x_1 = x_0 + h = 0 + 0.2 = 0.2$
$x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, $x_5 = 1.0$
To find $y(0.8)$ & $y(1)$ i.e $y(x_4)$ & $y(x_5)$.
i.e y_4 and y_5
& $y_0 = 2$, $y_1 = 1.92308$, $y_2 = 1.72414$, $y_3 = 1.47059$
& $f_0 = -x_0 y_0^2 = -(0)(2)^2 = 0$

$f_1 = -x_1 y_1^2 = -(0.2)(1.92308)^2 = -0.73965$

$f_2 = -x_2 y_2^2 = -(0.4)(1.72414)^2 = -1.18906$

$f_3 = -x_3 y_3^2 = -(0.6)(1.47059)^2 = -1.297581$

(i) By Milne's Method:-

Predictor Formula is

$$\tilde{y}_{n+1} = y_{n-3} + \frac{4h}{3}\left[2f_{n-2} - f_{n-1} + 2f_n\right] \longrightarrow (A)$$

And Corrector Formula is

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left[f_{n-1} + 4f_n + \tilde{f}_{n+1}\right] \longrightarrow (B)$$

Put $n=3$:- Put $n=3$ in Predictor formula

$$\Rightarrow \tilde{y}_4 = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$= 2 + \frac{4(0.2)}{3}\left[2(-0.73965) - (-1.18906) + 2(-1.297581)\right]$$

$$= 1.23056$$

$$\Rightarrow \tilde{f}_4 = -x_4 \tilde{y}_4^2 = -(0.8)(1.23056)^2 = -1.21142$$

Put $n=3$ in corrector formula

$$y_4 = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + \tilde{f}_4\right]$$

$$= 1.72414 + \frac{0.2}{3}\left[-1.18906 + 4(-1.297581) - 1.21142\right]$$

$$= 1.21809 \Rightarrow y(0.8) = 1.21809$$

& $f_4 = -x_4 y_4^2 = -(0.8)(1.21809)^2 = -1.187$

For $n=4$ Put $n=4$ in Predictor formula

$$\tilde{y}_5 = y_1 + \frac{4h}{3}\left[2f_2 - f_3 + 2f_4\right]$$

$$= 1.92308 + \frac{4(0.2)}{3}\left[2(-1.18906) - (-1.29758) + 2(-1.187)\right]$$

$$= 1.00187$$

$$\Rightarrow \tilde{f}_5 = -x_5 \tilde{y}_5^2 = -(1)(1.00187)^2 = -1.00187$$

& Put $n=4$ in corrector formula

$$\Rightarrow y_5 = y_3 + \frac{h}{3}[f_3 + 4f_4 + \tilde{f}_5]$$

$$= 1.47059 + \frac{0.2}{3}[-1.297581 + 4(-1.187) + (-1.00187)]$$

$$= 1.00076 \qquad \Rightarrow y(1.0) = 1.00076$$

2) By **Adam Bashforth Method:-**

Predictor formula is

$$\tilde{y}_{n+1} = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}] \longrightarrow ①$$

And corrector formula is

$$y_{n+1} = y_n + \frac{h}{24}[9\tilde{f}_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}]$$

For $n=3$:- Put $n=3$ in Predictor formula

$$\Rightarrow \tilde{y}_4 = y_3 + \frac{h}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0]$$

$$= 1.47059 + \frac{0.2}{24}[55(-1.297581) - 59(-1.18906) + 37(-0.73965) - 9(0)]$$

$$= 1.23243$$

$$\Rightarrow \tilde{f}_4 = -x_4 \tilde{y}_4^2 = -(0.8)(1.23243)^2 = -1.21511$$

Put $n=3$ in Corrector formula

$$\Rightarrow y_4 = y_3 + \frac{h}{24}[9\tilde{f}_4 + 19f_3 - 5f_2 + f_1]$$

$$= 1.47059 + \frac{0.2}{24}[9(-1.21511) + 19(-1.297581) - 5(-1.18906) + (-0.73965)]$$

$$= 1.21739$$

$$\Rightarrow y(0.8) = 1.21739$$

For $n=4$:- Put $n=4$ in Predictor formula

$\Rightarrow \tilde{y}_5 = y_4 + \frac{h}{24}[55 f_4 - 59 f_3 + 37 f_2 - 9 f_1]$

$= 1.21739 + \frac{0.2}{24}[55(-1.18563) - 59(-1.297581) + 37(-1.18906) - 9(-0.73965)]$

$= 1.0008$

$\Rightarrow \tilde{f}_5 = -x_5 \tilde{y}_5^2 = -(1.0)(1.0008)^2 = -1.0008$

Put $n=4$ in corrector formula

$\Rightarrow y_5 = y_4 + \frac{h}{24}[9\tilde{f}_5 + 19 f_4 - 5 f_3 + f_2]$

$= 1.21739 + \frac{0.2}{24}[9(-1.0008) + 19(-1.18563) - 5(-1.297581) + (-1.18906)]$

$= 0.99876$

$\Rightarrow y(1.0) = 0.99876$

** ———————— **

Q2:- Given $2\frac{dy}{dx} = (1+x^2)y^2$, $y(0)=1$

$\Rightarrow \frac{dy}{dx} = \frac{(1+x^2)y^2}{2}$; $x_0 = 0$, $y_0 = 1$

$y(0.1) = 1.06$, $y(0.2) = 1.12$, $y(0.3) = 1.21$

So $h = 0.1$

$\Rightarrow x_1 = x_0 + h = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$

$x_4 = 0.4$, $x_5 = 0.5$

To find $y(0.4)$ i.e $y(x_4)$ i.e y_4

$y_0 = 1$, $y_1 = 1.06$, $y_2 = 1.12$, $y_3 = 1.21$

Here $f(x_n, y_n) = \frac{(1+x_n^2)y_n^2}{2} = f_n$

$\Rightarrow f_0 = \frac{(1+x_0^2)y_0^2}{2} = \frac{[1+(0)^2](1)^2}{2} = 0.5$

$f_1 = \frac{(1+x_1^2)y_1^2}{2} = \frac{[1+(0.1)^2](1.06)^2}{2} = 0.56742$

$f_2 = \frac{(1+x_2^2)y_2^2}{2} = \frac{[1+(0.2)^2](1.12)^2}{2} = 0.65229$

$f_3 = \frac{(1+x_3^2)y_3^2}{2} = \frac{[1+(0.3)^2](1.21)^2}{2} = 0.79793$

1) <u>Milne's Method:-</u>

Predictor formula is

$$\tilde{y}_{n+1} = y_{n-3} + \frac{4h}{3}[2f_{n-2} - f_{n-1} + 2f_n] \longrightarrow ①$$

And corrector formula is

$$y_{n+1} = y_{n-1} + \frac{h}{3}[f_{n-1} + 4f_n + \tilde{f}_{n+1}] \longrightarrow ②$$

<u>For n=3:-</u> Put $n=3$ in predictor formula

$\Rightarrow \tilde{y}_4 = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$

$= 1 + \frac{4(0.1)}{3}[2(0.56742) - 0.65229 + 2(0.79793)]$

$= 1.27712$

$\Rightarrow \tilde{f}_4 = \frac{(1+x_4^2)\tilde{y}_4^2}{2} = \frac{[1+(0.4)^2](1.27712)^2}{2}$

$= 0.946$

Now Put $n=3$ in corrector formula

$$\Rightarrow y_4 = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$$= 1.12 + \frac{0.1}{3}[0.65229 + 4(0.79793) + 0.946]$$

$$\Rightarrow y(0.4) = 1.27967$$

2) By Adam Bashforth Method:-

Predictor formula is

$$\tilde{y}_{n+1} = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}] \rightarrow (3)$$

And corrector formula is

$$y_{n+1} = y_n + \frac{h}{24}[9\tilde{f}_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}] \rightarrow (4)$$

For $n=3$:- Put $n=3$ in Predictor formula

$$\Rightarrow \tilde{y}_4 = y_3 + \frac{h}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0]$$

$$= 1.21 + \frac{0.1}{24}[55(0.79793) - 59(0.65229) + 37(0.56742) - 9(0.5)]$$

$$= 1.30123$$

$$\Rightarrow \tilde{f}_4 = \frac{(1+x_4^2)\tilde{y}_4^2}{2} = \frac{[1+(0.4)^2](1.30123)^2}{2}$$

$$= 0.98206$$

Put $n=3$ in corrector formula.

$$y_4 = y_3 + \frac{h}{24}\left[9\tilde{f}_4 + 19f_3 - 5f_2 + f_1\right]$$

$$= 1.21 + \frac{0.1}{24}\left[9(0.98206) + 19(0.79793) - 5(0.65229) + 0.56742\right]$$

$$= 1.29877$$

$$\Rightarrow y(0.4) = 1.29877$$

* ——— ** ——— *

~~Q3:- Given $y' = y^2$, $y(1) = 1$~~

~~Let $h = 0.2$ & $y_0 = 1$, $x_0 = 1$~~

~~So $x_1 = x_0 + h = 1 + 0.2 \Rightarrow x_1 = 1.2$~~

~~$x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, $x_5 = 1$~~

* ——— ** ——— *

Q3:- Given $y' = y^2$, $y(1) = 1$

$\Rightarrow y_0 = 1$, $x_0 = 1$ & $h = 0.1$

So $x_1 = x_0 + h = 1 + 0.1 = 1.1$

$x_2 = 1.2$, $x_3 = 1.3$, $x_4 = 1.4$, $x_5 = 1.5$

To find $y(1.5)$ i.e $y(x_5)$

i.e y_5

Here $f(x_n, y_n) = y_n^2 = f_n$

$$f_0 = y_0^2 = (1)^2 = 1$$

Now By R-K method of order 4

$$y_{n+1} = y_n + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

where $k_1 = h\, f(x_n, y_n)$

$$k_2 = h\, f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

(A)

$$K_3 = hf\left[x_n + \frac{h}{2},\ y_n + \frac{K_2}{2}\right]$$

$$K_4 = hf(x_n + h,\ y_n + K_3)$$

<u>1st Iteration:-</u> Put $n = 0$ in (*)

$$\Rightarrow y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

where $K_1 = hf(x_0, y_0) = h(y_0^2)$

$$= 0.1(1)^2 = 0.1$$

$$K_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_1}{2}\right) = h\left[y_0 + \frac{K_1}{2}\right]^2$$

$$= 0.1\left[1 + \frac{0.1}{2}\right]^2 = 0.11025$$

$$K_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_2}{2}\right) = h\left[y_0 + \frac{K_2}{2}\right]^2$$

$$= 0.1\left[1 + \frac{0.11025}{2}\right]^2 = 0.11133$$

$$K_4 = hf(x_0 + h,\ y_0 + K_3) = h(y_0 + K_3)^2$$

$$= 0.1(1 + 0.11133)^2 = 0.12351$$

$$\Rightarrow y_1 = y_0 + \frac{1}{6}[K_1 + 2(K_2 + K_3) + K_4]$$

$$= 1 + \frac{1}{6}[0.1 + 2(0.11025 + 0.11133) + 0.12351]$$

$$= 1.11111 \Rightarrow y(1.1) = 1.11111$$

<u>2nd Iteration:-</u> Put $n = 1$ in (*)

$$\Rightarrow y_2 = y_1 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

where $K_1 = hf(x_1, y_1) = h(y_1^2)$

$$= 0.1(1.11111)^2 = 0.12346$$

$$K_2 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{K_1}{2}\right) = h\left(y_1 + \frac{K_1}{2}\right)^2$$

$$= 0.1\left[1.11111 + \frac{0.12346}{2}\right]^2 = 0.13756$$

$$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = h\left(y_1 + \frac{K_2}{2}\right)^2$$

$$= 0.1\left[1.11111 + \frac{0.13756}{2}\right]^2 = 0.13921$$

$$K_4 = hf(x_1 + h, y_1 + K_3) = h(y_1 + K_3)^2$$

$$= 0.1(1.11111 + 0.13921)^2 = 0.15633$$

$$\Rightarrow y_2 = y_1 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$= 1.11111 + \frac{1}{6}[0.12346 + 2(0.13756 + 0.13921) + 0.15633]$$

$$= 1.25 \Rightarrow y(1.2) = 1.25$$

<u>3rd Iteration:-</u> For $n = 2$

$$K_1 = hf(x_2, y_2) = h(y_2)^2 = 0.1(1.25)^2 = 0.15625$$

$$K_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{K_1}{2}\right) = h\left(y_2 + \frac{K_1}{2}\right)^2$$

$$= 0.1\left[1.25 + \frac{0.15625}{2}\right]^2 = 0.17639$$

$$K_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{K_2}{2}\right) = 0.17908$$

$$K_4 = hf(x_2 + h, y_2 + K_3) = h(y_2 + K_3)^2$$

$$= 0.1(1.25 + 0.17908)^2 = 0.20423$$

$$\Rightarrow y_3 = 1.25 + \frac{1}{6}[0.15625 + 2(0.17639 + 0.17908) + 0.20423]$$

$$= 1.42857 \Rightarrow y(1.3) = 1.42857$$

<u>4th Iteration:-</u> For $n = 3$

$$K_1 = hf(x_3, y_3) = h(y_3)^2 = 0.20408$$

$$K_2 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{K_1}{2}\right) = h\left[y_3 + \frac{K_1}{2}\right]^2$$

$$\Rightarrow K_2 = 0.1\left[1.42857 + \frac{0.20408}{2}\right]^2 = 0.23428$$

$$K_3 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{K_2}{2}\right) = 0.23892$$

$$K_4 = hf(x_3 + h, y_3 + K_3) = h(y_3 + K_3)^2$$

$$= 0.1(1.42857 + 0.23892)^2 = 0.27805$$

$$\Rightarrow y_4 = 1.42857 + \frac{1}{6}[0.20408 + 2(0.23428 + 0.23892) + 0.27805]$$

$$= 1.66666 \Rightarrow y(1.4) = 1.66666$$

<u>Now By Milnen Method:</u>

Predictor formula is

$$\tilde{y}_{n+1} = y_{n-3} + \frac{4h}{3}[2f_{n-2} - f_{n-1} + 2f_n] \longrightarrow (1)$$

& Corrector formula is

$$y_{n+1} = y_{n-1} + \frac{h}{3}[f_{n-1} + 4f_n + \tilde{f}_{n+1}] \longrightarrow (2)$$

& $f_0 = 1$

$f_1 = y_1^2 = 1.23456$, $f_2 = y_2^2 = 1.5625$

$f_3 = y_3^2 = 2.0408$, $f_4 = y_4^2 = 2.77775$

Put $n = 4$ in Predictor form

$$\Rightarrow \tilde{y}_5 = y_1 + \frac{4h}{3}[2f_2 - f_3 + 2f_4]$$

$$= 1.11111 + \frac{4(0.1)}{3}[2(1.5625) - 2.0408 + 2(2.77775)]$$

$$= 1.9964$$

$$\Rightarrow \tilde{f}_5 = (\tilde{y}_5)^2 = (1.9964)^2 = 3.9856$$

Put $n=4$ in corrector formula

$$\Rightarrow y_5 = y_3 + \frac{h}{3}[f_3 + 4f_4 + f_5]$$

$$= 1.42857 + \frac{0.1}{3}[2.0408 + 4(2.77775) + 3.9856]$$

$$= 1.9998 \quad \Rightarrow y(1.5) = 1.9998$$

By Adam Bashforth Method

Predictor formula is

$$\tilde{y}_{n+1} = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}]$$

& corrector formula is

$$y_{n+1} = y_n + \frac{h}{24}[9\tilde{f}_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}]$$

Put $n=4$ in predictor formula

$$\tilde{y}_5 = y_4 + \frac{h}{24}[55f_4 - 59f_3 + 37f_2 - 9f_1]$$

$$= 1.66666 + \frac{0.1}{2}[55(2.77775) - 59(2.0408) + 37(1.5625) - 9(1.23456)]$$

$$= 1.99612$$

$$\Rightarrow f_5 = \tilde{y}_5^2 = (1.99612)^2 = 3.9845$$

Now Put $n=4$ in corrector form

$$\Rightarrow y_5 = y_4 + \frac{h}{3}[9\tilde{f}_5 + 19f_4 - 5f_3 + f_2]$$

$$= 1.66666 + \frac{0.1}{3}[9(3.9845) + 19(2.77775) - 5(2.0408) + 1.5625]$$

$$= 1.99998$$

$$\Rightarrow y(1.5) = 1.99998$$

***

⇒ System of Differential Equations

A set of simultaneous equations that involves two or more unknown function and their derivatives is called system of differential equations.

for example

$$dx/dt = 2x + y + t$$

$$dy/dt = x - y + 2t$$

Here t is indipendent variable and x, y are dependent variables.

Runge-Kutta Method of order 4:-

To solve equations

$$\frac{dx}{dt} = f(t, x, y) \quad , \quad \frac{dy}{dt} = g(t, x, y)$$

$$x(t_0) = x_0 \quad , \quad y(t_0) = y_0$$

we find K_1, K_2, K_3, K_4 and l_1, l_2, l_3, l_4 in the order

$$K_1 = h f(t_n, x_n, y_n) \quad , \quad l_1 = h g(t_n, x_n, y_n)$$

$$K_2 = h f[t_n + h/2, x_n + K_1/2, y_n + l_1/2)$$

$$l_2 = h g(t_n + h/2, x_n + \frac{K_1}{2}, y_n + l_1/2)$$

$$K_3 = h f[t_n + h/2, x_n + K_2/2, y_n + l_2/2]$$

$$l_3 = h g[t_n + h/2, x_n + K_2/2, y_n + l_2/2]$$

$$K_4 = h f[t_n + h, x_n + K_3, y_n + l_3]$$

$$l_4 = h g[t_n + h, x_n + K_3, y_n + l_3]$$

finally we obtain

$$x_{n+1} = x_n + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$y_{n+1} = y_n + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$$

R-K Method of order 2:-

$$K_1 = hf(t_n, x_n, y_n)$$

$$l_1 = hg(t_n, x_n, y_n)$$

$$K_2 = hf[t_n + h, x_n + K_1, y_n + l_1]$$

$$l_2 = hg[t_n + h, x_n + K_1, y_n + l_1]$$

$$x_{n+1} = x_n + \frac{1}{2}[K_1 + K_2]$$

$$y_{n+1} = y_n + \frac{1}{2}[l_1 + l_2]$$

**

Example:- Solve initial value problem at $t = 1.2$ where

$$\frac{dy}{dt} = x - 2y, \quad \frac{dx}{dt} = 2x + y$$

$$x(1) = 2, \quad y(1) = 3$$

Solution

$$f(t, x, y) = 2x + y$$

$$g(t, x, y) = x - 2y$$

$$\therefore f(t_n, x_n, y_n) = 2x_n + y_n$$

$$g(t_n, x_n, y_n) = x_n - 2y_n$$

1st Iteration:- $t_0 = 1$, $x_0 = 2$, $y_0 = 3$

Let $h = 0.1$

$K_1 = h f(t_0, x_0, y_0)$

$= 0.1(2x_0 + y_0) = 0.1\{2(2) + 3\}$

$= 0.7$

$l_1 = h g(t_0, x_0, y_0) = h(x_0 - 2y_0)$

$= 0.1[2 - 2(3)] = -0.4$

$K_2 = h f[t_0 + \frac{h}{2}, x_0 + \frac{K_1}{2}, y_0 + \frac{l_1}{2}]$

$= h\{2(x_0 + \frac{K_1}{2}) + (y_0 + \frac{l_1}{2})\}$

$= 0.1\{2(2.35) + 2.8\} = 0.75$

$l_2 = h g[t_0 + \frac{h}{2}, x_0 + \frac{K_1}{2}, y_0 + \frac{l_1}{2}]$

$= 0.1\, g[1.05, 2.35, 2.8]$

$= 0.1[2.35 - 2(2.8)] = -0.325$

$K_3 = h f[t_0 + \frac{h}{2}, x_0 + \frac{K_2}{2}, y_0 + \frac{l_2}{2}]$

l_2 $= 0.1 f[1.05, 2.375, 2.8375]$

$= 0.1[2 \times 2.375 + 2.8375]$

$= 0.75875$

$l_3 = h g[t_0 + \frac{h}{2}, x_0 + \frac{K_2}{2}, y_0 + \frac{l_2}{2}]$

$= 0.1 g[1.05, 2.375, 2.8375]$

$= 0.1[2.375 - 2(2.8375)] = -0.33$

$k_4 = hf[t_0 + h, x_0 + k_3, y_0 + l_3]$

$= 0.1 f[1.1, 2.75875, 2.67]$

$= 0.1[2(2.7587) + 2.67] = 0.81875$

$l_4 = hg[t_0 + h, x_0 + k_3, y_0 + l_3]$

$= 0.1 g[1.1, 2.75875, 2.67]$

$= 0.1\{2.75875 - 2(2.67)\} = -0.25812$

$x_1 = x_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 2 + \frac{1}{6}[0.7 + 2(0.75 + 0.75875) + 0.81875]$

$= 2.75604$

$y_1 = y_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$

$= 3 + \frac{1}{6}[-0.4 + 2(-0.325 - 0.33) - 0.258125]$

$= 2.67198$

2nd Iteration:- $t_1 = t_0 + h = 1 + 0.1 = 1.1$

$x_1 = 2.75604, \quad y_1 = 2.67198$

$k_1 = hf(t_1, x_1, y_1), \quad l_1 = hg(t_1, x_1, y_1)$

So on same solve as in 1st iteration replace x_0, y_0 by x_1, y_1 and find k_1, k_2, k_3, k_4 and l_1, l_2, l_3, l_4 we get

$x_2 = 3.650467$

$y_2 = 2.47814$ Ans

Example:- Solve the equations

$$\frac{dy}{dx} = xz + 1 \quad , \quad \frac{dz}{dx} = -xy$$

for $x = 0.3, 0.6, 0.9$

Given $y = 0$, $z = 1$ when $x = 0$

Solution:- $x = 0.3, 0.6, 0.9$

$h = 0.3$

1st find value at 0.3 and at 0.6 and 0.9

Note that x is independent and y, z are dependent variables.

$$f(x,y,z) = xz + 1 \quad , \quad g(x,y,z) = -xy$$

$$f(x_n, y_n, z_n) = x_n z_n + 1 \quad , \quad g(x_n, y_n, z_n) = -x_n y_n$$

1st Iteration:- $x_0 = 0$, $y_0 = 0$, $z_0 = 1$

$h = 0.3$

$$K_1 = h f[x_0, y_0, z_0] = h[x_0 z_0 + 1]$$

$$= 0.3\{0+1\} = 0.3$$

$$l_1 = h g[x_0, y_0, z_0] = h[-x_0 y_0]$$

$$= 0.3\{-(0)(0)\} = 0$$

$$K_2 = h f\left[x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right]$$

$$= 0.3 f[0.15, 0.15, 1] = 0.3[0.15 \times 1 + 1]$$

$$= 0.3450$$

$$l_2 = h g\left[x_0 + h/2, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right]$$

$$= 0.3 g[0.15, 0.15, 1] = 0.3\{-0.15 \times 0.15\}$$

$\Rightarrow l_2 = -0.00675$

$K_3 = hf[x_0 + h/2, y_0 + \frac{K_2}{2}, z_0 + l_2/2]$

$= 0.3f[0.15, 0.1725, 0.9966]$

$= 0.3[0.15(0.9966) + 1] = 0.3448$

$l_3 = hg[x_0 + h/2, y_0 + \frac{K_2}{2}, z_0 + l_2/2]$

$= 0.3g[0.15, 0.1725, 0.9966)$

$= 0.3[-0.15 \times 0.1725] = -0.00776$

$K_4 = hf(x_0 + h, y_0 + K_3, z_0 + l_3)$

$= 0.3f(0.3, 0.3448, 0.99224)$

$= 0.3[0.3 \times 0.9924 + 1] = 0.3893$

$l_4 = hg[x_0 + h, y_0 + K_3, z_0 + l_3]$

$= 0.3g(0.3, 0.3448, 0.99224)$

$= 0.3\{-0.3 \times 0.3448\} = -0.031036$

$y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$= 0 + \frac{1}{6}[0.3 + 2(0.3450 + 0.3448) + 0.3893]$

$= 0.3448$

$z_1 = z_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$

$= 1 + \frac{1}{6}[0.0 + 2(-0.00675 + 0.3893) + (-0.031036)]$

$= 0.9899$

$\Rightarrow y(0.3) = 0.3448, \; z(0.3) = 0.9899$

2nd Iteration:- $x_1 = x_0 + h = 0.3$

$y_1 = 0.3448$, $z_1 = 0.9899$

By using R-K method of order 4 formula we have

$k_1 = 0.3891$, $l_1 = -0.031042$

$k_2 = 0.43155$, $l_2 = -0.07281$

$k_3 = 0.42873$, $l_3 = 0.07568$

$k_4 = 0.4646$, $l_4 = -0.1393$

$$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 0.3448 + \frac{1}{6}[0.3891 + 2(0.43155 + 0.4287) + 0.4646]$$

$$= 0.3448 + 0.4290$$

$$\Rightarrow y_2 = 0.7738$$

$$z_2 = z_1 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$$

$$= 0.9899 - 0.07788$$

$$= 0.9121$$

3rd Iteration:- $x_2 = x_0 + h = 0.6$

$y_2 = 0.7738$, $z_2 = 0.9121$

$k_1 = 0.4642$, $l_1 = -0.1393$

$k_2 = 0.4896$, $l_2 = -0.2263$

$k_3 = 0.47976$, $l_3 = -0.2292$

$k_4 = 0.4844$, $l_4 = -0.3386$

$$y_3 = y_2 + \frac{1}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

$\Rightarrow y_3 = 0.7738 + 0.4812$
$= 1.2550$

$z_3 = z_2 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$
$= 0.9121 - 0.23148 = 0.6806$

$\Rightarrow y(0.9) = 1.2550$ Ans

*——————— **** ——————— *

ASSIGNMENT

Solve the following system of equations find x and y two steps after the given value of t.

a) $\frac{dx}{dt} = x+y-t$, $\frac{dy}{dt} = 3x-y+2t$
$x(0)=1$, $y(0)=2$

b) $\frac{dx}{dt} = x^2+y$, $\frac{dy}{dt} = 2x-3y+t$
$x(0) = 0.2$, $y(0) = 0.1$

c) $\frac{dx}{dt} = 0.1x + 0.2y - 0.3t$
$\frac{dy}{dt} = 0.2x - 0.1y + 0.2t$
at $t=1.1$ where $x(1)=1$, $y(1)=2$

SOLUTIONS

a) Given $\frac{dx}{dt} = x+y-t$,

$$\frac{dy}{dt} = 3x-y+2t \quad ; x(0)=1, \quad y(0)=2$$

$\Rightarrow x_0 = 1$, $y_0 = 2$, $t_0 = 0$ Let $h = 0.2$

$\Rightarrow t_1 = t_0 + h = 0 + 0.2 \Rightarrow t_1 = 0.2$

$t_2 = 0.4$

Here $f(t,x,y) = x+y-t$

& $g(t,x,y) = 3x-y+2t$

$\therefore f(t_n, x_n, y_n) = x_n + y_n - t_n$

& $g(t_n, x_n, y_n) = 3x_n - y_n + 2t_n$

Now by R.K method of order 4

$x_{n+1} = x_n + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$y_{n+1} = y_n + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$

where $K_1 = hf(t_n, x_n, y_n)$, $l_1 = hg(t_n, x_n, y_n)$

$K_2 = hf(t_n + h/2, x_n + \frac{K_1}{2}, y_n + \frac{l_1}{2})$

$l_2 = hg(t_n + h/2, x_n + \frac{K_1}{2}, y_n + \frac{l_1}{2})$ $\rightarrow$ (★)

$K_3 = hf(t_n + h/2, x_n + \frac{K_2}{2}, y_n + \frac{l_2}{2})$

$l_3 = hg(t_n + h/2, x_n + K_2/2, y_n + l_2/2)$

$K_4 = hf(t_n + h, x_n + K_3, y_n + l_3$

$$l_4 = h\,g(t_n + h, x_n + k_3, y_n + l_3]$$

<u>1st Iteration:-</u> Put $n=0$ in ⊛

$$\Rightarrow x_1 = x_0 + \frac{1}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

$$y_1 = y_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$$

where $k_1 = h f(t_0, x_0, y_0) = h[x_0 + y_0 - t_0]$

$$= 0.2\{1 + 2 - 0\} = 0.6$$

$$l_1 = h\,g(t_0, x_0, y_0) = h[3x_0 - y_0 + 2t_0]$$

$$= 0.2\{3(1) - 2 + 2(0)\} = 0.2$$

$$k_2 = h f\left[t_0 + \frac{h}{2}, y_0 + \frac{l_1}{2}, x_0 + \frac{k_1}{2}\right]$$

$$= h\left[\left(x_0 + \frac{k_1}{2}\right) + \left(y_0 + \frac{l_1}{2}\right) - \left(t_0 + \frac{h}{2}\right)\right]$$

$$= 0.2\left[\left(1 + \frac{0.6}{2}\right) + \left(2 + \frac{0.2}{2}\right) - \left(0 + \frac{0.2}{2}\right)\right] = 0.66$$

$$l_2 = h\,g\left[t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}, y_0 + \frac{l_1}{2}\right] = h\left[3\left(x_0 + \frac{k_1}{2}\right) - \left(y_0 + \frac{l_1}{2}\right) + 2\left(t_0 + \frac{h}{2}\right)\right]$$

$$= 0.2\left[3\left(1 + \frac{0.6}{2}\right) - \left(2 + \frac{0.2}{2}\right) + 2\left(0 + \frac{0.2}{2}\right)\right] = 0.4$$

$$k_3 = h f\left[t_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}, y_0 + \frac{l_2}{2}\right]$$

$$= h\left[\left(x_0 + \frac{k_2}{2}\right) + \left(y_0 + \frac{l_2}{2}\right) - \left(t_0 + \frac{h}{2}\right)\right]$$

$$= 0.2\left[\left(1 + \frac{0.66}{2}\right) + \left(2 + \frac{0.4}{2}\right) - \left(0 + \frac{0.2}{2}\right)\right]$$

$$= 0.686$$

$$l_3 = h\,g\left(t_0 + \tfrac{h}{2},\, x_0 + \frac{K_2}{2},\, y_0 + \tfrac{l_2}{2}\right)$$

$$= h\left[3\left(x_0 + \frac{K_2}{2}\right) - \left(y_0 + \frac{l_2}{2}\right) + 2\left(t_0 + \tfrac{h}{2}\right)\right]$$

$$= 0.398$$

$$K_4 = h\,f\left[t_0 + h,\, x_0 + K_3,\, y_0 + l_3\right]$$

$$= h\left[(x_0 + K_3) + (y_0 + l_3) + (t_0 + h)\right]$$

$$= 0.2\left[(1 + 0.686) + (2 + 0.398) + (0.2 + 0.2)\right]$$

$$= 0.7768$$

$$l_4 = h\,g\left[t_0 + h,\, x_0 + K_3,\, y_0 + l_3\right]$$

$$= h\left[3(x_0 + K_3) - (y_0 + l_3) + 2(t_0 + h)\right]$$

$$= 0.2\left[3(1 + 0.686) - (2 + 0.398) + 2(0.2 + 0.2)\right]$$

$$= 0.612$$

$$\Rightarrow x_1 = 1 + \frac{1}{6}\left[0.6 + 2(0.66 + 0.686) + 0.7768\right]$$

$$= 1.6781$$

& $$y_1 = 1 + \frac{1}{6}\left[0.2 + 2(0.4 + 0.398) + 0.612\right]$$

$$= 2.4013$$

<u>2nd Iteration:-</u> Put $n = 1$ in (*)

$$\Rightarrow x_2 = x_1 + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$y_2 = y_1 + \frac{1}{6}\left[l_1 + 2l_2 + 2l_3 + l_4\right]$$

$$K_1 = h f(t_1, x_1, y_1) = h(x_1 + y_1 - t_1)$$
$$= 0.2\{1.6781 + 2.4013 - 0.2\} = 0.7759$$

$$l_1 = h g(t_1, x_1, y_1) = h(3x_1 - y_1 + 2t_1)$$
$$= 0.2\{3(1.6781) - 2.4013 + 2(0.2)\} = 0.6066$$

$$K_2 = h f\left(t_1 + \tfrac{h}{2}, x_1 + \tfrac{K_1}{2}, y_1 + \tfrac{l_1}{2}\right)$$
$$= h\left[\left(x_1 + \tfrac{K_1}{2}\right) + \left(y_1 + \tfrac{l_1}{2}\right) - \left(t_1 + \tfrac{h}{2}\right)\right]$$
$$= 0.2\left[\left(1.6781 + \tfrac{0.7759}{2}\right) + \left(2.4013 + \tfrac{0.6066}{2}\right) - \left(0.2 + \tfrac{0.2}{2}\right)\right]$$
$$= 0.8941$$

$$l_2 = h g\left(t_1 + \tfrac{h}{2}, x_1 + \tfrac{K_1}{2}, y_1 + \tfrac{l_1}{2}\right)$$
$$= h\left[3\left(x_1 + \tfrac{K_1}{2}\right) - \left(y_1 + \tfrac{l_1}{2}\right) + 2\left(t_1 + \tfrac{h}{2}\right)\right]$$
$$= 0.8187$$

$$K_3 = h f\left(t_1 + \tfrac{h}{2}, x_1 + \tfrac{K_2}{2}, y_1 + \tfrac{l_2}{2}\right)$$
$$= h\left[\left(x_1 + \tfrac{K_2}{2}\right) + \left(y_1 + \tfrac{l_2}{2}\right) - \left(t_1 + \tfrac{h}{2}\right)\right]$$
$$= 0.2\left[\left(1.681 + \tfrac{0.8941}{2}\right) + \left(2.4013 + \tfrac{0.8187}{2}\right) - \left(0.2 + \tfrac{0.2}{2}\right)\right]$$
$$= 0.9272$$

$$l_3 = h g\left[t_1 + \tfrac{h}{2}, x_1 + \tfrac{K_2}{2}, y_1 + \tfrac{l_2}{2}\right]$$
$$= h\left[3\left(x_1 + \tfrac{K_2}{2}\right) - \left(y_1 + \tfrac{l_2}{2}\right) + 2\left(t_1 + \tfrac{h}{2}\right)\right]$$
$$= 0.833$$

$k_4 = h f(t_1+h, x_1+k_3, y_1+l_3)$

$= h((x_1+k_3)+(y_1+l_3)-(t_1+h))$

$= 0.2[(1.6781+0.9272)+(2.4013+0.833)-(0.2+0.2)]$

$= 1.0879$

$l_4 = h g[t_1+h, x_1+k_3, y_1+l_3]$

$= h[3(x_1+k_3)-(y_1+l_3)+2(t_1+h)]$

$= 1.0763$

$\Rightarrow x_2 = 1.6781 + \frac{1}{6}[0.7756+2(0.8941+0.9272)$
$+1.0879]$

$= 2.5958$ &

$y_2 = 2.4013+\frac{1}{6}[0.6066+2(0.8187+0.833)+1.0763]$

$= 3.2324$

***

(b) $\frac{dx}{dt} = x^2+y$, $\frac{dy}{dt} = 2x-3y+t$

$x(0)=0.2$, $y(0)=0.1$ let $h=0.2$

$\Rightarrow x_0 = 0.2$, $y_0 = 0.1$, $t_0 = 0$

$t_1 = 0.2$, $t_2 = 0.4$

$f(t_n, x_n, y_n) = x_n^2 + y_n$

$g(t_n, x_n, y_n) = 2x_n - 3y_n + t_n$

Now By R-K method of order 4

$$x_{n+1} = x_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

& $$y_{n+1} = y_n + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$$

where $k_1 = hf(t_n, x_n, y_n)$

$$l_1 = hg(t_n, x_n, y_n)$$

$$k_2 = hf(t_n + h/2, x_n + \frac{k_1}{2}, y_n + l_1/2)$$

$$l_2 = hg(t_n + h/2, x_n + \frac{k_1}{2}, y_n + l_1/2)$$

$$k_3 = hf(t_n + h/2, x_n + \frac{k_2}{2}, y_n + l_2/2)$$

$$l_3 = hg(t_n + h/2, x_n + \frac{k_2}{2}, y_n + l_2/2)$$

$$k_4 = hf(t_n + h, x_n + k_3, y_n + l_3)$$

$$l_4 = hg(t_n + h, x_n + k_3, y_n + l_3)$$

$\rightarrow$ ★

<u>1st Iteration:-</u> Put $n = 0$ in ★

$\Rightarrow$ $x_1 = x_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

& $y_1 = y_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$

$k_1 = hf(t_0, x_0, y_0) \quad = h(x_0^2 + y_0)$

$= 0.2[(0.2)^2 + 0.1] = 0.028$

$l_1 = hg(t_0, x_0, y_0) = h[2x_0 - 3y_0 + t_0]$

$= 0.2[2(0.2) - 3(0.1) + 0] = 0.06$

$K_2 = hf\left(x_0 + \frac{K_1}{2}, y_0 + \frac{l_1}{2}, t_0 + \frac{h}{2}\right)$

$= h\left[\left(x_0 + \frac{K_1}{2}\right)^2 + \left(y_0 + \frac{l_1}{2}\right)\right] = 0.2\left[\left(0.2 + \frac{0.028}{2}\right)^2 + \left(0.1 + \frac{0.06}{2}\right)\right]$

$= 0.0352$

$l_2 = hg\left[t_0 + \frac{h}{2}, x_0 + \frac{K_1}{2}, y_0 + \frac{l_1}{2}\right]$

$= h\left[2\left(x_0 + \frac{K_1}{2}\right) - 3\left(y_0 + \frac{l_1}{2}\right) + \left(t_0 + \frac{h}{2}\right)\right]$

$= 0.2\left[2\left(0.2 + \frac{0.028}{2}\right) - 3\left(0.1 + \frac{0.06}{2}\right) + \left(0.2 + \frac{0.2}{2}\right)\right]$

$= 0.0276$

$K_3 = hf\left[t_0 + \frac{h}{2}, x_0 + \frac{K_2}{2}, y_0 + \frac{l_2}{2}\right)$

$= 0.03222$

$l_3 = hg\left[t_0 + \frac{h}{2}, x_0 + \frac{K_2}{2}, y_0 + \frac{l_2}{2}\right]$

$= 0.03876$

$K_4 = hf(t_0 + h, x_0 + K_3, y_0 + l_3)$

$= h\left((x_0 + K_3)^2 + (y_0 + l_3)\right)$

$= 0.2\left[(0.2 + 0.03222)^2 + (0.1 + 0.03876)\right]$

$= 0.03854$

$l_4 = hg(t_0 + h, x_0 + K_3, y_0 + l_3)$

$= h\left[2(x_0 + K_3) - 3(y_0 + l_3) + (t_0 + h)\right]$

$= 0.04963$

$\Rightarrow x_1 = 0.2 + \frac{1}{6}[0.028 + 2(0.0352 + 0.032222) + 0.03854]$

$= 0.2336$

$y_1 = 0.1 + \frac{1}{6}[0.06 + 2(0.0276 + 0.03876) + 0.04963]$

$= 0.1404$

<u>2nd Iteration:-</u> Put $n = 1$ in (*)

$\Rightarrow \quad x_2 = x_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$y_2 = y_1 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$

$k_1 = hf(t_1, x_1, y_1) \quad = h[x_1^2 + y_1]$

$= 0.2[(0.2336)^2 + 0.1404] \quad = 0.039$

$l_1 = hg(t_1, x_1, y_1) = h[2x_1 - 3y_1 + t_1]$

$= 0.2[2(0.2336) - 3(0.1404) + 0.2] = 0.0492$

$k_2 = hf[t_1 + \frac{h}{2}, x_1 + \frac{k_1}{2}, y_1 + \frac{l_1}{2}]$

$= h[(x_1 + \frac{k_1}{2})^2 + (y_1 + \frac{l_1}{2})]$

$= 0.2[(0.2336 + \frac{0.039}{2})^2 + (0.1404 + \frac{0.0492}{2})]$

$= 0.0458$

$l_2 = hg[t_1 + \frac{h}{2}, x_1 + \frac{k_1}{2}, y_1 + \frac{l_1}{2}]$

$= h[2(x_1 + \frac{k_1}{2}) - 3(y_1 + \frac{l_1}{2}) + (t_1 + h/2)]$

$= 0.2[2(0.2336 + \frac{0.039}{2}) - 3(0.1404 + \frac{0.0492}{2}) + (0.2 + \frac{0.2}{2})]$

$\Rightarrow \quad l_2 = 0.06224$

$k_3 = hf\left[t_1 + \frac{h}{2}, x_1 + \frac{k_2}{2}, y_1 + \frac{l_2}{2}\right]$

$= 0.04746$

$l_3 = hg\left[t_1 + \frac{h}{2}, x_1 + \frac{k_2}{2}, y_1 + \frac{l_2}{2}\right] = 0.05969$

$k_4 = hf\left[t_1 + h, x_1 + k_3, y_1 + l_3\right]$

$= h\left[(x_1 + k_3)^2 + (y_1 + l_3)\right]$

$= 0.2\left[(0.2336 + 0.04746)^2 + (0.1404 + 0.05969)\right]$

$= 0.05582$

$l_4 = hg\left[t_1 + h, x_1 + k_3, y_1 + l_3\right]$

$= h\left[2(x_1 + k_3) - 3(y_1 + l_3) + (t_1 + h)\right]$

$= 0.2\left[2(0.2336 + 0.04746) - 3(0.1404 + 0.5969) + (0.2 + 0.2)\right]$

$= 0.03237$

$\Rightarrow \quad x_2 = 0.2336 + \frac{1}{6}\left[0.039 + 2(0.0458 + 0.04746) + 0.05582\right]$

$= 0.2805$

& $y_3 = 0.1404 + \frac{1}{6}\left[0.0492 + 2(0.06224 + 0.05969) + 0.03237\right]$

$= 0.1947$

★★ ——————————————— ★★

(C) $\frac{dx}{dt} = 0.1x + 0.2y - 0.3t$

$$\frac{dy}{dt} = 0.2x - 0.1y + 0.2t$$

$x(1) = 1$, $y(1) = 2$,

To find at $t = 1.1$ let $h = 0.1$

So $x_0 = 1$, $y_0 = 2$, $t_0 = 1$

$\Rightarrow t_1 = t_0 + h = 1 + 0.1 = 1.1$

To find $y(t_1)$ i.e y_1

$$f(t_n, x_n, y_n) = 0.1x_n + 0.2y_n - 0.3t_n$$

$$g(t_n, x_n, y_n) = 0.2x_n - 0.1y_n + 0.2t_n$$

By R-K Method of order 4

$$x_{n+1} = x_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$y_{n+1} = y_n + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$$

where

$$\left.\begin{aligned}
k_1 &= hf(t_n, x_n, y_n) \\
l_1 &= hg(t_n, x_n, y_n) \\
k_2 &= hf\left(t_n + \frac{h}{2}, x_n + \frac{k_1}{2}, y_n + \frac{l_1}{2}\right) \\
l_2 &= hg\left(t_n + \frac{h}{2}, x_n + \frac{k_1}{2}, y_n + \frac{l_1}{2}\right) \\
k_3 &= hf\left(t_n + \frac{h}{2}, x_n + \frac{k_2}{2}, y_n + \frac{l_2}{2}\right) \\
l_3 &= hg\left(t_n + \frac{h}{2}, x_n + \frac{k_2}{2}, y_n + \frac{l_2}{2}\right) \\
k_4 &= hf(t_n + h, x_n + k_3, y_n + l_3) \\
l_4 &= hg(t_n + h, x_n + k_3, y_n + l_3)
\end{aligned}\right\} \rightarrow \star$$

Put $n=0$ in ⊛

$\Rightarrow x_1 = x_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$y_1 = y_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$

$K_1 = h f(t_0, x_0, y_0) = h[0.1x_0 + 0.2y_0 - 0.3t_0]$

$= 0.1[0.1(1) + 0.2(2) - 0.3(1)] = 0.02$

$l_1 = h g(t_0, x_0, y_0) = h[0.2x_0 - 0.1y_0 + 0.2t_0]$

$= 0.1[0.2(1) - 0.1(2) - 0.2(1)] = 0.02$

$K_2 = h f[t_0 + h/2, x_0 + \frac{K_1}{2}, y_0 + \frac{l_1}{2}]$

$= h[0.1(x_0 + \frac{K_1}{2}) + 0.2(y_0 + \frac{l_1}{2}) - 0.3(t_0 + h/2)]$

$= 0.1[0.1(1 + \frac{0.2}{2}) + 0.2(2 + \frac{0.2}{2}) - 0.3(1 + \frac{0.1}{2})]$

$= -0.2647$

$l_2 = h g(t_0 + h/2, x_0 + \frac{K_1}{2}, y_0 + \frac{l_1}{2})$

$= h[0.2(x_0 + \frac{K_1}{2}) - 0.1(y_0 + \frac{l_1}{2}) + 0.2(t_0 + h/2)]$

$= 0.1[0.2[1 + \frac{0.02}{2}) - 0.1(2 + \frac{0.2}{2}) + 0.2(1 + \frac{0.1}{2})]$

$= 0.0211$

$K_3 = h f[t_0 + h/2, x_0 + \frac{K_1}{2}, y_0 + l_1/2] = -0.2661$

$l_3 = h g[t_0 + \frac{h}{2}, x_0 + \frac{K_1}{2}, y_0 + \frac{l_1}{2}) = 0.01825$

$K_4 = h f(t_0 + h, x_0 + K_3, y_0 + l_3)$

$= h[0.1(x_0 + K_3) + 0.2(y_0 + l_3) - 0.3(t_0 + h)]$

$\Rightarrow K_4 = 0.1[0.1(1+(-0.2661)) + 0.2(2+0.01825) - 0.3(1+0.1)]$

$= 0.0147$

$l_4 = h g[t_0 + h, x_0 + k_3, y_0 + l_3]$

$= h[0.2(x_0 + k_3) - 0.1(y_0 + l_3) + 0.2(t_0 + h)]$

$= 0.0165$

$\Rightarrow x_1 = 1 + \frac{1}{6}[0.02 + 2(-0.2647 - 0.2661) + 0.0147]$

$= 0.8286$

& $y_1 = 2 + \frac{1}{6}[0.02 + 2(0.0211 + 0.01825) + 0.0165]$

$= 2.019$

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