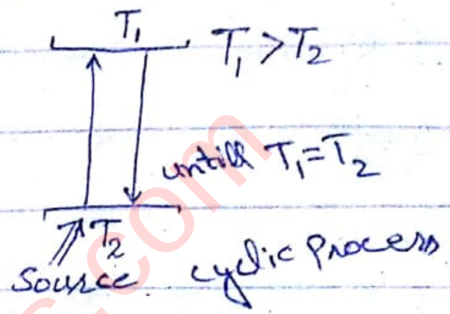
First Law of Thermodynamics 05-03-2015

When a system undergoes a cyclic process the first law of thermodynamics can be expressed as

$$\oint \delta Q = \oint \delta W \longrightarrow (1.19)$$



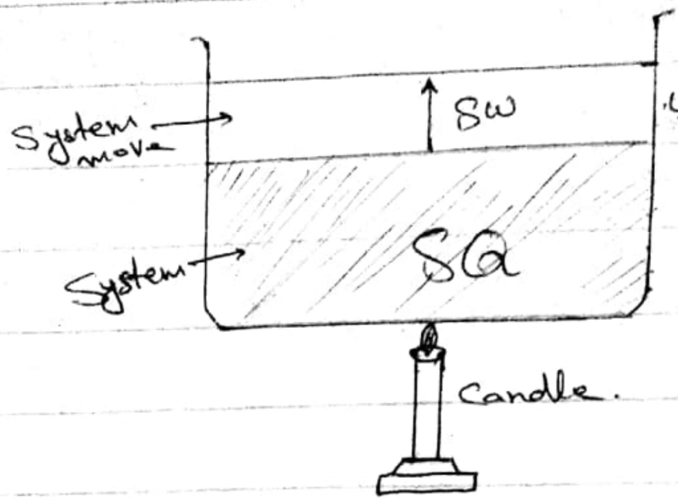
In universe cyclic process is an ideal process.

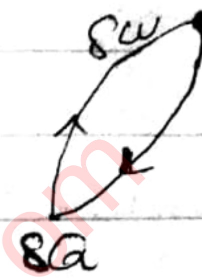
"Physical Significance"

The cyclic integral $\oint \delta Q$ is the amount of Heat added to the system & the cyclic integral $\oint \delta W$ is the work done by the system during the cyclic process.

Both amount of heat added to the system and work done by the system are path function. That is, the amount of heat transfer to the system & work done by the system both depends on path. That is why the differentials δQ & δW are inexact differential. We can write the first law of thermodynamics as

$$dE = \delta Q - \delta W \longrightarrow (1.20)$$





Where δQ is the amount of heat (energy) added to the system & δW is the work done by the system and dE is the corresponding increase in energy of the system. Energy like other properties is a thermodynamics property and is a point function. (i.e it depends on initial and final state of the system, not on the path of the system.).

Now first law of thermodynamics further can be written as

$$\frac{dE}{dt} = \frac{\delta Q}{dt} - \frac{\delta W}{dt} \longrightarrow (1.21a)$$

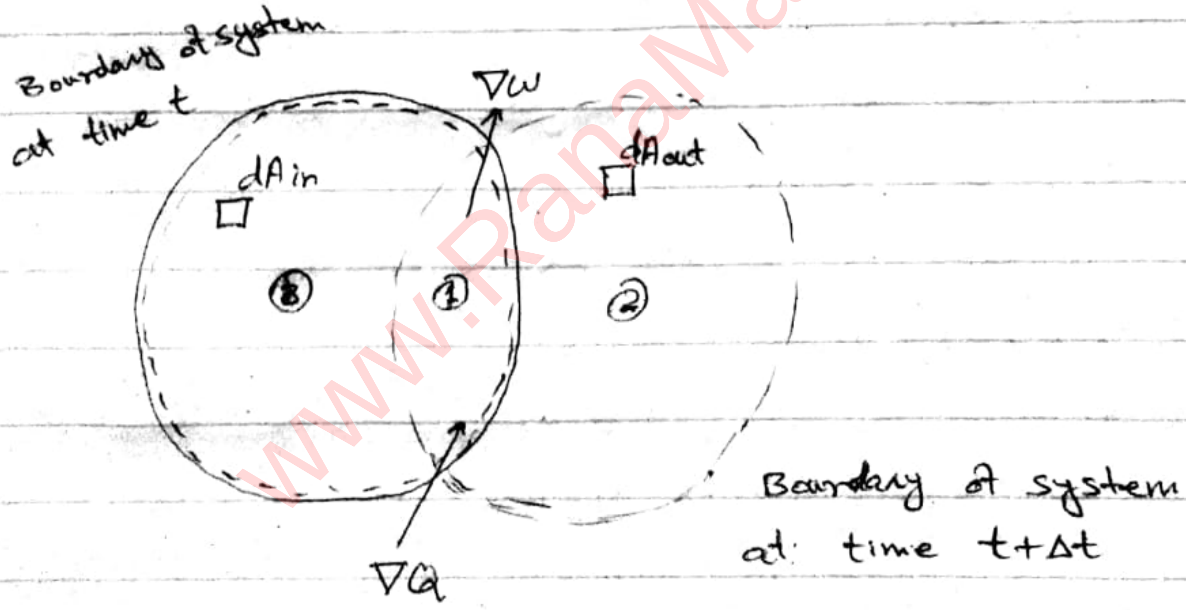
Increase in energy within small interval of time.

$$\Rightarrow \frac{dE}{dt} = \dot{q} - \dot{W} \longrightarrow (1.21b)$$

Where $\dot{q} = \frac{\delta Q}{dt}$ rate of heat added to the system and $\dot{W} = \frac{\delta W}{dt}$ is the rate of work done (or Power).

Now, we proceed to develope the control volume form of the first law of thermodynamics.

$$\Delta E = \nabla Q - \nabla W \longrightarrow (1.22)$$



Increase in energy during time interval Δt can be written as

$$\Delta E = E_1(t+\Delta t) + E_2(t+\Delta t) - E_3(t+\Delta t) - E_1(t) \longrightarrow (1.24)$$

Dividing by Δt on both sides we have

$$\frac{\Delta E}{\Delta t} = \frac{E_1(t+\Delta t) - E_1(t)}{\Delta t} + \frac{E_2(t+\Delta t)}{\Delta t} - \frac{E_3(t+\Delta t)}{\Delta t} \longrightarrow (1.25)$$

slope of the energy

$$\lim_{\Delta t \to 0} \frac{\Delta E}{\Delta t} = \lim_{\Delta t \to 0} \frac{E_1(t+\Delta t) - E_1(t)}{\Delta t} + \lim_{\Delta t \to 0} \frac{E_2(t+\Delta t)}{\Delta t}$$

$$- \lim_{\Delta t \to 0} \frac{E_3(t+\Delta t)}{\Delta t} \longrightarrow (A)$$

Let first term from the right hand side of the equation.

$$\lim_{\Delta t \to 0} \frac{E_1(t+\Delta t) - E_1(t)}{\Delta t} = \frac{\partial E_{cv}}{\partial t} = \frac{\partial}{\partial t} \int_{cv} e \rho \, dV \longrightarrow (1.26)$$

Where "e" is the specific energy of the system. Instantaneous value of the energy, this is the rate of change of energy within the control volume.

Now consider second & third terms,

$$\lim_{\Delta t \to 0} \frac{E_2(t+\Delta t)}{\Delta t} = \int_{dA_{out}} e \rho \vec{V} \cdot \hat{n} \, dA_{out} \longrightarrow (1.27)$$

Amount of energy leaving from the system

$$\lim_{\Delta t \to 0} \frac{E_3(t+\Delta t)}{\Delta t} = - \int_{dA_{in}} e \rho \vec{V} \cdot \hat{n} \, dA_{in} \longrightarrow (1.28)$$

Amount of energy intering to the system

Now eq. (A) can be expressed as,

$$\lim_{\Delta t \to 0} \frac{\Delta E}{\Delta t} = \frac{dE}{dt} = \frac{\partial}{\partial t} \int_{c.v.} e \rho \, dV + \int_{dA_{out}} e \rho \vec{V} \cdot \hat{n} \, dA_{out} + \int_{dA_{in}} e \rho \vec{V} \cdot \hat{n} \, dA_{in} \longrightarrow (1.29)$$

$$\Rightarrow \frac{dE}{dt} = \frac{\partial}{\partial t}\int_{c.v.} e\rho\, dV + \int_{c.s} e\rho\, \vec{V}\cdot\hat{n}\, dA \longrightarrow (1.29b)$$

Corresponding increase in energy of the system.

Now as

$$\frac{\Delta E}{dt} = \frac{\delta Q}{dt} - \frac{\delta W}{dt}$$

$$\Rightarrow \frac{dE}{dt} = q - \dot{W} \longrightarrow (B)$$

$\because \lim_{\Delta t \to 0} \left|\frac{\delta Q}{dt}\right| = q_{c.s} \rightarrow (1.30)$

& $\lim_{\Delta t \to 0} \left|\frac{\delta W}{dt}\right| = \dot{W} \rightarrow (1.31)$

By comparing (1.29b) & (B)

$$\frac{\partial}{\partial t}\int_{c.v} e\rho\, dV + \int_{c.s} e\rho\, \vec{V}\cdot\hat{n}\, dA = q - \dot{W} \longrightarrow (1.32)$$

Which is called control volume form of the first law of thermodynamics.

Second Law of thermodynamics 06-03-2015

The second law of thermodynamics leads to a property — entropy. (The loss of energy is known as entropy) For any reversible process that a system undergoes during cyclic process of time 't', the change in entropy S of the system is given by

$$dS = \left(\frac{\delta Q}{T}\right)_{rev.} \longrightarrow (1.51a)$$

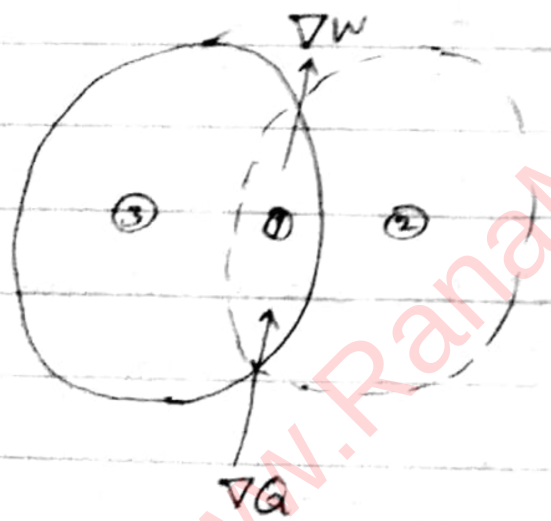
For an irreversible process.

$$dS > \left(\frac{\delta Q}{T}\right)_{irrev.} \longrightarrow (1.51b)$$

Where δQ is the amount of heat added to the system during time interval dt and T is the temperature of the system at the time of heat transform.

Above equation for the second law of thermodynamics can be expressed as

$$\frac{dS}{dt} \geq \frac{1}{T}\left(\frac{\delta Q}{dt}\right) \longrightarrow (1.52)$$



We want to check what is entropy within the control volume and around the control volume.

$$\Delta S = S_1(t+\Delta t) + S_2(t+\Delta t) - S_3(t+\Delta t) - S_1(t)$$

$$\frac{\Delta S}{\Delta t} = \frac{S_1(t+\Delta t) - S_1(t)}{\Delta t} + \frac{S_2(t+\Delta t)}{\Delta t} - \frac{S_3(t+\Delta t)}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} = \lim_{\Delta t \to 0} \frac{S_1(t+\Delta t) - S_1(t)}{\Delta t} + \lim_{\Delta t \to 0} \frac{S_2(t+\Delta t)}{\Delta t} - \lim_{\Delta t \to 0} \frac{S_3(t+\Delta t)}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{S(t+\Delta t) - S(t)}{\Delta t} = \frac{\partial S_{cv}}{\partial t} = \frac{\partial}{\partial t}\int_{C.V.} \rho s \, dV$$

↑ The rate change of entropy within the control volume.

$$\lim_{\Delta t \to 0} \frac{S_2(t+\Delta t)}{\Delta t} = \int_{dA_{out}} \rho s \vec{V}\cdot\hat{n} \, dA_{out}$$

$$\lim_{\Delta t \to 0} \frac{S_3(t+\Delta t)}{\Delta t} = -\int_{dA_{in}} \rho s \vec{V}\cdot\hat{n} \, dA_{in}$$

So,

$$\frac{dS}{dt} = \frac{\partial}{\partial t}\int_{C.V} \rho s \, dV + \int_{dA_{out}} \rho s \vec{V}\cdot\hat{n} \, dA_{out} + \int_{dA_{in}} \rho s \vec{V}\cdot\hat{n} \, dA_{in}$$

$$\frac{dS}{dt} = \frac{\partial}{\partial t}\int_{C.V} \rho s \, dV + \int_{C.S} \rho s \vec{V}\cdot\hat{n} \, dA$$

Now, we know that from the second law of thermodynamics,

$$\frac{dS}{dt} \geq \frac{1}{T}\left(\frac{\delta Q}{dt}\right)$$

So,
$$\frac{\partial}{\partial t}\int_{C.V} \rho s \, dV + \int_{C.S} \rho s \vec{V}\cdot\hat{n} \, dA \geq \frac{1}{T}\left(\frac{\delta Q}{dt}\right)$$

We can write it as,

$$\Delta S_{c.v} = \frac{\partial}{\partial t}\int_{cv} s\, \rho\, dV + \int_{cs} s\, \rho\, \vec{V}\cdot\hat{n}\, dA - \frac{1}{T}\left(\frac{\delta Q}{dt}\right)$$

which is the measure of entropy production within the control volume, moreover,

$$LW = T_0\, \Delta S_{c.v} \longrightarrow (1.55)$$

gives the lost work in an energy producing system due to irreversibility where T_0 is the absolute temperature of the system so called dead state at which the system is in thermodynamic equilibrium with it's environment.

Fourier law of Heat Conduction

For heat conduction in any direction "n" this law given by

$$q''_n = -K\frac{\partial T}{\partial n} \longrightarrow (1.56)$$

Temperature from higher to lower

where q''_n is the magnitude of the heat flux in the n-direction & $\frac{\partial T}{\partial n}$ is temperature gradiant in the same direction. Here

n → may be x, y, z
x mean in x-direction

K is the proportional constant and is known as thermal conductivity of the material through which heat is passing. Where negative sign indicates that heat flow is defined to be positive when it is in the direction of negative temperature gradient.

Thermal Conductivity is a thermophysical property and has unit $\frac{W}{m \cdot K}$ in S.I system. A medium is said to be homogeneous if it's thermal conductivity (K is constant) does not vary otherwise hetrogeneous. Further more a medium is said to be isotropic if it's thermal conductivity at any point in the medium is the same in all directions. otherwise anisotropic.

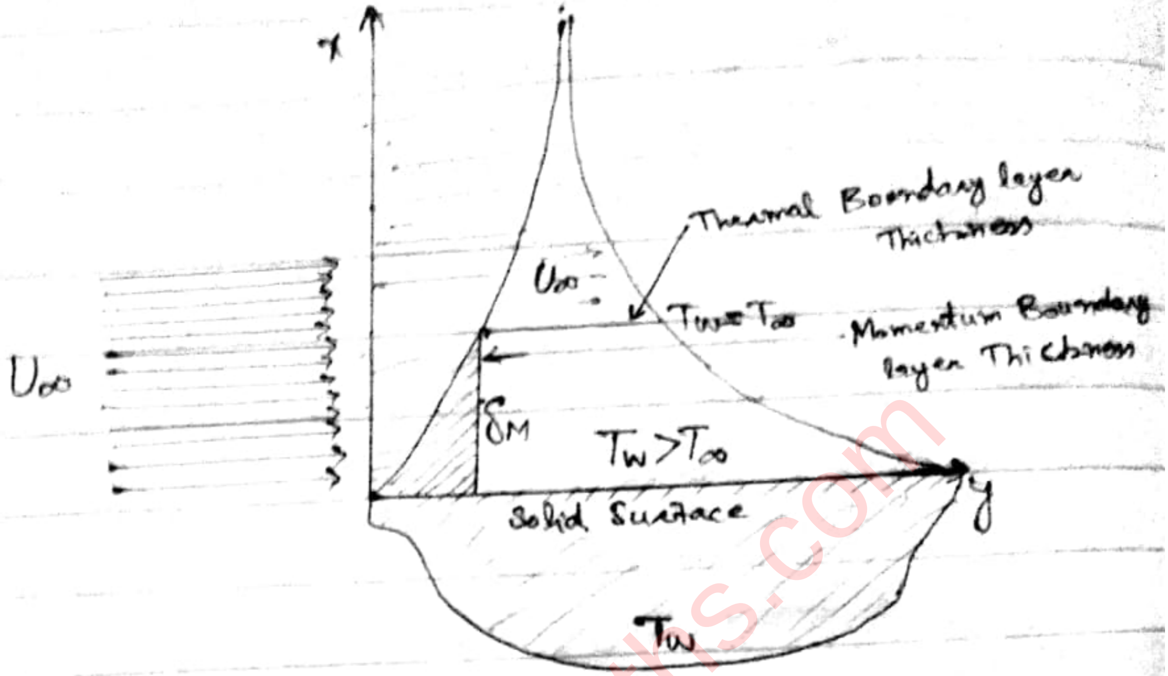
(1.57a) $q''_x = -K \frac{\partial T}{\partial x}$ for x-direction.

(1.57b) $q''_y = -K \frac{\partial T}{\partial y}$ for y-direction.

(1.57c) $q''_z = -K \frac{\partial T}{\partial z}$ for z-direction.

12-03-15

Newton's Law of Cooling



U_∞ = Free Stream Velocity.

T_∞ = Free stream Temperature Or Temperature of the srounding |OR| Ambient Fluid Temperature.

T_w OR T_s = Temperature of wall or Temperature of Surface.

δ_M =

Momentum Boundary Layer

When a fluid flows over a solid surface as illustrated in figure. We observed that the fluid particles adjecent to the solid surface stick to it and therefor have zero velocity relative to the surface. Other fluid particles attempting to slides over the stationary ones at the surface are retarded as a result of viscous forces between the fluid particles. The velocity of the fluid particles thus asymptotically approaches that of the undisturb free stream over a distance δ_M is called velocity Boundary layer or Momentum boundary layer.

Thermal Boundary Layer

If $T_w > T_\infty$, then heat will flow from the solid to the fluid particles at the surface. the energy thus transmitted increase the internal energy of the fluid particles and is carried away by the motion of the fluid. The temperature distribution in the fluid adjacent to the surface will them appear as shown in figure, asymptotically approaching the free stream values T_∞; In a short distance δ_T is called thermal Boundary layer thickness from the surface.

Since the fluid particles at the surface are stationary the heat flux from the surface to the fluid will be

$$q_n'' = -K_f \left(\frac{\partial T_f}{\partial n}\right)_w \longrightarrow (1.59)$$

where K_f is the thermal conductivity of the fluid, T_f is the temperature distribution of the fluid, the subscript w means the derivative is evaluated at the surface and 'n' denotes the normal direction (or any direction) from the surface.

Newton expressed the heat flux from a solid surface to a fluid by the equation.

$$q_n'' = h\,(T_w - T_\infty) \longrightarrow (1.60)$$ ← Newton law of cooling

where h is called coefficient of heat transfer.

$$\Rightarrow \quad h = \frac{q_n''}{(T_w - T_\infty)}$$ ← Temperature difference

For fluid

$$h = \frac{-K_f \left(\frac{\partial T_f}{\partial n}\right)_w}{(T_w - T_\infty)} \longrightarrow (1.61)$$

'h' has unit $\frac{W}{m^2 \cdot K}$

$\lambda = \dfrac{Gr_e}{Re^2}$

$\lambda \to 0$ F.C
$\lambda \to \infty$ N.C
$0 < \lambda < \infty$ Mixed C.

Forced Convection

If the fluid motion envolved in the process is induced by some external means such as a pump, blower or fan then the process is called as forced convection.

Natural or Free Convection

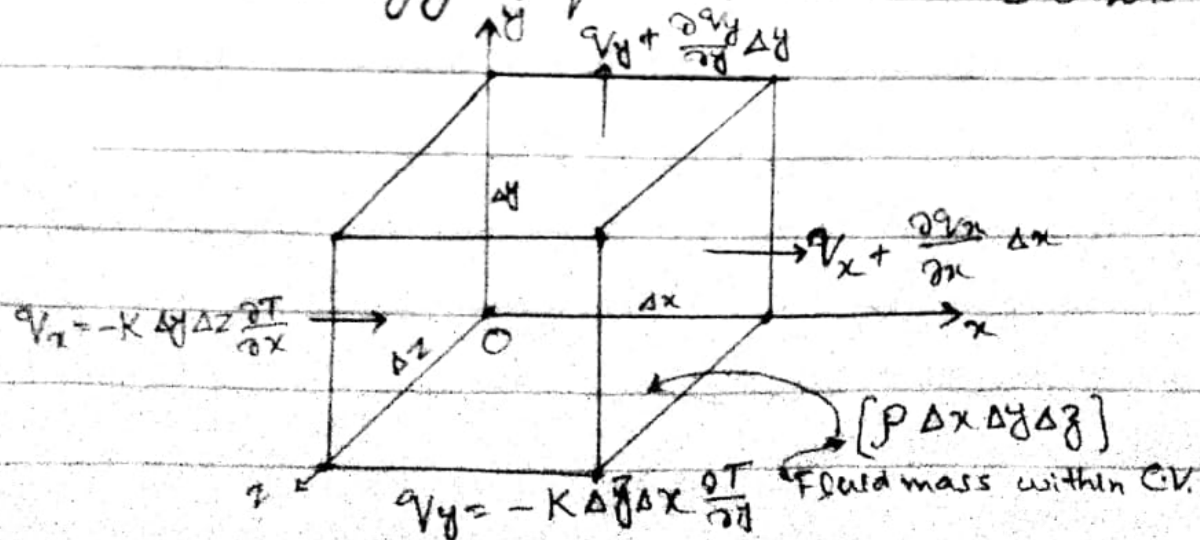
If the fluid motion caused by any body force within the system such as resulting from density gradiants near the surface then the process is called natural or free convection.

Forced & Free (or mixed) Convection

The combination of forced and free convection is called mixed convection.

Energy Equation

13-3-2015



$$\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} + \frac{\partial T}{\partial z} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

Temperature Gradient

$$\alpha = \frac{K}{\rho C_p}$$

Thermal conductivity $= K$

If K is variable then material is heterogeneous. (non linear graph).

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

convective Heat transfer.

When fluid moves in bulk form from one medium to other is called convection.

The Energy equation may be obtaind by applying first law of thermodynamics to an element of fluid of mass $(\rho \Delta x \Delta y \Delta z)$. Situated at the point (x, y, z) at time t as shown in figure. The first law of thermodynamics

$$\frac{dE}{dt} = q - \dot{W}$$

States that rate of

heat transfer to the system minus work done by the system (or element) is equal to the rate of increase of energy of the element.

The rate of heat transfer to the element (ignoring radiation heat transfer within the fluid element).

$$\left[\frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right)\right]\Delta x\,\Delta y\,\Delta z \rightarrow (2.37)$$

The net rate of work done by the system (fluid element) against the surface and body force is → (2.38)

$$-\left[\frac{\partial}{\partial x}\left(u\,\sigma_{xx}+v\,\tau_{xy}+w\,\tau_{xz}\right)+\frac{\partial}{\partial y}\left(u\,\tau_{yx}+v\,\sigma_{yy}+w\,\tau_{yz}\right)+\frac{\partial}{\partial z}\left(u\,\tau_{zx}+v\,\tau_{zy}+w\,\sigma_{zz}\right)+\rho\left(uf_x+vf_y+wf_z\right)\right]\Delta x\Delta y\Delta z$$

16/03/15

Now, The rate of increase of internal energy of the element can be written as,

$$\rho\,\Delta x\,\Delta y\,\Delta z\,\frac{D}{Dt}\left[\mathcal{U}+\frac{u^2+v^2+w^2}{2}\right] \longrightarrow (2.39)$$

Where $\mathcal{U}$ is the internal energy per unit mass of the fluid.

The first law of thermodynamics for the fluid particle becomes,

$$\rho\, \Delta x \cancel{\Delta y} \Delta z \frac{D}{Dt}\left[U + \frac{u^2+v^2+w^2}{2}\right] = \left[\frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right)\right.$$

$$+\frac{\partial}{\partial x}\left(u\sigma_{xx} + v\tau_{xy} + w\tau_{xz}\right) + \frac{\partial}{\partial y}\left(u\tau_{yx} + v\sigma_{yy} + w\tau_{yz}\right)$$

$$\left.+\frac{\partial}{\partial z}\left(u\tau_{zx} + v\tau_{zy} + w\sigma_{zz}\right) + \rho\left(uf_x + vf_y + wf_z\right)\right] \Delta x \cancel{\Delta y} \Delta z$$

$$\Rightarrow \rho\frac{D}{Dt}\left[U + \frac{u^2+v^2+w^2}{2}\right] = \frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right)$$

$$+\frac{\partial}{\partial x}\left(u\sigma_{xx} + v\tau_{xy} + w\tau_{xz}\right) + \frac{\partial}{\partial y}\left(u\tau_{yx} + v\sigma_{yy} + w\tau_{yz}\right)$$

$$+\frac{\partial}{\partial z}\left(u\tau_{zx} + v\tau_{zy} + w\sigma_{zz}\right) + \rho\left(uf_x + vf_y + wf_z\right) \longrightarrow (2.40)$$

which is also known as the total energy equation since it comprises both thermal and mechanical energies.

Now, (for mechanical energy)

$$\rho\frac{Du}{Dt} = \rho f_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

$$\rho\frac{Dv}{Dt} = \rho f_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$$

$$\rho\frac{Dw}{Dt} = \rho f_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}$$

Multiplying both sides of the momentum equations by u, v & w respectively.

$$\rho u\frac{Du}{Dt} = u\left(\rho f_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) \longrightarrow (2.41a)$$

$$\rho v \frac{Dv}{Dt} = v\left[\rho F_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right] \longrightarrow (2.41b)$$

$$\rho w \frac{Dw}{Dt} = w\left[\rho F_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right] \longrightarrow (2.41c)$$

Summing these equations we get

$$\rho\left[u\frac{Du}{Dt} + v\frac{Dv}{Dt} + w\frac{Dw}{Dt}\right] = u\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial x}\right)$$
$$+ v\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + w\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right)$$
$$+ \rho(uF_x + vF_y + wF_z)$$

$$\rho\frac{D}{Dt}\left[\frac{u^2+v^2+w^2}{2}\right] = u\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)$$
$$+ v\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + w\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right)$$
$$+ \rho(uF_x + vF_y + wF_z) \longrightarrow (2.42)$$

which is an energy equation obtained directly from laws of mechanics and appropriately called the mechanical energy equation.

Subtracting the mechanical energy equation (2.42) from the total energy equation (2.40) we obtain

$$\rho \frac{DU}{Dt} = \frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right)$$

$$+ \sigma_{xx}\frac{\partial u}{\partial x} + \sigma_{yy}\frac{\partial v}{\partial y} + \sigma_{zz}\frac{\partial w}{\partial z}$$

$$+ \tau_{xy}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \tau_{yz}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) + \tau_{zx}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \longrightarrow (2.43)$$

which is referred to as the thermal energy equation.

For a viscous, Newtonian fluid we get,

$$\sigma_{xx}\frac{\partial u}{\partial x} + \sigma_{yy}\frac{\partial v}{\partial y} + \sigma_{zz}\frac{\partial w}{\partial z} = -p\,\nabla\cdot V - \frac{2}{3}\mu(\nabla\cdot V)^2$$

$$+ 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] \longrightarrow (2.44)$$

and making the use of the continuity equation.

$$\frac{D\rho}{Dt} + \rho\,\nabla\cdot V = 0 \longrightarrow (2.45)$$

$$\Rightarrow \frac{D\rho}{Dt} = -\rho\,\nabla\cdot V$$

So, eq. (2.44) becomes

$$\sigma_{xx}\frac{\partial u}{\partial x} + \sigma_{yy}\frac{\partial v}{\partial y} + \sigma_{zz}\frac{\partial w}{\partial z} = \frac{p}{\rho}\frac{D\rho}{Dt} - \frac{2}{3}\mu(\nabla\cdot V)^2$$

$$+ 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] \longrightarrow (2.46)$$

We know that

$$\tau_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

So, we can write

$$\tau_{xy}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2 \longrightarrow (2.47a)$$

Similarly

$$\tau_{yz}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)=\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2 \longrightarrow (2.47b)$$

&

$$\tau_{zx}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2 \longrightarrow (2.47c)$$

When these relations are substituted into the energy eq. 2.43, then

$$\rho\frac{D\mathcal{U}}{Dt}=\frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right)$$
$$+\frac{p}{\rho}\frac{D\rho}{Dt}-\frac{2}{3}\mu(\nabla\cdot V)^2+2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]$$
$$+\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2+\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2+\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2$$

$$\rho\frac{D\mathcal{U}}{Dt}=\frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right)$$
$$+\frac{p}{\rho}\frac{D\rho}{Dt}+2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right.$$
$$\left.+\frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2-\frac{1}{3}(\nabla\cdot V)^2\right]$$

$$\rho\frac{D\mathcal{U}}{Dt}=\frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right)+\frac{p}{\rho}\frac{D\rho}{Dt}+\phi \longrightarrow (2.48)$$

Where

$$\phi=2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right.$$
$$\left.+\frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2-\frac{1}{3}(\nabla\cdot V)^2\right] \longrightarrow (2.49)$$

which is called the dissipation function and is the rate at which the viscous force do irreversible work on the fluid particles per unit volume.

19-03-2015

Finally, the energy equation (2.48) may also be written in terms of the fluid enthalpy defined by

$$i = U + \frac{P}{\rho}$$

$$\frac{DU}{Dt} = \frac{Di}{Dt} - \left[\frac{\rho \frac{DP}{Dt} - P\frac{D\rho}{Dt}}{\rho^2}\right]$$

xing by ρ.

$$\Rightarrow \quad U = i - \frac{P}{\rho}$$

$$\rho\frac{DU}{Dt} = \rho\frac{Di}{Dt} - \frac{DP}{Dt} + \frac{P}{\rho}\frac{D\rho}{Dt}$$

as follows:

$$\rho\frac{Di}{Dt} = \frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right) + \frac{DP}{Dt} + \Phi \longrightarrow (2.50a)$$

or

$$\rho\frac{Di}{Dt} = \nabla\cdot(K\nabla T) + \frac{DP}{Dt} + \Phi \longrightarrow (2.50b)$$

For a perfect gas,

$$di = C_p\, dT \longrightarrow (2.51)$$

Where C_p is the specific heat at constant pressure. Hence, for a gas the energy equation (2.50b) takes the form

$$\rho c_p \frac{DT}{Dt} = \nabla\cdot(K\nabla T) + \frac{Dp}{Dt} + \Phi \longrightarrow (2.52)$$

Convective terms

The work of compression, $\frac{Dp}{Dt}$, is usually negligible except above sonic velocities.

$$\frac{DT}{Dt} = \alpha \nabla^2 T + \frac{1}{\rho C_p}\Phi \longrightarrow (2.53)$$

where $\alpha = \frac{K}{\rho C_p} \longrightarrow (2.54)$ is the thermal diffusivity of the fluid and

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \longrightarrow (2.55)$$

For incompressible fluid

$$du = c\,dT \longrightarrow (2.56)$$

where $c = C_v \cong C_p$. Hence for an incompressible fluid the energy Eq. (2.48) takes the form

$$\rho c \frac{DT}{Dt} = \frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right) + \Phi \longrightarrow (2.57)$$

where

$$\Phi = 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2\right] \longrightarrow (2.58)$$

When the thermal conductivity is constant, the energy equation (2.57) for incompressible fluid reduces to

$$\frac{DT}{Dt} = \alpha \nabla^2 T + \frac{1}{\rho c} \Phi \longrightarrow (2.59)$$

This eq. for steady flow

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{1}{\rho c} \Phi \longrightarrow (2.60)$$

This eq. for two-dimensions.

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \frac{1}{\rho c} \Phi \longrightarrow (2.61)$$

where $\Phi = 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right]$

Heat eq. with ~~with~~ energy production is

$$\frac{DT}{Dt} = \alpha \nabla^2 T + \frac{\dot{q}}{K}$$

"We want to make surface homogeneous this process is called Homogenity".

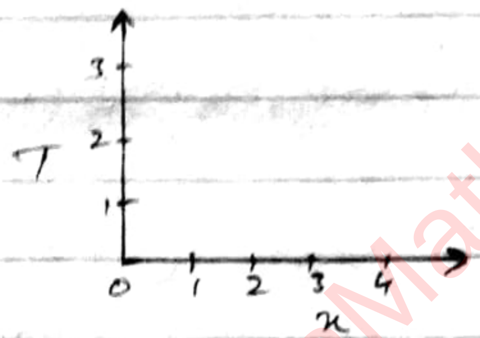
Chap# 7

Dimensional Analysis and Similitude.

Method of solving models of the problem by using numbers is called numerical analysis.

$\frac{\partial u}{\partial x} = 0$ x - variable.

but $\pi = \frac{x}{l} = \frac{m}{m} = 1$



20-03-15

The Principle of Dimensional Homogeneity.

"Every team in a complete physical equation has the same measure formula"

Thus, an eq. is complete when every essential factor is explicitly represented.

Let there exist n quantities Q $(Q_1, Q_2, \ldots, Q_n)$, that are involved in an experiment, and postulate that they are related by a dimensionally homogeneous equation $F(Q_1, Q_2, \ldots, Q_n) = 0 \longrightarrow (7.4)$

This equation is dimensionally homogeneous, We can express it in dimensionless form by normalizing each term by the dimensions of any other term in eq(7.4).

For example, consider the dimensionally homogeneous algebraic Eq.

$$F(x,y,z,l) = x^3 y + \frac{x y^5}{z^2} + z^4 \sin \frac{x}{l} = 0 \longrightarrow (7.5)$$

The dimension of eq. 7.5 is L^4.

The eq. can be made dimensionless by dividing each term by some refference length l^4, which must be a constant:

$$F\left(\frac{x}{l}, \frac{y}{l}, \frac{z}{l}\right) = \frac{x^3 y}{l^4} + \frac{x y^5}{z^2 l^4} + \left(\frac{z}{l}\right)^4 \sin \frac{x}{l} = 0 \longrightarrow 7.6$$

We have expressed a relationship among four quantities (x, y, z, l) in terms of three dimensionless quantities x/l, y/l, z/l. In General Eq. (7.4) can be expressed in terms of dimensionless quantities $\Pi_1, \Pi_2, \Pi_3, \ldots$ formed with the quantities $Q_1, Q_2, \ldots Q_n$.

The number of quantities that are involved can be reduced not necessary by one, but perhaps two or more.

Buckingham Pi Theorem

Protagonists of the method use it to solve dimensional analysis problems ranging from the difficult to the impossible. Antagonists expound the dangers of using the method under any condition. The central issue arises from the inability of the Buckingham pi theorem to provide either a unique or a complete solution to a problem. The theorem requires experiments involving models that provides solutions which can never be free from inaccuracies or procedural errors. Thus, complete similarity is impossible ~~to~~ achieve.

Theoretically, the Buckingham pi theorem is mathematically correct though not necessarily accurate. For the user, it has the advantage of speed and a certain ease. If an accidental slip occurs in the analysis it can easily be corrected by examining the dimensionless quantity. Its dangerous weakness is that it can be learned by following some simple steps but without grasping the essence of modeling or similitude.

The basic dimensional system used in mechanics is M, L & T. i.e mass, length and time in a physical system. Thus, each mechanical quantity, say Q, dimensionally denoted by [Q], can be represented as a power product of the basic physical dimensions

$$[Q] = M^a L^b T^c \longrightarrow (7.7)$$

Or, we can represent the mechanical quantity [Q] as a power product of the technical system dimensions

$$[Q] = F^a L^b T^c \longrightarrow (7.8)$$

Let there exist n quantities $Q_1, Q_2, \ldots, Q_n$, involved in an experimental fluid dynamics problems. We wish to determine the dimensions of each quantity Q such that there are m dimensions involved in the n quantities. We denote the fundamental dimensions by D (e.g., $D_1 = F$, $D_2 = L$, $D_3 = T$, using the technical system in mechanics) and then construct a dimensional formula for each quantity Q in terms of dimensions D:

$$[Q_1] = D_1^{a_1} D_2^{b_1} \cdots D_m^{m_1} \longrightarrow (7.9)$$

$$[Q_2] = D_1^{a_2} D_2^{b_2} \cdots D_m^{m_2} \longrightarrow 7.10$$

$$[Q_n] = D_1^{a_n} D_2^{b_n} \cdots D_m^{m_n} \longrightarrow 7.11$$

where the exponents of Eqs. (7.9)–(7.11) are known values of a dimension D.

The question arises if there exist dimensionless terms of the form

$$\Pi = Q_1^{K_1} Q_2^{K_2} \cdots Q_m^{K_n} \longrightarrow 7.12$$

Since Π is dimensionless, it can be expressed in the form

$$\Pi = D_1^0 D_2^0 \cdots D_m^0 \longrightarrow 7.13$$

Substituting Eqs (7.13) & (7.9)–(7.11) into Eq. (7.12) yields

$$D_1^0 D_2^0 \cdots D_m^0 = (D_1^{a_1} D_2^{b_1} \cdots D_m^{m_1})^{K_1} \times (D_1^{a_2} D_2^{b_2} \cdots D_m^{m_2})^{K_2} \cdots (D_1^{a_n} D_2^{b_n} \cdots D_m^{m_n})^{K_n} \longrightarrow (7.14)$$

Equating like powers of the dimensions $D_1, D_2, \ldots, D_m$, we obtain the following set of linear algebraic equations:

For D_1: $0 = a_1 K_1 + a_2 K_2 + \cdots + a_n K_n \longrightarrow 7.15$

For D_2: $0 = b_1 K_1 + b_2 K_2 + \cdots + b_n K_n \longrightarrow 7.16$

For D_m: $0 = m_1 K_1 + m_2 K_2 + \cdots + m_n K_n \longrightarrow (7.17)$

where the coefficients $a_n, b_n, \ldots, m_n$ are known and K_n is to be determined. Equations (7.15)–(7.17) form a set of homogeneous linear system of m equations in "n" unknown variables $K_1, K_2, \ldots, K_n$.

$$\Rightarrow \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ \vdots & & & \\ m_1 & m_2 & \cdots & m_n \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

According to Cramer's rule, "If the determinant $\mathcal{D}$.

$$\mathcal{D} = \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ b_1 & b_2 & b_3 & \cdots & b_n \\ \vdots & & & & \\ m_1 & m_2 & m_3 & \cdots & m_n \end{vmatrix} \longrightarrow 7.18$$

does not vanish, the simultaneous equations of Eqs. (7.15)–(7.17) are satisfied by one & only one set of values of the unknowns K_n. The theory of algebraic equations tells us that a homogeneous linear system of equations in n unknown variables whose matrix of the coefficients exhibits the rank r, has exactly $(n-r)$ linearly independent solutions. So, given n physical quantities $Q_1, Q_2, \dots, Q_n$ with a relation among them, there exist exactly $(n-r)$ independent dimensionless π terms, with $r \le m \le n$ being the rank of the dimensional matrix. We can then mathematically pose this theorem as

$$F(\pi_1, \pi_2, \dots, \pi_{n-r}) = 0 \longrightarrow 7.19$$

calling it the Buckingham pi theorem. In words the Buckingham pi theorem states that "any complete physical relationship can be represented as one subsisting between a set of independent non-dimensional product combinations of the physical quantities concerned."

To solve problems using the Buckingham pi theorem, follow these steps:-

Step(I):- Isolate the physical quantities in the given problem. Identify the quantities Q_n and the number n.

Step II:- Select the M, L, T or F, L, T system. For each quantity Q, select the appropriate dimensions, and determine the number m.

Step III:- Construct the dimensional matrix and evaluate the rank r.

Step IV:- Evaluate the $(n-r)$ dimensionless π coefficients.

Step (V):- Apply Eq. $F(\pi_1, \dots, \pi_{n-r}) = 0$ to obtain the desired empirical relationship governing the physical problem.

Example 7.1 30-03-15

Consider a cylindrical pipe of uniform cross section, so that the geometry of the pipe is completely defined by the inner diameter D of the pipe and its axial length l, which is sufficiently large to negled end effects. Let the fluid motion be steady and consider the inertia and viscosity as represented by the fluid density ρ and dynamic viscosity μ. In addition, the experimental measurements show that the composition of the inner surface affects the flow, particularly the pressure drop along the axis of the pipe. Let the composition of the inner surface be denoted by an absolute roughness ϵ. Last the average fluid velocity $\bar{V}$ over the pipe cross section can vary in the experiment. Using the Buckingham pi theorem, find the empirical relationship for the experimental problem. Let the pipe be inclined w.r.t the horizontal.

Solution:- Inner diameter $= D$

Axial length $= l$

density $= \rho$

viscosity $= \mu$, Absolute Roughness $= \epsilon$

Acceleration due to gravity $= g$

Average velocity $= \bar{V}$

Step I:-

Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7
D	l	$\tilde{V}$	g	ϵ	μ	ρ

Step II:-

D_1	D_2	D_3
F	L	T

$m = 3$

$[D] = L$

$[l] = L$

$[\tilde{V}] = LT^{-1}$

$[g] = LT^{-2}$

$[\epsilon] = L$

$[\mu] = FL^{-2}T$

$[\rho] = FL^{-4}T^{2}$

Step III:-

	D	l	$\tilde{V}$	g	ϵ	μ	ρ
F	0	0	0	0	0	1	1
L	1	1	1	1	1	-2	-4
T	0	0	-1	-2	0	1	2

$r = 3$ $\qquad n - r = 7 - 3 = 4$

Step IV:- Thus there are 4 π terms. Let us arbitrarily select $\tilde{V}$, D, and μ as the repeating variables:

$$\Pi_1 = \bar{V}^{K_1} D^{K_2} \mu^{K_3} \rho \longrightarrow (i)$$

$$\Pi_2 = \bar{V}^{K_4} D^{K_5} \mu^{K_6} g \longrightarrow (ii)$$

$$\Pi_3 = \bar{V}^{K_7} D^{K_8} \mu^{K_9} \epsilon \longrightarrow (iii)$$

$$\Pi_4 = \bar{V}^{K_{10}} D^{K_{11}} \mu^{12} \ell \longrightarrow (iv)$$

From (i)

$$\Pi_1 = F^0 L^0 T^0 = (LT^{-1})^{K_1} (L)^{K_2} (FL^{-2}T)^{K_3} (FL^{-4}T^2)^1$$

Equating exponents of like dimensions, we obtain

$F:\quad K_3 + 1 = 0$

$L:\quad K_1 + K_2 - 2K_3 - 4 = 0$

$T:\quad -K_1 + K_3 + 2 = 0$

$$\Rightarrow \boxed{K_3 = -1}, \quad \boxed{K_1 = K_2 = 1}$$

$$\Rightarrow \Pi_1 = \bar{V} D \mu^{-1} \rho$$

$$= \frac{\bar{V} D \rho}{\mu} = \frac{\bar{V} D}{\nu} \qquad \because \nu = \mu/\rho$$

From (ii)

$$\Pi_2 = F^0 L^0 T^0 = (LT^{-1})^{K_4} (L)^{K_5} (FL^{-2}T)^{K_6} (LT^{-2})^1$$

Equating exponents of like dimensions, we obtain

$F:\quad K_6 = 0$

$L:\quad K_4 + K_5 - 2K_6 + 1 = 0$

$$T: \quad -K_4 + \cancel{K_6} - 2 = 0$$

$$\Rightarrow \boxed{K_6 = 0}, \quad \boxed{K_4 = -2} \quad \boxed{K_5 = 1}$$

$$\Pi_2 = \tilde{V}^{-2} D^1 \mu^0 g^1$$

$$\Pi_2 = \frac{gD}{\tilde{V}^2}$$

From (iii)

$$\Pi_3 = F^0 L^0 T^0 = (LT^{-1})^{K_7} (L)^{K_8} (FL^{-2}T)^{K_9} (L)^1$$

Equating exponents of like dimensions we obtain

$$F: \quad K_9 = 0$$

$$L: \quad \cancel{K_7} + K_8 - 2\cancel{K_9} + 1 = 0$$

$$T: \quad -K_7 + \cancel{K_9} = 0$$

$$\Rightarrow \boxed{K_9 = K_7 = 0} \quad \boxed{K_8 = -1}$$

$$\Pi_3 = \tilde{V}^0 D^{-1} \mu^0 \in$$

$$\Pi_3 = \frac{\in}{D}$$

From (iv).

$$\Pi_4 = F^0 L^0 T^0 = (LT^{-1})^{K_{10}} (L)^{K_{11}} (F L^{-2} T)^{K_{12}} (L)^1$$

Equating exponents of like dimensions, we obtain

$F: \quad K_{12} = 0$

$L: \quad K_{10} + K_{11} - 2K_{12} + 1 = 0$

$T: \quad -K_{10} + K_{12} = 0$

$$\Rightarrow \boxed{K_{10} = K_{12} = 0} \qquad K_{11} = -1$$

$$\Rightarrow \Pi_4 = \tilde{V}^0 D^{-1} \mu^0 \ell^1$$

$$\Pi_4 = \frac{\ell}{D}$$

Step (IV):-

$$F(\Pi_1, \Pi_2, \Pi_3, \Pi_4) = 0$$

$$\Rightarrow F\left(\frac{\rho \tilde{V} D}{\mu}, \frac{Dg}{\tilde{V}^2}, \frac{\epsilon}{D}, \frac{\ell}{D}\right) = 0$$

Which is the solution of the problem.

A dimensionless friction factor f is defined as

$$f = f\left(\frac{\rho \tilde{V} D}{\mu}, \frac{\epsilon}{D}\right)$$

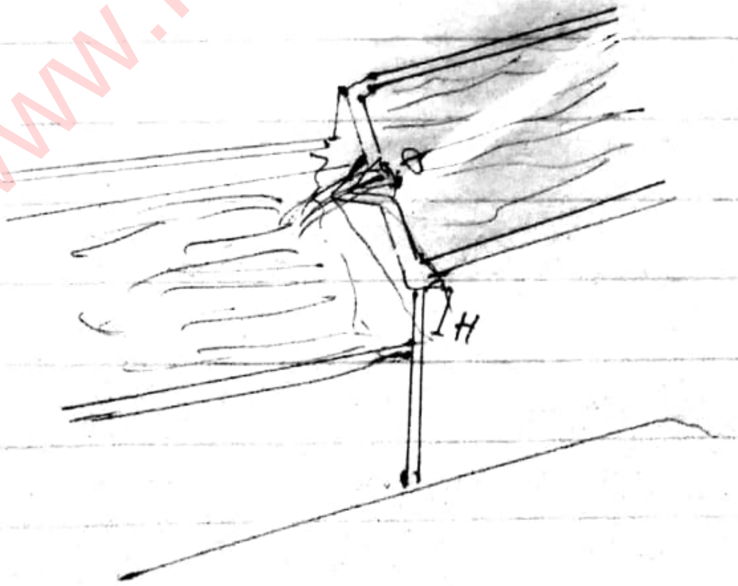
So,

$$F\left(f, \frac{\tilde{V}^2}{Dg}, \frac{\ell}{D}\right) = 0$$

$$\Rightarrow h_f = f \frac{\ell}{D} \cdot \frac{\tilde{V}^2}{2g} \quad \text{Darcy's Eq.}$$

Example: 7.2 02-04-2015

Consider a vertical plate with a cutout having a notch of angle ϕ cut into the top of it and placed across an open channel containing water. The plate backs up the water in the channel until it flows through the notch. The volume rate of flow Q is some function of the elevation H of (opposite to the flow) upstream liquid surface above the bottom of the notch. In addition, the discharge depends upon gravity and upon the velocity of approach $\tilde{V}$ to the vertical plate. Determine the form of discharge equation.



Solution. Step 1:- Identify the quantities in the problem. A functional relationship

$$F(Q, H, g, \tilde{V}, \phi) = 0$$

is to be grouped into dimensionless parameters; ϕ is considered dimensionless. Hence it is one of the π parameters.

Step 2:-

Determine the number of dimensions in the problem. Only two dimensions are needed L and T.

Step 3:-

Select the repeating variables. We need one geometric quantity, so we will choose elevation H. For the second category, we need a fluid property. We notice that no fluid property is given: no density, viscosity or surface tension.

So we advance to the third category and select a kinematic property. We have Q, g and $\tilde{V}$ to select from. Any would be appropriate, so let us choose acceleration g. Thus our repeating quantities will be H and g.

Step 4:-

Follow steps i – v. As in example 7.1, we construct the solution.

Step (i).

Q_1	Q_2	Q_3	Q_4
Q	H	g	$\tilde{V}$

$$n = 4$$

Step II.

D_1	D_2
L	T

$m = 2$

$$[Q] = L^3T^{-1}, \quad [H] = L, \quad [g] = LT^{-2}$$

$$[\tilde{V}] = LT^{-1}$$

Step (III):-

	Q	H	g	$\tilde{V}$
L	3	1	1	1
T	-1	0	-2	-1

$$r = 2$$

Step (IV):-

Thus there are two pi terms, and we shall use H and g as the repeating variables:

$$\Pi_1 = H^{k_1} g^{k_2} Q \longrightarrow (i)$$

$$\Pi_2 = H^{k_3} g^{k_4} \tilde{V} \longrightarrow (ii)$$

From (i)

$$\pi_1 = L^0 T^0 = L^{K_1} (LT^{-2})^{K_2} (L^3 T^{-1})^1$$

Equating exponents of like dimensions. we obtain

$$L: \quad K_1 + K_2 + 3 = 0$$

$$T: \quad -2K_2 - 1 = 0$$

$$\Rightarrow K_2 = \frac{-1}{2}, \quad K_1 = \frac{-5}{2}$$

$$\Rightarrow \pi_1 = H^{-5/2} g^{1/2} Q$$

$$= \frac{Q}{\sqrt{g}\, H^{5/2}}$$

From (ii)

$$\pi_2 = L^0 T^0 = L^{K_3} (LT^{-2})^{K_4} (LT^{-1})^1$$

$$\Rightarrow L: \quad K_3 + K_4 + 1 = 0$$

$$T: \quad -2K_4 - 1 = 0$$

$$\Rightarrow K_4 = -\tfrac{1}{2}, \quad K_3 = \frac{-1}{2}$$

$$\Rightarrow \pi_2 = \frac{\bar{v}}{\sqrt{gH}}$$

We have stated that ϕ is dimensionless, So

$$\pi_3 = \phi$$

We have

$$F(\pi_1, \pi_2, \pi_3) = 0$$

$$\Rightarrow F\left(\frac{Q}{\sqrt{g}\,H^{5/2}}, \frac{\bar{V}}{\sqrt{gH}}, \phi\right) = 0$$

an alternate form

$$\frac{Q}{\sqrt{g}\,H^{5/2}} = f\left(\frac{\bar{V}}{\sqrt{gH}}, \phi\right)$$

$$\Rightarrow Q = \sqrt{g}\,H^{5/2}\, f\left(\frac{\bar{V}}{\sqrt{gH}}, \phi\right)$$

The Rayleigh Method 03-04-2015

In 1899 Lord Rayleigh proposed an easy method for analyzing the behavior of fluid motion. He proposed that the dimensions of any term in a homogeneous equation be the same. Let f be a function of properties Q_n as in Eq. (7.4) Since the equation is to be homogeneous we can write as

$$f \propto Q_1^a\, Q_2^b\, Q_3^c \cdots Q_n^r \longrightarrow (7.20)$$

where $a, b, c, \ldots, r$ are to be determined from the fact that the arrangement of quantities Q_n must be reducible to the dimensions of f.

The Rayleigh method requires no formal rules.

Example 7.3

Using the Rayleigh method, determine an expression for the drag on a missile in supersonic flow. Consider the primary quantities in the problem as the density, dynamic viscosity, bulk modulus, Reference length l, and flight speed V.

Solution:- Given ρ, μ, K, l, V

Step I. The Drag force D is expressed as

$$D = C \rho^a \mu^b K^c l^d V^e \longrightarrow (i)$$

C ← Dimensionless constant

where both sides of the equation have the units of force.

Step 2 Choose the FLT system

$$F = F^1 L^0 T^0 = (FT^2L^{-4})^a (FTL^{-2})^b (FL^{-2})^c (L)^d (LT^{-1})^e$$

Step 3:- Equate exponents of like dimensions and solve.

$$F: \quad 1 = a + b + c$$

$$L: \quad 0 = -4a - 2b - 2c + d + e$$

$$T: \quad 0 = 2a + b - e$$

By solving these, we have

$$a = 1 - b - c$$

$$e = 2 - b - 2c$$

$$d = 2 - b$$

Step 4:-

$$\Rightarrow D = C\rho^{1-b-c}\mu^{b}K^{c}l^{2-b}V^{2-b-2c}$$

$$D = C\rho\rho^{-b}\rho^{-c}\mu^{b}K^{c}l^{2}l^{-b}V^{2}V^{-b}V^{-2c}$$

Step-5:-

$$D = C(\rho l^2 V^2)\left(\frac{\mu}{\rho l V}\right)^{b}\left(\frac{K}{\rho V^2}\right)^{c}$$

$$D = C(\rho l^2 V^2)\left(\frac{1}{R_e}\right)^{b}\left(\frac{1}{M^2}\right)^{c}$$

Mibe number controlls sound

Renold number

Both Renold number & Mibe no. are dimensionless.

last expression can be written as

$$D = \rho l^2 V^2 [g(M, R_e)] \leftarrow \boxed{f(x,y)}$$

$$C_D = C_D(M, R_e) \rightarrow \text{The coefficient}$$

of Drag force. It is like the function as

$$f(x,y) = x^2 + 2y + 2y + 1$$

A Critique of the Two Methods

Read by yourself

Dimensionless Navier-Stokes Equation

NSE is given as

$$\frac{\partial \vec{V}}{\partial t} + \vec{V}(\nabla \cdot \vec{V}) = -\frac{1}{\rho}\nabla p + \nu \nabla^2 V + g$$

Let us suppose that the flow is steady.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = g - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

⇒ One-D dimensioned problem.

⇒ For viscous incompressible fluid flow.

By introducing dimensionless variables like this

$$\bar{x} = \frac{x}{L} \qquad \bar{w} = \frac{w}{U_0}$$

$$\bar{y} = \frac{y}{L} \qquad \bar{z} = \frac{z}{L}$$

$$\bar{u} = \frac{u}{U_0}$$

$$\bar{v} = \frac{v}{U_0}$$

$$\bar{P} = \frac{P}{\rho U_0^2}$$

$$\frac{U_0}{L}\left(\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}}\right) = 0 \qquad \because \frac{U_0}{L} \neq 0$$

$$\Rightarrow \frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0$$

$$\frac{U_0^2}{L}\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right) = \frac{1}{\rho}\frac{\rho U_0^2}{L}\frac{\partial \bar{P}}{\partial \bar{x}}$$

$$+ \frac{\nu U_0}{L^2}\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{z}^2}\right)$$

$$\because Re = \frac{UL}{\nu}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{\partial \bar{P}}{\partial \bar{x}} + \frac{1}{R_e}\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{z}^2}\right)$$

i.e. dimensionless Eq. of given model of problem.

Dimensionlization of Energy Eq. 09-04-2015

Consider two dimensions, viscous inconcompressible flow with a constant thermophysical properties

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = K\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

Pag: 22

$$+2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right]$$

Group of dimensionless variables

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}, \quad \bar{u} = \frac{u}{U_0}, \quad \bar{V} = \frac{V}{U_0}$$

$$\bar{T} = \frac{T - T_0}{\Delta T}$$

$$\Rightarrow \quad T = \Delta T\,\bar{T} + T_0 \Rightarrow \frac{\partial T}{\partial x} = \Delta T\frac{\partial \bar{T}}{\partial x}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left[\left(\frac{\partial^2 T}{\partial x^2}\right) + \frac{\partial^2 T}{\partial y^2}\right]$$

$$+\frac{2\mu}{\rho C_p}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right]$$

where $\alpha = \dfrac{K}{\rho C_p}$

$$\Rightarrow \bar{u}\frac{U_0\,\Delta T}{L}\cdot\frac{\partial\bar{T}}{\partial\bar{x}} + \bar{V}\frac{U_0\Delta T}{L}\cdot\frac{\partial\bar{T}}{\partial\bar{y}} = \frac{\alpha\,\Delta T}{L^2}\left[\frac{\partial^2\bar{T}}{\partial\bar{x}^2} + \frac{\partial^2\bar{T}}{\partial\bar{y}^2}\right]$$

$$+\frac{2\mu}{\rho C_p}\cdot\frac{U_0^2}{L^2}\left[\left(\frac{\partial\bar{u}}{\partial\bar{x}}\right)^2 + \left(\frac{\partial\bar{V}}{\partial\bar{y}}\right)^2 + \frac{1}{2}\left(\frac{\partial\bar{u}}{\partial\bar{y}} + \frac{\partial\bar{V}}{\partial\bar{x}}\right)^2\right]$$

$$\bar{u}\frac{\partial\bar{T}}{\partial\bar{x}} + \bar{V}\frac{\partial\bar{T}}{\partial\bar{y}} = \frac{\alpha\cancel{\Delta T}}{L^{\cancel{2}}}\cdot\frac{\cancel{L}}{U_0\cancel{\Delta T}}\left[\frac{\partial^2\bar{T}}{\partial\bar{x}^2} + \frac{\partial^2\bar{T}}{\partial\bar{y}^2}\right]$$

$$+\frac{\cancel{L}}{U_0\Delta T}\cdot\frac{2\mu U_0^{\cancel{2}}}{\cancel{L^2}\rho C_p}\left[\left(\frac{\partial\bar{u}}{\partial\bar{x}}\right)^2 + \left(\frac{\partial\bar{V}}{\partial\bar{y}}\right)^2 + \frac{1}{2}\left(\frac{\partial\bar{u}}{\partial\bar{y}} + \frac{\partial\bar{V}}{\partial\bar{x}}\right)^2\right]$$

$$\bar{u}\frac{\partial\bar{T}}{\partial\bar{x}} + \bar{V}\frac{\partial\bar{T}}{\partial\bar{y}} = \frac{\alpha}{\nu}\cdot\frac{1}{\frac{U_0 L}{\nu}}\left[\frac{\partial^2\bar{T}}{\partial\bar{x}^2} + \frac{\partial^2\bar{T}}{\partial\bar{y}^2}\right]$$

$$+2\frac{\nu}{U_0 L}\cdot\frac{U_0^2}{C_p\Delta T}\left[\left(\frac{\partial\bar{u}}{\partial\bar{x}}\right)^2 + \left(\frac{\partial\bar{V}}{\partial\bar{y}}\right)^2 + \frac{1}{2}\left(\frac{\partial\bar{u}}{\partial\bar{y}} + \frac{\partial\bar{V}}{\partial\bar{x}}\right)^2\right]$$

where $\nu = \frac{\mu}{\rho}$

$$\bar{u}\frac{\partial\bar{T}}{\partial\bar{x}} + \bar{V}\frac{\partial\bar{T}}{\partial\bar{y}} = \frac{1}{P_r}\cdot\frac{1}{Re_L}\left[\frac{\partial^2\bar{T}}{\partial\bar{x}^2} + \frac{\partial^2\bar{T}}{\partial\bar{y}^2}\right]$$

$$+2\frac{E_c}{Re_L}\left[\left(\frac{\partial\bar{u}}{\partial\bar{x}}\right)^2 + \left(\frac{\partial\bar{V}}{\partial\bar{y}}\right)^2 + \frac{1}{2}\left(\frac{\partial\bar{u}}{\partial\bar{y}} + \frac{\partial\bar{V}}{\partial\bar{x}}\right)^2\right]$$

where $Re_L = \frac{U_0 L}{\nu}$, $P_r = \frac{\nu}{\alpha}$.

Eckert No. $\Rightarrow E_c = \frac{U_0^2}{C_p\,\Delta T}$

also $R_a = Re_L P_r = \frac{U_0 L}{\alpha}$

Peclet No. $\rightarrow P_E = \frac{E_c}{Re_L} = \frac{U_0\,\nu}{C_p\Delta T\,L}$

Reynolds Number, R_L

In 1908 Sommerfeld suggested that O. Reynolds be credited with the dimensionless quantity

$$R_L = \frac{U_0 L}{\nu} = \frac{\rho U_0 L}{\mu} \longrightarrow 7.29$$

It is the ratio of the momentum flux to the shearing stress.

The Reynolds number measures the relative importance of the fluid's inertia and viscosity. Thus, if the vicous force play a predominant role, (as in the case of a flow very near a body) then R_L is small. If, on the other hand, inertia effects are predominant, then R_L is very large.

Problems where R_L is usually numerically very large (or infinite) are

1):- Turbulent flows
2):- Inviscid flows
3):- Potential flows
4):- Flows far removed from boundaries.

Cases where R_L is numerically very small are:

1)- Creeping flows

2)- Laminar flows

3)- Stokes flow and Lubrication Theory

4)- Bubble flows

5)- Flows very close to a boundary

Froude Number, F_r

$$F_r = \frac{U}{\sqrt{gL}}$$

It was Froude in 1868 who was the first to determine the drag of ships by using laws of similarity.

$U(x,y)$ ← PDE

$\Rightarrow U(\eta)$ ← ODE

x, y is similar to η

If $F_r < 1$, open channel flow indicates a subcritical condition, & if $F_r > 1$, we have supercritical flow corresponding to tranquil flow or rapid flow in a channel.

Mach Number, M

10-04-2015

Mach number M is a measure of compressibility of a fluid. It is the ratio of the speed of the fluid flow U to the speed of sound c in the fluid:

$$M = \frac{U}{c} \longrightarrow 7.36$$

where

$$c = \sqrt{\frac{dP}{d\rho}}$$

For an isentropic process, the pressure P is a function of density ρ only so that we can write

$$c = \sqrt{\frac{KP}{\rho}} \longrightarrow (7.38)$$

Where K is a measure of the internal complexities of the molecules and is related to the ratio of two constant specific heat:

$$K = \frac{C_p}{C_v} \longrightarrow (7.39)$$

Heat at constant volume

For a perfect gas, Eq (7.38) can be written as

$$c = \sqrt{K g_c R T} \longrightarrow (7.40)$$

Where T is the absolute temperature

g_c is a universal constant & R is the gas constant.

For very small Mach numbers, the variation of density due to the variation of the flow field is negligible & the fluid may be considered incompressible.

When $0.3 \leq M \leq 1$, the flow is termed **subsonic flow**, & when $M > 1$, the flow is termed **supersonic**.

Cauchy number, C

It is the ratio of the compressibility force to the inertial force, or

$$C = \frac{K}{\rho U^2}$$

$$\Rightarrow M = \frac{1}{\sqrt{C}}$$

So, that $M = 0$, or $C = \infty$ defines incompressible flow.
fluid

Weber Number, W 13-04-2015

There is a large class of fluid flows that deals with a free-surface where surface tension force exist and are important. The surface tension is denoted by σ.

The weber number W is defined as the ratio of the inertial force to the surface tension force. Mathematically speaking

$$W = \frac{\rho U^2 L}{\sigma}$$

Large values of the weber no. indicate that surface tension is unimportant as compared to the inertial force.

Euler Number, E

This number can be defined as the ratio of the pressure force P_∞ (ambient fluid pressure) to the inertial force & can be written as

$$E = \frac{P_\infty}{\rho U^2}$$

Specific Pressure, C_p

(or Pressure Coefficient)

The specific pressure $\tilde{C}_p$ is defined as the ratio of pressure difference ΔP to the dynamic pressure ρU^2 & mathematically can be be expressed as

$$\tilde{C}_p = \frac{\Delta P}{\rho U^2}$$

Similitude (Simulation, Similarity)

It enables us to take the model in practicale life.

It is a process in which we transform the PDE to ODE.

Example 7.4:-

Consider the problem of calculating the distribution of the axial velocity of a real fluid's circular jet. The jet flows through a small circular orifice in a wall, and the resultant fluid motion is axisymmetric about the axial axis of the jet, as shown in figure. Let r denote the normal to the axis. The appropriate equation of linear momentum is

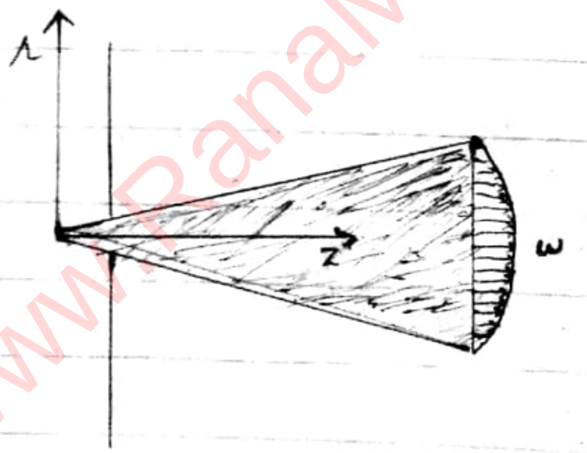
$$w\frac{\partial w}{\partial z}+V_r\frac{\partial w}{\partial r}=\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) \longrightarrow (i)$$

with boundary conditions,

$$\frac{\partial w}{\partial r}=0, \quad \text{at} \quad r=0 \longrightarrow (ii)$$

$$V_r=0, \quad \text{at} \quad r=0 \longrightarrow (iii)$$

$$V_r=0, \quad \text{at} \quad r=\infty \longrightarrow (iv)$$



Let us define a stream function $\psi(r,z)$ that is uniquely related to the velocity components V_r and w so that it satisfies the continuity Eq.

$$w=\frac{1}{r}\cdot\frac{\partial\psi}{\partial r} \longrightarrow (v)$$

$$V_r=-\frac{1}{r}\frac{\partial\psi}{\partial z} \longrightarrow (vi)$$

We introduce a similarity variable η as

$$\eta = \frac{1}{\sqrt{\nu}}\left(\frac{r}{z}\right) \longrightarrow \text{(vii)}$$

$$\psi = \nu z f(\eta) \longrightarrow \text{(viii)}$$

We seek to find expressions for the velocity components v_r & w if the rate of flow of momentum $\dot{M}$ across any cross section of the jet is constant, where

$$\dot{M} = 2\pi\rho \int_0^\infty r w^2 \, dr = \text{constant} \rightarrow \text{(ix)}$$

Solution:-

$$w = \frac{1}{r}\frac{\partial \psi}{\partial r}$$

$$w = \frac{1}{r}\frac{\partial}{\partial r}(\nu z f(\eta))$$

$$w = \frac{\nu z}{r}\left(\frac{\partial f}{\partial \eta} \cdot \frac{\partial \eta}{\partial r}\right)$$

$$w = \frac{\nu z}{r}\left(f' \cdot \frac{1}{\sqrt{\nu}\, z}\right) \qquad \because r = \eta z \sqrt{\nu}$$

$$w = \frac{\cancel{\sqrt{\nu}}}{\eta \cancel{\sqrt{\nu}}\, z}(f')$$

$$w = \frac{f'}{\eta z} \longrightarrow \text{(x)}$$

Now

$$v_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}$$

$$v_r = -\frac{1}{r} \cdot \frac{\partial}{\partial z}(\nu z f(\eta))$$

$$v_r = \frac{-\nu}{r}\left(z\,\frac{\partial f}{\partial \eta}\cdot\frac{\partial \eta}{\partial z} + f\right)$$

$$v_r = \frac{-\nu}{r}\left(z\cdot f'\left(\frac{-\eta}{z}\right) + f\right)$$

$$v_r = \frac{-\nu}{\eta z\sqrt{\nu}}\left[-\eta f' + f\right]$$

$$v_r = \frac{\sqrt{\nu}}{z}\left[f' - \frac{f}{\eta}\right] \longrightarrow (\text{xi})$$

$\because \eta = \frac{1}{\sqrt{\nu}}\left(\frac{r}{z}\right)$

$\Rightarrow \frac{\partial \eta}{\partial z} = \frac{-1}{\sqrt{\nu}}\cdot\frac{r}{z^2}$

$\frac{\partial \eta}{\partial z} = \frac{-\eta}{z}$

It is radial component of velocity.

Now

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}\left(\frac{f'}{\eta z}\right)$$

$$\frac{\partial w}{\partial z} = \frac{\eta z\left(\frac{\partial f'}{\partial \eta}\cdot\frac{\partial \eta}{\partial z}\right) - f'\left[\eta + z\,\frac{\partial \eta}{\partial z}\right]}{(\eta z)^2}$$

$$\frac{\partial w}{\partial z} = \frac{1}{(\eta z)^2}\left[\eta z\, f''\left(\frac{-\eta}{z}\right) - f'\eta - f'z\left(\frac{-\eta}{z}\right)\right]$$

$$\frac{\partial w}{\partial z} = \frac{1}{\eta^2 z^2}\left[-\eta^2 f'' - \cancel{f'\eta} + \cancel{f'\eta}\right]$$

$$\frac{\partial w}{\partial z} = \frac{-f''}{z^2}$$

$$w\,\frac{\partial w}{\partial z} = \left(\frac{-f''}{z^2}\right)\left(\frac{f'}{\eta z}\right)$$

$$\boxed{w\,\frac{\partial w}{\partial z} = \frac{-f' f''}{\eta z^3}}$$

$$\frac{\partial w}{\partial r} = \frac{\partial}{\partial r}\left(\frac{f'}{\eta z}\right)$$

$$\frac{\partial w}{\partial r} = \frac{1}{Z}\frac{\partial}{\partial r}\left(\frac{f'}{\eta}\right)$$

$$\frac{\partial w}{\partial r} = \frac{1}{Z}\cdot\frac{\eta\left(\frac{\partial f'}{\partial \eta}\cdot\frac{\partial \eta}{\partial r}\right) - f'\frac{\partial \eta}{\partial r}}{(\eta)^2}$$

$$\frac{\partial w}{\partial r} = \frac{1}{\eta^2 Z}\left[\eta f''\frac{1}{\sqrt{\nu}Z} - f'\frac{1}{\sqrt{\nu}Z}\right]$$

$$\frac{\partial w}{\partial r} = \frac{1}{\eta^2 Z^2\sqrt{\nu}}\left[\eta f'' - f'\right]$$

$$r\frac{\partial w}{\partial r} = \frac{\eta Z\cancel{\sqrt{\nu}}}{\eta^2 Z^2\cancel{\sqrt{\nu}}}\left[\eta f'' - f'\right]$$

$$r\frac{\partial w}{\partial r} = \frac{1}{\eta Z}\left[\eta f'' - f'\right]$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = \frac{\partial}{\partial r}\left[\frac{1}{Z}\left[f'' - \frac{f'}{\eta}\right]\right]$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = \frac{1}{Z}\left[f'''\frac{\partial \eta}{\partial r} - \left[\frac{\eta f''\frac{\partial \eta}{\partial r} - f'\frac{\partial \eta}{\partial r}}{\eta^2}\right]\right]$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = \frac{1}{Z}\cdot\frac{1}{\eta^2}\left[\eta^2 f'''\frac{1}{\sqrt{\nu}Z} - \eta f''\frac{1}{\sqrt{\nu}Z} + f'\frac{1}{\sqrt{\nu}Z}\right]$$

$$\boxed{\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = \frac{1}{\eta^2 Z^2\sqrt{\nu}}\left[\eta^2 f''' - \eta f'' + f'\right]}$$

Now

$$V_r\cdot\frac{\partial w}{\partial r} = \frac{\cancel{\sqrt{\nu}}}{Z}\left[f' - \frac{f}{\eta}\right]\cdot\frac{1}{\eta^2 Z^2\cancel{\sqrt{\nu}}}\left[\eta f'' - f'\right]$$

$$\boxed{V_r\frac{\partial w}{\partial r} = \frac{1}{\eta^2 Z^3}\left[\eta f' f'' - f'^2 - f f'' + \frac{f f'}{\eta}\right]}$$

as $\quad w\frac{\partial w}{\partial Z} + V_r\frac{\partial w}{\partial r} = \frac{\nu}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)$

So,

$$-\frac{f'f''}{\eta Z^3} + \frac{1}{\eta^2 Z^3}\left[\eta f' f'' - f'^2 - f f'' + \frac{f f'}{\eta}\right] = \frac{\nu}{r}\cdot\frac{1}{\eta^2 Z^2\sqrt{\nu}}\left[\eta^2 f''' - \eta f'' + f'\right]$$

$$-\cancel{\frac{f'f''}{\eta Z^3}} + \cancel{\frac{f'f''}{\eta Z^3}} - \frac{f'^2}{\eta^2 Z^3} - \frac{ff''}{\eta^2 Z^3} + \frac{ff'}{\eta^3 Z^3} = \frac{\sqrt{\nu}}{\eta Z\sqrt{\nu}}\,\frac{1}{\eta^2 Z^2}\left[\eta^2 f''' - \eta f'' + f'\right]$$

$$-\frac{f'^2}{\eta^2 Z^3} - \frac{ff''}{\eta^2 Z^3} + \frac{ff'}{\eta^3 Z^3} = \frac{f'''}{\eta Z^3} - \frac{f''}{\eta^2 Z^3} + \frac{f'}{\eta^3 Z^3}$$

Multiplying by ηZ^3.

$$-\frac{f'^2}{\eta} - \frac{ff''}{\eta} + \frac{ff'}{\eta^2} = f''' - \frac{f''}{\eta} + \frac{f'}{\eta^2}$$

$$f''' - \frac{f''}{\eta} + \frac{f'}{\eta^2} + \frac{f'^2}{\eta} + \frac{ff''}{\eta} - \frac{f'f}{\eta^2} = 0$$

$$\frac{d}{d\eta}\left[f'' - \frac{f'}{\eta}\right] + \frac{d}{d\eta}\left(\frac{ff'}{\eta}\right) = 0 \longrightarrow (xiv)$$

$$f'' - \frac{f'}{\eta} + \frac{ff'}{\eta} = 0 \qquad \frac{d}{d\eta}\left[f'' - \frac{f'}{\eta} + \frac{ff'}{\eta}\right] = 0$$

$$\underbrace{\eta f'' - f' + ff' = 0}_{\text{ODE in terms of } \eta} \longrightarrow (xv)$$

Boundary Conditions.

$\frac{\partial w}{\partial r} = 0$ at $r = 0$

$$\Rightarrow \frac{1}{\eta^2 Z^2\sqrt{\nu}}\left[\eta f'' - f'\right] = 0 \quad \text{at } \eta = 0$$

$$\Rightarrow \frac{1}{Z^2\sqrt{\nu}} \neq 0 \quad \& \quad \frac{f''}{\eta} - \frac{f'}{\eta^2} = 0$$

$$\Rightarrow \frac{f''}{\eta} - \frac{f'}{\eta^2} = 0$$

$$\Rightarrow \frac{\eta f'' - f'}{\eta^2} = 0$$

$$\frac{d}{d\eta}\left(\frac{f'}{\eta}\right) = 0 \longrightarrow (xiv)$$

$V_r = 0$ at $r = 0$

$$\frac{\sqrt{\nu}}{2}\left[f' - \frac{f}{\eta}\right] = 0, \quad \eta = 0$$

$$\frac{\sqrt{\nu}}{2} \neq 0 \Rightarrow f' - \frac{f}{\eta} = 0 \longrightarrow (xiii)$$

Eq. (xv) has the solution 17-4-2015

$$f = \frac{\xi^2}{1 + \xi^2/4} \longrightarrow (x\,vi)$$

where $\xi = a\eta \rightarrow (x\,vii)$ and 'a' is an arbitrary constant.

$$\Rightarrow f = \frac{a^2\eta^2}{1 + \frac{a^2\eta^2}{4}}$$

$$f' = a^2\left[\frac{\left(1 + \frac{a^2\eta^2}{4}\right)(2\eta) - (\eta^2)\left(\frac{a^2\eta}{2}\right)}{\left(1 + \frac{a^2\eta^2}{4}\right)^2}\right]$$

$$f' = \frac{a^2\left[2\eta + \cancel{\frac{a^2\eta^3}{2}} - \cancel{\frac{a^2\eta^3}{2}}\right]}{\left(1 + \frac{a^2\eta^2}{4}\right)^2}$$

$$f' = \frac{2a^2\eta}{\left(1 + \xi^2/4\right)^2}$$

Eq. (X) $\Rightarrow$ $w = \dfrac{\gamma'}{z\eta}$

$$w = \frac{1}{z\cancel{\eta}}\left(\frac{2a^2\cancel{\eta}}{(1+\xi^2/4)^2}\right)$$

$$w = \frac{2a^2}{z(1+\xi^2/4)^2} \longrightarrow (xviii)$$

We know $\xi^2 = a^2\eta^2 = \dfrac{a^2 r^2}{\nu z^2}$ $\qquad \because \xi = a\eta$ & $\eta = \dfrac{1}{\sqrt{\nu}}\cdot\left(\dfrac{r}{z}\right)$

$$\Rightarrow w = \frac{2a^2}{z\left(1+\frac{a^2r^2}{4\nu z^2}\right)^2}$$

Put w in Eq. (IX)

Eq. (IX) $\Rightarrow$ $\dot{M} = 2\pi\rho\int_0^\infty r\left(\dfrac{4a^4}{z^2\left(1+\frac{a^2r^2}{4\nu z^2}\right)^4}\right)dr$

$$\dot{M} = 8\times 2\pi\rho a^2\nu\int_0^\infty\left(1+\frac{a^2r^2}{4\nu z^2}\right)^{-4}\left(\frac{a^2 r}{2\nu z^2}\right)dr$$

$$\dot{M} = 16\pi a^2(\rho\nu)\left.\frac{\left(1+\frac{a^2r^2}{4\nu z^2}\right)^{-3}}{-3}\right|_0^\infty$$

$$\dot{M} = \frac{16}{-3}\pi a^2\mu \left.\frac{1}{\left(1+\frac{a^2r^2}{4\nu z^2}\right)^3}\right|_0^\infty$$

$$\dot{M} = -\frac{16}{3}\pi a^2\mu\left[\frac{1}{\infty} - \frac{1}{(1-0)}\right]$$

$$\dot{M} = \frac{16}{3}\pi a^2\mu \longrightarrow (xix)$$

$$Eq\,(XIX) \Rightarrow \quad a^2 = \frac{3\dot{M}}{16\pi\mu} \longrightarrow (\star)$$

$$\Rightarrow 2a^2 = \frac{3\dot{M}}{8\pi\mu}$$

Put in Eq. (XVIII)

$$\omega = \frac{3\dot{M}}{8\pi\mu z\left(1+\frac{\xi^2}{4}\right)^2} \longrightarrow (XX)$$

Now Eq. (XI)

$$V_r = \frac{\sqrt{\nu}}{z}\left[f' - \frac{f}{\eta}\right]$$

By putting the values of "f" & "f'"

$$V_r = \frac{\sqrt{\nu}}{z}\left[\frac{2a^2\eta}{\left(1+\xi^2/4\right)^2} - \frac{a^2\eta}{\left(1+\xi^2/4\right)}\cdot\frac{1}{\eta}\right]$$

$$= \frac{\sqrt{\nu}}{z}(a^2\eta)\left[\frac{2}{\left(1+\xi^2/4\right)^2} - \frac{1}{1+\xi^2/4}\right]$$

$$= \frac{\sqrt{\nu}}{z}(a\cdot a\eta)\left[\frac{2-\left(1+\xi^2/4\right)}{\left(1+\xi^2/4\right)^2}\right]$$

$$= \frac{\sqrt{\nu}}{z}\cdot a(\xi)\left[\frac{1-\xi^2/4}{\left(1+\xi^2/4\right)^2}\right] \qquad \because \xi = a\eta$$

From Eq. $(\star)$ $\quad a = \sqrt{\frac{3\dot{M}}{16\pi\mu}}$

$$\Rightarrow \quad V_r = \frac{\sqrt{\nu}}{z}\sqrt{\frac{3\dot{M}}{16\pi\mu}}\,\frac{\xi\left(1-\xi^2/4\right)}{\left(1+\xi^2/4\right)^2} \qquad \because \rho = \frac{\mu}{\nu}$$

$$V_r = \sqrt{\frac{3\dot{M}}{16\pi\rho}}\cdot\frac{\xi\left(1-\xi^2/4\right)}{z\left(1+\xi^2/4\right)^2} \longrightarrow (XXI)$$

where $a = \sqrt{\dfrac{3\dot{M}}{16\pi\mu}}$

x'ing η.

$$\eta a = \sqrt{\frac{3\dot{M}}{16\pi\mu}}\cdot\eta \qquad \because \xi = a\eta$$

$$\because \mu = \rho\nu$$

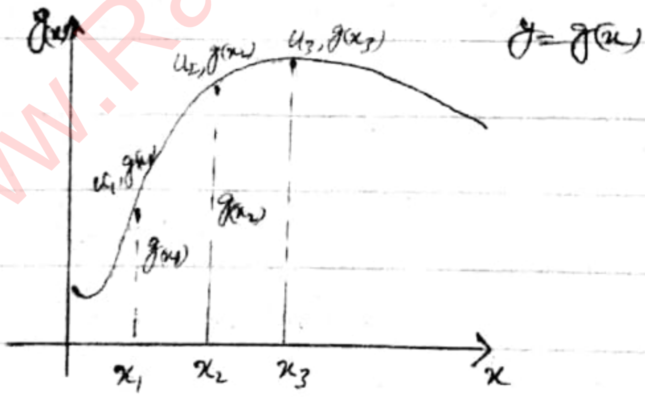
$$\xi = \sqrt{\frac{3\dot{M}}{16\pi\rho\nu}}\cdot\frac{1}{\sqrt{\nu}}\cdot\left(\frac{r}{z}\right) \qquad \because \eta = \frac{1}{\sqrt{\nu}}\cdot\left(\frac{r}{z}\right)$$

$$\boxed{\xi = \sqrt{\frac{3\dot{M}}{16\pi\rho\nu^2}}\cdot\left(\frac{r}{z}\right)}$$

This problem was initially solved by Schlichting in 1933.

Similarity Solutions & Transformations

20-04-2015



$$\frac{u\{x_1, [y/g(x_1)]\}}{U(x_1)} = \frac{u\{x_2, [y/g(x_2)]\}}{U(x_2)}$$

This provides solely the similarity. The results are expressed in terms of parameters only.

Geometric and Dynamic Similitude

For Geometric Similitude, the ratio of characteristic lengths is

$$\frac{r_2}{r_1} = \text{constant} \longrightarrow (7.48)$$

Dynamic Similitude

In dynamic similitude, the ratio of corresponding forces F_2/F_1 in both flow fields is the same, or there is a similarity of forces.

For complete similitude to exist in two flow fields, the ratio of forces of the same nature must be the same in both flow fields. Some of the forces that influence fluid behavior are inertial, pressure, body or gravitational, magnetic & electrical, viscous, surface tension & compressible forces. Similarity of forces is necessary because the direction taken by any fluid particle is determined by the resultant force acting on it. These forces are in a fixed ratio of magnitude.

Kinematic Similitude

It is the ratio of corresponding velocities $\frac{V_2}{V_1}$ is a constant. i.e

$$\frac{V_2}{V_1} = \text{Constant}$$

Let the subscript 1 denote the model system and the subscript 2 denote the prototype. For Geometric Similarity,

$$\frac{r_2}{r_1} = C_1 \longrightarrow (7.49)$$

where r_2 & r_1 are two position vectors to the same characteristic point in the flow, and C_1 is a constant. The geometries of both flow fields are identical if all space dimensions in flow 2 are made dimensionless with r_2 and all space dimensions in flow 1 are made dimensionless with r_1.

Now that the linear dimensions have been made similar, real time also possesses similarity.

From kinematic similitude,

$$\frac{V_2}{V_1} = c_2 \longrightarrow (7.50)$$

So that from Eq. (7.49) a similarity in time is

$$\frac{T_2}{T_1} = c_3 = \frac{c_1}{c_2} \longrightarrow (7.51)$$

In case of density, there is a similarity between all homogeneous Newtonian fluids.

$$\frac{\rho_2}{\rho_1} = c_4 \longrightarrow (7.52)$$

Similar ratios hold for pressure

$$\frac{P_2}{P_1} = c_5 \longrightarrow (7.53)$$

The scale of accelerat $\vec{a}$ stems from kinematic similitude, so that from Eqs (7.50) & (7.51) we obtain

$$\frac{a_2}{a_1} = \frac{c_2}{c_3} = \frac{c_2^2}{c_1} \longrightarrow (7.54)$$

$\nu \Rightarrow m^2/s$
$\Rightarrow \frac{c_1^2}{c_3}$

The scale for kinematic viscosity ν is obtained from Newton's law of friction, so that from Eqs (7.49) – (7.51) we find,

$$\frac{\nu_2}{\nu_1} = \frac{c_1^2}{c_3} = c_1 c_2 \longrightarrow (7.55)$$

and so on.

Consider now two flows around similar bodies moving in a real fluid. For the flows to be dynamically similar, the forces must be proportional. From Eq. (7.26)

$$\frac{\partial V_1^*}{\partial T_1} + (V_1^* \cdot \nabla_1^*) V_1^* = -\nabla_1^* \bar{P}_1^* + \frac{1}{R_{L_1}} \nabla_1^{*2} V_1^* \longrightarrow (7.56)$$

Representing the dimensionless Navier-Stokes equation for the model and

$$\frac{\partial V_2^*}{\partial T_2} + (V_2^* \cdot \nabla_2^*) V_2^* = -\nabla_2^* \bar{P}_2^* + \frac{1}{R_{L_2}} \nabla_2^{*2} V_2^* \longrightarrow (7.57)$$

representing the dimensionless Navier-Stokes equation for the prototype

For these two Eqs to yield similar solutions, the two Eqs must differ by a proportionality constant. Thus we can show, by ratioing the force per unit mass of each term of Eq. (7.56) with that of Eq. (7.57), that

$$\frac{\left(\frac{\partial V_1^*}{\partial T_1}\right)}{\left(\frac{\partial V_2^*}{\partial T_2}\right)} = \frac{(V_1^* \cdot \nabla_1^*)V_1^*}{(V_2^* \cdot \nabla_2^*)V_2^*} = \frac{\nabla_1^* \bar{P}_1^*}{\nabla_2^* \bar{P}_2^*}$$

$$= \left(\frac{\nabla_1^{*2} V_1^*}{R_{L_1}}\right) \cdot \left(\frac{R_{L_2}}{\nabla_2^{*2} V_2^*}\right) = 1$$

with the result that each force ratio is equal to the same constant, and the flows of the prototype and model are dynamically and geometrically similar.

Chap # 8 **Flow Visualization** 24-04-2015

"One picture is worth a thousand words"

To visualize flow.

This the demonstration of fundamentals of fluid flow mechanism.

Flow of smoke is steady, then we cannot compute its consentration.

Concentration depends on

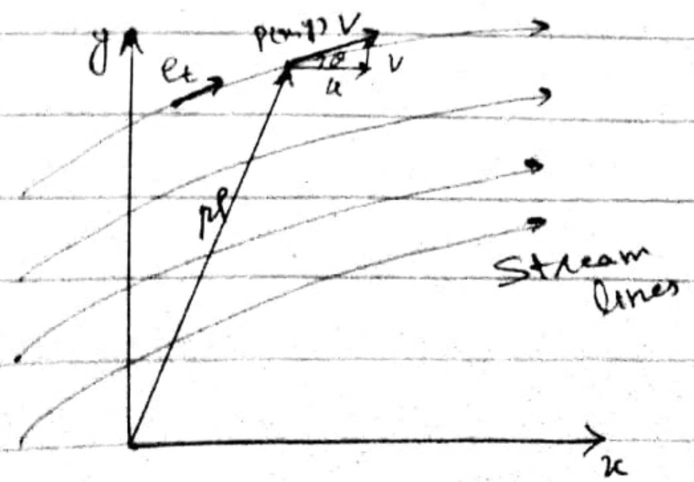
(i) Streamline (ii) Path line

(iii) Streak line.

These lines are important for visualization.

Stream line:-

"A stream line is defined as a line whose tangent at any point is in the direction of the velocity at that point."

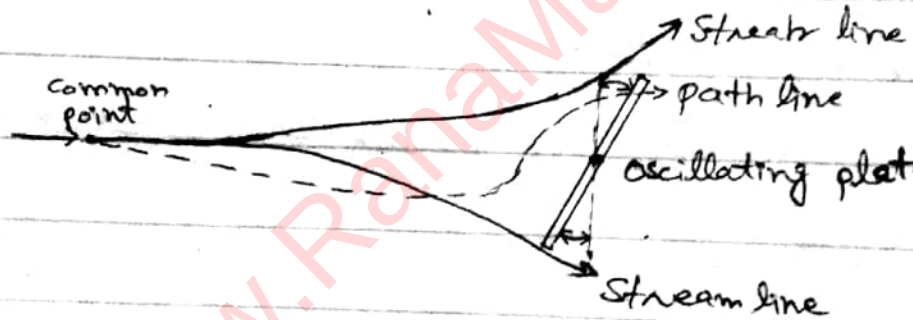


Path line:-

A path line, on the other hand, is the trajectory of a single particle of fluid.

Streak line:-

A streak line is a line joining the instantaneous positions of a succession of particles which have issued from one source or passed through one point.



Note:-

(i):- For steady flow, steam lines, path lines & streaklines coincide.

(ii):- In flow visualization we usually see streak lines.

(iii):- In some unsteady flows, however, the three lines can be distinguished. (as shown in above fig.)

8-06-2015

Flow Net:-

The aggregate of all streamlines is called the flow net.

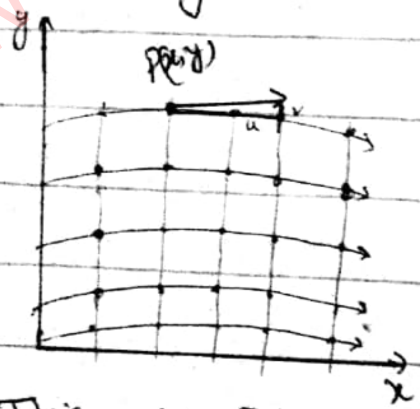
Mathematical Expression:-

Consider the steady two-dimensional flow pattern at a point $P(x,y)$ one & only one streamline can pass. By definition, stream line is tangent to the velocity vector $\vec{V}$ at point P. Using Cartesian coordinates, we obtain from geometry

$$\frac{dy}{dx} = \tan\theta = \frac{v}{u} \longrightarrow (8.1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{v}{u}$$

$$u\,dy - v\,dx = 0 \longrightarrow (8.2)$$



This is flow net.

{Along normal direction change ⇒ 2-Dim flow}

Five streamlines are under consideration.

{Change is normal to the plane ⇒ 3-dim flow}

For 3-dimensional flow, we then find

$\vec{V} \times d\underline{r} = 0$ → (8.3) → which is known as the Eq. of a stream line

Velocity of field.

$\{ (U, V, W) \times (r_1, r_2, r_3) = 0$

plane

Rotation

$\perp$ to plane $\}$

where $\underline{r}$ is a distance but dr is a small distance.

$\vec{V} \times d\underline{r} = 0$

Either $\vec{V} = 0$ or $d\underline{r} = 0$

or The flow $\vec{V}$ must be tangent to the streamline $d\underline{r}$.

When $\vec{V} = 0$, Then stream line will become a solid line.

S.Q Stream Surface:-

A surface across which no flow passes is called stream surface.

$$\frac{dx}{\rho u} = \frac{dy}{\rho v} = \frac{dz}{\rho w} \longrightarrow (8.4)$$

ratio of small distance to dynamic force.

This is 3-dimensional & ~~symetrical~~.

Example 8.1:-

Given the velocity

$\vec{V} = (1+t)x\hat{i} + (2+t)y\hat{j}$,

find the Eq. of the

(a):- stream line (b):- pathline (c):- Streak line, given that the common point for all three is $x=1$, $y=2$, & $z=0$ at $t=0$.

Solution:-

Step# 1 (i) Two dimensional flow.

(ii) Because t involve, So flow is unsteady.

Where $u=(1+t)x$, $v=(2+t)y$

Continuity:-

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Streamline:-

$$\frac{dx}{u} = \frac{dy}{v}$$

{velocity vector is tangent to the point.

The equation $\vec{\nabla}\cdot\vec{V}=0$ is not satisfied.

$$\frac{\partial u}{\partial x} = (1+t) \quad , \quad \frac{\partial v}{\partial y} = (2+t)$$

$$(1+t)+(2+t) \neq 0$$

The components u & v does not satisfy the Eq. of continuity of mass. So it contradicts the conservation of mass.

(a).

To find the eq. of streamline.

$$\frac{dx}{(1+t)x} = \frac{dy}{(2+t)y}$$

$$(2+t)\int \frac{dx}{x} = (1+t)\int \frac{dy}{y}$$

$$(2+t)\ln x = (1+t)\ln y + \ln c$$

We use given initial conditions;

$y=2$, $x=1$ at $t=0$

$$\Rightarrow \quad (2+0)\ln 1 = (1+0)\ln(2) + \ln c$$

$$-\ln(2) = \ln c$$

So,

$$(2+t)\ln x = (1+t)\ln y - \ln 2$$

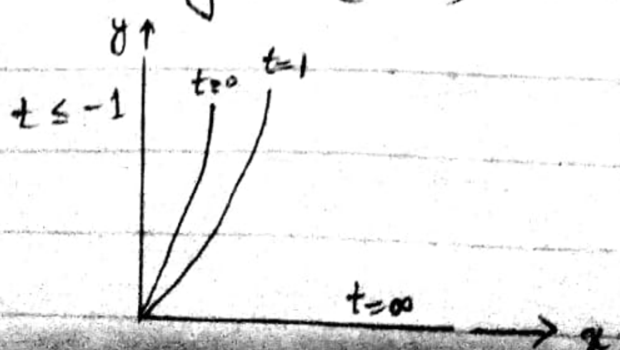
$$\ln x^{(2+t)} = \ln y^{(1+t)} - \ln 2$$

$$\ln 2 = \ln\left(\frac{y^{(1+t)}}{x^{(2+t)}}\right)$$

$$2 = \frac{y^{1+t}}{x^{2+t}}$$

or $\quad 2x^{2+t} = y^{1+t}$

or $\quad y = \left(2x^{2+t}\right)^{\frac{1}{1+t}}$



b):- We obtain the Eq. for pathlines using the definition of the velocity :

$$u = \frac{dx}{dt} , \quad v = \frac{dy}{dt}$$

$$\Rightarrow u = (1+t)x = \frac{dx}{dt} , \quad (2+t)y = \frac{dy}{dt}$$

$$\int (1+t)dt = \int \frac{1}{x} dx , \quad \int (2+t)dt = \int \frac{1}{y} dy$$

$$t + \frac{t^2}{2} + \ln c_1 = \ln x , \quad 2t + \frac{t^2}{2} + \ln c_2 = \ln y$$

$$\Rightarrow x = c_1 e^{t+\frac{t^2}{2}} , \quad y = c_2 e^{2t+\frac{t^2}{2}}$$

Using initial conditions

$$x = 1 , y = 2 \text{ at } t = 0$$

$$\Rightarrow 1 = c_1 , \quad 2 = c_2$$

$$\Rightarrow x = e^{t+\frac{t^2}{2}} , \quad y = 2e^{2t+\frac{t^2}{2}}$$

$$\Rightarrow \ln x = t + \frac{t^2}{2} \rightarrow (i) , \quad \ln(y/2) = 2t + \frac{t^2}{2} \rightarrow (ii)$$

$$\text{Eq. (ii)} \Rightarrow \ln(y/2) = t + \left(t + \frac{t^2}{2}\right)$$

Put (i) here,

$$\ln(y/2) = t + \ln(x)$$

$$\Rightarrow \ln\left(\frac{y}{2x}\right) = t$$

Put t in Eq (i).

$$\ln x = \ln\left(\frac{y}{2x}\right) + \frac{\left[\ln\left(\frac{y}{2x}\right)\right]^2}{2}$$

$$\Rightarrow 2\ln x = 2\ln\left(\frac{y}{2x}\right) + \left[\ln\left(\frac{y}{2x}\right)\right]^2$$

$$\Rightarrow 2\ln x = \left[\ln\left(\frac{y}{2x}\right)\right]^2 + (1)^2 + 2\ln\left(\frac{y}{2x}\right) - 1$$

$$\Rightarrow 2\ln x = \left[\ln\left(\frac{y}{2x}\right) + 1\right]^2 - 1$$

$$\Rightarrow 2\ln x - \left(\ln\left(\frac{y}{2x}\right) + 1\right)^2 + 1 = 0$$

C):-

To find the Eq. of streak lines,

$$x = c_1 e^{t + t^2/2}$$

$$y = c_2 e^{2t + t^2/2}$$

Let $t = k$ and put $x = 1, y = 2$

$$1 = c_1 e^{k + k^2/2}, \quad 2 = c_2 e^{2k + k^2/2}$$

$$\Rightarrow c_1 = e^{-k - k^2/2}, \quad c_2 = 2e^{-2k - k^2/2}$$

$$\Rightarrow x = \exp(-k - k^2/2), \quad y = 2\exp(-2k - k^2/2)$$

$$\Rightarrow \ln x = -k - \frac{k^2}{2}, \quad \ln(y/2) = -2k - k^2/2$$

$$\ln(y/2) = -k \underbrace{- k - \frac{k^2}{2}}$$

$$\ln(y/2) = -k + \ln x$$

$$\Rightarrow \ln\left(\frac{y}{2x}\right) = -k$$

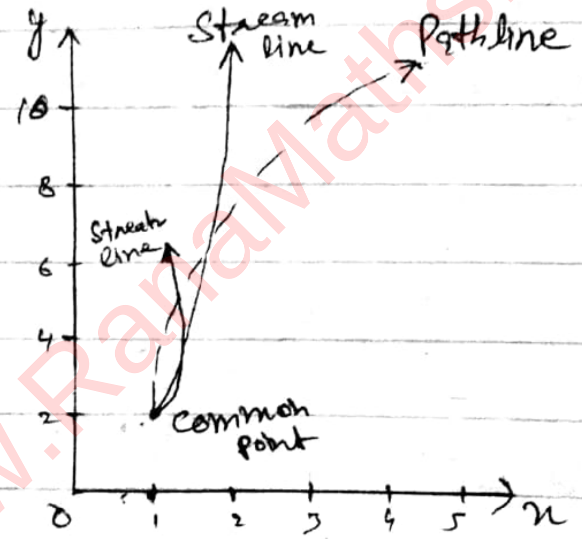
$$\Rightarrow \ln x = \ln\left(\frac{y}{2x}\right) - \frac{\left(\ln\left(\frac{y}{2x}\right)\right)^2}{2}$$

$$2\ln x = -\left[\ln\left(\frac{y}{2x}\right)\right]^2 + 2\ln\left(\frac{y}{2x}\right)$$

$$2\ln x = -\left[\left[\ln\left(\frac{y}{2x}\right)\right]^2 - 2\ln\left(\frac{y}{2x}\right) + 1\right] + 1$$

$$2\ln x = -\left[\ln\left(\frac{y}{2x}\right) - 1\right]^2 + 1$$

$$2\ln x + \left[\ln\left(\frac{y}{2x}\right) - 1\right]^2 - 1 = 0$$



Stream function.

Consider an arbitrary scalar field function $\psi(x,y)$ and its total differential $d\psi$. If the function depends upon the location x, y.

$$\psi = \psi(x,y)$$

then the differential can be

written as

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \longrightarrow 8.6$$

compare with

$$u dy - v dx = 0$$

$$\Rightarrow \quad u = \frac{\partial \psi}{\partial y} \longrightarrow (8.7)$$

$$v = -\frac{\partial \psi}{\partial x} \longrightarrow (8.8)$$

$$\& \quad d\psi = 0 \longrightarrow (8.9)$$

On integrating
$\Rightarrow \psi =$ constant

11-06-2015

ψ is the Lagrange stream function of a two-dimensional fluid flow.

$$u = \frac{\partial \psi}{\partial y} \rightarrow (1), \quad v = -\frac{\partial \psi}{\partial x} \longrightarrow (2)$$

It is necessary for x & y components of velocities to satisfy the differential form of the continuity equation.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \longrightarrow (3)$$

Using (1) & (2) in (3). we get

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

Since the order of the differentiation

is inconsequential, the above Eq shows that continuity is automatically satisfied.

Other forms of velocities components in the form Stream Functions:-

1):- Steady compressible two-dimensional flow in the x, y plane:

$$u = \frac{1}{\rho} \cdot \frac{\partial \psi}{\partial y} \longrightarrow (8.12)$$

$$v = -\frac{1}{\rho} \cdot \frac{\partial \psi}{\partial x} \longrightarrow (8.13)$$

2):- Unsteady incompressible axisymmetric flow in the r, z plane.

$$V_r = \frac{1}{r} \cdot \frac{\partial \psi}{\partial z} \longrightarrow (8.14)$$

$$V_z = -\frac{1}{r} \cdot \frac{\partial \psi}{\partial r} \longrightarrow (8.15)$$

These Eqs define a Stoke stream function

3):- Steady compressible axisymmetric flow in r, z plane:

$$V_r = \frac{1}{\rho r} \cdot \frac{\partial \psi}{\partial z} \longrightarrow (8.16)$$

$$V_z = -\frac{1}{\rho r} \cdot \frac{\partial \psi}{\partial r} \longrightarrow (8.17)$$

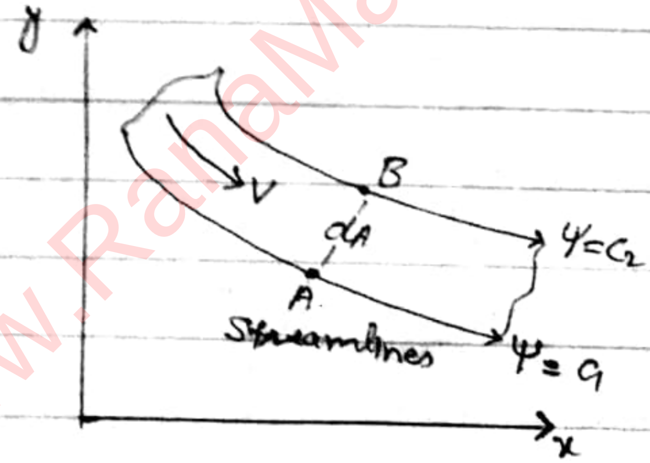
The equations also define a stoke stream function.

For Incompressible flow in the r, θ plane:

$$V_r = \frac{1}{r} \cdot \frac{\partial \psi}{\partial \theta} \longrightarrow (8.18)$$

$$V_\theta = -\frac{\partial \psi}{\partial r} \longrightarrow (8.19)$$

These Eqs define a Lagrangian stream function.



Consider two stream lines, $\psi = C_1$ & $\psi = C_2$ as shown in fig.

We evaluate the volume rate of flow Q in terms of the stream function. We see that the elemental area dA can be expressed as,

$$dA = i\, dy\, dz - j\, dx\, dz \longrightarrow (8.20)$$

Using the integrated form of the continuity Eq.

$$Q = \int dQ = \int_A^B V \cdot dA \longrightarrow (8.21)$$

We obtain after substitution of the elemental area

$$Q = \iint_A^B (ui + vj) \cdot (i\,dy\,dz - j\,dx\,dz)$$

$$Q = \iint_A^B (u\,dy\,dz - v\,dz\,dx)$$

$$Q = \iint_A^B (u\,dy - v\,dx)\,dz$$

If we let the depth be unity, then the volume rate of flow per unit depth becomes

$$\frac{Q}{\text{unit depth}} = \int_A^B (u\,dy - v\,dx) \longrightarrow (8.23)$$

The integrands of Eq. (8.23) represents the change of the stream function $d\psi$.

$$\frac{Q}{\text{unit depth}} = \int_A^B d\psi = \psi_B - \psi_A$$

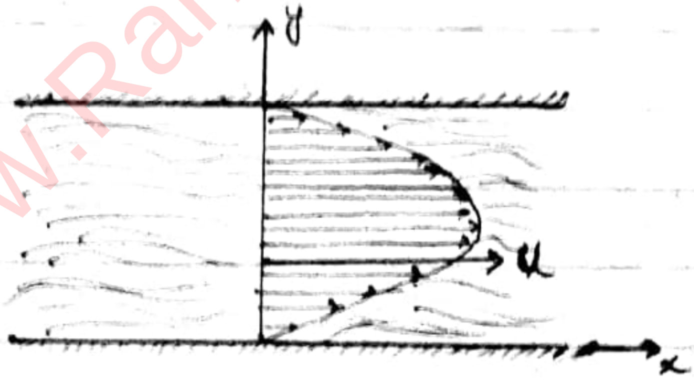
Thus the flow rate Q per unit depth between any two streamlines in a two-dimensional incompressible flow is numerically equal to the difference in the values of the stream function ψ.

Example 8.2

Find the stream function of the two-dimensional incompressible flow

$$u = U\left[\left(\frac{y}{l}\right)^2 - \left(\frac{y}{l}\right)\right]$$

$$v = w = 0$$



Solution:- The fluid is incompressible and the flow is steady and two-dimensional.

Continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \longrightarrow (i)$

Stream functions:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \longrightarrow (ii)$$

Substituting the given velocity components into Eq. (ii), we obtain

$$u = \frac{\partial \psi}{\partial y} = U\left[\left(\frac{y}{l}\right)^2 - \left(\frac{y}{l}\right)\right] \longrightarrow \text{(iii)}$$

$$v = -\frac{\partial \psi}{\partial x} = 0 \longrightarrow \text{(iv)}$$

On integrating.

$$\text{(iii)} \Rightarrow \quad \psi = U\left[\frac{y^3}{3l^2} - \frac{y^2}{2l} + f_1(x)\right] \longrightarrow \text{(v)}$$

&

$$\text{(iv)} \Rightarrow \quad \psi = f_2(y) \longrightarrow \text{(vi)}$$

Diff. Eq. (v) w.r.t "x"

$$\frac{\partial \psi}{\partial x} = U\left[f_1'(x)\right]$$

By using (iv).

$$U\left[f_1'(x)\right] = 0$$

$$f_1'(x) = 0$$

On Integrating

$$f_1(x) = \text{constant} \longrightarrow \text{(vii)}$$

So,

$$f_2(y) = U\left[\frac{y^3}{3l^2} - \frac{y^2}{2l}\right] + C$$

$$\psi = Ul\left[\frac{1}{3}\left(\frac{y}{l}\right)^3 - \frac{1}{2}\left(\frac{y}{l}\right)^2\right] + C$$

The value of c is immaterial & depends upon that choice of the streamline which one wishes to designate $\psi=0$. It is convenient to select the streamline along the bottom of the channel for $\psi=0$. Then c must be zero so that $\psi=0$ at $y=0$. So

$$\psi = Ul\left[\frac{1}{3}\left(\frac{y}{l}\right)^3 - \frac{1}{2}\left(\frac{y}{l}\right)^2\right]$$

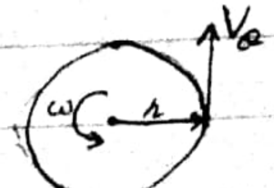
Example 8.3 14-05-2015

Find the stream function for the vortex whose flow field is given by

$$V_r = W = 0$$

$$V_\theta = \omega r$$

where ω is the angular speed & is a constant. Let the $\psi=0$ streamline be at $r=0$.



Solution:- Assume that the fluid is incompressible. The flow is planar and axisymmetric.

Continuity Eq:-

$$\frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\cdot\frac{\partial V_\theta}{\partial \theta} = 0 \longrightarrow (i)$$

Stream Function:-

$$V_r = \frac{1}{r}\cdot\frac{\partial \psi}{\partial \theta} \longrightarrow (ii)$$

$$V_\theta = -\frac{\partial \psi}{\partial r} \longrightarrow (iii)$$

as $V_r = 0$, Eq. (ii) $\Rightarrow \frac{1}{r}\cdot\frac{\partial \psi}{\partial \theta} = 0$

$$\Rightarrow \frac{\partial \psi}{\partial \theta} = 0 \longrightarrow (\star)$$

On Integrating

$$\psi = f_1(r) + c \longrightarrow (iv)$$

As $V_\theta = \omega r$ So $-\frac{\partial \psi}{\partial r} = \omega r$

$$\Rightarrow \frac{\partial \psi}{\partial r} = -\omega r$$

On Integrating, $\psi = -\frac{\omega r^2}{2} + f_2(\theta) + c \longrightarrow (v)$

Diff. (v) w.r.t θ.

$$\frac{\partial \psi}{\partial \theta} = f_2'(\theta)$$

From $(\star)$

$$f_2'(\theta) = 0$$

On integrating

$$f_2(\theta) = \text{Constant} \longrightarrow (vii)$$

Comparing (iv) & (v)

$$f_1(r) = -\frac{\omega r^2}{2} + f_2(\theta)$$

This constant $f_2(\theta)$ will be zero at initial conditions. So

$$f_1(r) = \frac{-\omega r^2}{2} \longrightarrow (vi)$$

So, Eq (iv) $\Rightarrow$

$$\psi = -\frac{\omega r^2}{2} + c \longrightarrow (viii)$$

Put $\psi = 0$ at $r = 0$

$$\Rightarrow c = 0 \longrightarrow (ix)$$

So that $\psi = -\frac{\omega r^2}{2} \longrightarrow (x)$ is the stream function for this vortex flow. Note that at a particular radius, the stream function is constant, which means that the path is circular. This was rather obvious since we were given no radial inflow or outflow, only a circumferential velocity which signifies that the fluid particles are traveling in circular paths.

We could also obtain the Eq of the streamline by using a different approach. Since the Eq. of a streamline is given by

$$V \times dr = 0 \longrightarrow (8.3)$$

Where $\vec{V} = V_r \hat{e}_r + V_\theta e_\theta$

$$\underline{V} = \omega r e_\theta \longrightarrow (xi)$$

&

$$\underline{r} = r e_r \longrightarrow (xii)$$

So that

$$d\underline{r} = dr\, e_r + r d\theta\, e_\theta \longrightarrow (xiii)$$

So Eq. (8.3) $\Rightarrow$

$$\omega r e_\theta \times (dr\, e_r + r d\theta\, e_\theta) = 0$$

$$\Rightarrow -\omega r dr \hat{k} = 0 \qquad \because e_\theta \times e_\theta = 0 \ \& \ e_\theta \times e_r = -\hat{k}$$

On integrating.

$$-\frac{\omega r^2}{2} = \text{constant}$$

This example shows a flow that is both axisymmetric & 2-dimensional so that either the Lagrange or Stokes stream function can be used.

The only restriction we must keep in mind when using the stream function is that the flow must be two-dimensional. It can be unsteady or viscous, compressible or incompressible, but it must be planar or axisymmetric.

Cauchy Riemann Conditions.

Consider an irrotational two-dimensional fluid flow. For a fluid to be irrotational we have shown that a necessary & sufficient condition is that the curl of the velocity vector vanish, such that a scalar potential function ϕ exists. s.t

$$V = \nabla \phi \longrightarrow (8.26)$$

Now we have to find a relationship between the velocity potential ϕ and the stream function ψ.

$$Eq\,(8.26) \Rightarrow \quad u\hat{i} + v\hat{j} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j}$$

$$\Rightarrow \quad u = \frac{\partial \phi}{\partial x} \quad , \quad v = \frac{\partial \phi}{\partial y}$$

But we know from Eq (8.7) & Eq. (8.8)

$$u = \frac{\partial \psi}{\partial y} \quad , \quad v = -\frac{\partial \psi}{\partial x}$$

So, we can write,

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \longrightarrow (8.27)$$

$$\& \quad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \longrightarrow (8.28)$$

Now from Eq. (8.18) & (8.19) we know that

$$V_r = \frac{1}{r} \cdot \frac{\partial \psi}{\partial \theta}$$

&

$$V_\theta = \frac{-\partial \psi}{\partial r}$$

Now from Eq. (8.26) we can write,

$$V_r = \frac{\partial \phi}{\partial r}$$

&

$$V_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta}$$

So

$$V_r = \frac{\partial \phi}{\partial r} = \frac{1}{r} \cdot \frac{\partial \psi}{\partial \theta} \longrightarrow (8.29)$$

&

$$V_\theta = \frac{1}{r} \cdot \frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} \longrightarrow (8.30)$$

The above relationships are called the Cauchy-Reimann conditions, and always exist so long as the flow is two-dimensional & irrotational.

Example 8.4

Given $\phi = x^2 - 2y - y^2$ for a two-dimensional irrotational incompressible flow field, (a):- find the stream function ψ & (b):- identify the type of flow.

Solution:-

The fluid is incompressible and the flow is steady, two-dimensional and irrotational.

We know

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

& $$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

a):- $$\frac{\partial \phi}{\partial x} = 2x$$

& $$\frac{\partial \phi}{\partial y} = -2 - 2y$$

$$\Rightarrow \frac{\partial \psi}{\partial y} = 2x \longrightarrow (i)$$

& $$\frac{\partial \psi}{\partial x} = 2(1+y) \longrightarrow (ii)$$

On integrating (i) & (ii) $\Rightarrow$

$$\psi = 2xy + f_1(x) \longrightarrow (iii)$$

& $$\psi = 2x + 2xy + f_2(y) \longrightarrow (iv)$$

Diff. (iv) w.r.t y.

$$\Rightarrow \frac{\partial \psi}{\partial y} = 2x + f_2'(y)$$

Compare it with (i)

$$\cancel{2x} = \cancel{2x} + f_2'(y)$$

$$\Rightarrow f_2'(y) = 0$$

On integrating.

$$f_2(y) = c$$

at initial condition "c" will be 0.

So, $f_2(y) = 0 \longrightarrow$ (vi)

Now compare (ii) & (iv) and $f_2(y) = 0$

$$\Rightarrow \cancel{2xy} + f_1(x) = 2x + \cancel{2xy} + 0$$

$$\Rightarrow f_1(x) = 2x \longrightarrow (v)$$

So $\psi = 2x(1+y) \longrightarrow$ (vii)

b):- To find what the flow looks like, we first set $\psi = 0$

$$\Rightarrow 2x(1+y) = 0$$

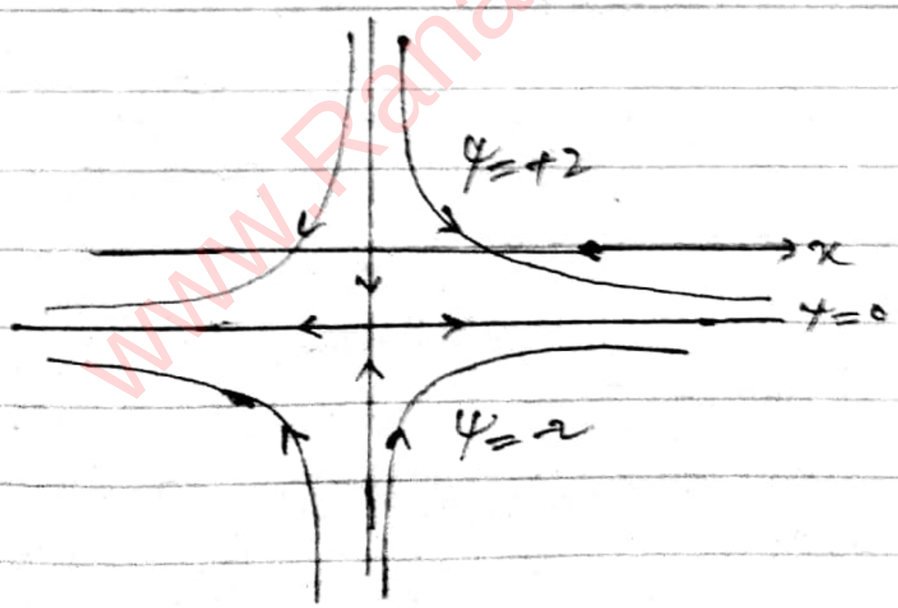
$$\Rightarrow x = 0, \quad y = -1$$

Which represent the zero streamline

To obtain the relationship for $\psi = 2$ streamline, we construct a data table,

$$\psi = \pm 2$$

x	y
± 1	$0, -2$
± 2	$-\frac{1}{2}, -\frac{3}{2}$
± 3	$-\frac{2}{3}, -\frac{4}{3}$
$\pm \infty$	-1
$\pm \frac{1}{2}$	$1, -3$
$\pm \frac{1}{3}$	$2, -4$



The direction of the flow can be found by noting the sign of x, y components of velocity. Since

$$u = \frac{\partial \psi}{\partial y} = 2x$$

and $v = -\frac{\partial \psi}{\partial x} = -2(1+y)$

Then $(x>0, y>0)$ in the first quadrant such that $u>0, v<0$. Thus the slope of the streamline is negative in the first quadrant. This flow is called Stagnation point flow or flow in a corner.

Orthogonality of ϕ & ψ.

$$\phi = \phi(x,y)$$

$$\Rightarrow \quad d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \longrightarrow 8.32$$

We know from Eq. (4.128)

$$d\phi = u\,dx + v\,dy \longrightarrow (8.33)$$

If we define equipotential as having velocity potential ϕ as constant then $d\phi = 0$. Such that

$$\left.\frac{dy}{dx}\right|_{\phi = \text{const.}} = -\frac{u}{v} \longrightarrow (8.34)$$

A similar operation is performed on the stream function. We express the total change of stream function as

$$d\psi = u\,dy - v\,dx \longrightarrow (8.35)$$

Thus, the slope of a streamline is found to be the ratio of the velocities v/u i.e

$$\left.\frac{dy}{dx}\right|_{\psi = const.} = \frac{v}{u} \longrightarrow (8.36)$$

So, by using Eq. (8.34) & (8.36)

$$\frac{dy}{dx} \cdot \frac{dy}{dx} = \frac{-u}{v} \cdot \frac{v}{u} = -1$$

So, the slopes of equipotential lines are orthogonal to the slopes of streamlines.

Visualization Techniques 15-05-2015

Flow visualization is an extremely powerful experimental technique for gaining insight into the fluid flow.

⇒ F.V. by flame. (cylindrical vortex flow)

⇒ F.V. by using solid particles

⇒ F.V. by using differen colours

in fluid flow, density should be same.

⇒ F.V. of dust particles in air.

⇒ F.V. of different gasses by different densities.

⇒ F.V. of some liquid metals.

⇒ F.V. of missible fluid flow.

⇒ F.V. of immissible fluid flow.

⇒ F.V. of two phase flow.

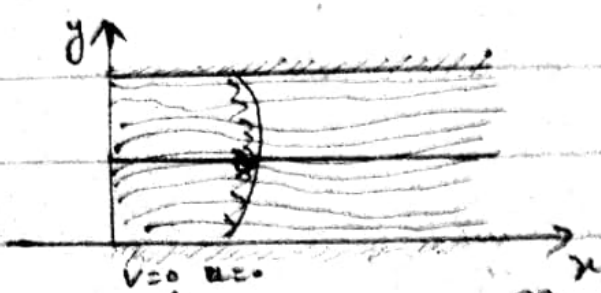
Chap 9 Viscous Fluid Flows 19-05-15

In this chapter, we shall discuss the behavior of a real fluid flowing through objects that are confined in a particular manner. The confinement may be at some finite geometric domain, or, it may be at some infinite or semi-infinite $[0,\infty)$ location.

For this we will study viscous flows between:

- Flat or curved boundaries.
- Between parallel plates or flat plates.
- Past a flat plate
- B/w rough surfaces
- B/w cocentric shells
- B/w cocentric cylinders.

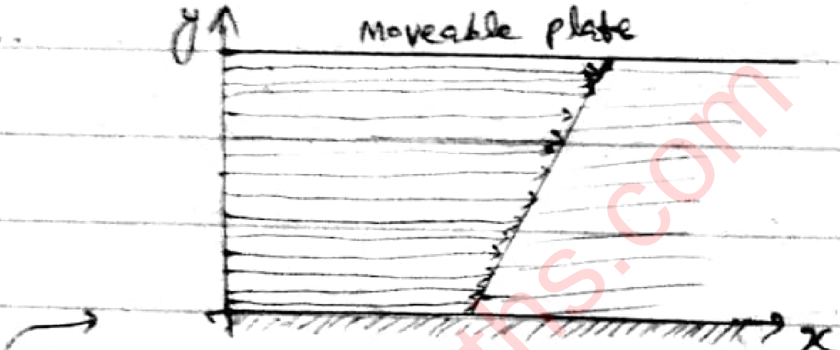
For this purpose we will consider the fluid flow.



These are effects of no slip condition.

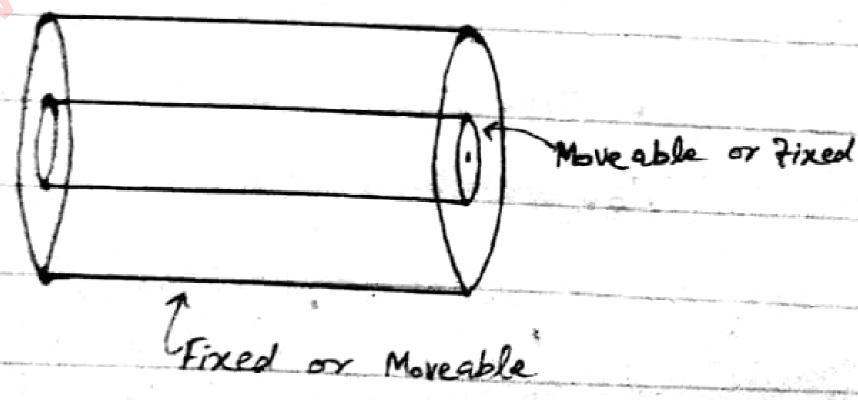
If $\frac{\partial p}{\partial y} = 0 \Rightarrow$ pressure is always along the direction of fluid flow.

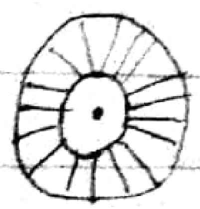
When velocity components (u & v) are relative to boundary of fluid flow is zero then these are no slip condition.



Couette flow.

Cocentric Cylinders:- If the center of two cylinders is at same point then these are co centric cylinders.





Rectilinear Flow Between Parallel Plates

well-behaved

Consider the incompressible flow contained between two infinite, parallel plates that move steadily in their own plane, such that we have rectilinear flow in the fully developed sense, i.e we neglect inlet effects etc. In order to observe & describe the fluid flow between the two plates, we choose an inertial rectangular Cartesian system with the z-axis normal to the plates and the xy-plane lying midway between the plates so that plates are located at $z = \pm h$.

Based on symmetry, the flow is assumed to be steady & parallel to the plate. The boundary conditions for no-slip would then be

$$\left.\begin{array}{l} u = U \\ v = V \end{array}\right\} z = h \qquad \longrightarrow (9.1)$$

$$\left.\begin{array}{l} u = U' \\ v = V' \end{array}\right\} z = -h$$

Where the velocity of the upper plate is $U\hat{i} + V\hat{j}$ and the velocity of the lower plate is $U'i + V'j$.

We seek solutions for the velocity in the form

$$V = u(z)\,i + V(z)\,j \longrightarrow (9.2)$$

& where $\frac{\partial P}{\partial y} = 0$.

$\Rightarrow P(x, z)$.

The velocity field given by Eq. 9.2 satisfies the (D. form) continuity Eq. since the z-component of velocity w does not exist, and the other velocity components are solely functions of z.

$$\frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} = 0$$

The Navier-Stokes Eq. will be written as

$$\left(-\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{d^2 u}{dz^2}\right)i + \nu\frac{d^2 v}{dz^2}j$$

$$+\underbrace{\left(-g - \frac{1}{\rho}\frac{\partial P}{\partial z}\right)}_{\text{Body force within the control volume}}k = 0 \longrightarrow (9.3)$$

This is the Navier-Stokes Eq for the flow b/w two parallel plates.

The boundary conditions will be

$$\vec{V}(x,y,h) = Ui + Vj \longrightarrow (9.4a)$$

$$\vec{V}(x,y,-h) = U'i + V'j \longrightarrow (9.4b)$$

The z-component of linear momentum of Eq. (9.3) is

$$-g - \frac{1}{\rho}\frac{\partial p}{\partial z} = 0$$

$$\Rightarrow \quad \frac{\partial p}{\partial z} = -\rho g$$

On integrating

$$p = -\rho g z + f_1(x,y) \rightarrow 9.5$$

The x-components of linear momentum of Eq (9.3) is

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{d^2 u}{dz^2} = 0$$

$$\nu \frac{d^2 u}{dz^2} = \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \because \mu = \nu\rho$$

$$\mu \frac{d^2 u}{dz^2} = \frac{\partial p}{\partial x} = \frac{\partial f_1}{\partial x} \longrightarrow 9.6$$

Since the L.H.S of this Eq. is a function of z only & the R.H.S a function of x & y only, both sides must be equal to a constant G.

$$G = \frac{\partial \mathcal{P}}{\partial x} \longrightarrow (9.7a)$$

or $$G = \mu \frac{d^2 u}{dz^2} \longrightarrow (9.7b)$$

or $$G = \frac{\partial p}{\partial x} \longrightarrow (9.7c)$$

On Integrating

$$p = Gx + f_2(z) \longrightarrow (9.8)$$

Combining the two pressure expressions of Eqs (9.5) & (9.8) results in

$$p = Gx - \rho g z + p_0 \longrightarrow (9.9)$$

where p_0 is taken to be the pressure at the origin of our coordinate system.

x-component of linear momentum of Eq. (9.3) is

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$$-\frac{1}{\rho} \cdot \frac{\partial p}{\partial x} + \nu \frac{d^2 u}{dz^2} = 0$$

$$\nu \frac{d^2 u}{dz^2} = + \frac{1}{\rho} \cdot \frac{\partial p}{\partial x}$$

$$\because \mu = \rho \nu$$

$$\frac{d^2 u}{dz^2} = \frac{1}{\mu} G$$

On two time integrating.

$$u = \frac{G}{2\mu} z^2 + c_1 z + c_2 \longrightarrow (9.10)$$

where c_1 & c_2 are abitrary constants of integration.

By using

$u = U$ for $z = h$

$u = U'$ for $z = -h$

$$U = \frac{G}{2\mu} h^2 + c_1 h + c_2$$

&

$$U' = \frac{G}{2\mu} h^2 - c_1 h + c_2$$

$$U + U' = \frac{G}{\mu} h^2 + 2c_2$$

$$\boxed{c_2 = \frac{U+U'}{2} - \frac{G}{2\mu} h^2}$$

&

$$U - U' = 2hc_1$$

$$\boxed{c_1 = \frac{U-U'}{2h}}$$

So,

$$u = \frac{G}{2\mu} z^2 + \frac{U-U'}{2} z + \frac{U+U'}{2} - \frac{G}{2\mu} h^2 \longrightarrow 9.12$$

The y-component of Eq. 8.3

is
$$\nu \frac{d^2 v}{dz^2} = 0$$

as $\nu \neq 0$, so, $\frac{d^2 v}{dz^2} = 0$

On two time integration.

$$v = C_3 z + C_4 \longrightarrow (9.11)$$

By using

$v = V$ for $z = h$

$v = V'$ for $z = -h$

$$V = C_3 h + C_4$$

& $V' = -C_3 h + C_4$

$$V - V' = 2 C_3 h$$

$$\boxed{C_3 = \frac{V - V'}{2h}}$$

& $V + V' = 2 C_4$

$$\Rightarrow \boxed{C_4 = \frac{V + V'}{2}}$$

So,
$$v = \frac{V - V'}{2h} z + \frac{V + V'}{2} \longrightarrow (9.13)$$

The velocity components as

given by Eqs (9.14) & (9.13) and the pressure field as given by Eq 9.9 are a complete closed form analytic solution of the Navier-Stokes Eq.

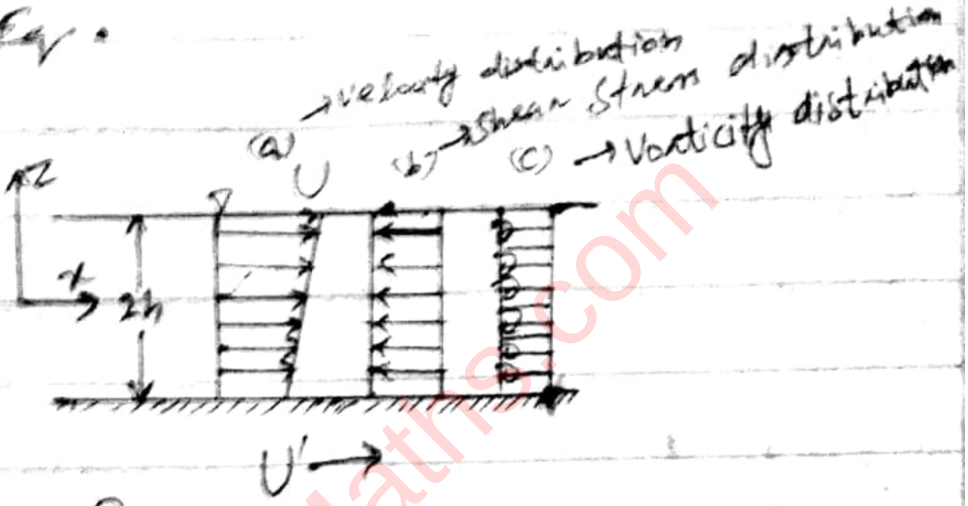


Fig (9.2)

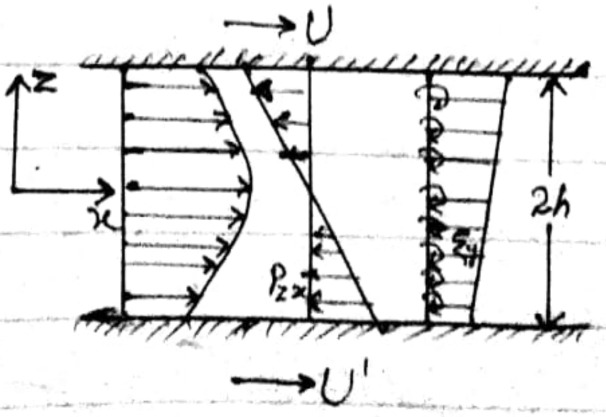
The velocity field given in Eq (9.2) resulted in the convective acceleration vanishing. This linearized the Navier-Stokes equation as well as decoupled the three resultant scalar equations which allowed a simple integration of each scalar form of the Navier-Stokes equation. Physically, the decoupling of the variables means that the flow along the x-axis does not influence the flow along the y-axis and vice versa.

Hence we can look at the velocity profiles independently.

Figure 9.2 depicts the velocity profile, shear stress distribution and vorticity distribution in the xz-plane for the case where the upper plate is the free-surface moving at velocity U and the lower plate moves at velocity U'. When the flow along the x-axis has no pressure gradient, we call the flow uniform shear flow or Couette flow.

Note that the normal stress distribution P_{zx} & vorticity ξ_y are constant throughout the flow.

Now, consider a flow that has no free-surface.



Poiseuille Flow $\frac{dP}{dx} \neq 0$

Here we are viewing the velocity profile, shear stress distribution and vorticity distribution in the xz-plane for $U > U' > 0$. Notice that the velocity profile in this case is parabolic owing to the existence of the pressure gradient. The shear stress & the vorticity distribution across the flow are linear, since for laminar flows they are always one order lower than the velocity distribution.

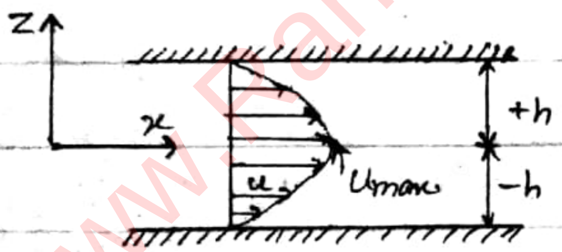


Fig. 9.4 (a) $U = U' = 0$

$$G = \frac{dp}{dx} < 0$$

In figure 9.4 (a) the flow is symmetric about the x-axis, because both upper and lower plates are at rest. The characteristic of this flow is that the pressure decreases along

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the direction of flow. Without this pressure drop, the flow would not exist. Also the maximum velocity is at the center of the channel.

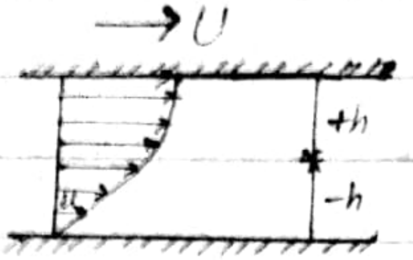


Fig 9.4(b)

$$G = \frac{dP}{dx} = -\frac{\mu U}{2h^2}$$

In figure 9.4(b) the lower plate is at rest and the upper plate is moving at uniform velocity U. The characteristic of this flow is that the pressure decreases along the direction of flow with the specific value $\frac{dP}{dx} = -\left(\frac{U\mu}{2h^2}\right)$. For this flow, the maximum velocity occurs at the top plate.

at $U' = 0$

$$\text{Since } u = \frac{G}{2\mu}z^2 + \frac{U}{2h}z + \frac{U}{2} - \frac{Gh^2}{2\mu}$$

$$\frac{du}{dz} = \frac{G}{\mu}z + \frac{U}{2h}$$

$$\text{Put } \frac{du}{dz} = 0$$

$$0 = \frac{Gz}{\mu} + \frac{U}{2h}$$

$$\frac{Gz}{\mu} = -\frac{U}{2h}$$

$$G = \frac{-\mu U}{2hz}$$

$$\frac{\partial p}{\partial x} = \frac{-\mu U}{2hz} \longrightarrow *$$

put $z = h$

$$\frac{\partial p}{\partial x} = \frac{-\mu U}{2h^2}$$

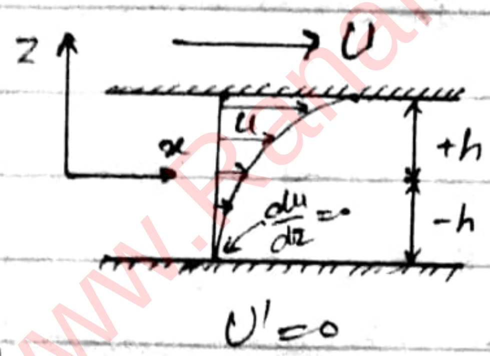


Fig. 9.4 (c)

$$G = \frac{dp}{dx} = \frac{\mu U}{2h^2}$$

Same as above we put $z = -h$ in Eq(*)

$$G = \frac{dp}{dx} = \frac{\mu U}{2h^2}$$

Fig. 9.4 (c) the pressure increases along the direction of flow with the pressure gradient $\frac{dp}{dx} = \frac{U\mu}{2h^2}$. This unique value of the pressure gradient causes both the velocity u & its gradient $\frac{du}{dz}$ to vanish at the lower stationary plate.

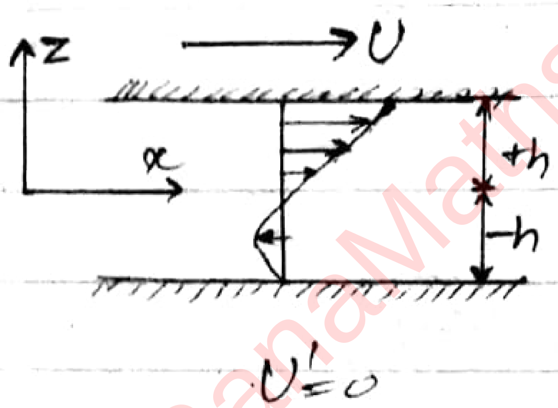


Fig. 9.4(d) $G = \frac{dp}{dx} > \frac{\mu U}{2h^2}$.

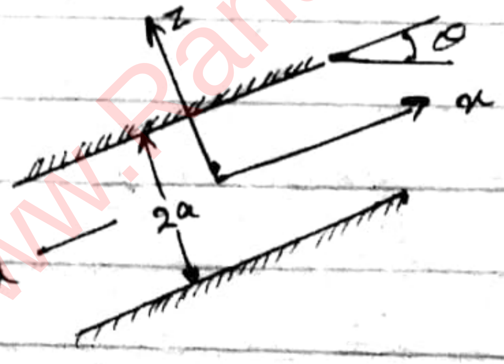
In Fig. 9.4(d) the pressure increases along the direction of flow causing a back flow like a wake near the lower stationary plate.

If P is +ve ⇒ back flow.

If P is -ve ⇒ developed flow.

Example 9.1

consider two fixed parallel flat plates that are a distance $2a$ apart. Let the plates be inclined at an angle θ and have a fluid flowing between them of viscosity μ at a sufficiently low velocity so that the flow can be assumed laminar. Express the angle θ of the slope in terms of viscosity, flow rate Q/L, specific weight γ, and distance 'a', if the pressure gradient $\frac{\partial p}{\partial s}$ is zero.



Solution:- The fluid is viscous & incompressible. The flow is steady, laminar and one-dimensional.

Since

$$u = \frac{Gz^2}{2\mu} + \frac{U-U'}{2h}z + \frac{U+U'}{2} - \frac{Gh^2}{2\mu} \longrightarrow \text{(i)}$$

&

$$Q = \int v \cdot dA \longrightarrow \text{(ii)}$$

Applying the assumptions used in Fig. 9.4(a) (i.e $U = U' = 0$ & $h = a$) we have

$$u = \frac{G z^2}{2\mu} - \frac{G a^2}{2\mu}$$

$$u = \frac{G}{2\mu}(z^2 - a^2) \longrightarrow \text{(iii)}$$

Where

$$G = -\frac{\partial(p + \gamma z)}{\partial s} \longrightarrow \text{(iv)}$$

The volume rate of flow per unit length L is

$$\frac{Q}{L} = \int_{-a}^{a} u \, dz$$

$$= \frac{G}{2\mu} \int_{-a}^{a} (z^2 - a^2) \, dz$$

$$= \frac{G}{2\mu} \left(\frac{z^3}{3} - a^2 z \right) \Big|_{-a}^{a}$$

$$= \frac{G}{2\mu} \left[\frac{a^3}{3} - a^3 + \frac{(a)^3}{3} - a^3 \right]$$

$$= \frac{G}{2\mu} \left[\frac{2a^3}{3} - 2a^3 \right]$$

$$= \frac{1}{\cancel{2}\mu} \left[-\frac{\partial(p + \gamma z)}{\partial s} \right] \left(-\frac{\cancel{4}^{2}}{3} a^3 \right)$$

If the pressure gradient $\frac{\partial P}{\partial s}$ is zero then

$$\frac{Q}{L} = \frac{2a^3}{3\mu} \frac{\partial(\gamma z)}{\partial s}$$

$$\frac{\partial(\gamma z)}{\partial s} = \frac{3\mu Q}{2a^3 L} \longrightarrow \text{(vii)}$$

Such that the slope θ becomes

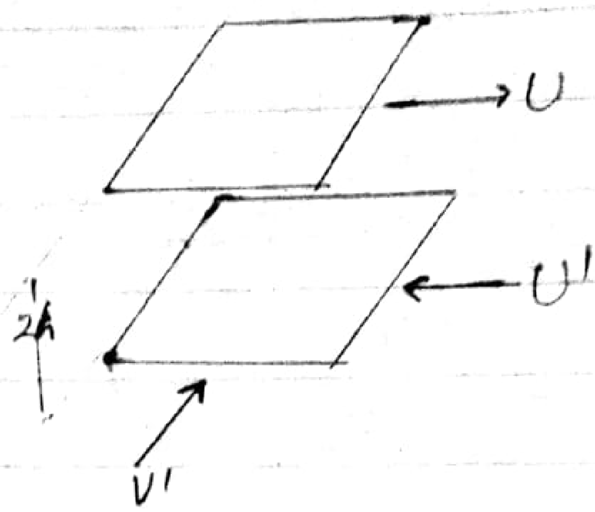
$$\theta = \frac{\partial z}{\partial s} = \frac{3\mu Q}{2a^3 L}$$

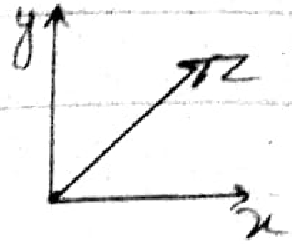
Example 9.2 29-05-15

Two parallel flat plates are a distance $2h$ apart. The top plate is moving with a velocity U to the right & the lower plate is moving with a velocity U' to the left & V' in the positive y-direction as shown in Fig.

(a):- Calculate the location & the value of the maximum velocity of the flow between the two plates &

b):- Calculate the distribution of vorticity ξ.





Solution:-

The fluid is viscous & incompressible. The flow is steady, laminar and two-dimensional.

Since

$$u = \frac{G}{2\mu}(z^2 - h^2) + \frac{U+U'}{2h} z + \frac{U-U'}{2} \longrightarrow (i)$$

$$v = -\frac{V'}{2h} z + \frac{V'}{2} \longrightarrow (ii) \quad \because V = 0$$

a:- The maximum velocity u_{max} is found by taking the derivative of Eq.(i) and setting the result equal to zero.

$$\frac{\partial u}{\partial z} = \frac{Gz}{\mu} + \frac{U+U'}{2h} = 0 \longrightarrow (iii)$$

$$\frac{Gz}{\mu} = -\frac{U+U'}{2h}$$

$$z = -\mu\left(\frac{U+U'}{2hG}\right) \longrightarrow (iv)$$

Which is where the velocity component u is a maximum. Put Eq. (iv) in (i).

$$u_{max} = \frac{G}{2\mu}\left[\mu^2\left(\frac{U+U'}{2hG}\right)^2 - h^2\right] - \frac{\mu}{G}\left(\frac{U+U'}{2h}\right)^2 + \frac{U-U'}{2} \longrightarrow (V)$$

is the expression for the max. velocity components in the x-direction.

The maximum value of the y-component of velocity v occurs at $z=-h$ and thus

eq. (ii) $\Rightarrow$ $$v_{max} = -\frac{V'}{2h}(-h) + \frac{V'}{2}$$

$$v_{max} = \frac{V'}{2} + \frac{V'}{2}$$

$$v_{max} = V'$$

b):- The vorticity ξ_y is quite easy to obtain:

$$\xi_y = \frac{\partial u}{\partial z}$$

$$\xi_y = \frac{Gz}{\mu} + \frac{U+U'}{2h}$$

The vorticity ξ_x is

$$\xi_x = -\frac{\partial V}{\partial z}$$

$$\xi_x = \frac{V'}{2h}$$

Hence, the vorticity distribution is

$$\xi = \xi_x \hat{i} + \xi_y \hat{j}$$

$$= \frac{V'}{2h}\hat{i} + \left(\frac{Gz}{\mu} + \frac{U+U'}{2h}\right)\hat{j}$$

Temperature Distribution for Couette and Poiseuille Flows

Once we know the velocity for couette or Poiseuille flow, the temperature distribution may be calculated using the energy Eq. If we have no heat transfer and there are constant values of thermal conductivity and specific heats for steady incompressible flow, then we can express the energy Eq. as

$$\rho C_v \frac{DT}{Dt} = k\nabla^2 T + \Phi \longrightarrow (9.14)$$

If there is one dimensional flow,

$$\rho C_v u \frac{\partial T}{\partial x} = K\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right) + \mu\left(\frac{\partial u}{\partial z}\right)^2 \longrightarrow (9.15)$$

Further if we assume that the boundary plates maintain constant temperature then Eq. (9.15) reduces to

$$K \frac{\partial^2 T}{\partial z^2} + \mu\left(\frac{\partial u}{\partial z}\right)^2 = 0 \longrightarrow (9.16)$$

$$\because \frac{\partial T}{\partial x} = 0$$

If we know u then we can find $T(z)$.

Couette Flow Temperature Distribution

Couette flow is defined as $G=0$.

Since $u = \frac{Gz^2}{2\mu} + \frac{U-U'}{2h}z + \frac{U+U'}{2} - \frac{Gh^2}{2\mu}$

Put $G=0$

$$\Rightarrow u = \frac{U-U'}{2h}z + \frac{U+U'}{2}$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{U-U'}{2h}$$

Put in Eq. (9.16).

$$K \frac{d^2 T}{dz^2} = -\mu\left(\frac{U-U'}{2h}\right)^2$$

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On integrating two times we get

$$T(z) = -\mu \frac{(U-U')^2}{8h^2 K} z^2 + c_1 z + c_2 \longrightarrow (i)$$

Using the boundary conditions.

$$T = T_w \quad at \quad z = -h$$

$$T_w = -\mu \frac{(U-U')^2}{8\cancel{h^2} K} \cancel{h^2} + c_1(-h) + c_2$$

$$T_w = -\mu \frac{(U-U')^2}{8K} - c_1 h + c_2 \longrightarrow (ii)$$

$T = T_1$ at $z = +h$

$$T_1 = -\mu \frac{(U-U')^2}{8\cancel{h^2} K} \cancel{h^2} + c_1 h + c_2$$

$$T_1 = -\mu \frac{(U-U')^2}{8K} + c_1 h + c_2 \longrightarrow (iii)$$

By adding (ii) & (iii).

$$T_w + T_1 = -2\mu \frac{(U-U')^2}{8K} + 2c_2$$

$$\Rightarrow \boxed{c_2 = \frac{T_w + T_1}{2} + \mu \frac{(U-U')^2}{8K}}$$

By subtracting (iii) from (ii)

$$T_w - T_1 = -2c_1 h$$

$$\Rightarrow \boxed{c_1 = \frac{T_1 - T_w}{2h}}$$

So,

$$T(z) = \frac{T_w + T_1}{2} + \mu\frac{(U-U')^2}{8K} + \frac{T_1 - T_w}{2h}z - \mu\frac{(U-U')^2}{8h^2K}z^2$$

It is sometimes worthwhile to investigate whether there is a cooling or heating of the flow, which can be easily checked by examining the sign of the temperature gradient $\frac{dT}{dz}$:

$$\frac{dT}{dz} = \frac{T_1 - T_w}{2h} - \mu\frac{(U-U')^2}{4h^2K}z$$

The sign of the heat transfer depends on whether $T_1 > T_w$ or $T_1 < T_w$ as well as

$$\frac{T_1 - T_w}{2h} > \mu\frac{(U-U')^2}{4h^2K}z$$

$$\Rightarrow T_1 - T_w > \mu\frac{(U-U')^2}{2hK}z$$

Remember that z may +ive or -ve, depending upon which plate has heat transfer

Poiseuille Flow Temperature Distribution.

Here $G \neq 0 \Rightarrow \frac{dP}{dx} \neq 0$

Since $K\frac{\partial^2 T}{\partial z^2} + \mu\left(\frac{\partial u}{\partial z}\right)^2 = 0$

$$\Rightarrow K\frac{\partial^2 T}{\partial z^2} = -\mu\left(\frac{\partial u}{\partial z}\right)^2 \longrightarrow (A)$$

We know

$$u = \frac{GZ^2}{2\mu} + \frac{U-U'}{2h}z + \frac{U+U'}{2} - \frac{Gh^2}{2\mu}$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{GZ}{\mu} + \frac{U-U'}{2h}$$

$$\left(\frac{\partial u}{\partial z}\right)^2 = \frac{G^2Z^2}{\mu^2} + \frac{(U-U')^2}{4h^2} + \frac{G(U-U')Z}{\mu h}$$

Put in (A)

$$K\frac{\partial^2 T}{\partial z^2} = -\mu\left[\frac{G^2Z^2}{\mu^2} + \frac{G(U-U')}{\mu h}Z + \frac{(U-U')^2}{4h^2}\right]$$

$\partial \Rightarrow d$

$$\Rightarrow \frac{d^2T}{dz^2} = -\frac{G^2Z^2}{\mu K} - \frac{G(U-U')}{hK}Z - \frac{\mu(U-U')^2}{4h^2K} \longrightarrow \text{O.D.E}$$

On integrating twice, we get.

$$T(z) = -\frac{G^2z^4}{12\mu K} - \frac{G(U-U')z^3}{6hK} - \frac{\mu(U-U')^2}{8h^2K}z^2 + c_1z + c_2$$

Using Boundary conditions.

When $z=-h \Rightarrow T = T_w$

So,

$$T_w = -\frac{G^2h^4}{12\mu K} + \frac{G(U-U')h^3}{6hK} - \mu\frac{(U-U')^2\cancel{h^2}}{8\cancel{h^2}K} + c_1(-h) + c_2$$

$$\Rightarrow \quad T_w = -\frac{G^2h^4}{12K\mu} + \frac{G(U-U')}{6K}h^2 - \frac{\mu(U-U')^2}{8K}$$
$$-C_1h + C_2$$

When $z = +h \Rightarrow T = T_1$

So,
$$T_1 = -\frac{G^2h^4}{12\mu K} - \frac{G(U-U')h^{\cancel{3}\,2}}{6\cancel{h}K} - \frac{\mu(U-U')^2\cancel{h^2}}{8\cancel{h^2}K}$$
$$+C_1h + C_2$$

$$T_1 = -\frac{G^2h^4}{12K\mu} - \frac{G(U-U')h^2}{6K} - \mu\frac{(U-U')^2}{8K}$$
$$+C_1h + C_2$$

Adding T_1 & T_w $\Rightarrow$

$$T_1 + T_w = -\frac{G^2h^4}{6K\mu} - \mu\frac{(U-U')^2}{4K} + 2C_2$$

$$C_2 = \frac{T_1+T_w}{2} + \frac{G^2h^4}{12K\mu} + \frac{\mu(U-U')^2}{8K}$$

By Subtracting T_w from T_1.

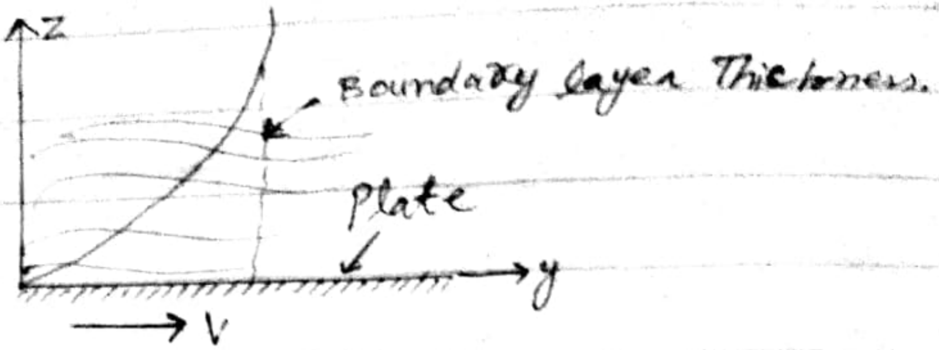
$$T_1 - T_w = -\frac{G(U-U')}{3K}h^2 + 2C_1h$$

$$C_1 = \frac{T_1-T_w}{2h} + \frac{G(U-U')h}{6K}$$

So,
$$T(z) = \frac{T_1+T_w}{2} + \frac{G^2h^2}{12K\mu} + \frac{\mu(U-U')^2}{8K} + \left[\frac{T_1-T_w}{2h} + \frac{G(U-U')h}{6K}\right]z - \mu\frac{(U-U')^2z^2}{8Kh^2}$$
$$-\frac{G(U-U')z^3}{6Kh} - \frac{G^2z^4}{12\mu K}$$

$T_1 = T_w$
$T > T_w$
$U = U'$

Suddenly Accelerated Flat Plate in a Viscous Fluid



Consider a semi-infinite flat plate immersed in a viscous fluid. The inertial coordinate system is shown in fig.

At time $t<0$ the plate is stationary. At $t=0$ the plate is suddenly given a velocity V in the positive y-direction. We seek the resultant velocity and pressure field in the fluid as a result of this sudden acceleration of the flat plate.

Clearly, the plate induces a velocity only in the y-direction, since the plate is assumed to have no have no appreciable thickness. Hence, the velocity components $u=w=0$ & $v=v(z,t)$. Such a flow is seen to satisfy the continuity equation. The Navier-Stokes Eq.s reduce to the simple form

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial z^2} \longrightarrow (9.24)$$

$$\frac{dP}{dz} = -\rho g \longrightarrow (9.25)$$

Our pressure is thus the hydrostatic pressure.

The boundary conditions are

$$v = v' \text{ at } z = 0 \text{ for } t \geq 0 \longrightarrow (9.26)$$

$$v \to 0 \text{ as } z \to \infty, \text{ for all } t. \longrightarrow (9.27)$$

Eq. (9.24) is a well-known linear P.D.E called the diffusion equation.

The diffusion Eq. is a parabolic P.D.E. and a number of analytic methods enable us to solve this equation. Thus we seek a solution of the form,

$$v = v(z, \nu, t) = V f(z^{k_1} \nu^{k_2} t^{k_3})$$

$$v = V f(\eta) \longrightarrow (9.28)$$

Where η is a dimensionless variable calle the similarity variable.

We consider finding a dimensionless parameter

$$\Pi_1 = z^{k_1} \nu^{k_2} t^{k_3}$$

such that $[\Pi_1] = L^{k_1} (L^2 T^{-1})^{k_2} (T)^{k_3} = F^0 L^0 T^0$

Equating exponents of like dimensions.

$L: \quad K_1 + 2K_2 = 0$

$\Rightarrow K_2 = \frac{-K_1}{2}$

also

$T: \quad -K_2 + K_3 = 0$

$\Rightarrow K_3 = K_2 = \frac{-K_1}{2}$

$9.28 \Rightarrow \quad \eta = A\left(z^{K_1} \nu^{K_2} t^{K_3}\right)$

Using $K_3 = K_2 = \frac{-K_1}{2}$

$\eta = A\left(z^{K_1} \nu^{-K_1/2} t^{-K_1/2}\right)$

$$\eta = A\left(\frac{z}{\sqrt{\nu t}}\right)^{K_1} \longrightarrow (9.32)$$

For convenience we take K_1 equal to unity and set $A = \frac{1}{2}$.

So,

$$\eta = \frac{1}{2} \cdot \frac{z}{\sqrt{\nu t}} \longrightarrow (9.33)$$

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial z^2} \longrightarrow (9.24)$$

$$v = V f(\eta)$$

$$\frac{\partial v}{\partial t} = V \frac{\partial f}{\partial \eta} \cdot \frac{\partial \eta}{\partial t}$$

$$\frac{\partial \eta}{\partial t} = \frac{1}{2}\left(-\frac{1}{2}(\nu t)^{-\frac{3}{2}}\nu\right) z$$

$$\frac{\partial \eta}{\partial t} = -\frac{1}{2}\cdot\left(\frac{1}{2}\frac{z}{\sqrt{\nu t}}\right)\left(\frac{1}{t}\right)$$

$$\frac{\partial \eta}{\partial t} = \frac{-\eta}{2t}$$

$$\Rightarrow \frac{\partial v}{\partial t} = \frac{-\eta V}{2t}\cdot\frac{\partial f}{\partial \eta}$$

Now $\quad v = V f(\eta)$

$$\frac{\partial v}{\partial z} = V \frac{\partial f}{\partial \eta}\cdot\frac{\partial \eta}{\partial z}$$

$\because \frac{\partial \eta}{\partial z} = \left(\frac{1}{2\sqrt{\nu t}}\right)$

$$\frac{\partial v}{\partial z} = V \frac{\partial f}{\partial \eta}\cdot\left(\frac{1}{2\sqrt{\nu t}}\right)$$

$$\frac{\partial v}{\partial z} = \frac{V}{2\sqrt{\nu t}}\cdot\frac{\partial f}{\partial \eta}$$

Again diff.

$$\frac{\partial^2 v}{\partial z^2} = \frac{V}{2\sqrt{\nu t}}\cdot\frac{\partial^2 f}{\partial \eta^2}\cdot\frac{\partial \eta}{\partial z}$$

$$\frac{\partial^2 v}{\partial z^2} = \frac{V}{2\sqrt{\nu t}}\cdot\frac{\partial^2 f}{\partial \eta^2}\cdot\frac{1}{2\sqrt{\nu t}}$$

$$\frac{\partial^2 v}{\partial z^2} = \frac{V}{4(\nu t)}\cdot\frac{\partial^2 f}{\partial \eta^2}$$

Eq. (9.24) $\Rightarrow$ $$\frac{-\eta \cancel{V}}{2\cancel{t}}\cdot\frac{\partial f}{\partial \eta} = \cancel{\nu}\,\frac{\cancel{V}}{4\cancel{\nu t}}\cdot\frac{\partial^2 f}{\partial \eta^2}$$

$$\Rightarrow -2\eta\frac{\partial f}{\partial \eta} = \frac{\partial^2 f}{\partial \eta^2}$$

$$\frac{d^2 f}{d\eta^2} + 2\eta \frac{df}{d\eta} = 0 \longrightarrow 7.34$$

What is the advantage of stream function formulation?

$\Rightarrow$ To convert the P.D.E into O.D.E to find the analytic or numerical solution of the phenomenon.

when $z=0 \Rightarrow v=V$

& then

$v = V f(\eta)$

$\Rightarrow V = V f(\eta)$

$\Rightarrow f = 1$

$z=0 \Rightarrow \eta = \frac{1}{2}\frac{z}{\sqrt{\nu t}} \Rightarrow \eta = 0$

So, $\boxed{f=1 \text{ at } \eta=0}$

also

$v \to 0$ as $z \to \infty$

$v = Vf$

$\Rightarrow 0 \leftarrow Vf$

$V \neq 0,\ f \to 0$

$\eta = \frac{1}{2}\frac{z}{\sqrt{\nu t}}$

as $z \to \infty$ then $\eta \to \infty$

So, $f \to 0$ as $\eta \to \infty$

To the differential Eq (9.24) Let

$$g(\eta) = \frac{d\theta}{d\eta}$$

$$\Rightarrow \frac{dg}{d\eta} + 2\eta g = 0$$

$$\Rightarrow \frac{1}{g} dg = -2\eta \, d\eta$$

On integrating

$$\ln g = -\eta^2 + \ln c_1$$

$$\Rightarrow g = c_1 e^{-\eta^2} = \frac{d\theta}{d\eta}$$

$$\Rightarrow \theta(\eta) = c_1 \int_0^{\eta} e^{-\eta^2} d\eta + c_2$$

Using Boundary conditions

$\theta = 1$ at $\eta = 0$

$$1 = 0 + c_2$$

$$\Rightarrow \boxed{c_2 = 1}$$

$\theta \to 0$ as $\eta \to \infty$ and $c_2 = 1$

$$\Rightarrow 0 = c_1 \int_0^{\infty} e^{-\eta^2} d\eta + 1$$

$$\Rightarrow \boxed{c_1 = \frac{-1}{\int_0^{\infty} e^{-\eta^2} d\eta}}$$

So,

$$f(\eta) = 1 - \frac{\int_0^\eta e^{-\eta^2}\,d\eta}{\int_0^\infty e^{-\eta^2}\,d\eta}$$

← Error function

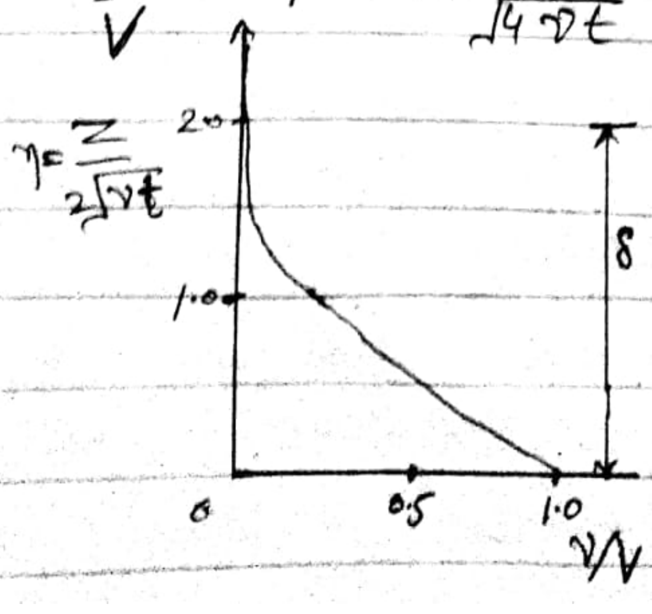
$$f(\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta^2}\,d\eta$$

$$\because \operatorname{erf} \eta = \frac{2}{\sqrt{\pi}} \int_0^\eta \exp(-\eta^2)\,d\eta$$

The ratio of the two integrals shown above is the well-know error function, erf η, found in probability theory. The velocity distribution for V can be obtained by substituting the similarity function $f(\eta)$ into the velocity expression of Eq. (9-28)

$$v = V\left(1 - \operatorname{erf} \frac{z}{\sqrt{4\nu t}}\right) \qquad \because \eta = \frac{z}{\sqrt{4\nu t}}$$

$$\frac{v}{V} = 1 - \operatorname{erf} \frac{z}{\sqrt{4\nu t}}$$



$$\eta = \frac{\delta}{2\sqrt{\nu t}} = 2 \longrightarrow (9.42)$$

$$\Rightarrow \delta = 4\sqrt{\nu t} \longrightarrow (9.43)$$

δ is the thickness.

Now we ask the question: How far past the plate will the fluid be influenced by the movement of the plate?

We can answer this question by setting $t = \frac{L}{V}$, which is the time it takes a fluid particle to go a distance L at a velocity V.

Thus

$$\delta = 4\sqrt{\nu \cdot \frac{L}{V}}$$

$$\delta = 4\sqrt{\frac{\nu}{LV} \cdot L^2}$$

$$\delta = 4L\sqrt{\frac{\nu}{LV}}$$

$$\delta = \frac{4L}{\sqrt{R_L}} \longrightarrow (9.44)$$

Chap#10 Laminar Pipe Flow or Internal Flow. 1-06-15

We will discuss the problems of transferring energy from one location to another by means of fluid flow through pipes. For this purpose we treat the differential form of pipe flow equations that help us to calculate the pressure & velocity distribution. Later we will discuss the integral form pipe flow equations that give us the Hagen–Poiseuille result, which expresses the head loss representing loss of energy in a pipe.

Description of the Physical Phenomenon

When fluid moves through a pipe, its behavior can be described by the Reynolds number

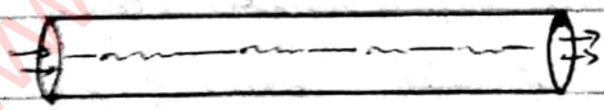
$$Re_D = \frac{\tilde{V} D}{\nu} \longrightarrow (10.1)$$

Where the characteristic length is the inner diameter D of the pipe, $\tilde{V}$ is the mean velocity of the flow, and ν is the kinematic viscosity of the fluid. Three basic types of flow are possible in the pipe, each possessing different characteristics of behaviour

Uniform flow or Laminar flow

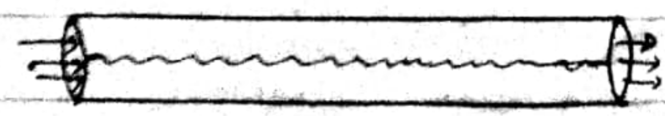
$100 \leq Re_D \leq 2300$

Laminar flow is one where the trajectory of a fluid particle is predictable, not random or unstable.



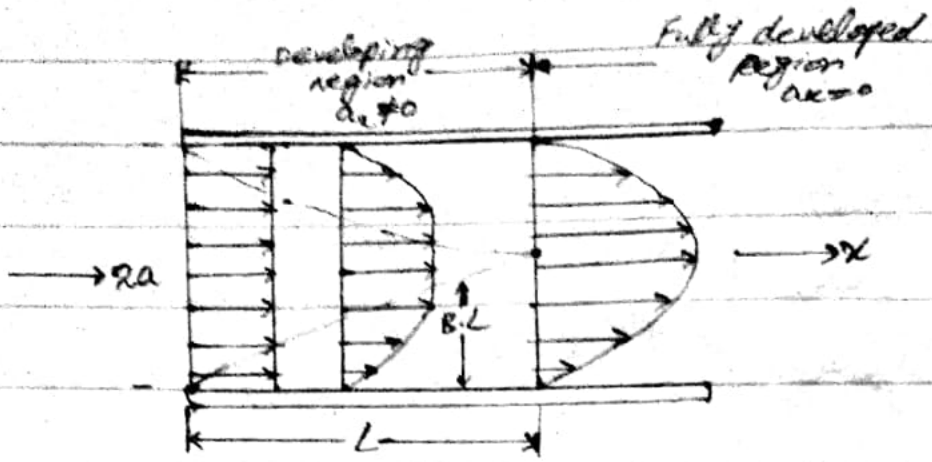
Transitional flow

$2300 \leq Re_D \leq 2500$



Turbulent flow.

$2500 \leq Re_D \leq 4400$



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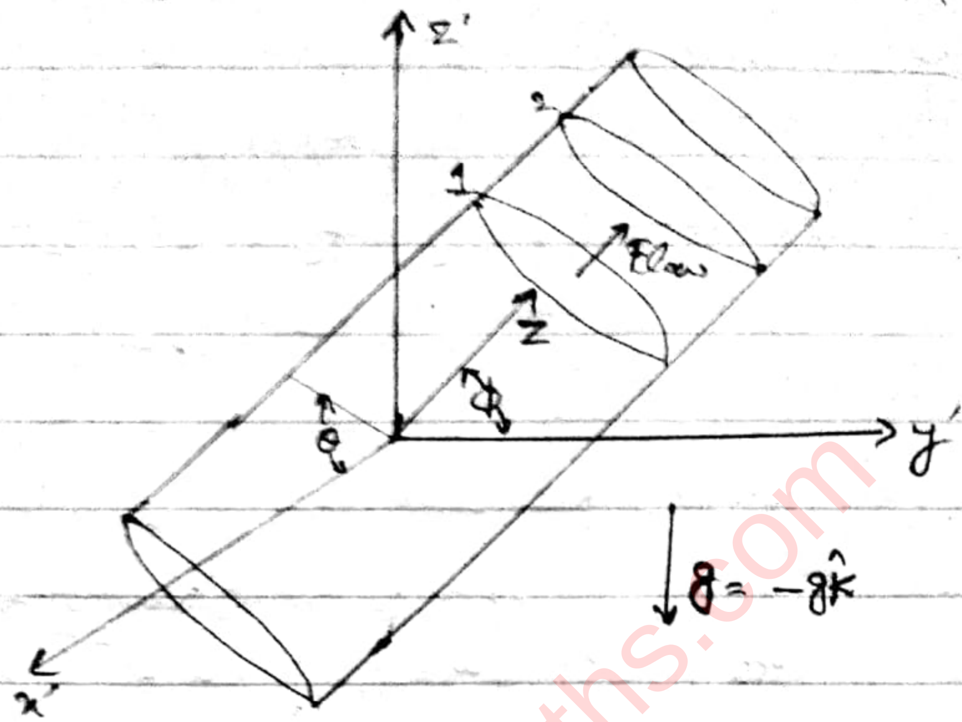
Equations of Motion for Laminar Flow in a Pipe

Statement:-

Let us consider a steady flow of viscous fluid through straight round pipe. Let us consider a pipe of inside radius "a" located in a gravitational field. The center line of the pipe lies in the $y'z'$-plane. We assume that fluid completely fills the pipe, allowing no cavities or free-surface in the pipe. Since the outer boundary of the flow is cylindrical, we will use cylindrical coordinates to describe the flow. A boundary condition of the flow is that there is no slip at the pipe wall:

$V_r = V_\theta = w = 0 \quad \text{at} \quad r = a \longrightarrow (10.5)$



The flow is assumed axisymmetric, fully developed and steady. Thus we seek solutions of the form

$$\left.\begin{aligned} V_\theta &= 0 \\ w &= w(r) \end{aligned}\right\} \longrightarrow (10.6)$$

It is obvious thi this flow field satisfies the Eq. of continuity.

The Navier-stokes Eq's written in cylindrical coordinates are

r-component is

$$\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} + w\frac{\partial V_r}{\partial z} = g_r - \frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left(\frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r}\frac{\partial V_r}{\partial r} - \frac{V_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial^2 V_r}{\partial z^2}\right)$$

θ-component:

$$\frac{\partial V_\theta}{\partial t} + V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} + \frac{W \partial V_\theta}{\partial z} = g_\theta - \frac{1}{r\rho}\frac{\partial p}{\partial \theta}$$

$$+ \nu\left(\frac{\partial^2 V_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial^2 V_\theta}{\partial z^2}\right)$$

Z-component

$$\frac{\partial W}{\partial t} + V_r \frac{\partial W}{\partial r} + \frac{V_\theta}{r}\cdot\frac{\partial W}{\partial \theta} + W\frac{\partial W}{\partial z} = g_z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$+ \nu\left(\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} + \frac{\partial^2 W}{\partial z^2}\right)$$

By using the Boundary conditions then

So

r-component $\to$ $g_r - \frac{1}{\rho}\cdot\frac{\partial p}{\partial r} = 0 \longrightarrow (10.7)$

θ-component $\to$

$$g_\theta - \frac{1}{r\rho}\frac{\partial p}{\partial \theta} = 0 \longrightarrow 10.8$$

Z-component

$$g_z - \frac{1}{\rho}\cdot\frac{\partial p}{\partial z} + \frac{\mu}{\rho r}\left(r\frac{\partial^2 W}{\partial r^2} + \frac{\partial W}{\partial r}\right) = 0$$

or $$g_z - \frac{1}{\rho}\cdot\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\cdot\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right) = 0 \to 10.9$$

Where

$$g = g_r e_r + g_\theta e_\theta + g_z \hat{k}$$

$$\Rightarrow g_r = -g\cos\phi \sin\theta \longrightarrow 10.10$$

$$g_\theta = -g \cos\phi \cos\theta \longrightarrow 10.11$$

$$g_z = -\frac{\partial}{\partial z}\, g \sin\phi\, z \longrightarrow 10.12$$

Put Eq.(10.10) into Eq.(10.7), we get

$$-g\cos\phi\sin\theta - \frac{1}{\rho}\cdot\frac{\partial p}{\partial r} = 0$$

$$\frac{\partial p}{\partial r} = -\rho g \cos\phi \sin\theta$$

On integrating,

$$p = -\rho g r \cos\phi \sin\theta + f_1(\theta, z) \rightarrow 10.13$$

Put Eq.(10.11) into Eq.(10.8), we get

$$-g\cos\phi\cos\theta - \frac{1}{r\rho}\cdot\frac{\partial p}{\partial \theta} = 0$$

$$\frac{\partial p}{\partial \theta} = -r\rho g\cos\phi \cos\theta$$

On integrating,

$$p = -r\rho g\cos\phi \sin\theta + f_2(r, z) \rightarrow (10.14)$$

By comparing Eq (10.13) with Eq (10.14)

$$p = -r\rho g\cos\phi\sin\theta + f(z) \rightarrow (10.15)$$

Put Eq.(10.12) & (10.15) in Eq.(10.9)

Eq(10.9) $\Rightarrow$ $$-g\sin\phi - \frac{1}{\rho}\frac{df}{dz} + \frac{\mu}{\rho}\cdot\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = 0$$

$\times$ing ρ $$\Rightarrow \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = -\left(-g\rho\sin\phi - \frac{df}{dz}\right)$$

$$\Rightarrow \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = -G \longrightarrow (10.16)$$

where $G = -\rho g \sin\phi - \frac{dp}{dz} \longrightarrow (10.17)$

Eq.(10.16) $\Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = -\frac{G r}{\mu}$

On integrating,

$$r\frac{\partial w}{\partial r} = -\frac{G r^2}{2\mu} + c_1$$

$$\frac{\partial w}{\partial r} = -\frac{G r}{2\mu} + \frac{c_1}{r}$$

On integrating.

$$w = -\frac{G r^2}{4\mu} + c_1 \ln r + c_2 \longrightarrow (10.18)$$

Integrating Eq. (10.17) w.r.t z.

$$p(z) = -G z - \rho g \sin\phi\, z + c_3 \longrightarrow (10.19)$$

Note that in Eq. (10.18), c_1 must be zero otherwise the velocity w would become infinite at the center of the pipe. The constant $c_2 = \frac{G a^2}{4\mu}$ at boundary condition $w(a) = 0$.

then $w = -\frac{G r^2}{4\mu} + 0 + \frac{G a^2}{4\mu}$

$$w = \frac{G a^2}{4\mu}\left(1 - \frac{r^2}{a^2}\right) \longrightarrow (10.20)$$

at $r=0$, $w = w_{max}$

$$w_{max} = \frac{Ga^2}{4\mu}$$

$$\Rightarrow \quad G = \frac{4\mu w_{max}}{a^2}$$

$\because D = 2a$, $a = \frac{D}{2}$

$$G = \frac{16\mu w_{max}}{D^2} \longrightarrow (10.21)$$

05-06-2015

The flow rate Q is evaluated as

$$Q = A\bar{w}$$

$$G = \pi a^2 \frac{Ga^2}{8\mu} \qquad \because \bar{w} = \frac{w_{max}}{2}$$

$$G = \frac{\pi G a^4}{8\mu} \longrightarrow (10.22)$$

Now put Eq. (10.18) into Eq. (10.15)

$$P = -\rho g r \cos\phi \sin\theta - GZ - \rho g \sin\phi Z + C_3 \rightarrow (10.24)$$

Using Eq. (10.15) in Eq. (10.17)

$$(10.17) \Rightarrow \quad -G = \rho g \sin\phi + \frac{\partial P}{\partial Z}$$

$$-G = \frac{\partial}{\partial Z}(\rho g \sin\phi Z + P) \longrightarrow (10.25)$$

Integrating the above Eq. along the axial extent of the pipe from station

1 to 2.

$$-Gz\Big|_{z_1}^{z_2} = \rho g \sin\phi \Big|_{z_1}^{z_2} + P\Big|_{P_1}^{P_2}$$

$$-G(z_2 - z_1) = \rho g \sin\phi (z_2 - z_1) + (P_2 - P_1)$$

$$G(z_2 - z_1) = -\rho g \sin\phi\, z_2 + \rho g \sin\phi\, z_1 - P_2 + P_1$$

$$GL = (P_1 + \rho g \sin\phi\, z_1) - (P_2 + \rho g \sin\phi\, z_2)$$

where $L = z_2 - z_1$

Put $z' = z \sin\phi$, $\gamma = \rho g$

$$G = \frac{16\mu W_{max}}{D^2} = \frac{32\mu\bar{\omega}}{D^2} \qquad \because \bar{\omega} = \frac{W_{max}}{2}$$

$$\Rightarrow \frac{32\mu\bar{\omega}}{D^2} L = (P_1 + \gamma z_1') - (P_2 + \gamma z_2') \longrightarrow (10.28)$$

This result is the famous Hagen-Poiseuille expression for the pressure drop in a pipe due to viscosity. It is however applicable only for laminar flow. Note that there is no obstruction in the pipe between stations 1 & 2 in the Eq. The Eq. also assume that the pipe is of constant diameter.

The Hagen-Poiseuill equation is a simple statement that the loss in the energy per unit volume h_f between Stations 1 & 2 is given by.

$$h_f = \frac{32 \mu \bar{w} L}{D^2 \gamma} \longrightarrow (10.29)$$

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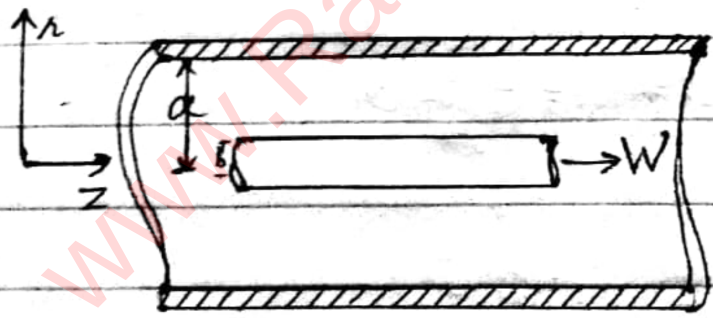
Example 10.1

Consider fluid of kinematic viscosity ν between two circular pipes shown in figure



Let the outer cylinder of radius "a" be stationary and the inner cylinder of radius "b" moves in the positive z-direction with uniform velocity W. Determine a) the radial distribution of axial velocity component w;

b):- The shear stress at the outer wall $r=a$ & the shear stress at the inner wall $r=b$.

c):- The average velocity W.

d):- The maximum velocity w_{max}.

Solution:-

The fluid is viscous & incompressible. The flow is steady & full established.

- Linear momentum:

$$\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\cdot\frac{\partial w}{\partial r}\right)=\frac{1}{\mu}\cdot\frac{\partial P}{\partial z}\longrightarrow(i)$$

$$\frac{\partial}{\partial r}\left(r\cdot\frac{\partial w}{\partial r}\right)=-\frac{r}{\mu}G \qquad \because G=\frac{\partial P}{\partial z}\rightarrow(iii)$$

Integrating

$$r\frac{\partial w}{\partial r}=-\frac{Gr^2}{2\mu}+c_1$$

$$\frac{\partial w}{\partial r}=-\frac{Gr}{2\mu}+\frac{c_1}{r}$$

Again integrating

$$w=-\frac{Gr^2}{4\mu}+c_1\ln r+c_2\longrightarrow(ii)$$

Using Boundary conditions

$$w=W \text{ at } r=b\longrightarrow(iv)$$

$$\Rightarrow \quad W = -\frac{Gb^2}{4\mu} + C_1 \ln b + C_2 \longrightarrow (a)$$

at $r = a \Rightarrow W = 0 \longrightarrow (V)$

$$0 = -\frac{Ga^2}{4\mu} + C_1 \ln a + C_2 \longrightarrow (b)$$

By (a) $-$ (b)

$$W = -\frac{Gb^2}{4\mu} + C_1 \ln b + \frac{Ga^2}{4\mu} - C_1 \ln a$$

$C_1 \ln a - C_1 \ln b =$

Using the values of c_1 & c_2 in Eq.(ii):- $\because G = \frac{\partial P}{\partial z}$

$$w = \frac{r^2}{4\mu}\frac{\partial P}{\partial z} + \frac{\ln r}{\ln(a/b)}\left[\mu W - \frac{1}{4}\frac{\partial P}{\partial z}(a^2-b^2)\right]$$

$$- \frac{b^2}{4}\frac{\partial P}{\partial z} - \frac{\ln b}{\ln(a/b)}\left[\mu W - \frac{1}{4}\frac{\partial P}{\partial z}(a^2-b^2)\right]$$

$$w = \frac{1}{4}\frac{\partial P}{\partial z}\left[\frac{r^2}{\mu} - \frac{\ln r}{\ln(\frac{a}{b})}(a^2-b^2) - b^2 + \frac{\ln b}{\ln(\frac{a}{b})}(a^2-b^2)\right]$$

$$+ \frac{\mu W}{\ln(\frac{a}{b})}\ln r - \frac{\mu W}{\ln(a/b)}\ln b$$

$$w = \frac{1}{4}\frac{\partial P}{\partial z}\left[\frac{r^2}{\mu} - \frac{(a^2-b^2)\ln r}{\ln(\frac{a}{b})} - b^2 + \frac{(a^2-b^2)}{\ln(\frac{a}{b})}\ln b\right]$$

$$+ \frac{\mu W}{\ln(\frac{a}{b})}\ln\left(\frac{r}{b}\right) \longrightarrow (viii)$$

b):- We evaluate shear stress P_{rz} at the outer wall $r=a$ using

$$P_{rz}\Big|_{r=a} = \mu\frac{\partial w}{\partial r}\Big|_{r=a} \longrightarrow (ix)$$

Since $w = \frac{r^2}{4\mu}\frac{\partial P}{\partial z} + c_1\ln r + c_2$

$$\Rightarrow \frac{\partial w}{\partial r} = \frac{r}{2\mu}\frac{\partial P}{\partial z} + \frac{c_1}{r}$$

So, $$P_{rz}\Big|_{r=a} = \left(\frac{c_1}{r} + \frac{r}{2}\frac{\partial P}{\partial z}\right)\Bigg|_{r=a}$$

$$\Rightarrow P_{rz}\Big|_{r=a} = \frac{C_1}{a} + \frac{a}{2}\frac{\partial P}{\partial z} \longrightarrow (x)$$

Similarly for $r=b$

$$P_{rz}\Big|_{r=b} = \frac{C_1}{b} + \frac{b}{2}\cdot\frac{\partial P}{\partial z} \longrightarrow (xii)$$

C):- The one-dimensional (I.F) continuity equation is

$$Q = \bar{V}A = \int_0^{2\pi}\int_0^b w r\,dr\,d\theta \longrightarrow (xiii)$$

Substituting the velocity distribution for w of Eq (ii) into Eq. (xiii) & integrating result in

$$\bar{V} = \frac{1}{8\mu}\cdot\frac{\partial P}{\partial z}\left[a^2+b^2+\frac{(a^2-b^2)}{\ln b/a}\right]$$

d):- The max. velocity is where $\frac{\partial w}{\partial r}=0$

so,
$$\frac{\partial w}{\partial r} = \frac{\partial P}{\partial z}\cdot\frac{r}{2\mu} + \frac{C_1}{r} = 0$$

~~xing~~ we get

$$r = \sqrt{\frac{2C_1}{\mu\left(\partial P/\partial z\right)}}$$

then eq-(ii) $\Rightarrow$

$$w_{max} = \frac{C_1}{2\mu}\left[1+\ln 2C_1 - \ln\mu\left(\frac{\partial P}{\partial z}\right)\right]$$

08-06-2015

Example 1st

Consider fluid confined between two cylinders as

The radius of the inner stationary cylinder is 'a' and the radius of the outer cylinder is "b". The outer cylinder rotates at a uniform circumferential velocity U.

Let the fluid between two cylinders be steady, axisymmetric and invariant in the axial direction, so that the fluid motion is purely circular.

Determine

a): The velocity components v_r, v_θ & w.

b): The shear stresses on the inner & outer walls.

Solution:

The fluid is viscous & incompressible. The flow is steady, axisymmetric & fully established.

The Navier-Stokes Eqns written in cylindrical coordinates are;

z-component:

$$\frac{\partial w}{\partial t}+V_r\frac{\partial w}{\partial r}+\frac{V_\theta}{r}\frac{\partial w}{\partial \theta}+w\frac{\partial w}{\partial z}=g_z-\frac{1}{\rho}\frac{\partial p}{\partial z}$$
$$+\nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}+\frac{\partial^2 w}{\partial z^2}\right)\longrightarrow(i)$$

θ-component:-

$$\frac{\partial V_\theta}{\partial t}+V_r\frac{\partial V_\theta}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta}+\frac{V_r V_\theta}{r}+w\frac{\partial V_\theta}{\partial z}$$
$$=g_\theta-\frac{1}{\rho r}\frac{\partial p}{\partial \theta}+\nu\left(\frac{\partial^2 V_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial V_\theta}{\partial r}-\frac{V_\theta}{r^2}+\frac{1}{r}\frac{\partial^2 V_\theta}{\partial \theta^2}\right.$$
$$\left.+\frac{2}{r^2}\frac{\partial V_r}{\partial \theta}+\frac{\partial^2 V_\theta}{\partial z^2}\right)\longrightarrow(ii)$$

Eq. of Continuity:

$$\frac{\partial V_r}{\partial r}+\frac{V_r}{r}+\frac{1}{r}\frac{\partial V_\theta}{\partial \theta}+\frac{\partial w}{\partial z}=0\longrightarrow(iii)$$

a):- For a flow to be invariant in the axial direction.

$$\frac{\partial}{\partial z}=0\longrightarrow(iv)$$

and for a flow to be axisymmetric

$$\frac{\partial}{\partial \theta}=0\longrightarrow(v)$$

Eq. (iii) becomes

$$\frac{\partial V_r}{\partial r} + \frac{V_r}{r} = 0$$

$$\frac{1}{r}\left[r\frac{\partial V_r}{\partial r} + V_r\right] = 0$$

$\frac{1}{r} \neq 0$, $\quad r\frac{\partial V_r}{\partial r} + V_r = 0$

$$\frac{\partial}{\partial r}(rV_r) = 0 \longrightarrow (vi)$$

On integrating,

$$rV_r = C_1$$

$$\Rightarrow V_r = \frac{C_1}{r} \longrightarrow (vii)$$

at $r = b \Rightarrow V_r = 0$

So, $\quad C_1 = 0$

$$\Rightarrow V_r = 0 \longrightarrow (viii)$$

Now

Eq (i) implies that

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial w}{\partial r} = 0$$

$$\frac{1}{r}\left[r\frac{\partial^2 w}{\partial r^2} + \frac{\partial w}{\partial r}\right] = 0$$

as $\frac{1}{r} \neq 0$ then $\quad r\frac{\partial^2 w}{\partial r^2} + \frac{\partial w}{\partial r} = 0$

$$\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = 0 \longrightarrow (ix)$$

On integrating

$$r \frac{\partial w}{\partial r} = C_2$$

$$\frac{\partial w}{\partial r} = \frac{C_2}{r}$$

again $w = C_2 \ln r + C_3 \longrightarrow (x)$

$w = 0$ at $r = a \longrightarrow (xi)$

$\Rightarrow 0 = C_2 \ln a + C_3 \longrightarrow (a)$

$w = 0$ at $r = b \longrightarrow (xii)$

$0 = C_2 \ln b + C_3 \longrightarrow (b)$

by Eq. (a) − Eq. (b)

$$0 = C_2(\ln a - \ln b)$$

$\Rightarrow C_2 \ln\left(\frac{a}{b}\right) = 0 \longrightarrow (xiii)$

as $a \neq b$ so $\ln\left(\frac{a}{b}\right) \neq 0$

then $\boxed{C_2 = 0}$

Eq. (a) $\Rightarrow 0 = 0 + C_3$

$\Rightarrow \boxed{C_3 = 0}$

So

$w = 0 \longrightarrow (xiv)$

Now Eq. (ii) $\Rightarrow$

$$\frac{\partial^2 V_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r^2} = 0$$

$\times$ing $r \Rightarrow r\frac{\partial^2 V_\theta}{\partial r^2} + \frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r} = 0$

$$\frac{\partial}{\partial r}\left(r\frac{\partial V_\theta}{\partial r}\right) = \frac{V_\theta}{r} \longrightarrow (xv)$$

$$\Rightarrow \quad V_\theta = \frac{c_4}{r} + c_5 r \longrightarrow (xvi)$$

$$V_\theta = 0 \quad at \quad r = a \longrightarrow (xvii)$$

$$\Rightarrow \quad 0 = \frac{c_4}{a} + c_5 a \longrightarrow (a)$$

$$V_\theta = U \quad at \quad r = b \longrightarrow (xviii)$$

$$U = \frac{c_4}{b} + c_5 b \longrightarrow (b)$$

$$\Rightarrow \frac{c_4}{b} = U - c_5 b$$

$$\Rightarrow c_4 = bU - c_5 b^2 \longrightarrow (c)$$

Put in eq (a)

$$0 = \frac{1}{a}(bU - c_5 b^2) + c_5 a$$

xing by a.

$$0 = bU - c_5 b^2 + c_5 a^2$$

$$-bU = c_5(a^2 - b^2)$$

$$\frac{-bU}{a^2-b^2} = c_5 \longrightarrow (xx)$$

put in (c).

$$c_4 = bU + \frac{b^3 U}{a^2-b^2}$$

$$c_4 = \frac{a^2bU - \cancel{b^3U} + \cancel{b^3U}}{a^2-b} = \frac{a^2bU}{a^2-b^2} \longrightarrow (xx)$$

So

$$V_\theta = \frac{1}{r}\frac{a^2bU}{a^2-b^2} + r\left(-\frac{bU}{a^2-b^2}\right)$$

$$V_\theta = \frac{Ub}{a^2-b^2}\left(\frac{a^2}{r} - r\right) \longrightarrow (xxi)$$

b):- The shear stress found using Eq. (4.74). For axisymmetric flow

$$P_{r\theta} = \mu\left(r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) + \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right)$$

$$P_{r\theta} = \mu r \frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) \longrightarrow (xxii)$$

For steady motion, the total moment on any annular element must vanish, so that

$$(P_{r\theta})_{wall}\, a^2 = (P_{r\theta})\, r^2$$

or

$$P_{r\theta} = \left(\frac{a}{r}\right)^2 (P_{r\theta})_{wall} \longrightarrow (xxiii)$$

$$\text{Eq. } (xxi) \Rightarrow \frac{V_\theta}{r} = \frac{Ub}{a^2-b^2}\cdot\left(\frac{a^2}{r^2} - 1\right)$$

$$\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) = \frac{Ub}{a^2-b^2}\left(\frac{-2a^2}{r^3}\right)$$

$$\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) = \frac{-2a^2bU}{(a^2-b^2)r^3}$$

xing μr

$$\mu r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) = \frac{-2b\mu U}{(a^2-b^2)}\cdot\frac{a^2}{r^2}$$

using Eq. (XXii).

$\Rightarrow$
$$P_{r\theta} = \frac{-2b\mu U}{(a^2-b^2)} \cdot \frac{a^2}{r^2}$$

Put in Eq. (XXiii).

$$\frac{-2b\mu U}{a^2-b^2} \cdot \frac{\cancel{a^2}}{\cancel{a^2}} = \cancel{\frac{a^2}{r^2}} (P_{r\theta})_{wall}$$

$$\Rightarrow (P_{r\theta})_{wall} = -\frac{2b\mu U}{a^2-b^2} \longrightarrow (XXiV).$$

This For inner wall.

For outer wall.

$$(P_{r\theta})_{wall} = \frac{a^2}{b^2}\left(-\frac{2b\mu U}{a^2-b^2}\right)$$

$$= \frac{-2a^2\mu U}{b(a^2-b^2)} \longrightarrow (XXV)$$

Department of Mathematics

Fluid Dynamics-II (Mid Term Exams)

(BS-VIII)

177

Max Marks 20 Time allowed 1 hr

Q. No. 1. Answer the following questions

(i). What is the advantage of Buckingham Pi Theorem?

(ii). Give the brief description of the Principle of Dimensional Homogeneity.

(iii). Differentiate between mechanical quantity and the technical system of dimensions.

(iv). What is the advantage of Rayleigh Method over Buckingham Pi Theorem?

(v). Define role of local acceleration in fluid flow domain

(vi). Model the equation for the conservation of mass

(vii). Define dynamic similitude

(viii). Give the dimensionless form of Navier Stokes equation

(ix). Give the relation between Mach number M and Cauchy number C

(x). Explain the physical significance of pressure coefficient C_p $\frac{\Delta P}{\rho u}$

Q. No.2. (a). Arrange the following into dimensionless parameters

$F, \gamma, A, D, \nabla p$ where F is force, γ is specific weight, A is area, D is diameter and ∇p is pressure difference

(b) Using the Rayleigh method, determine an expression for the drag on a missile in supersonic flow. Consider the primary quantities in the problem as the density ρ, dynamic viscosity μ, bulk modulus K, reference length l, and flight speed V.

Fluid Dynamics-II

Total Marks(60) **Final term exams-2015** Time Allowed **02 hrs**

Q. No. 1. Consider a vertical flat plate having a notch of angle $\emptyset$ cut into the top of it and placed across an open channel containing water. The plate backs up the water in the channel until it flows through the notch. The volume rate of flow Q is some function of the elevation H of upstream liquid surface above the bottom of the notch. In addition, the discharge depends upon gravity and upon the velocity of approach $\bar{V}$ in the vertical plate. Determine the form of discharge equation.

Q. No. 2. Find the stream function for the vortex whose flow field is given by

$$v_r = w = 0$$

$$v_\theta = \omega r$$

where ω is the angular speed and is constant. Let $\psi = 0$ streamline be at r=0

Q. No. 3. Two parallel flat plates are a distance 2h apart. The top plate is moving with a velocity U to the right and the lower plate is moving with a velocity U' to the left and V' in the positive y-direction. Calculate the (a) location and the value of the maximum velocity of the flow between the two plates. and (b). Calculate the distribution of vorticity ζ.

Q. No. 4. State and prove equation of motion for laminar flow in pipe, and calculate

$$h_f = \frac{32\mu\bar{w}L}{D^2\gamma}$$

Q. No.5 Given $u = 2x, v = -2y$, determine the volume rate of flow Q per unit depth between the points (1,2) and (-1,-2).

Q. No. 6. Using Buckingham Pi theorem, obtain a relationship for pump's power in terms of flow rate Q, pressure rise ΔP, density, efficiency, dynamic viscosity and pipe diameter D g