ELASTO-DYNNAMICS

1

INSTRUCTOR:- Prof. Dr. Aftab Khan.

★ We use tensor notations in this course.

⇒ **Introduction:-** If $\underline{A}$ is a vector, then

$$\underline{A} = A_1\underline{i} + A_2\underline{j} + A_3\underline{k} = (A_1, A_2, A_3)$$

In tensor form it is written as

$$\underline{A} = A_i \quad ; \quad i = 1, 2, 3$$

double انڈیسز پہ understood Σ ہوتی ہے

$$\underline{A}\cdot\underline{B} = A_1B_1 + A_2B_2 + A_3B_3$$

$$\left.\begin{aligned} &= \sum_{i=1}^{3} A_iB_i \quad ; \quad i = 1,2,3 \\ &= A_iB_i \quad ; \quad i = 1,2,3 \end{aligned}\right\} \text{In the tensor form}$$

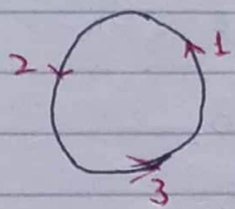
$$\underline{A}\times\underline{B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \underline{i}\begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix} - \underline{j}\begin{vmatrix} A_1 & A_3 \\ B_1 & B_3 \end{vmatrix} + \underline{k}\begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix}$$

$$= \underline{i}(A_2B_3 - B_2A_3) - \underline{j}(A_1B_3 - B_1A_3) + \underline{k}(A_1B_2 - B_1A_2)$$

& In the form of tensor

$\underline{A} \times \underline{B} = \epsilon_{ijk} A_j B_k$, where

$\epsilon_{ijk} = 0$ if any two indices are equal
$= 1$ if i, j, k are cyclic.
$= -1$ if i, j, k are anti-cyclic.



$\epsilon_{112} = 0$ $\qquad$ $\epsilon_{132} = -1$
$\epsilon_{123} = 1$ $\qquad$ $\epsilon_{131} = 0$
$\epsilon_{321} = -1$ $\qquad$ $\epsilon_{213} = -1$

Now, $\underline{A} \times \underline{B} = \epsilon_{ijk} A_j B_k$

For $i = 1 \Rightarrow$ $(\underline{A} \times \underline{B})_1 = \epsilon_{1jk} A_j B_k$

Put $i=1$ کر کے پہلا component نکل آئے گا اسی طرح $i=2,3$ کرنے سے دوسرا اور تیسرا آئے گا

j اور k دونوں double ہیں اس لیے indices تینوں ہی کے لازمی آئے گا

$$\Rightarrow (\underline{A} \times \underline{B})_1 = \cancel{\epsilon_{11k}} A_1 B_k + \epsilon_{12k} A_2 B_k + \epsilon_{13k} A_3 B_k$$

$$= \cancel{\epsilon_{121}} A_2 B_1 + \cancel{\epsilon_{122}} A_2 B_2 + \epsilon_{123} A_2 B_3 + \cancel{\epsilon_{131}} A_3 B_1 + \epsilon_{132} A_3 B_2 + \cancel{\epsilon_{133}} A_3 B_3$$

$$= \epsilon_{123} A_2 B_3 + \epsilon_{132} A_3 B_2$$

$$= (A_2 B_3 - B_2 A_3) \quad \because \epsilon_{123} = 1 \ \& \ \epsilon_{132} = -1$$

For $i = 2 \Rightarrow$ $(\underline{A} \times \underline{B})_2 = \epsilon_{2jk} A_j B_k$

$$\Rightarrow (\underline{A} \times \underline{B})_2 = \epsilon_{21k} A_1 B_k + \cancel{\epsilon_{22k}} A_2 B_k + \epsilon_{23k} A_3 B_k$$

$$= \cancel{\epsilon_{211}} A_1 B_1 + \cancel{\epsilon_{212}} A_1 B_2 + \epsilon_{213} A_1 B_3 + \epsilon_{231} A_3 B_1 + \cancel{\epsilon_{232}} A_3 B_2 + \cancel{\epsilon_{233}} A_3 B_3$$

$$\Rightarrow (\underline{A} \times \underline{B})_2 = \epsilon_{213} A_1 B_3 + \epsilon_{231} A_3 B_1$$

$$= (A_3 B_1 - A_1 B_3) \quad \begin{cases} \because \epsilon_{231} = 1 & \text{cyclic} \\ \& \ \epsilon_{213} = -1 & \text{a-cyclic} \end{cases}$$

For $i = 3 \Rightarrow$ $(\underline{A} \times \underline{B})_3 = \epsilon_{3jk} A_j B_k$

$$\Rightarrow (\underline{A} \times \underline{B})_3 = \epsilon_{31k} A_1 B_k + \epsilon_{32k} A_2 B_k + \cancel{\epsilon_{33k} A_3 B_k}^{0}$$

$$= \cancel{\epsilon_{311} A_1 B_1}^{0} + \epsilon_{312} A_1 B_2 + \cancel{\epsilon_{313} A_1 B_3}^{0}$$

$$+ \epsilon_{321} A_2 B_1 + \cancel{\epsilon_{322} A_2 B_2}^{0} + \cancel{\epsilon_{323} A_2 B_3}^{0}$$

$$= \epsilon_{312} A_1 B_2 + \epsilon_{321} A_2 B_1$$

$$= (A_1 B_2 - B_1 A_2) \quad \begin{cases} \because \epsilon_{312} = 1 \ \& \\ \epsilon_{321} = -1 \end{cases}$$

$$\Rightarrow \underline{A} \times \underline{B} = (A_2 B_3 - B_2 A_3)\underline{i} + (A_3 B_1 - A_1 B_3)\underline{j}$$

$$+ (A_1 B_2 - B_1 A_2)\underline{k}$$

★ If i, j, k have bars i.e. ($\underline{i}, \underline{j}, \underline{k}$) then they represent unit vectors otherwise they represent indices.

⇒ Partial Derivatives:-

★ If $u = u(x_1, x_2, x_3)$ is a scaler function then,

$$\frac{\partial u}{\partial x_1} = u_{,1} \quad , \quad \frac{\partial u}{\partial x_2} = u_{,2} \quad , \quad \frac{\partial u}{\partial x_3} = u_{,3}$$

★ For a vector $\underline{A} = (A_1, A_2, A_3)$

$$\Rightarrow \underline{A} = A_i \quad ; \quad i = 1, 2, 3$$

$$\frac{\partial \underline{A}}{\partial x} = A_{i,1} = (A_{1,1}, A_{2,1}, A_{3,1})$$

* $\phi(x,y,z)$

$$\Rightarrow \frac{\partial \phi}{\partial x} = \phi_{,1} \; ; \; \frac{\partial \phi}{\partial y} = \phi_{,2} \; ; \; \frac{\partial \phi}{\partial z} = \phi_{,3}$$

Similarly we can define double derivatives

$$\frac{\partial^2 \phi}{\partial x^2} = \phi_{,11} \; , \; \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \phi_{,21}$$

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial}{\partial y}(\phi_{,1}) = \phi_{,12}$$

And similarly for 3rd, 4th and so on derivatives.

⇒ **Gradient** :- (of a scaler function)

$$\nabla \phi = \frac{\partial \phi}{\partial x}\underline{i} + \frac{\partial \phi}{\partial y}\underline{j} + \frac{\partial \phi}{\partial z}\underline{k}$$

$$= \phi_{,1}\underline{i} + \phi_{,2}\underline{j} + \phi_{,3}\underline{k}$$

$$= (\phi_{,1} , \phi_{,2} , \phi_{,3})$$ in component form

$$= \phi_{,i} \; ; \; i = 1,2,3$$ (Tensor form)

⇒ **Divergence** :- (of a vector function)

$$\nabla \cdot \underline{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}$$

$$= f_{1,1} + f_{2,2} + f_{3,3}$$

$$= f_{i,i} \; ; \; i = 1,2,3$$

⇒ **Curl :-** (of a vector function)

$$\nabla \times \underline{f} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ f_1 & f_2 & f_3 \end{vmatrix} = \epsilon_{ijk} \nabla_j f_k$$

$$= \underline{i}\left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) - \underline{j}\left(\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}\right) + \underline{k}\left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)$$

$$= \epsilon_{ijk} f_{k,j}$$

Results:- 1) $\underline{A} = A_i$

2) $\underline{A} \cdot \underline{B} = A_i B_i = A_j B_j = A_k B_k$

3) $\underline{A} \times \underline{B} = \epsilon_{ijk} A_j B_k$

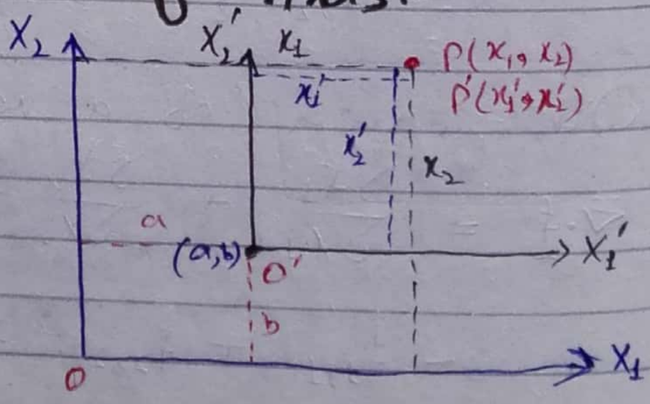
4) Grad $\phi = \phi_{,i}$

5) Div $\underline{f} = f_{i,i}$

6) Curl $\underline{f} = \epsilon_{ijk} f_{k,j}$

⇒ **Transformation of Axis:-**

If P is any point Then,
$P = P(x_1, x_2)$ w.r.t $X_1 X_2$ and
$P = (x_1', x_2')$ w.r.t X_1' X_2' axis

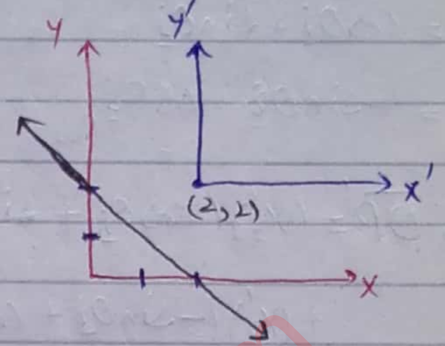


Transformations are

$$x_1 = a + x_1' \longrightarrow (1)$$

$$x_2 = b + x_2' \longrightarrow (2)$$

Example:- ★ Let $a = 2 = b$



$$x + y = 2$$

As $x = a + x'$

& $y = b + y'$

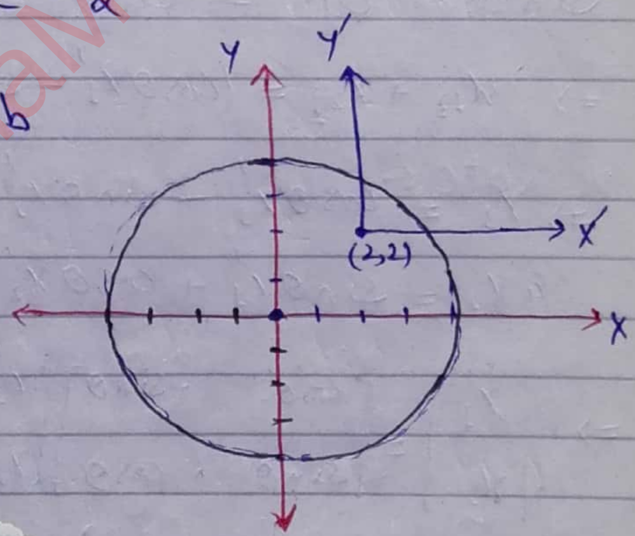
$$\Rightarrow \quad x + y = 2$$

$$\Rightarrow \quad (2 + x') + (2 + y') = 2 \qquad \because a = 2 = b$$

$$\Rightarrow \quad x' + y' + 4 = 2$$

$$\Rightarrow \quad x' + y' = -2$$

★ Again Let $a = 2 = b$



$$x^2 + y^2 = 4^2$$

$$\Rightarrow (2 + x')^2 + (2 + y')^2 = 4^2$$

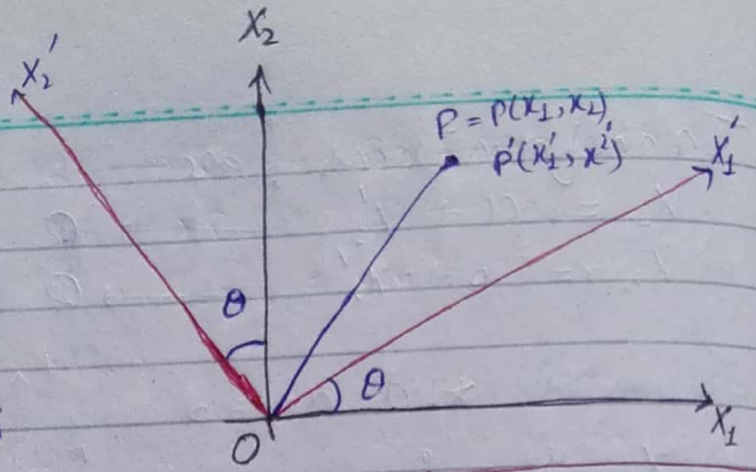
$$\Rightarrow (x' + 2)^2 + (y' + 2)^2 = 4^2$$

is an equation of circle having centre at $(-2, -2)$ w.r.t $X'Y'$ (while original equation has centre at $(0,0)$ w.r.t XY.

$\Rightarrow$ **Rotation of Axis:-**

$$\overline{OP} = \overline{OP'} \longrightarrow (1)$$

$$\overline{OP} = x_1 \underline{i} + x_2 \underline{j} \longrightarrow (2)$$

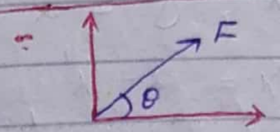


& $OP' = x_1' \underline{i}' + x_2' \underline{j}'$

Now,

$\underline{i}' = \cos\theta \underline{i} + \sin\theta \underline{j}$

$\underline{j} = -\sin\theta \underline{i} + \cos\theta \underline{j}$



$F = |F| \cos\theta \underline{i} + |F| \sin\theta \underline{j}$

$\underline{i}' = |\underline{i}'| \cos\theta \underline{i} + |\underline{i}'| \sin\theta \underline{j}$

$\underline{j}' = |\underline{j}'| \cos\theta \underline{i} + |\underline{j}'| \sin\theta \underline{j}$

$\Rightarrow \overline{OP}' = (x_1')(\cos\theta \underline{i} + \sin\theta \underline{j})$

$+ (x_2')(-\sin\theta \underline{i} + \cos\theta \underline{j})$

$\Rightarrow \overline{OP}' = (\cos\theta x_1' - \sin\theta x_2') \underline{i}$

$+ (\sin\theta x_1' + \cos\theta x_2') \underline{j} \longrightarrow (3)$

Since $\overline{OP} = \overline{OP}'$

$\Rightarrow x_1 \underline{i} + x_2 \underline{j} = (\cos\theta x_1' - \sin\theta x_2') \underline{i} + (\sin\theta x_1' + \cos\theta x_2') \underline{j}$

$\Rightarrow x_1 = \cos\theta x_1' - \sin\theta x_2'$

& $x_2 = \sin\theta x_1' + \cos\theta x_2'$

If $\theta = 90^\circ$

$\Rightarrow x_1 = -x_2'$ &

$x_2 = x_1'$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

where $a_{11} = \cos\theta$

$a_{12} = -\sin\theta$, $a_{21} = \sin\theta$

$a_{22} = \cos\theta$

$\Rightarrow x_1 = a_{11} x_1' + a_{12} x_2'$ and

$x_2 = a_{21} x_1' + a_{22} x_2'$

$\Rightarrow x_1 = a_{1i} x_i'$ & $x_2 = a_{2i} x_i'$; $i = 1, 2$

$\Rightarrow \quad x_i = a_{ij} x_j' \longrightarrow ⊛ \quad i,j = 1,2$

⊛ is the equation of rotation of axis.

where a_{ij} = cos of angle between x_i and x_j'

$\Rightarrow a_{11} = \cos\theta \quad , \quad a_{12} = \cos(90+\theta) = -\sin\theta$

$a_{21} = \cos(90-\theta) = \sin\theta$

similarly $\quad a_{22} = \cos\theta$

Results :- 1) Transformation $\Rightarrow x_i = x_i' + a_i$

2) Rotation $\Rightarrow \quad x_i = a_{ij} x_j'$

$\Rightarrow$ **ELASTIC BODIES :-** A body is said to be elastic if Hook's law holds in it, i.e. stress $\propto$ strain, where stress is force per unit area and strain is measure of deformation

$\Rightarrow \sigma_{ij} \propto \epsilon_{ij}$

σ_{ij} is stress & ϵ_{ij} is strain tensor

$\Rightarrow \sigma_{ij} = C_{ijkl} \epsilon_{kl}$

σ_{ij} is a tensor of order 2.
Similarly ϵ_{ij} is a tensor of order 2.
and C_{ijkl} is a tensor / elastic constant of order 4.
It has $3^4 = 81$ Components.
Also it is symmetric w.r.t. indices

A_i has 3 Component
A_{ij} has $9 = 3^2$ "
A_{ijk} has 3^3 "
A_{ijkl} " 3^4 "

$\Rightarrow$ By Viogt's Notation has 36 independent Components

$[9\times 9]$ $9, 8, 7, \ldots, 1$ $= 45 - 9 = 36$
Σ - rows

$\Rightarrow$ $11 \to 1$, $22 \to 2$, $33 \to 3$
$23 \to 4$, $13 \to 5$; $12 \to 6$

$\Rightarrow$ $C_{1122} = C_{12}$, $C_{1223} = C_{64}$
$C_{25} = C_{2213}$

$$\Rightarrow C_{ij} = \begin{array}{c} \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \end{array} \begin{array}{c} 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ \hline 21 \end{array}$$

Also C_{ij} is symmetric due to double commutative property of C_{ijkl},
ie $C_{ijkl} = C_{klij} \Rightarrow C_{ij} = C_{ji}$

$\Rightarrow$ C_{ij} has 21 indipendent components.

★ Note:- C_{ijkl} has 21 components.

$\Rightarrow$ **Isotropic Material:-** A material which has same properties in all directions is known as Isotropic.

Now by using equation of Transformation

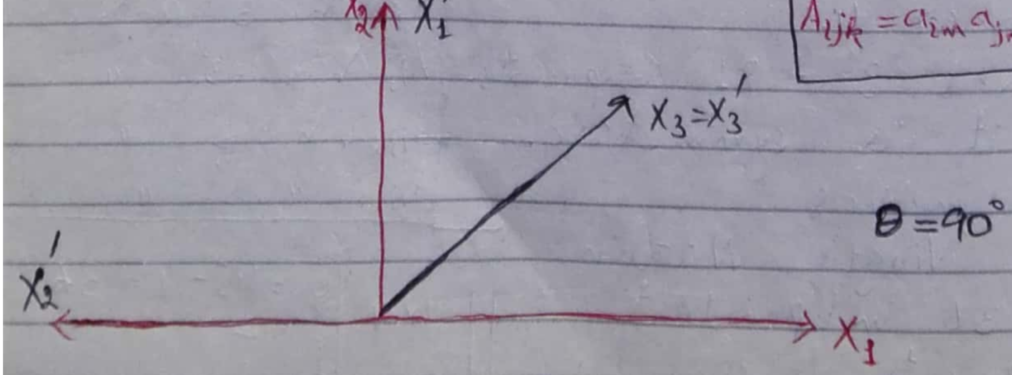
$$C_{ij} = a_{im}\, a_{jn}\, C'_{mn}$$

Equ of Transfor

$x_i = a_{ij}\, x'_j$,

$A_{ij} = a_{im}\, a_{jn}\, A'_{mn}$

$A_{ijk} = a_{im}\, a_{jn}\, a_{kp}\, A'_{mnp}$



$$C_{ij} = a_{im}\, a_{jn}\, C'_{mn}$$

a_{ij} - cos of angle b/w a_i and a'_j

$$\Rightarrow C_{11} = a_{1m}\, a_{1n}\, C'_{mn}$$

$$= a_{11} a_{1n} C'_{1n} + a_{12} a_{1n} C'_{2n} + a_{13} a_{1n} C'_{3n}$$

$$= \cos 90\, \cancel{a_{1n} C'_{1n}} + \cos 180\, C_{1n} C'_{2n} + \cos 90\, \cancel{a_{1n} C'_{3n}}$$

$\because \cos 90 = 0$

$$= -1\, a_{1n} C'_{2n}$$

$$= -1 \{ a_{11} C'_{21} + a_{12} C'_{22} + a_{13} C'_{23} \}$$

$$= -1 \{ \cancel{\cos 90\, C'_{21}} + \cos 180\, C'_{22} + \cancel{\cos 90\, C'_{23}} \}$$

$$= -1 \{ -1\, C'_{22} \}$$

$$\Rightarrow C_{11} = C'_{22} \quad \Rightarrow \boxed{C_{11} = C_{22}}$$

$\because$ for isotropic $C_{22} = C'_{22}$

Now

$a_{11} = \cos 90^\circ = 0$, $a_{12} = \cos 180^\circ = -1$

$a_{13} = \cos 90^\circ = 0$, $a_{21} = \cos 0 = 1$

$a_{22} = \cos 90^\circ = 0$, $a_{23} = \cos 90^\circ = 0$

$a_{31} = \cos 90^\circ = 0$, $a_{32} = \cos 90^\circ = 0$

$a_{33} = \cos 0 = 1$

$$\Rightarrow C_{12} = a_{1m}\, a_{2n}\, C'_{mn}$$

$$= \cancel{a_{11} a_{2n} C'_{1n}} + a_{12} a_{2n} C'_{2n} + \cancel{a_{13} a_{2n} C'_{3n}}$$

$$= -1 \{ a_{21} C'_{21} + \cancel{a_{22} C'_{22}} + \cancel{a_{23} C'_{23}} \}$$

$$C_{12} = -C'_{21} \quad \Rightarrow \boxed{C_{11} = +C_{21}}$$

$$C_{13} = a_{1m}\, a_{3n}\, C'_{mn}$$

$$= \cancel{a_{11} a_{3n} C'_{1n}} + a_{12} a_{3n} C'_{2n} + \cancel{a_{13} a_{3n} C'_{3n}}$$

$$\Rightarrow C_{13} = -1\{a_{31} C'_{21} + a_{32} C'_{22} + a_{33} C'_{23}\}$$

$$= -1\{C'_{23}\}$$

$$\Rightarrow \boxed{C_{13} = C'_{23}}$$

$$C_{14} = C_{1123}$$

& $C_{1123} = a_{1m} a_{1n} a_{2p} a_{3q} C'_{mnpq}$

$= a_{12} a_{12} a_{2p} a_{3q} C_{22pq}$ ∵ all other components are zero

$= (-1)(-1)\{a_{21} a_{3q} C'_{221q} +$

$a_{22} a_{3q} C'_{222q} + a_{23} a_{3q} C'_{223q}\}$

$$\Rightarrow C_{1123} = 0 \Rightarrow \boxed{C_{14} = 0}$$

$$C_{15} = C_{1113}$$

& $C_{1113} = a_{1m} a_{1n} a_{1p} a_{3q} C'_{mnpq}$ ∵ all other combinations are zero.

$= a_{12} a_{12} a_{12} a_{3q} C'_{111q}$

$= (-1)(-1)(-1)\{a_{31} C'_{1111} + a_{32} C'_{1112}$

$+ a_{33} C'_{1113}\}$

$= -1\{1 C'_{1113}\} = -C'_{1113}$

$$\Rightarrow C_{1113} = -C'_{1113} \Rightarrow C_{15} = -C'_{15}$$

$$\Rightarrow C_{15} = -C_{15} \Rightarrow 2C_{15} = 0$$

$$\Rightarrow \boxed{C_{15} = 0}$$

$C_{16} = C_{1112}$ $\quad c_4$

$$C_{1112} = a_{1m}a_{1n}a_{1p}a_{2q}C'_{mnpq}$$

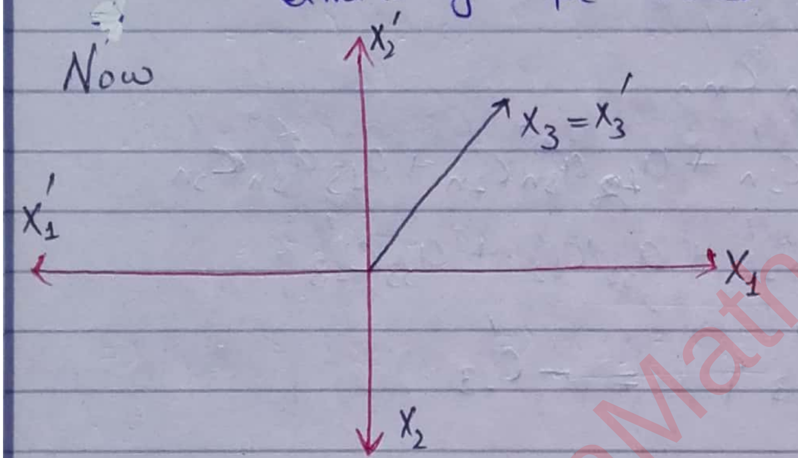
$$= a_{12}a_{12}a_{12}a_{2q}C'_{111q}$$

$$= -1\{a_{21}C'_{1111} + \overset{0}{\cancel{a_{22}}}C'_{1112} + \overset{0}{\cancel{a_{23}}}C'_{1113}\}$$

$C_{1112} = -C'_{1111} \Rightarrow C_{16} = -C'_{11}$? should be zero

Similarly for others.

Now



$\theta = 180^\circ$

$a_{11} = \cos 180^\circ = -1$, $\quad a_{12} = \cos 90^\circ = 0$
$a_{13} = \cos 90^\circ = 0$, $\quad a_{21} = \cos 90^\circ = 0$
$a_{22} = \cos 180^\circ = -1$, $\quad a_{23} = \cos 90^\circ = 0$
$a_{31} = \cos 90^\circ = 0$, $\quad a_{32} = \cos 90^\circ = 0$
$a_{33} = \cos(0) = 1$

$$C_{ij} = a_{im}a_{jn}C'_{mn}$$

$$\Rightarrow C_{11} = a_{1m}a_{1n}C'_{mn}$$

$$= a_{11}a_{1n}C'_{1n} + a_{12}\overset{0}{\cancel{a_{1n}}}C'_{2n} + a_{13}\overset{0}{\cancel{a_{1n}}}C'_{3n}$$

$$= -1\{a_{11}C'_{11} + \overset{0}{\cancel{a_{12}}}C'_{12} + \overset{0}{\cancel{a_{13}}}C'_{13}\}$$

$$= -1\{-C'_{11}\} = C'_{11}$$

$\Rightarrow \boxed{C_{11} = C_{11}}$ $\quad \because$ By Isotropic $C'_{11} = C_{11}$

$C_{12} = a_{1m} a_{2n} C'_{mn}$

$= a_{11} a_{2n} C'_{1n} + a_{12} a_{2n} C'_{2n} + a_{13} a_{2n} C'_{3n}$

$= -1 \{ a_{21} C'_{11} + a_{22} C'_{12} + a_{23} C'_{13} \}$

$= -1 \{ -C'_{12} \} = C'_{12}$

$\Rightarrow \boxed{C_{12} = C'_{12}}$ $\because$ By Isotropi $C_{12} = C'_{12}$

$C_{13} = a_{1m} a_{3n} C'_{mn}$

$= a_{11} a_{3n} C'_{1n} + a_{12} a_{3n} C'_{2n} + a_{13} a_{3n} C'_{3n}$

$= -1 \{ a_{31} C'_{11} + a_{32} C'_{12} + a_{33} C'_{13} \}$

$= -1 \{ C'_{13} \} = -C'_{13}$

$\Rightarrow C_{13} = -C_{13} \Rightarrow C_{13} = ?$

$C_{14} = C_{1123}$

$\Rightarrow C_{1123} = a_{1m} a_{1n} a_{2p} a_{3q} C'_{mnpq}$

$= a_{11} a_{11} a_{22} a_{33} C'_{1123}$

$= (-1)(-1)(-1)(1) C'_{1123}$

$C_{1123} = -C'_{1123}$

$\Rightarrow C_{1123} = -C_{1123} \Rightarrow C_{1123} + C_{1123} = 0$

$\Rightarrow 2C_{1123} = 0$

$\Rightarrow C_{1123} = 0 \Rightarrow \boxed{C_{14} = 0}$

$$\Rightarrow C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix}$$

Lame's Put :- $\left.\begin{aligned} C_{11} &= \lambda + 2\mu \\ \text{\&} \quad C_{12} &= \lambda \end{aligned}\right\} \Rightarrow \frac{C_{11}-C_{12}}{2} = \mu$

$$\Rightarrow C_{ij} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

λ and μ are known as Lame's Constant.

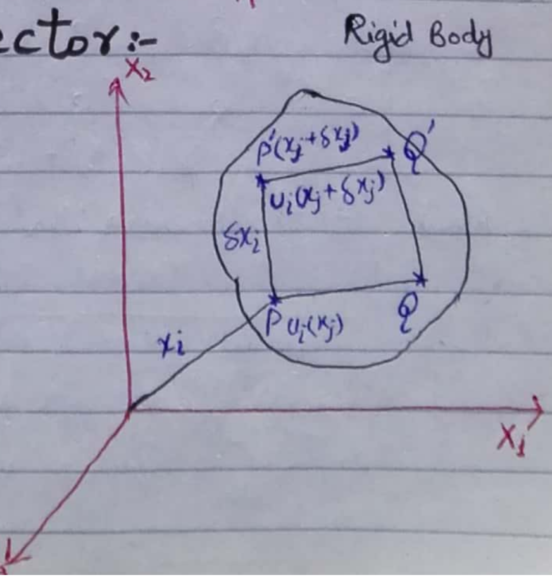
⇒ Displacement Vector:-

$$\delta\overline{PQ} = \overline{P'Q'} - \overline{PQ}$$

$$= U_i(x_j + \delta x_j) - U_i(x_j)$$

$$= U_i(x_j) + \frac{\partial u_i}{\partial x_j}\delta x_j + \dots - U_i(x_j)$$

∵ By Taylor series



We consider the linear transformation.
Therefore neglecting higher terms

$$\delta\overline{PQ} = u_i(x_j) + \frac{\partial u_i}{\partial x_j}\delta x_j - u_i(x_j)$$

$$= \frac{\partial u_i}{\partial x_j}\delta x_j$$

$$\Rightarrow \delta\overline{PQ} = \left\{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\right\}\delta x_j$$

$$\Rightarrow \delta\overline{PQ} = \epsilon_{ij}\delta x_j + \xi_{ij}\delta x_j$$

where $\epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ This is symmetric part and

$\xi_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right)$ is anti-symmetri part due to rotation.

For Isotropic material

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix}$$

And Hook's Law is a follows

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} \longrightarrow (\star)$$

Equation of Motion is as follows

$$\sigma_{ij,j} = \rho \ddot{u}_i$$

where ρ is mass density and σ_{ij} is the stress.

For $i=1$:- $\quad \sigma_{1j,j} = \rho \ddot{u}_1$ (Solving for 2D)

$$\Rightarrow \sigma_{11,1} + \sigma_{12,2} = \rho \ddot{u}_1 \longrightarrow (1)$$

And $\quad \sigma_{11} = C_{11}\epsilon_{11} + C_{12}\epsilon_{22}$ from $(\star)$

But $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$$\Rightarrow \epsilon_{11} = \frac{1}{2}(u_{1,1} + u_{1,1}) = u_{1,1}$$

$$\epsilon_{22} = u_{2,2}$$

$$\Rightarrow \sigma_{11} = C_{11}u_{1,1} + C_{12}u_{2,2}$$

$$\Rightarrow \sigma_{11,1} = C_{11}u_{1,11} + C_{12}u_{2,21}$$

Also $\quad \sigma_{12} = \frac{1}{2}(C_{11} - C_{12}) \cdot 2\epsilon_{12}$ from $(\star)$

$$= (C_{11} - C_{12})\epsilon_{12}$$

$$= \frac{1}{2}(C_{11} - C_{12})(u_{1,2} + u_{2,1})$$

$$\Rightarrow \sigma_{12,2} = \frac{1}{2}(C_{11}-C_{12})(u_{1,22}+u_{2,12})$$

Thus equ ① becomes

$$C_{11}u_{1,11} + C_{12}u_{2,12} + \frac{1}{2}(C_{11}-C_{12})(u_{1,22}+u_{2,12}) = \rho\ddot{u}_1 \longrightarrow (A)$$

For $i=2$:-

$$\sigma_{2j,j} = \rho\ddot{u}_2$$

$$\Rightarrow \sigma_{21,1} + \sigma_{22,2} = \rho\ddot{u}_2 \longrightarrow ②$$

Now

$$\sigma_{21} = (C_{11}-C_{12})\epsilon_{12} \qquad \because \sigma_{21} = \sigma_{12}$$

$$\Rightarrow \sigma_{21} = \frac{1}{2}(C_{11}-C_{12})(u_{1,2}+u_{2,1}) \qquad \because \epsilon_{12} = \frac{1}{2}(u_{1,2}+u_{2,1})$$

$$\Rightarrow \sigma_{21,1} = \frac{1}{2}(C_{11}-C_{12})(u_{1,21}+u_{2,11})$$

And

$$\sigma_{22} = C_{12}\epsilon_{11} + C_{11}\epsilon_{22} \qquad \because \text{for 2D}$$

$$= \frac{1}{2}C_{12}(u_{1,1}+u_{1,1}) + \frac{1}{2}C_{11}(u_{2,2}+u_{2,2})$$

$$= C_{12}u_{1,1} + C_{11}u_{2,2}$$

$$\Rightarrow \sigma_{22,2} = C_{12}u_{1,12} + C_{11}u_{2,22}$$

Thus equ ② becomes

$$\frac{1}{2}(C_{11}-C_{12})(u_{1,21}+u_{2,11}) + C_{12}u_{1,12} + C_{11}u_{2,22} = \rho\ddot{u}_2 \longrightarrow (B)$$

Let

$$u_i = Ae^{ik(x_jn_j - ct)}p_i \longrightarrow ③$$

Wave equation

Wave equation $\Rightarrow u_{xx} + u_{yy} = \frac{1}{c^2}u_{tt}$

$$\Rightarrow u_{xx} = (ikn_1)^2 Ae^{\cdots} p_i$$

$$u_{yy} = (ikn_2)^2 Ae^{\cdots} p_i$$

$$u_{tt} = (-ikc)^2 Ae^{\cdots} p_i$$

Since x_1 for x & x_2 for y

Here A is the amplitude of the wave, k is the wave number, n_j is the propagation vector. P_i is the polarization vector and c is the speed of wave.

By using ③ equ Ⓐ becomes

$$C_{11}(ikn_1)^2 P_1 Ae^{ik(x_jn_j-ct)} + C_{12}(ikn_1)(ikn_2)P_2 Ae^{ik(x_jn_j-ct)}$$
$$+\frac{1}{2}(C_{11}-C_{12})\left\{(ikn_2)^2 P_1 Ae^{ik(x_jn_j-ct)} + (ikn_1)(ikn_2)P_2 Ae^{ik(x_jn_j-ct)}\right\}$$
$$= \rho(-ikc)^2 P_1 Ae^{ik(x_jn_j-ct)}$$

$$\Rightarrow \cancel{Ae^{ik(x_jn_j-ct)}}\left[C_{11}(ikn_1)^2 P_1 + C_{12}(ikn_1)(ikn_2)P_2 + \frac{1}{2}(C_{11}-C_{12})\left\{(ikn_2)^2 P_1 + (ikn_1)(ikn_2)P_2\right\}\right] = \rho(-ikc)^2 P_1 \cancel{Ae^{ik(\ldots)}}$$

$$\Rightarrow \cancel{k^2}\left[C_{11}(in_1)^2 P_1 + C_{12}(in_1)(in_2)P_2 + \frac{1}{2}(C_{11}-C_{12})\left\{(in_2)^2 P_1 + (in_1)(in_2)P_2\right\}\right] = -\rho \cancel{k^2} c^2 P_1$$

$$\Rightarrow -C_{11}n_1^2 P_1 - C_{12}n_1n_2P_2 - \frac{1}{2}(C_{11}-C_{12})n_2^2 P_1 - \frac{1}{2}(C_{11}-C_{12})n_1n_2P_2 = -\rho c^2 P_1$$

$$\Rightarrow \left\{C_{11}n_1^2 + \frac{1}{2}(C_{11}-C_{12})n_2^2 - \rho c^2\right\}P_1 + \left\{C_{12}n_1n_2 + \frac{1}{2}(C_{11}-C_{12})n_1n_2\right\}P_2 = 0 \longrightarrow ④$$

Now substitute values of equ ③ in Ⓑ

$$\Rightarrow \frac{1}{2}(C_{11}-C_{12})\left\{n_1n_2P_1 + n_1^2 P_2\right\} + C_{12}n_1n_2P_1 + C_{11}n_2^2 P_2 = \rho c^2 P_2$$

$$\Rightarrow \left\{\tfrac{1}{2}(C_{11}+C_{12})n_1n_2\right\}P_1 + \left\{\tfrac{1}{2}(C_{11}-C_{12})n_1^2 + C_{11}n_2^2 - \rho c^2\right\}P_2 = 0 \quad \rightarrow (5)$$

$$\Rightarrow \begin{vmatrix} C_{11}n_1^2 + \tfrac{1}{2}(C_{11}-C_{12})n_2^2 - \rho c^2 & C_{12}n_1n_2 + \tfrac{1}{2}(C_{11}-C_{12})n_1n_2 \\ \tfrac{1}{2}(C_{11}+C_{12})n_1n_2 & \tfrac{1}{2}(C_{11}-C_{12})n_1^2 + C_{11}n_2^2 - \rho c^2 \end{vmatrix} = 0$$

$a_1x_1 + a_2x_2 = 0$ &

$b_1x_1 + b_2x_2 = 0$

from (1) $x_1 = \frac{-a_2}{a_1}x_2$

Put in (2)

$b_1\left(\frac{-a_2}{a_1}\right)x_2 + b_2x_2 = 0$

$\Rightarrow -b_1a_2x_2 = a_1b_2x_2$

$a_1b_2 - a_2b_1 = 0$

$\Rightarrow \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

for simplicity choose

$\underline{n} = (1, 0)$

$$\Rightarrow \begin{vmatrix} C_{11} - \rho c^2 & 0 \\ 0 & \tfrac{1}{2}(C_{11}-C_{12}) - \rho c^2 \end{vmatrix} = 0$$

$$\Rightarrow (C_{11} - \rho c^2)\left[\tfrac{1}{2}(C_{11}-C_{12}) - \rho c^2\right] = 0$$

$\Rightarrow \rho c^2 = C_{11}$

$\Rightarrow c^2 = \frac{C_{11}}{\rho}$

$\Rightarrow c = \sqrt{\frac{C_{11}}{\rho}}$

$\rho c^2 = \frac{1}{2}(C_{11} - C_{12})$

$\Rightarrow c = \sqrt{\frac{(C_{11}-C_{12})/2}{\rho}}$

Lame's constant $\Rightarrow C_{11} = \lambda + 2\mu$

and $C_{12} = \lambda$

$$\Rightarrow c = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \sqrt{\frac{\mu}{\rho}}$$

$\sigma_{ij} = \lambda \epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij}$ (By Lame's)

$\Rightarrow \sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i})$

$\because \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

Equation of motion is

$$\sigma_{ij,j} = \rho \ddot{u}_i$$

$$\Rightarrow \lambda u_{k,kj}\delta_{ij} + \mu(u_{i,jj} + u_{j,ij}) = \rho \ddot{u}_i$$

$$\Rightarrow \lambda u_{k,ki} + \mu(u_{i,jj} + u_{j,ij}) = \rho \ddot{u}_i$$

$\delta_{ij} f_i = f_j$

$\delta_{ij} f_j = f_i$

for $i = 1$

$\delta_{1j} f_j = f_1$

$\Rightarrow \delta_{11} f_1 + \delta_{12} f_2 + \delta_{13} f_3 = f_1$

$\Rightarrow f_1 = f_1 \quad \because \delta_{ij} = 1; i=j, \ 0; i \neq j$

Let $u_i = Ae^{ik(x_j n_j - ct)} P_i$

$u_{i,1} = (ikn_1) Ae^{\dots} P_i$

$u_{i,2} = (ikn_2) Ae^{\dots} P_i$

$u_{i,j} = ikn_j Ae^{\dots} P_i$

$$\Rightarrow \lambda P_k (in_k)(in_i) Ae^{ik(x_j n_j - ct)}$$

$$+ \mu\{P_i (in_j)^2 Ae^{ik(x_j n_j - ct)} + P_j (in_i)(in_j) Ae^{ik(x_j n_j - ct)}\}$$

$$= (-ick)^2 \rho Ae^{ik(x_j n_j - ct)} P_i$$

$$\Rightarrow \lambda P_k (in_k)(in_i) + \mu\{P_i (in_j)^2 + P_j (in_i)(in_j)\} = -c^2 \rho P_i$$

$\because n_j^2 = 1$

$$\Rightarrow \lambda P_k n_k n_i + \mu\{P_i n_j^2 + P_j n_i n_j\} = \rho c^2 P_i$$

$$\Rightarrow \lambda (n_k P_k) n_i + \mu\{P_i + (n_j P_j) n_i\} = \rho c^2 P_i$$

For Longitudinal wave $n_i \parallel P_i$

For Transverse wave $n_i \perp P_i$

* For $\perp$ wave $n_i \perp P_i \Rightarrow n_i P_i = 0$

$$\Rightarrow \mu P_i = \rho c^2 P_i \Rightarrow \rho c^2 = \mu$$

$$\Rightarrow c = \sqrt{\frac{\mu}{\rho}}$$

* For $n_i \parallel P_i \Rightarrow n_i = P_i$

$\because (n_i)^2 = 1$

$$\Rightarrow \lambda n_i + \mu(n_i + n_i) = \rho c^2 n_i$$

$$\Rightarrow (\lambda + 2\mu) n_i = \rho c^2 n^2$$

$$\Rightarrow c = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

★ Equation of Motion :-

$$\sigma_{ij,j} = \rho \ddot{u}_i$$

where ρ is mass density and σ_{ij} is the stress.

For $i=1$ $\quad \sigma_{1j,j} = \rho \ddot{u}_1$ (solving for 31)

$$\Rightarrow \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} = \rho \ddot{u}_1 \longrightarrow (1)$$

Now

$$\sigma_{11} = C_{11}\epsilon_{11} + C_{12}\epsilon_{22} + C_{12}\epsilon_{33}$$

$$= C_{11}\cdot\frac{1}{2}(u_{1,1}+u_{1,1}) + C_{12}\frac{1}{2}(u_{2,2}+u_{2,2})$$

$$+ C_{11}\frac{1}{2}(u_{3,3}+u_{3,3}) \quad \because \epsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i})$$

$$= C_{11}u_{1,1} + C_{12}u_{2,2} + C_{12}u_{3,3}$$

$$\Rightarrow \sigma_{11,1} = C_{11}u_{1,11} + C_{12}u_{2,21} + C_{12}u_{3,31}$$

$$\sigma_{12,2} = \frac{1}{2}(C_{11}-C_{12})(u_{1,22}+u_{2,12})$$

And

$$\sigma_{13} = (C_{11}-C_{12})\epsilon_{13} = \frac{1}{2}(C_{11}-C_{12})(u_{1,3}+u_{3,1})$$

$$\Rightarrow \sigma_{13,3} = \frac{1}{2}(C_{11}-C_{12})(u_{1,33}+u_{3,13})$$

So equation (1) becomes

$$C_{11}u_{1,11} + C_{12}u_{2,21} + C_{12}u_{3,31} + \frac{1}{2}(C_{11}-C_{12})(u_{1,22}+u_{2,12})$$

$$+\frac{1}{2}(C_{11}-C_{12})(u_{1,33}+u_{3,13}) \longrightarrow (A)$$

For $i=2$,- $\quad \sigma_{2j,j} = \rho \ddot{u}_2$

$$\Rightarrow \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} = \rho \ddot{u}_2 \longrightarrow ②$$

$$\sigma_{21,1} = \frac{1}{2}(c_1 - c_{12})(u_{2,11} + u_{1,21})$$

Now $\quad \sigma_{22} = c_{12}\epsilon_{11} + c_1\epsilon_{22} + c_{12}\epsilon_{33}$

$$= \frac{1}{2}c_{12}(u_{1,1}+u_{1,1}) + \frac{1}{2}c_1(u_{2,2}+u_{2,2}) + \frac{1}{2}c_{12}(u_{3,3}+u_{3,3})$$

$$= c_{12}u_{1,1} + c_1 u_{2,2} + c_{12}u_{3,3}$$

$$\Rightarrow \sigma_{22,2} = c_{12}u_{1,12} + c_1 u_{2,22} + c_{12}u_{3,32}$$

And $\quad \sigma_{23} = (c_1 - c_{12})\epsilon_{23} = \frac{1}{2}(c_1 - c_{12})(u_{2,3}+u_{3,2})$

$$\Rightarrow \sigma_{23,3} = \frac{1}{2}(c_1 - c_{12})(u_{2,33} + u_{3,23})$$

So eqn ② becomes

$$\frac{1}{2}(c_1 - c_{12})(u_{1,21} + u_{2,11}) + c_{12}u_{1,12} + c_1 u_{2,22} + c_{12}u_{3,32}$$
$$+ \frac{1}{2}(c_1 - c_{12})(u_{2,33} + u_{3,23}) = \rho \ddot{u}_2 \longrightarrow (B)$$

For $i=3$,- $\quad \sigma_{3j,j} = \rho \ddot{u}_3$

$$\Rightarrow \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = \rho \ddot{u}_3 \longrightarrow ③$$

Now $\quad \sigma_{31} = (c_1 - c_{12})\epsilon_{13} = \frac{1}{2}(c_1 - c_{12})(u_{1,3} + u_{3,1})$

$$\Rightarrow \sigma_{31,1} = \frac{1}{2}(c_1 - c_{12})(u_{1,31} + u_{3,11})$$

$$\sigma_{32} = \frac{1}{2}(c_1 - c_{12})(u_{2,3} + u_{3,2})$$

$$\Rightarrow \sigma_{32,2} = \frac{1}{2}(C_{11} - C_{12})(u_{2,32} + u_{3,22})$$

And

$$\sigma_{33} = C_{12}\epsilon_{11} + C_{12}\epsilon_{22} + C_{11}\epsilon_{33}$$

$$= C_{12}u_{1,1} + C_{12}u_{2,2} + C_{11}u_{3,3}$$

$$\Rightarrow \sigma_{33,3} = C_{12}u_{1,13} + C_{12}u_{2,23} + C_{11}u_{3,33}$$

So equ ③ becomes

$$\frac{1}{2}(C_{11} - C_{12})(u_{1,31} + u_{3,11}) + \frac{1}{2}(C_{11} - C_{12})(u_{2,32} + u_{3,22})$$
$$+ C_{12}u_{1,13} + C_{12}u_{2,23} + C_{11}u_{3,33} = \rho\ddot{u}_3 \longrightarrow (C)$$

Now Wave Equation

$$u_i = Ae^{ik(x_j n_j - ct)}P_i \longrightarrow ④$$

By using ④ equ Ⓐ becomes

$$C_{11}(ikn_1)^2 P_1 Ae^{ik(x_j n_j - ct)} + C_{12}(ikn_1)(ikn_2)P_2 Ae^{ik(x_j n_j - ct)}$$
$$+ C_{12}(ikn_3)(ikn_1)P_3 Ae^{ik(x_j n_j - ct)} + \frac{1}{2}(C_{11} - C_{12})\{(ikn_2)^2 P_1 Ae^{ik\cdots}$$
$$+ (ikn_1)(ikn_2)P_2 Ae^{ik(x_j n_j - ct)}\} + \frac{1}{2}(C_{11} - C_{12})\{(ikn_3)^2 P_1$$
$$Ae^{ik(x_j n_j - ct)} + (ikn_1)(ikn_3)P_3 Ae^{ik(x_j n_j - ct)}\}$$
$$= (-ikc)^2 \rho Ae^{ik(x_j n_j - ct)} P_1$$

$$\Rightarrow C_{11}n_1^2 P_1 + C_{12}n_1 n_2 P_2 + C_{12}n_1 n_3 P_3 + \frac{1}{2}(C_{11} - C_{12})$$
$$\{n_2^2 P_1 + n_1 n_2 P_2\} + \frac{1}{2}(C_{11} - C_{12})\{n_3^2 P_1 + n_1 n_3 P_3\}$$
$$= \rho c^2 P_1$$

$$\Rightarrow \left\{C_{11} n_1^2 + \tfrac{1}{2}(C_1 - C_2) n_2^2 + \tfrac{1}{2}(C_1 - C_2) n_3^2 - \rho c^2 \right\} P_1$$

$$+ \left\{C_{12} n_1 n_2 + \tfrac{1}{2}(C_1 - C_2) n_1 n_2 \right\} P_2 + \left\{C_{12} n_1 n_3 + \tfrac{1}{2}(C_1 - C_2) n_1 n_3 \right\} P_3$$

$$= 0 \longrightarrow (5)$$

Now by using equ (4) equ (B) becomes

$$\tfrac{1}{2}(C_1 - C_2)\left\{(ikn_2)(ikn_1) P_1 A e^{ik(x_j n_j - ct)} + (ikn_1)^2 P_2 A e^{ik(x_j n_j - ct)}\right\}$$

$$+ C_{12}(ikn_1)(ikn_2) P_1 A e^{ik(x_j n_j - ct)} + C_1 (ikn_2)^2 P_2 A e^{ik(x_j n_j - ct)}$$

$$+ C_{12}(ikn_3)(ikn_2) P_3 A e^{ik(x_j n_j - ct)} + \tfrac{1}{2}(C_1 - C_2)\left\{(ikn_3)^2 P_2\right.$$

$$\left. A e^{ik(x_j n_j - ct)} + (ikn_2)(ikn_3) P_3 A e^{ik(x_j n_j - ct)}\right\} = \rho(-ikc)^2 P_2 A e^{ik(x_j n_j - ct)}$$

$$\Rightarrow \tfrac{1}{2}(C_1 - C_2)\left\{n_1 n_2 P_1 + n_1^2 P_2\right\} + C_{12} n_1 n_2 P_1 + C_1 n_2^2 P_2 + C_{12} n_2 n_3 P_3$$

$$+ \tfrac{1}{2}(C_1 - C_2)\left\{n_3^2 P_2 + n_2 n_3 P_3\right\} = \rho c^2 P_2$$

$$\Rightarrow \left\{\tfrac{1}{2}(C_1 - C_2) n_1 n_2 + C_{12} n_1 n_2\right\} P_1 + \left\{\tfrac{1}{2}(C_1 - C_2) n_1^2 + C_1 n_2^2\right.$$

$$\left. + \tfrac{1}{2}(C_1 - C_2) n_3^2 - \rho c^2 \right\} P_2 + \left\{C_{12} n_2 n_3 + \tfrac{1}{2}(C_1 - C_2)\right.$$

$$\left. n_2 n_3 \right\} P_3 = 0 \longrightarrow (6)$$

Now by using equ (4) equ (C) becomes

$$\tfrac{1}{2}(C_1 - C_2)\left\{(ikn_3)(ikn_1) P_1 A e^{(\cdots)} + (ikn_1)^2 P_3 A e^{(\cdots)}\right\} +$$

$$\tfrac{1}{2}(C_1 - C_2)\left\{(ikn_2)(ikn_3) P_2 A e^{(\cdots)} + (ikn_2)^2 P_3 A e^{(\cdots)}\right\} +$$

$$C_{12}(ikn_1)(ikn_3) P_1 A e^{(\cdots)} + C_{12}(ikn_2)(ikn_3) P_2 A e^{(\cdots)}$$

$$+ C_1 (ikn_3)^2 P_3 A e^{(\cdots)} = \rho(-ikc)^2 P_3 A e^{(\cdots)}$$

$$\Rightarrow \frac{1}{2}(C_{11}-C_{12})\{n_1 n_3 P_1 + n_1^2 P_3\} + \frac{1}{2}(C_{11}-C_{12})\{n_2 n_3 P_2$$
$$+ n_2^2 P_3\} + C_{12} n_1 n_3 P_1 + C_{12} n_2 n_3 P_2 + C_{11} n_3^2 P_3$$
$$= \rho c^2 P_3$$

$$\Rightarrow \{\frac{1}{2}(C_{11}-C_{12}) n_1 n_3 + C_{12} n_1 n_3\} P_1 + \{\frac{1}{2}(C_{11}-C_{12}) n_2 n_3$$
$$+ C_{12} n_2 n_3\} P_2 + \{\frac{1}{2}(C_{11}-C_{12}) n_1^2 + \frac{1}{2}(C_{11}-C_{12}) n_2^2$$
$$+ C_{11} n_3^2 - \rho c^2\} P_3 = 0 \longrightarrow (7)$$

Solving eqn (5), (6) & (7) we get the value of c. i.e (For simple take $\underline{n}=(1,0,0)$)

For simplicity Put $\underline{n} = n(1,0,0)$

So (5) $\Rightarrow (C_{11} n_1^2 - \rho c^2) P_1 = 0 \Rightarrow (C_{11} - \rho c^2) P_1 = 0$

(6) $\Rightarrow (\frac{1}{2}(C_{11}-C_{12}) - \rho c^2) P_2 = 0$

(7) $\Rightarrow (\frac{1}{2}(C_{11}-C_{12}) - \rho c^2) P_3 = 0$

$$\Rightarrow \begin{vmatrix} C_{11}-\rho c^2 & 0 & 0 \\ 0 & \frac{1}{2}(C_{11}-C_{12})-\rho c^2 & 0 \\ 0 & 0 & \frac{1}{2}(C_{11}-C_{12})-\rho c^2 \end{vmatrix} = 0$$

$$\Rightarrow (C_{11}-\rho c^2)\left[\frac{1}{2}(C_{11}-C_{12})-\rho c^2\right]\left[\frac{1}{2}(C_{11}-C_{12})-\rho c^2\right] = 0$$

$\Rightarrow C_{11} - \rho c^2 = 0$

$\Rightarrow \rho c^2 = C_{11}$

$\Rightarrow \boxed{c = \sqrt{\frac{C_{11}}{\rho}}}$

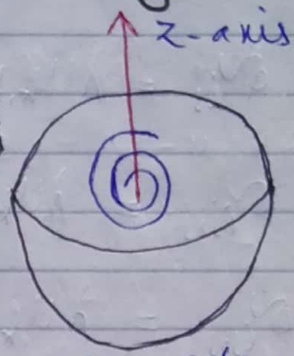
$\frac{1}{2}(C_{11}-C_{12}) - \rho c^2 = 0$

$\Rightarrow \rho c^2 = \frac{1}{2}(C_{11}-C_{12})$

$\Rightarrow c^2 = \frac{C_{11}-C_{12}}{2\rho} \Rightarrow \boxed{c = \sqrt{\frac{C_{11}-C_{12}}{2\rho}}}$

& $\boxed{c = \sqrt{\frac{C_{11}-C_{12}}{2\rho}}}$

⇒ Speed of Wave in a Rotating Isotropic Medium:-



$\dot{u}_i \to \dot{u}_i + \underline{\Omega} \times \underline{u}$ (from linear → Angular)

Similarly

$$\frac{d^2 u}{dt^2} = \ddot{u}_i$$

$\Omega = \Omega(0,0,1)$

Ω is the angular velocity & $\dot{u}_i$ is rate of change of displacement.

$$\frac{d}{dt}(\dot{u}_i) \xrightarrow{\text{rotating medium}} \frac{d}{dt}\left\{\dot{u}_i + \underline{\Omega} \times \underline{u}\right\}$$

$$\Rightarrow \left\{\ddot{u}_i + \underline{\Omega} \times \dot{u}\right\} + \left\{\underline{\Omega} \times (\dot{u}_i + \underline{\Omega} \times u)\right\}$$

And $\frac{df}{dt} \Rightarrow \frac{d}{dt}\underline{f} + \underline{\Omega} \times \underline{f}$

And
$$\ddot{u}_i \to \ddot{u}_i + 2\underline{\Omega} \times \dot{u} + \underline{\Omega} \times (\underline{\Omega} \times \underline{u})$$

$$= \ddot{u}_i + 2\epsilon_{ijk}\Omega_j \dot{u}_k + (\underline{\Omega} \cdot \underline{u})\underline{\Omega} - (\underline{\Omega} \cdot \underline{\Omega})\underline{u}$$

$$= \ddot{u}_i + 2\epsilon_{ijk}\Omega_j \dot{u}_k + \Omega_j u_j \Omega_i - \Omega^2 u_i$$

Thus the equation of motion in a rotating medium becomes

$$\sigma_{ij,j} = \rho\left\{\ddot{u}_i + 2\epsilon_{ijk}\Omega_j \dot{u}_k + \Omega_j u_j \Omega_i - \Omega^2 u_i\right\} \longrightarrow (*)$$

For Isotropic Medium

$$\sigma_{ij} = \lambda \epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij},$$

where $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$\Rightarrow \sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i})$

$\Rightarrow \sigma_{ij,j} = \lambda u_{k,kj}\delta_{ij} + \mu(u_{i,jj} + u_{j,ij})$

$\Rightarrow \sigma_{ij,j} = \lambda u_{k,ki} + \mu(u_{i,jj} + u_{j,ij})$

So equation ⊛ becomes as

$$\lambda u_{k,ki} + \mu(u_{i,jj} + u_{j,ij}) = \rho\left(\ddot{u}_i + 2\epsilon_{ijk}\Omega_j\dot{u}_k + \Omega_j u_j \Omega_i - \Omega^2 u_i\right) \longrightarrow ①$$

Let $u_i = Ae^{ik(x_j n_j - ct)} p_i$ put $k=1$

So equ ① becomes

$$\lambda(in_k)(in_i)p_k Ae^{i(x_j n_j - ct)} + \mu\left\{(in_j)(in_j)p_i Ae^{i(x_j n_j - ct)} + (in_i)(in_j)p_j Ae^{i(x_j n_j - ct)}\right\} = \rho\left[(-ic)^2 p_i Ae^{i(x_j n_j - ct)} + 2\epsilon_{ijk}\Omega_j(-ic)p_k Ae^{ik(x_j n_j - ct)} + \Omega_j p_j Ae^{i(x_j n_j - ct)}\Omega_i - \Omega^2 p_i Ae^{i(x_j n_j - ct)}\right]$$

$$\Rightarrow -\lambda(n_i n_k)p_k - \mu\{n_j n_j p_i + n_i n_j p_j\} = \rho\left[-c^2 p_i + 2\epsilon_{ijk}\Omega_j(-ic)p_k + \Omega_j p_j \Omega_i - \Omega^2 p_i\right]$$

$$\Rightarrow \{\lambda n_i n_k p_k + \mu p_i + \mu n_j p_j n_i\} = \rho c^2 p_i + 2\rho i c\epsilon_{ijk}\Omega_j p_k - \rho\Omega_j p_j \Omega_i + \rho\Omega^2 p_i$$

$$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2) p_i - 2\rho i c \epsilon_{ijk} \Omega_j p_k + (\lambda+\mu) n_k p_k n_i + \rho \Omega_j p_j \Omega_i = 0 \longrightarrow (2)$$

For Longitudinal Waves $n_i \parallel p_i \Rightarrow n_i p_i = 1$
$\Rightarrow n_i = p_i$

$$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2) p_i - 2\rho i c \epsilon_{ijk} \Omega_j p_k + (\lambda+\mu) p_i + \rho \Omega_j p_j \Omega_i = 0$$

For $i=1$

$$(\mu - \rho c^2 - \rho \Omega^2) p_1 - 2i\rho c \epsilon_{1jk} \Omega_j p_k + (\lambda+\mu) p_1 + \rho \Omega_j p_j \Omega_1 = 0$$

Let $\Omega = \Omega(0,0,1)$

i.e medium is rotating about z-axis

$$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2) p_1 - 2i\rho c \epsilon_{1jk} \Omega_j p_k + (\lambda+\mu) p_1 + 0 = 0$$

$$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2) p_1 - 2i\rho c \epsilon_{132} \Omega p_2 + (\lambda+\mu) p_1 = 0$$

$\because \epsilon_{132} = -1$

$$\Rightarrow (\lambda + 2\mu - \rho c^2 - \rho \Omega^2) p_1 + 2i\rho c \Omega p_2 = 0 \longrightarrow (A)$$

For $i=2$

$$(\mu - \rho c^2 - \rho \Omega^2) p_2 - 2\rho i c \epsilon_{2jk} \Omega_j p_k + (\lambda+\mu) p_2 + \rho \Omega_j p_j \Omega_2 = 0$$

Let $\Omega = \Omega(0,0,1)$

$$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2) p_2 - 2\rho i c \epsilon_{2jk} \Omega_j p_k + (\lambda+\mu) p_2 + 0 = 0$$

$$\Rightarrow (\lambda + 2\mu - \rho c^2 - \rho \Omega^2) p_2 - 2i\rho c \epsilon_{231} \Omega_3 p_1 = 0$$

$\because$ for $i=2$, $\Omega_j = 0$

$$\Rightarrow (\lambda+2\mu-\rho c^2-\rho\Omega^2)p_2 - 2i\rho c\Omega p_1 = 0 \longrightarrow (B)$$

From equ (1) & (2)

$$\begin{vmatrix} \lambda+2\mu-\rho c^2-\rho\Omega^2 & 2i\rho c\Omega \\ -2i\rho c\Omega & \lambda+2\mu-\rho c^2-\rho\Omega^2 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+2\mu-\rho c^2-\rho\Omega^2)^2 - 4\rho^2c^2\Omega^2 = 0$$

$$\Rightarrow (\lambda+2\mu-\rho c^2-\rho\Omega^2)^2 = 4\rho^2c^2\Omega^2$$

$$\Rightarrow \lambda+2\mu-\rho c^2-\rho\Omega^2 = \pm 2\rho\Omega c$$

$$\lambda+2\mu-\rho c^2-\rho\Omega^2 = 2\rho\Omega c$$

$$\rho c^2+\rho\Omega^2+2\rho\Omega c = \lambda+2\mu$$

$$\Rightarrow c^2+\Omega^2+2\Omega c = \frac{\lambda+2\mu}{\rho}$$

$$\Rightarrow (c+\Omega)^2 = \frac{\lambda+2\mu}{\rho}$$

$$\Rightarrow c+\Omega = \sqrt{\frac{\lambda+2\mu}{\rho}}$$

$\because c$ is +ve always

$$\Rightarrow \boxed{c = \sqrt{\frac{\lambda+2\mu}{\rho}} - \Omega}$$

$$\lambda+2\mu-\rho c^2-\rho\Omega^2 = -2\rho\Omega c$$

$$\rho c^2+\rho\Omega^2-2\rho\Omega c = \lambda+2\mu$$

$$c^2+\Omega^2-2\Omega c = \frac{\lambda+2\mu}{\rho}$$

$$\Rightarrow (c-\Omega)^2 = \frac{\lambda+2\mu}{\rho}$$

$$\Rightarrow c-\Omega = \sqrt{\frac{\lambda+2\mu}{\rho}}$$

$$\Rightarrow \boxed{c = \sqrt{\frac{\lambda+2\mu}{\rho}} + \Omega}$$

Now for Transverse Waves $n_i \perp p_i$

$$\Rightarrow n_i p_i = 0$$

$$(\mu-\rho c^2-\rho\Omega^2)p_i - 2\rho i c\,\epsilon_{ijk}\Omega_j p_k + \rho\Omega_j p_j \Omega_i = 0$$

for $i=1$ $(\mu - \rho c^2 - \rho \Omega^2) P_1 - 2\rho i c \epsilon_{1jk} \Omega_j P_k +$

$\rho \Omega_j P_j \Omega_1 = 0$

Let $\underline{\Omega} = \Omega(0,0,1)$

i.e medium is rotating about z-axis

$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2) P_1 - 2\rho i c \epsilon_{132} \Omega P_2 + 0 = 0$

$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2) P_1 + 2\rho i c \underbrace{\epsilon_{132}}_{=-1} \Omega P_2 = 0 \longrightarrow (3)$

for $i=2$ $\Rightarrow (\mu - \rho c^2 - \rho \Omega^2) P_2 - 2i\rho c \epsilon_{2jk} \Omega_j P_k +$

$\rho \Omega_j P_j \Omega_2 = 0$

Let $\underline{\Omega} = \Omega(0,0,1)$

$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2) P_2 - 2i\rho c \epsilon_{231} \Omega P_1 + 0 = 0$

$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2) P_2 - 2i\rho c \Omega P_1 = 0 \longrightarrow (4)$

From equation (3) & (4)

$$\begin{vmatrix} (\mu - \rho c^2 - \rho \Omega^2) & 2\rho i c \Omega \\ -2\rho i c \Omega & \mu - \rho c^2 - \rho \Omega^2 \end{vmatrix} = 0 \quad \left\{ \because \text{for trivial solution} \right.$$

$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2)^2 - 4\rho^2 c^2 \Omega^2 = 0$

$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2)^2 = 4\rho^2 c^2 \Omega^2$

$\Rightarrow \mu - \rho c^2 - \rho \Omega^2 = \pm 2\rho c \Omega$

$\Rightarrow \mu - \rho c^2 - \rho \Omega^2 = 2\rho c \Omega$	$\mu - \rho c^2 - \rho \Omega^2 = -2\rho c \Omega$
$\Rightarrow \rho c^2 + \rho \Omega^2 + 2\rho c \Omega = \mu$	$\Rightarrow \rho c^2 + \rho \Omega^2 - 2\rho c \Omega = \mu$

$\Rightarrow c^2+\Omega^2+2c\Omega = \mu/\rho$

$\Rightarrow (c+\Omega)^2 = \mu/\rho$

$\Rightarrow c+\Omega = \sqrt{\mu/\rho}$

$\Rightarrow \boxed{c = \sqrt{\mu/\rho} - \Omega}$

$\Rightarrow c^2+\Omega^2-2c\Omega = \mu/\rho$

$\Rightarrow (c-\Omega)^2 = \mu/\rho$

$\Rightarrow c-\Omega = \sqrt{\mu/\rho}$

$\Rightarrow \boxed{c = \sqrt{\mu/\rho} + \Omega}$

Now For 3D:- For Longitudinal Waves $n_i \perp p_i \Rightarrow n_i p_i = 0$

$\Rightarrow (\mu - \rho c^2 - \rho\Omega^2)p_i - 2\rho i c \epsilon_{ijk}\Omega_j p_k + (\lambda+\mu)p_i + \rho\Omega_j p_j \Omega_i = 0$

for $i=3$ $(\mu-\rho c^2-\rho\Omega^2)p_3 - 0 + (\lambda+\mu)p_3 + \rho\Omega_3 p_3 \Omega_3 = 0$

$\because \underline{\Omega} = \Omega(0,0,1) \Rightarrow \epsilon_{ijk} = 0$ for $j=1,2$ & $i=j=3$

$\Rightarrow (\mu - \rho c^2 - \cancel{\rho\Omega^2} + \lambda + \mu + \cancel{\rho\Omega^2})p_3 = 0$

$\Rightarrow (\lambda+2\mu-\rho c^2)p_3 = 0 \Rightarrow \lambda+2\mu-\rho c^2 = 0$

$\Rightarrow \rho c^2 = \lambda+2\mu \Rightarrow \boxed{c = \sqrt{\frac{\lambda+2\mu}{\rho}}}$

For Transverse Waves $n_i \parallel p_i \Rightarrow n_i p_i = 1 \Rightarrow n_i = p_i$

equ ① $\Rightarrow (\mu-\rho c^2-\rho\Omega^2)p_i - 2\rho i c\epsilon_{ijk}\Omega_j p_k + \rho\Omega_j p_j \Omega_i = 0$

for $i=3$ $\Rightarrow (\mu-\rho c^2-\rho\Omega^2)p_3 - 2\rho i c\epsilon_{3jk}\Omega_j p_k + \rho\Omega_j p_j \Omega_3 = 0$

Let $\underline{\Omega} = \Omega(0,0,1)$

$\Rightarrow (\mu-\rho c^2-\rho\Omega^2)p_3 - 0 + \rho\Omega_3 p_3 \Omega_3 = 0$

$\Rightarrow (\mu-\rho c^2-\cancel{\rho\Omega^2}+\cancel{\rho\Omega^2})p_3 = 0$

$\Rightarrow (\mu-\rho c^2)p_3 = 0 \Rightarrow \mu-\rho c^2 = 0$

$\Rightarrow \rho c^2 = \mu \Rightarrow \boxed{c = \sqrt{\frac{\mu}{\rho}}}$

Remaining all same as in 2D upto equ ② & for $i=1,2$. In 3D only $i=3$ is extra.

Scanned by CamScanner

⇒ **Transversely Isotropic Medium:** A body / medium is said to be transversely isotropic medium if there exit an axis s.t all the planes ⊥ to that axis are plane of isotropic. In Transversely isotropic there are 5 independent elastic constants.

$$c_{ij} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{c_{11}-c_{12}}{2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{c_{11}-c_{12}}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix}$$

Equation of Motion is

$$\sigma_{ij,j} = \rho \ddot{u}_i$$

For $i=1$ $\quad \sigma_{1j,j} = \rho \ddot{u}_1$

$$\Rightarrow \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} = \rho \ddot{u}_1 \longrightarrow (1)$$

Now

$$\sigma_{11} = c_{11}\epsilon_{11} + c_{12}\epsilon_{22} + c_{13}\epsilon_{33}$$

$$\Rightarrow \sigma_{11} = C_{1} \cdot \frac{1}{2}(u_{1,1}+u_{1,1}) + \frac{1}{2} C_{2}(u_{2,2}+u_{2,2}) + C_{3} \cdot \frac{1}{2}(u_{3,3}+u_{3,3})$$

$$\Rightarrow \sigma_{11} = C_{1} u_{1,1} + C_{2} u_{2,2} + C_{3} u_{3,3}$$

$$\Rightarrow \sigma_{11,1} = C_{11} u_{1,11} + C_{2} u_{2,21} + C_{3} u_{3,31}$$

And $\sigma_{12} = (C_{1} - C_{2})\epsilon_{12} = \frac{1}{2}(C_{1}-C_{2})(u_{1,2}+u_{2,1})$

$$\Rightarrow \sigma_{12,2} = \frac{1}{2}(C_{1}-C_{2})(u_{1,22}+u_{2,12})$$

And $\sigma_{13} = 2C_{55}\epsilon_{13} = 2C_{55} \cdot \frac{1}{2}(u_{1,3}+u_{3,1})$

$$\Rightarrow \sigma_{13} = C_{55}(u_{1,3}+u_{3,1})$$

$$\Rightarrow \sigma_{13,3} = C_{55}(u_{1,33}+u_{3,13})$$

So equ ① becomes

$$C_{11} u_{1,11} + C_{12} u_{2,21} + C_{3} u_{3,31} + \frac{1}{2}(C_{1}-C_{2})(u_{1,22}+u_{2,12}) + C_{55}(u_{1,33}+u_{3,13}) = \rho \ddot{u}_{1} \longrightarrow \text{(A)}$$

For $i=2$

$$\sigma_{2j,j} = \rho \ddot{u}_{2}$$

$$\Rightarrow \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} = \rho \ddot{u}_{2} \longrightarrow ②$$

Now $\sigma_{21} = (C_{1}-C_{2})\epsilon_{12}$ $\quad \because \sigma_{ij} = \sigma_{ji}$

$$= \frac{1}{2}(C_{1}-C_{2})(u_{1,2}+u_{2,1})$$

$$\Rightarrow \sigma_{21,1} = \frac{1}{2}(C_{1}-C_{2})(u_{1,21}+u_{2,11})$$

And $\sigma_{22} = C_{12}\epsilon_{11} + C_{11}\epsilon_{22} + C_{13}\epsilon_{33}$

$$= C_{12}U_{1,1} + C_{11}U_{2,2} + C_{13}U_{3,3}$$

$$\Rightarrow \sigma_{22,2} = C_{12}U_{1,12} + C_{11}U_{2,22} + C_{13}U_{3,32}$$

And $\sigma_{23} = 2C_{44}\epsilon_{23} = \frac{1}{2}\cdot 2C_{44}(U_{2,3} + U_{3,2})$

$$= C_{44}(U_{2,3} + U_{3,2})$$

$$\Rightarrow \sigma_{23,3} = C_{44}(U_{2,33} + U_{3,23})$$

So equ ② becomes

$$\frac{1}{2}(C_{11} - C_{12})(U_{1,21} + U_{2,11}) + C_{12}U_{1,12} + C_{11}U_{2,22} + C_{13}U_{3,32}$$
$$+ C_{44}(U_{2,33} + U_{3,23}) = \rho\ddot{U}_2 \longrightarrow (B)$$

For $i=3$

$$\sigma_{3j,j} = \rho\ddot{U}_3$$

$$\Rightarrow \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = \rho\ddot{U}_3 \longrightarrow ③$$

Now $\sigma_{31} = 2C_{55}\epsilon_{13} = 2\cdot\frac{1}{2}C_{55}(U_{1,3} + U_{3,1})$

$$= C_{55}(U_{1,3} + U_{3,1})$$

$$\sigma_{32} = 2C_{44}\epsilon_{23} = 2\cdot\frac{1}{2}C_{44}(U_{2,3} + U_{3,2})$$

$$= C_{44}(U_{2,3} + U_{3,2})$$

And $\sigma_{33} = C_{13}\epsilon_{11} + C_{13}\epsilon_{22} + C_{11}\epsilon_{33}$

$$= C_{13}U_{1,1} + C_{13}U_{2,2} + C_{11}U_{3,3}$$

$$\Rightarrow \sigma_{31,1} = C_{55}(U_{1,31} + U_{3,11})$$

$$\sigma_{32,2} = C_{44}(U_{2,32} + U_{3,22})$$

$$\sigma_{33,3} = C_{13} U_{1,13} + C_{13} U_{2,23} + C_{11} U_{3,33}$$

So equation ③ becomes

$$C_{55}(U_{1,31} + U_{3,11}) + C_{44}(U_{2,32} + U_{3,22}) + C_{13} U_{1,13} + C_{13} U_{2,23} + C_{11} U_{3,33} = \rho \ddot{U}_3 \longrightarrow (C)$$

Let $U_i = A e^{ik(x_j n_j - ct)} P_i$

Using U_i equ (A) becomes

$$C_{11}(ikn_1)(ikn_1) P_1 A e^{ik(x_j n_j - ct)} + C_{12}(ikn_2)(ikn_1) P_2 A e^{-} + C_{13}(ikn_3)(ikn_1) P_3 A e^{ik(-)} + \frac{1}{2}(C_{11} - C_{12})\{(ikn_2)^2 P_1 A e^{ik(-)} + (ikn_1)(ikn_2) P_2 A e^{ik(-)}\} + C_{55}\{(ikn_3)^2 P_1 A e^{ik(-)} + (ikn_1)(ikn_3) P_3 A e^{ik(-)}\} = \rho(-ikc)^2 P_1 A e^{ik(-)}$$

$$\Rightarrow C_{11} n_1^2 P_1 + C_{12} n_1 n_2 P_2 + C_{13} n_1 n_3 P_3 + \frac{1}{2}(C_{11} - C_{12})\{n_2^2 P_1 + n_1 n_2 P_2\} + C_{55}\{n_3^2 P_1 + n_1 n_3 P_3\} = \rho c^2 P_1$$

$$\Rightarrow \{C_{11} n_1^2 + \frac{1}{2}(C_{11} - C_{12}) n_2^2 + C_{55} n_3^2 - \rho c^2\} P_1 + \{C_{12} n_1 n_2 + \frac{1}{2}(C_{11} - C_{12}) n_1 n_2\} P_2 + \{C_{13} n_1 n_3 + C_{55} n_1 n_3\} P_3 = 0 \longrightarrow (4)$$

Now

using U_i equ (B) becomes

$$\frac{1}{2}(C_{11} - C_{12})\{(ikn_1)(ikn_2) P_1 + (ikn_1)^2 P_2\} + C_{12}(ikn_1)(ikn_2) P_1$$

$$+C_{11}(ikn_2)^2P_2 + C_{13}(ikn_2)(ikn_3)P_3 + C_{44}\{(ikn_3)^2P_2 + (ikn_2)(ikn_3)P_3\} = (-ikc)^2\rho P_2$$

$$\Rightarrow \frac{1}{2}(C_{11}-C_{12})\{n_1n_2P_1 + n_1^2P_2\} + C_{12}n_1n_2P_1 + C_{11}n_2^2P_2 + C_{13}n_2n_3P_3 + C_{44}\{n_3^2P_2 + n_2n_3P_3\} = \rho c^2 P_2$$

$$\Rightarrow \{\tfrac{1}{2}(C_{11}-C_{12})n_1n_2 + C_{12}n_1n_2\}P_1 + \{\tfrac{1}{2}(C_{11}-C_{12})n_1^2 + C_{11}n_2^2 + C_{44}n_3^2 - \rho c^2\}P_2 + \{C_{13}n_2n_3 + C_{44}n_2n_3\}P_3 = 0 \longrightarrow (5)$$

Using u_i equ (c) becomes

$$C_{55}\{(ikn_3)(ikn_1)P_1 + (ikn_1)^2P_3\} + C_{44}\{(ikn_3)(ikn_2)P_2 + (ikn_2)^2P_3\} + C_{13}(ikn_1)(ikn_3)P_1 + C_{13}(ikn_2)(ikn_3)P_2 + (ikn_3)^2P_3 = \rho(-ikc)^2P_3$$

$$\Rightarrow \{C_{55}n_1n_3 + C_{13}n_1n_3\}P_1 + \{C_{44}n_2n_3 + C_{13}n_2n_3\}P_2 + \{C_{55}n_1^2 + C_{44}n_2^2 + C_{33}n_3^2 - \rho c^2\}P_3 = 0 \longrightarrow (6)$$

For simplicity Put $n = n(1,0,0)$

so

$$(4) \Rightarrow (C_{11} - \rho c^2)P_1 = 0 \longrightarrow (7)$$

$$(5) \Rightarrow \{\tfrac{1}{2}(C_{11}-C_{12}) - \rho c^2\}P_2 = 0 \longrightarrow (8)$$

$$(6) \Rightarrow (C_{55} - \rho c^2)P_3 = 0 \longrightarrow (9)$$

from (7), (8) & (9)
we have

$$\begin{vmatrix} C_{11}-\rho c^2 & 0 & 0 \\ 0 & \frac{1}{2}(C_{11}-C_{12})-\rho c^2 & 0 \\ 0 & 0 & C_{55}-\rho c^2 \end{vmatrix} = 0$$

$$\Rightarrow (C_{11}-\rho c^2)\left\{\frac{1}{2}(C_{11}-C_{12})-\rho c^2\right\}(C_{55}-\rho c^2) = 0$$

$\Rightarrow C_{11}-\rho c^2 = 0$

$\rho c^2 = C_{11}$

$\Rightarrow c^2 = \dfrac{C_{11}}{\rho}$

$\Rightarrow \boxed{c = \sqrt{\dfrac{C_{11}}{\rho}}}$

$\frac{1}{2}(C_{11}-C_{12})-\rho c^2 = 0$

$\Rightarrow \rho c^2 = \frac{1}{2}(C_{11}-C_{12})$

$\Rightarrow c^2 = \dfrac{C_{11}-C_{12}}{2\rho}$

$\Rightarrow \boxed{c = \sqrt{\dfrac{C_{11}-C_{12}}{2\rho}}}$

$C_{55}-\rho c^2 = 0$

$\Rightarrow \rho c^2 = C_{55}$

$\Rightarrow c^2 = \dfrac{C_{55}}{\rho}$

$\Rightarrow \boxed{c = \sqrt{\dfrac{C_{55}}{\rho}}}$

★ For Isotropic Medium:-

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix}$$

Two Independent C_{ij}

(★ 2D کیلئے solve کرنے کیلئے ون پہلے دو انٹری (دیکھنی ہیں) Rows میں)
نان زیرو ہوگی اس کے بعد والی ساری زیرو ہونگی (ہر میٹرکس میں)

* For Transversely Isotropic Medium:-

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix}$$

"5 independent"

* For Orthotropic Medium:-

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}$$

"There are 9 independent"

* For Monoclinic Medium:-

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{12} & C_{22} & C_{23} & C_{24} & 0 & 0 \\ C_{13} & C_{23} & C_{33} & C_{34} & 0 & 0 \\ C_{14} & C_{24} & C_{34} & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & C_{56} \\ 0 & 0 & 0 & 0 & C_{56} & C_{66} \end{bmatrix}$$

"There are 13 independent"

⇒ Speed of Waves in a Non-Homogeneous Isotropic Medium:-

Equation of Motion is

$$\sigma_{ij,j} = \rho \ddot{u}_i \longrightarrow (*)$$

Here $\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$

$$\Rightarrow \sigma_{ij} = \lambda \cdot \frac{1}{2}(u_{k,k} + u_{k,k})\delta_{ij} + 2 \cdot \frac{1}{2} \mu (u_{i,j} + u_{j,i})$$

$$\because \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$\Rightarrow \sigma_{ij} = \lambda u_{k,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i})$$

$$\Rightarrow \sigma_{ij,j} = \{\lambda u_{k,k} \delta_{ij}\}_{,j} + \{\mu (u_{i,j} + u_{j,i})\}_{,j}$$

$$= \{\lambda u_{k,k}\}_{,i} + \{\mu (u_{i,j} + u_{j,i})\}_{,j}$$

$$\Rightarrow \sigma_{ij,j} = \lambda_{,i} u_{k,k} + \lambda u_{k,ki} + \mu_{,j}(u_{i,j} + u_{j,i}) + \mu (u_{i,jj} + u_{j,ij})$$

So $(*)$ becomes

$$\lambda_{,i} u_{k,k} + \lambda u_{k,ki} + \mu_{,j}(u_{i,j} + u_{j,i}) + \mu(u_{i,jj} + u_{j,ij}) = \rho \ddot{u}_i \longrightarrow (1)$$

Since for Non Homogeneous medium

$\lambda = \lambda_0 e^{-mx_2}$, $\mu = \mu_0 e^{-mx_2}$ & $\rho = \rho_0 e^{-mx_2}$ { Non-Homo w.r.t x_2

So equation ① becomes

$$(\lambda_o e^{-mx_2})_{,i}\, u_{k,k} + (\lambda_o e^{-mx_2}) u_{k,ki} + (\mu_o e^{-mx_2})_{,j}(u_{i,j} + u_{j,i}) + (\mu_o e^{-mx_2})\{u_{i,jj} + u_{j,ij}\} = \rho_o e^{-mx_2}\ddot{u}_i \longrightarrow ②$$

Let $u_i = A e^{i(x_j n_j - ct)} P_i \quad \because k=1$

So equ ② becomes

$$(\lambda_o e^{-mx_2})_{,i}\, i n_k A e^{i(x_j n_j - ct)} P_k + \lambda_o e^{-mx_2}\{(i n_k)(i n_i) A e^{i(x_j n_j - ct)} P_k\} + (\mu_o e^{-mx_2})_{,j}\{(i n_j) A e^{i(x_j n_j - ct)} P_i + (i n_i) A e^{i(x_j n_j - ct)} P_j\} + (\mu_o e^{-mx_2})\{(i n_j)^2 A e^{i(x_j n_j - ct)} P_i + (i n_i)(i n_j) A e^{i(x_j n_j - ct)} P_j\} = \rho_o e^{-mx_2}(-ic)^2 A e^{i(x_j n_j - ct)} P_i$$

$$\Rightarrow (\lambda_o e^{-mx_2})_{,i}\, i n_k P_k - \lambda_o e^{-mx_2} n_k n_i P_k + (\mu_o e^{-mx_2})_{,j}\{i n_j P_i + i n_i P_j\} - \mu_o e^{-mx_2}\{n_j^2 P_i + n_i n_j P_j\} = -\rho_o e^{-mx_2} c^2 P_i \longrightarrow ③$$

$\{2D\}$

For $i=1$

$$\{\lambda_o e^{-mx_2}\}_{,1}\, i n_k P_k - \lambda_o e^{-mx_2} n_k n_1 P_k + (\mu_o e^{-mx_2})_{,j}\{i n_j P_1 + i n_1 P_j\} - \mu_o e^{-mx_2}\{n_j^2 P_1 + n_1 n_j P_j\} = -\rho_o e^{-mx_2} c^2 P_1$$

$$\Rightarrow -\lambda_o e^{-mx_2}\{n_1^2 P_1 + n_2 n_1 P_2\} + (-m)\mu_o e^{-mx_2}\{i n_2 P_1 +$$

$i n_1 P_2\} - \mu_0 e^{-mx_2}\{n_1^2 P_1 + n_1^2 P_1 + n_2^2 P_1 + n_1 n_2 P_2\}$

$$= -c^2 \rho_0 e^{-mx_2} P_1$$

Put $\underline{n} = n(1,0)$

$\Rightarrow -\lambda_0 e^{-mx_2} P_1 - i m \mu_0 e^{-mx_2} P_2 - 2\mu_0 e^{-mx_2} P_1$

$$= -c^2 \rho_0 e^{-mx_2} P_1$$

$\Rightarrow \{(\lambda_0 + 2\mu_0 - \rho_0 c^2) P_1 + (i m \mu_0) P_2\} e^{-mx_2} = 0$

$\Rightarrow (\lambda_0 + 2\mu_0 - \rho_0 c^2) P_1 + (i m \mu_0) P_2 = 0$

choose $\underline{P} = P(1,0)$

$\Rightarrow \lambda_0 + 2\mu_0 - \rho_0 c^2 = 0$

$\Rightarrow \rho_0 c^2 = \lambda_0 + 2\mu_0$

$$\Rightarrow \boxed{c = \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho_0}}}$$

for $i=2$

$\Rightarrow (\lambda_0 e^{-mx_2})_{,2}\, i n_k P_k - \lambda_0 e^{-mx_2}\{n_k n_2 P_k\}$

$+ (\mu_0 e^{-mx_2})_{,j}\{i n_j P_2 + i n_2 P_j\} - \mu_0 e^{-mx_2}\{n_j^2 P_2$

$+ n_2 n_j P_j\} = -\rho_0 c^2 e^{-mx_2} P_2$

$\Rightarrow (-m)\lambda_0 e^{-mx_2}\{i n_1 P_1 + i n_2 P_2\} - \lambda_0 e^{-mx_2}\{n_1 n_2 P_1 + n_2^2 P_2\}$

$+ (-m)\mu_0 e^{-mx_2}\{i n_2 P_2 + i n_2 P_2\} - \mu_0 e^{-mx_2}\{n_1^2 P_2 +$

$$n_2^2 P_2 + n_2 n_1 P_1 + n_2^2 P_2\} = -\rho_0 c^2 e^{-mx_2} P_2$$

Put $\underline{n} = n(1,0)$

$$\Rightarrow -im\lambda_0 e^{-mx_2} P_1 - \mu_0 e^{-mx_2} P_2 = -\rho_0 c^2 e^{-mx_2} P_2$$

$$\Rightarrow \{+im\lambda_0 e^{-mx_2} P_1\} + \{(\mu_0 - \rho_0 c^2) P_2\} e^{-mx_2} = 0$$

$$\Rightarrow im\lambda_0 P_1 + (\mu_0 - \rho_0 c^2) P_2 = 0$$

Choose $P = \underline{P}(0,1)$

$$\Rightarrow \mu_0 - \rho_0 c^2 = 0 \Rightarrow \rho_0 c^2 = \mu_0$$

$$\Rightarrow \boxed{c = \sqrt{\frac{\mu_0}{\rho_0}}}$$

⇒ **Orthotropic:-** Having three mutually perpendicular planes of at each point.

⇒ **Monoclinic:-** Having three unequal axis with two perpendicular and one oblique intersection.

MUHAMMAD TAHIR

FA15-RMT-007

M.S MATH

Non-Homogeneous Isotropic Medium :-
(w.r.t x_2 in 2D)

Equation of Motion is

$$\sigma_{ij,j} = \rho \ddot{u}_i \longrightarrow (*)$$

where for isotropic medium

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$$

$$= \lambda u_{k,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i})$$

$$\Rightarrow \sigma_{ij,j} = \{\lambda u_{k,k} \delta_{ij}\}_{,j} + \{\mu (u_{i,j} + u_{j,i})\}_{,j}$$

$$= \{\lambda u_{k,k}\}_{,i} + \{\mu (u_{i,j} + u_{j,i})\}_{,j}$$

$$\Rightarrow \sigma_{ij,j} = \lambda_{,i} u_{k,k} + \lambda u_{k,ki} + \mu_{,j} (u_{i,j} + u_{j,i}) + \mu (u_{i,jj} + u_{j,ij})$$

So (*) becomes

$$\lambda_{,i} u_{k,k} + \lambda u_{k,ki} + \mu_{,j} (u_{i,j} + u_{j,i}) + \mu (u_{i,jj} + u_{j,ij}) = \rho \ddot{u}_i \longrightarrow (1)$$

for $i=1$ $\Rightarrow$ $\lambda_{,1} u_{k,k} + \lambda u_{k,k1} + \mu_{,j} (u_{1,j} + u_{j,1}) + \mu (u_{1,jj} + u_{j,1j}) = \rho \ddot{u}_1$

$$\because \lambda = \lambda_0 e^{-mx_2}, \quad \mu = \mu_0 e^{-mx_2}$$

$$\rho = \rho_0 e^{-mx_2}$$

$$\Rightarrow \lambda_{,1}(U_{1,1}+U_{2,2}) + \lambda\{U_{1,11}+U_{2,21}\} + \mu_{,1}\{U_{1,1}+U_{1,1}\}$$

$$+\mu_{,2}\{U_{1,2}+U_{2,1}\} + \mu\{U_{1,11}+U_{1,11}+U_{1,22}+U_{2,12}\}$$

$$= \rho\ddot{U}_1$$

Let $U_i = A e^{i(x_j n_j - ct)} P_1$

$$\Rightarrow 0 + \lambda_0 e^{-m x_2}\{(in_1)^2 P_1 + (in_1)(in_2)P_2\} + 0 +$$

$$(-m)\mu_0 e^{-m x_2}\{(in_2)P_1 + (in_1)P_2\} + \mu_0 e^{-m x_2}\{2(in_1)^2 P_1$$

$$+(in_2)^2 P_1 + (in_1)(in_2)P_2\} = (-ic)^2 \rho_0 e^{-m x_2}$$

$$\Rightarrow +\lambda_0 n_1^2 P_1 + n_1 n_2 \lambda_0 P_2 + im\, n_2 \mu_0 P_1 + im\, n_1 \mu_0 P_2$$

$$+2\mu_0 n_1^2 P_1 + \mu_0 n_2^2 P_1 + n_1 n_2 \mu_0 P_2 = c^2 \rho_0$$

$$\Rightarrow \{\lambda_0 n_1^2 + im\, n_2 \mu_0 + 2\mu_0 n_1^2 + \mu_0 n_2^2 - \rho_0 c^2\} P_1$$

$$\{\lambda_0 n_1 n_2 + im\, n_1 \mu_0 + n_1 n_2 \mu_0\} P_2 = 0$$

$$\Rightarrow \{(\lambda_0 + 2\mu_0) n_1^2 + \mu_0 n_2^2 + im\, \mu_0 n_1 - \rho_0 c^2\} P_1$$

$$+\{(\lambda_0+\mu_0) n_1 n_2 + im\, n_1 \mu_0\} P_2 = 0 \longrightarrow (A)$$

For $i=2$

$$\lambda_{,2} U_{k,k} + \lambda U_{k,k2} + \mu_{,j}(U_{2,j}+U_{j,2}) +$$

$$\mu(U_{2,jj}+U_{j,2j}) = \rho\ddot{U}_2$$

$$\Rightarrow \lambda_{,2}(U_{1,1}+U_{2,2}) + \lambda(U_{1,12}+U_{2,22}) + \mu_{,1}(U_{2,1}+U_{1,2})$$

$$+\mu_{,2}(U_{2,2}+U_{2,2}) + \mu(U_{2,11}+U_{1,21}+U_{2,22}+U_{2,22})$$

$$= \rho\ddot{U}_2$$

Since $\lambda = \lambda_0 e^{-mx_2}$, $\mu = \mu_0 e^{-mx_2}$

$\rho = \rho_0 e^{-mx_2}$

And $U_i = A e^{ik(x_j n_j - ct)}$

$\Rightarrow (-m)\lambda_0((in_1)P_1 + (in_2)P_2) + \lambda_0((in_1)(in_2)P_1 + (in_2)^2 P_2)$

$+ (-m)\mu_0(2(in_2)P_2) + \mu_0((in_1)^2 P_2 + (in_1)(in_2)P_1 +$

$2(in_2)^2 P_2) = (-ic)^2 \rho_0 P_2$

$\Rightarrow (+imn_1\lambda_0 + n_1n_2\lambda_0 + n_1n_2\mu_0)P_1 + (+imn_2\lambda_0$

$+ n_2^2\lambda_0 + 2in_2m\mu_0 + n_1^2\mu_0 + 2n_2^2\mu_0 - \rho c_0^2)P_2 = 0$

$\Rightarrow ((\lambda_0+\mu_0)n_1n_2 + imn_1\lambda_0)P_1 + \{(\lambda_0+2\mu_0)n_2^2$

$+ \mu_0 n_1^2 + im(2\mu_0+\lambda_0)n_2 - \rho c^2)P_2 = 0$ — (B)

For Non trivial solution of eqn A & B

$$\Rightarrow \begin{vmatrix} (\lambda_0+2\mu_0)n_1^2 + \mu_0 n_2^2 + im\mu_0 n_1 - \rho_0 c^2 & (\lambda_0+\mu_0)n_1n_2 + imn_1\mu_0 \\ (\lambda_0+\mu_0)n_1n_2 + im\lambda_0 n_1 & (\lambda+2\mu_0)n_2^2 + \mu_0 n_1^2 + im(2\mu_0+\lambda_0)n_2 - \rho c^2 \end{vmatrix} = 0$$

$\Rightarrow \{(\lambda_0+2\mu_0)n_1^2 + \mu_0 n_2^2 + im\mu_0 n_1 - \rho_0 c^2\}\{(\lambda_0+2\mu_0)n_2^2 + \mu_0 n_1^2$

$+ im(2\mu_0+\lambda_0)n_2 - \rho c^2\} - \{(\lambda_0+\mu_0)n_1n_2 + imn_1\mu_0\}$

$\{(\lambda_0+\mu_0)n_1n_2 + im\lambda_0 n_1\} = 0$

Scanned by CamScanner

$$\Rightarrow \Big\{(\lambda_0+2\mu_0)^2 n_1^2 n_2^2 + (\lambda_0+2\mu_0)\mu_0 n_1^4 + im(\lambda_0+2\mu_0)n_1^2 n_2 - (\lambda_0+2\mu_0)\rho_0 n_1^2 c^2 + (\lambda_0+2\mu_0)\mu_0 n_2^4 + \mu_0^2 n_1^2 n_2^2 + im(\lambda_0+2\mu_0)\mu_0 n_2^3 - \mu_0\rho_0 c^2 n_2^2 + im(\lambda_0+2\mu_0)\mu_0 n_1 n_2^2 + im\mu_0^2 n_1 n_1^2 - m^2(\lambda_0+2\mu_0)\mu_0 n_1 n_2 - im\mu_0\rho_0 c^2 n_1 - (\lambda_0+2\mu_0)\rho_0 c^2 n_2^2 - \mu_0\rho_0 c^2 n_1^2 - im(\lambda_0+\mu_0)\rho_0 c^2 n_2 + (\rho_0 c^2)^2\Big\} - \Big\{(\lambda_0+\mu_0)^2 n_1^2 n_2^2 + im(\lambda_0+\mu_0)\lambda_0 n_1^2 n_2 + im(\lambda_0+\mu_0)\mu_0 n_1^2 n_2 - m^2\mu_0\lambda_0 n_1^2\Big\} = 0$$

$$\Rightarrow (\rho_0 c^2)^2 + (\cdots)\rho_0 c^2 + (\quad) = 0$$

$$\Rightarrow \rho_0 c^2 = \frac{\lambda n_1^2}{2\rho_0} + \frac{3\mu n_1^2}{2\rho_0} + \frac{m\lambda_0 n_2}{2\rho_0} + \frac{m\mu_0 n_2}{2\rho_0} + \frac{\lambda n_2^2}{2\rho_0} + \frac{3\mu n_2^2}{2\rho_0} \pm A$$

$$\Rightarrow \rho_0 c^2 = \left\{\frac{\lambda_0+3\mu_0}{2\rho_0} + \frac{m}{2\rho_0}(\lambda_0+\mu_0)n_2\right\} \pm A$$

where $A = \frac{1}{2\rho_0^2}\Big[\big(-\lambda_0\rho_0 n_1^2 - 3\mu n_1^2 - m\lambda_0 n_2 - m\mu\rho_0 n_2 - \lambda_0\rho_0 n_2^2 - 3\mu\rho_0 n_2^2\big)^2 - 4\rho_0^2\big(m\lambda_0^2 n_1 + m\lambda_0\mu_0 n_1 + m\lambda_0 n_1^2 + \lambda_0\mu_0 n_1^4 + 2\mu_0^2 n_1^4 - \lambda_0^2 n_1 n_2 - 2\lambda_0\mu_0 n_1 n_2 - \mu_0^2 n_1 n_2 - \lambda_0 n_1^2 n_2 + m\lambda_0^2 n_1^2 n_2 - \mu_0 n_1^2 n_2 + 4m\lambda_0\mu_0 n_1^2 n_2 + 3m\mu_0 n_1^2 n_2 - m^2\lambda_0\mu_0 n_2^2 - 2m^2\mu_0^2 n_2^2 + \lambda^2 n_1^2 n_2^2 + 4\lambda_0\mu_0 n_1^2 n_2^2 + 5\mu_0^2 n_1^2 n_2^2 + \lambda_0 n_2^2 + 2\mu_0^2 n_2^2\big)\Big]^{1/2}$

Particular Solutions

(i) $n\,(1,0)$ $\quad P=(1,0)$

(A) $\Rightarrow \lambda_0+2\mu_0+im\mu_0 n_1-\rho_0 c^2=0$

Put $m=0$

$\Rightarrow \lambda_0+2\mu_0=\rho_0 c^2$

$$\Rightarrow c=\sqrt{\frac{\lambda_0+2\mu_0}{\rho_0}}$$

(B) $\Rightarrow \mu_0-\rho_0 c^2=0$

$\Rightarrow \mu_0=\rho_0 c^2 \quad \Rightarrow c=\sqrt{\mu_0/\rho_0}$

(ii) $n=(0,1)$ $\quad P=(0,1)$

(A) $\Rightarrow \mu_0-\rho_0 c^2=0$

$\Rightarrow \mu_0=\rho_0 c^2 \quad \Rightarrow c=\sqrt{\mu_0/\rho_0}$

(B) $\Rightarrow (\lambda_0+2\mu_0)+im(2\mu_0+\lambda_0)-\rho_0 c^2=0$

Put $m=0$

$\Rightarrow \lambda_0+2\mu_0-\rho_0 c^2=0$

$\Rightarrow \rho_0 c^2=\lambda_0+2\mu_0$

$$\Rightarrow c=\sqrt{\frac{\lambda_0+2\mu_0}{\rho_0}}$$

③ $\underline{n}(1,0)$ & $\underline{P} = (0,1)$

Ⓐ $\Rightarrow$ $i m \mu_0 = 0 \Rightarrow$ imaginary

Ⓑ $\Rightarrow$ $\mu_0 - \rho_0 c^2 = 0$

$\Rightarrow \mu_0 = \rho_0 c^2 \Rightarrow c = \sqrt{\mu_0/\rho_0}$

④ $\underline{n} = (0,1)$ $\underline{P} = (1,0)$

Ⓐ $\Rightarrow$ $\mu_0 - \rho_0 c^2 = 0$

$\Rightarrow c = \sqrt{\mu_0/\rho_0}$

Ⓑ $\Rightarrow 0$

Question:- For a material it is given that $\lambda + 2\mu = 42.80 \times 10^9$ N/m²
$\mu = 10.70 \times 10^9$ N/m², $\rho = 5.35 \times 10^3$ Kg/m³
$\Omega = 2 \times 10^4$, 2×10^6 ; $\Omega = (0,0,1)$

(i) $\underline{n} = (1,0,0)$, $P = (1,0,0)$

(ii) $\underline{n} = (0,1,0)$, $P = (1,0,0)$

find speed of Waves.

Solution

★ For a Non Homogeneous material along x_2 - axis

$$\lambda = \lambda_0 e^{-mx_2}, \quad \mu = \mu_0 e^{-mx_2}$$

$$\rho = \rho_0 e^{-mx_2}$$

★ Non - Homogeneous along x_i axis

$$\lambda = \lambda_0 e^{-m_i x_i}, \quad \mu = \mu_0 e^{-m_i x_i}$$

$$\rho = \rho_0 e^{-m_i x_i}$$

★ For 3D

$$\lambda = \lambda_0 e^{-m_1x_1 - m_2x_2 - m_3x_3}$$

$$\mu = \mu_0 e^{-m_1x_1 - m_2x_2 - m_3x_3}$$

$$\rho = \rho_0 e^{-m_1x_1 - m_2x_2 - m_3x_3}$$

★———★

Assignment:- Find speed of waves in a Non-Homogeneous, w.r.t x_1, x_2, x_3 axis, isotropic medium.

Solution Equation of motion is

$$\sigma_{ij,j} = \rho \ddot{u}_i \longrightarrow (1)$$

where $\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$

$$= \lambda u_{k,k} \delta_{ij} + \mu(u_{i,j} + u_{j,i})$$

$$\Rightarrow \sigma_{ij,j} = \{\lambda u_{k,k} \delta_{ij}\}_{,j} + \{\mu(u_{i,j} + u_{j,i})\}_{,j}$$

$$\Rightarrow \sigma_{ij,j} = (\lambda u_{k,k})_{,i} + \{\mu(u_{i,j}+u_{j,i})\}_{,j}$$

$$= \lambda_{,i} u_{k,k} + \lambda u_{k,ki} + \mu_{,j}(u_{i,j}+u_{j,i})$$

$$+ \mu(u_{i,jj} + u_{j,ij})$$

Thus equation ① becomes

$$\lambda_{,i} u_{k,k} + \lambda u_{k,ki} + \mu_{,j}(u_{i,j}+u_{j,i}) + \mu(u_{i,jj} + u_{j,ij}) = \rho \ddot{u}_i \longrightarrow ②$$

for $i=1$

$$\Rightarrow \lambda_{,1} u_{k,k} + \lambda u_{k,k1} + \mu_{,j}(u_{1,j}+u_{j,1}) + \mu(u_{1,jj} + u_{j,1j}) = \rho \ddot{u}_1$$

$$\Rightarrow \lambda_{,1}(u_{1,1}+u_{2,2}+u_{3,3}) + \lambda(u_{1,11}+u_{2,21}+u_{3,31})$$

$$+\mu_{,1}(u_{1,1}+u_{1,1}) + \mu_{,2}(u_{1,2}+u_{2,1}) + \mu_{,3}(u_{1,3}+u_{3,1})$$

$$+\mu(u_{1,11}+u_{1,11}+u_{1,22}+u_{2,12}+u_{1,33}+u_{3,13}) = \rho\ddot{u}_1 \longrightarrow ③$$

Since for 3D Non-Homogeneous medium

$$\lambda = \lambda_0 e^{-m_1x_1 - m_2x_2 - m_3x_3}, \quad \mu = \mu_0 e^{-m_1x_1 - m_2x_2 - m_3x_3}$$

$$\rho = \rho_0 e^{-m_1x_1 - m_2x_2 - m_3x_3}$$

And also $u_i = A e^{i(x_j n_j - ct)} p_i$

So eqn ③ becomes

$$(-m_1)\lambda_0(in_1 p_1 + in_2 p_2 + in_3 p_3) + \lambda_0\{(in_1)^2 p_1 - n_1 n_2 p_2 - n_1 n_3 p_3\} + (-m_1)\mu_0(2in_1 p_1) - (m_2)\mu_0(in_2 p_1 + in_1 p_2)$$

$+(-m_3)\mu_0(in_3P_1+in_1P_3)+\mu_0\{2n_1^2P_1+n_2^2P_1+n_1n_2P_2+n_3^2P_1+n_1n_3P_3\}=\rho_0c^2P_1$

$\Rightarrow\{im_1n_1\lambda_0+2im_1n_1\mu_0+im_2n_2\mu_0+im_3n_3\mu_0+2n_1^2\mu_0+n_3^2\mu_0+n_2^2\mu_0-\rho_0c^2\}P_1+\{im_1n_2\lambda_0+im_2n_1\mu_0+n_1n_2\mu_0+n_1n_2\lambda_0\}P_2+\{im_1n_3\lambda_0+in_1m_3\mu_0+n_1n_3\mu_0+n_1n_3\lambda_0\}P_3=0 \rightarrow (A)$

for $i=2$ $\Rightarrow$

$\lambda_{,2}u_{k,k}+\lambda u_{k,k2}+\mu_{,j}\{u_{2,j}+u_{j,2}\}+\mu\{u_{2,jj}+u_{j,2j}\}=\rho\ddot{u}_2$

$\Rightarrow\lambda_{,2}(u_{1,1}+u_{2,2}+u_{3,3})+\lambda(u_{1,12}+u_{2,22}+u_{3,32})+\mu_{,1}(u_{2,1}+u_{2,2})+\mu_{,2}(u_{2,12}+u_{2,2})+\mu_{,3}(u_{2,3}+u_{3,2})+\mu(u_{2,11}+u_{1,21}+u_{2,22}+u_{2,22}+u_{2,33}+u_{3,23})=\rho\ddot{u}_2$

$\Rightarrow+m_2\lambda_0\{in_1P_1+in_2P_2+in_3P_3\}+\lambda_0\{n_1n_2P_1+n_2^2P_2+n_2n_3P_3\}+m_1\mu_0(in_1P_2+in_2P_1)+2im_2\mu_0P_2+im_3\mu_0n_3P_2+im_3n_2\mu_0P_3+\mu_0\{n_1^2P_2+n_1n_2P_1+2n_2^2P_2+n_3^2P_2+n_2n_3P_3\}$
$=\rho_0(-ic)^2P_2$

$\Rightarrow\{im_2n_1\lambda_0+n_2^2\lambda_0+im_1n_1\mu_0+im_2n_2\mu_0+im_3n_3\mu_0+n_1^2\mu_0+2n_2^2\mu_0+n_3^2\mu_0-\rho c^2\}P_2+\{in_1m_2\lambda_0+$

$$n_1 n_2 \lambda_0 + i m_1 n_2 \mu_0 + n_1^2 \mu_0 \} P_1 + \{ i m_2 n_3 \lambda_0 + n_2 n_3 \lambda_0 + i n_2 m_3 \mu_0 + n_2 n_3 \mu_0 \} P_3 = 0 \longrightarrow (B)$$

For $i=3$

$$\Rightarrow \lambda_{,3} u_{k,k} + \lambda u_{k,k3} + \mu_{,j}(u_{3,j} + u_{j,3}) + \mu(u_{3,jj} + u_{j,3j}) = \rho \ddot{u}_3$$

$$\Rightarrow \lambda_{,3}(u_{1,1} + u_{2,2} + u_{3,3}) + \lambda(u_{1,13} + u_{2,23} + u_{3,33}) + \mu_{,1}(u_{3,1} + u_{1,3}) + \mu_{,2}(u_{3,2} + u_{2,3}) + \mu_{,3}(u_{3,3} + u_{3,3}) + \mu(u_{3,11} + u_{1,31} + u_{3,22} + u_{2,32} + u_{3,33} + u_{3,33}) = \rho \ddot{u}_3 \longrightarrow (4)$$

Since $\lambda = \lambda_0 e^{-m_q x_q}$, $\mu = \mu_0 e^{-m_q x_q}$ &

$\rho = \rho_0 e^{-m_q x_q}$ where $q = 1,2,3$

And $u_i = A e^{i(x_j n_j - ct)} P_i$

So equation (4) becomes

$$-m_3 \lambda_0 \{(in_1)P_1 + (in_2)P_2 + i(n_3)P_3\} + \lambda_0 \{(in_1)(in_3)P_1 + (in_2)(in_3)P_2 + (in_3)^2 P_3\} + (-m_1)\mu_0(in_1 P_3 + in_3 P_1) + (-m_2)\mu_0(in_2 P_3 + in_3 P_2) + (-m_3)\mu_0(2 i n_3 P_3) + \mu_0((in_1)^2 P_3 + (in_1)(in_3)P_1 + (in_2)^2 P_3 + (in_2)(in_3)P_2 + 2(in_3)^2 P_3\} = -\rho_0 c^2 P_3$$

$$\Rightarrow \{i n_1 m_3 \lambda_0 + n_1 n_3 \lambda_0 + i m_1 n_3 \mu_0 + n_1 n_3 \mu_0\} P_1 + \{i n_2 m_3 \lambda_0 + n_1 n_3 \lambda_0 + i m_2 n_3 \mu_0 + n_2 n_3 \mu_0\} P_2 + \{i m_3 n_3 \lambda_0 + n_3^2 \lambda_0 + i m_1 n_1 \mu_0 + i m_2 n_2 \mu_0 + 2 i m_3 n_3 \mu_0 + n_1^2 \mu_0 + n_2^2 \mu_0 + 2 n_3^2 \mu_0 - \rho_0 c^2\} P_3 = 0 \longrightarrow (C)$$

Now let $n = (1,0,0)$ & $P = P(1,0,0)$ for equ (A)

So equ (A) becomes

$$\{(i m_1 \lambda_0) + 2 i m_1 \mu_0 + 2\mu_0 + \lambda_0 - \rho_0 c^2\} = 0$$

Let $m_1 = 0$

$\Rightarrow \quad 2\mu_0 + \lambda_0 - \rho_0 c^2 = 0 \quad \Rightarrow \rho_0 c^2 = \lambda_0 + 2\mu_0$

$$\Rightarrow \boxed{c = \sqrt{\frac{2\mu_0 + \lambda_0}{\rho_0}}}$$

Now put $n = n(0,1,0)$ & $P = P(0,1,0)$

Equ (B) $\Rightarrow$

$(i m_2 \lambda_0 + \lambda_0 + i m_2 \mu_0 + (-\rho_0 c^2)) = 0$

put $m_2 = 0$

$\Rightarrow \lambda_0 + 2\mu_0 - \rho_0 c^2 = 0 \Rightarrow \rho_0 c^2 = \lambda_0 + 2\mu_0$

$$\Rightarrow \boxed{c = \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho_0}}}$$

Now put $n = n(0,0,1)$ & $P = P(0,0,1)$

So equ (B) implies

$i m_3 \lambda_0 + \lambda_0 + i m_3 \mu_0 + 2\mu_0 - \rho_0 c^2 = 0$

$\Rightarrow i m_3 (\lambda_0 + \mu_0) + \lambda_0 + 2\mu_0 - \rho_0 c^2 = 0$

put $m_3 = 0$

$\Rightarrow \lambda_0 + 2\mu_0 - \rho_0 c^2 = 0$

$\Rightarrow \rho_0 c^2 = \lambda_0 + 2\mu_0$

$\Rightarrow c^2 = \frac{\lambda_0 + 2\mu_0}{\rho_0}$

$$\Rightarrow \boxed{c = \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho_0}}}$$

$\Rightarrow$ Prove that: $\text{Div}(\text{Curl}\,\underline{A}) = 0$

Proof) $\text{Div}(\text{Curl}\,\underline{A}) = \text{Div}(\epsilon_{ijk} A_{k,j})$

$\Rightarrow \text{Div}(\text{Curl}\,\underline{A}) = (\epsilon_{ijk} A_{k,j})_{,i}$ $\quad \left\{ \begin{array}{l} \because \text{Curl}\,\underline{A} = \epsilon_{ijk} A_{k,j} \\ \text{Div}\,\underline{A} = A_{i,i} \end{array} \right.$

$= \epsilon_{ijk} A_{k,ji}$

$\Rightarrow \text{Div}(\text{Curl}\,\underline{A}) = \epsilon_{1jk} A_{k,j1} + \epsilon_{2jk} A_{k,j2} + \epsilon_{3jk} A_{k,j3} \longrightarrow (\star)$

Now

$\epsilon_{1jk} A_{k,j1} = \cancel{\epsilon_{11k} A_{k,11}}^{\,0} + \epsilon_{12k} A_{k,21} + \epsilon_{13k} A_{k,31}$

$= \cancel{\epsilon_{121} A_{1,21}}^{\,0} + \cancel{\epsilon_{122} A_{2,21}}^{\,0} + \epsilon_{123} A_{3,21} + \cancel{\epsilon_{131} A_{1,31}}^{\,0} + \epsilon_{132} A_{2,31} + \cancel{\epsilon_{133} A_{3,31}}^{\,0}$

$= A_{3,21} - A_{2,31} \quad \because \epsilon_{123} = 1 \;\&\; \epsilon_{132} = -1$

Now

$\epsilon_{2jk} A_{k,j2} = \epsilon_{21k} A_{k,12} + \cancel{\epsilon_{22k} A_{k,22}}^{\,0} + \epsilon_{23k} A_{k,32}$

$= \cancel{\epsilon_{211} A_{1,12}}^{\,0} + \cancel{\epsilon_{212} A_{2,12}}^{\,0} + \epsilon_{213} A_{3,12} + \epsilon_{231} A_{1,32} + \cancel{\epsilon_{232} A_{2,32}}^{\,0} + \cancel{\epsilon_{233} A_{3,32}}^{\,0}$

$= -A_{3,12} + A_{1,32}$

And

$\epsilon_{3jk} A_{k,j3} = \epsilon_{31k} A_{k,13} + \epsilon_{32k} A_{k,23} + \cancel{\epsilon_{33k} A_{k,33}}^{\,0}$

$= \cancel{\epsilon_{311} A_{1,13}}^{\,0} + \epsilon_{312} A_{2,13} + \cancel{\epsilon_{313} A_{3,13}}^{\,0} + \epsilon_{321} A_{1,23} + \cancel{\epsilon_{322} A_{2,23}}^{\,0} + \cancel{\epsilon_{323} A_{3,23}}^{\,0}$

$= A_{2,13} - A_{1,23}$

$\Rightarrow \text{Div}(\text{Curl}\,\underline{A}) = A_{3,21} - A_{2,31} - A_{3,12} + A_{1,32} + A_{2,13} - A_{1,23}$

$= \cancel{A_{3,21}} - \cancel{A_{2,31}} - \cancel{A_{3,21}} + \cancel{A_{1,32}} + \cancel{A_{2,31}} - \cancel{A_{1,32}}$

$\because A_{3,12} = A_{3,21}$; $A_{2,13} = A_{2,31}$ &
$A_{1,23} = A_{1,32}$

$\Rightarrow \text{Div}(\text{Curl}\,\underline{A}) = 0$

Remark:- $\epsilon_{ijk}\,\psi_{k,ji} = 0$

Wave Equation:-

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \longrightarrow (1)$$

where 'c' is the speed of wave.
The solution of wave equation can be written as

$$u = Ae^{ik(x_j n_j - ct)} \longrightarrow (*)$$

Now, we verify that it satisfies the wave equation.

Now $\dfrac{\partial^2 u}{\partial x_1^2} = (ik n_1)^2 A e^{ik(x_j n_j - ct)}$

$= -k^2 n_1^2 A e^{ik(x_j n_j - ct)}$

$= -k^2 A e^{ik(x_j n_j - ct)}$

⇒ Solution of Wave Equation By Decomposition Method:-

Wave equation in elastic medium may be written as

$$\sigma_{ij,j} = \rho \ddot{u}_i \longrightarrow (1)$$

where for Isotropic medium

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$$

$$\Rightarrow \sigma_{ij,j} = \lambda u_{k,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i})$$

Let
$$u_i = \text{Grad}\,\phi + \text{Curl}\,\psi$$
i.e. $u_i = \phi_{,i} + \epsilon_{ijk} \psi_{k,j}$

Since $\sigma_{ij,j} = \lambda u_{k,ki} + \mu (u_{i,jj} + u_{j,ij})$

Thus wave equation becomes

$$\lambda u_{k,ki} + \mu (u_{i,jj} + u_{j,ij}) = \rho \ddot{u}_i \longrightarrow (*)$$

Now
$$u_k = \phi_{,k} + \epsilon_{kpq} \psi_{q,p}$$

$$\Rightarrow u_{k,ki} = \phi_{,kki} + \epsilon_{kpq} \psi_{q,pki}$$

Similarly
$$u_{i,jj} = \phi_{,ijj} + \epsilon_{ipq} \psi_{q,pjj}$$

$$\& \quad u_{j,ij} = \phi_{,jij} + \epsilon_{jpq} \psi_{q,pij}$$

Thus equ (*) becomes

$$\lambda \{ \phi_{,kki} + \epsilon_{kpq} \psi_{q,pki} \} + \mu (\phi_{,ijj} + \epsilon_{ipq} \psi_{q,pjj})$$
$$+ \mu (\phi_{,jij} + \epsilon_{jpq} \psi_{q,pij}) = \rho (\ddot{\phi}_{,i} + \epsilon_{ipq} \ddot{\psi}_{q,p})$$

$$\Rightarrow \lambda\{\phi_{,kki} + (\cancel{\epsilon_{kpq}\psi_{q,pk}})_{,i}\} + \mu(\phi_{,ijj} + \epsilon_{ipq}\psi_{q,pjj})$$

$$+\mu\{\phi_{,jij} + (\cancel{\epsilon_{jpq}\psi_{q,pj}})_{,i}\} = \rho\{\ddot{\phi}_{,i} + \epsilon_{ipq}\ddot{\psi}_{q,p}\}$$

$$\Rightarrow \frac{\partial}{\partial x_i}\{\lambda\phi_{,kk} + \mu\phi_{,jj} + \mu\phi_{,jj} - \rho\ddot{\phi}\} +$$

$$\{\mu\epsilon_{ipq}\psi_{q,pjj} - \rho\epsilon_{ipq}\ddot{\psi}_{q,p}\} = 0$$

$$\Rightarrow \frac{\partial}{\partial x_i}\{\underbrace{\lambda\phi_{,kk} + 2\mu\phi_{,jj} - \rho\ddot{\phi}}_{\text{scaler}}\} +$$

$$\underbrace{\{\mu\epsilon_{ipq}\psi_{q,jj} - \rho\epsilon_{ipq}\ddot{\psi}_q\}_{,p}}_{\text{vector}} = 0$$

$\because$ Scaler + Vector = 0

$\Rightarrow$ scaler = 0 &

vector = 0

$$\Rightarrow \lambda\phi_{,kk} + 2\mu\phi_{,jj} = \rho\ddot{\phi} \longrightarrow (2)$$

$$\& \quad \epsilon_{ipq}\{\mu\psi_{q,jj} - \rho\ddot{\psi}_q\} = 0 \longrightarrow (3)$$

Equ (2) $\Rightarrow (\lambda+2\mu)\nabla^2\phi = \rho\ddot{\phi}$

$$\Rightarrow \nabla^2\phi - \frac{\rho}{\lambda+2\mu}\ddot{\phi}$$

$$\Rightarrow c = \sqrt{\frac{\lambda+2\mu}{\rho}} \longrightarrow \text{Longitudinal waves}$$

Equ (3) $\Rightarrow \nabla^2\psi_q - \frac{\rho}{\mu}\ddot{\psi}_q = 0$

$$\Rightarrow \nabla^2\psi_q = \frac{\rho}{\mu}\ddot{\psi}$$

$$\Rightarrow c = \sqrt{\mu/\rho} \longrightarrow \text{Transverse Waves}$$

Since:- Wave Equation

$$\nabla^2 u = \frac{1}{c^2}\ddot{u}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

where c is the speed of Waves.

★ If Wave equation is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 10\frac{\partial^2 u}{\partial t^2}$$

$\Rightarrow$ The speed of waves $= 1/\sqrt{10}$

⇒ Surface Waves / Rayleigh Waves / Plane Waves :-

Equation of motion is

$$\sigma_{ij,j} = \rho\ddot{u}_i \longrightarrow (\star)$$

$$\Rightarrow \lambda u_{k,ki} + \mu(u_{i,jj} + u_{j,ij}) = \rho\ddot{u}_i$$

They are always in 2D, So

for $i=1$

$$\Rightarrow \lambda u_{k,k1} + \mu(u_{1,jj} + u_{j,1j}) = \rho\ddot{u}_1$$

$$\Rightarrow \lambda(u_{1,11} + u_{2,21}) + \mu(u_{1,11} + u_{1,11} + u_{1,22} + u_{2,12}) = \rho\ddot{u}_1$$

$$\Rightarrow (\lambda+2\mu)u_{1,11} + \mu(u_{1,22}) + (\lambda+\mu)u_{2,12} = \rho\ddot{u}_1 \longrightarrow (1)$$

for $i=2$

$$\Rightarrow \lambda u_{k,k2} + \mu(u_{2,jj} + u_{j,2j}) = \rho\ddot{u}_2$$

$$\Rightarrow \lambda(u_{1,12} + u_{2,22}) + \mu(u_{2,11} + u_{1,21} + u_{2,22} + u_{2,22}) = \rho\ddot{u}_2$$

$$\left\{ \begin{array}{l} \mathcal{L}(y'') = s^2 \bar{y}(s) - s y(0) - y'(0) \\ \mathcal{L}(e^{at}) = \frac{1}{s-a} \end{array} \right\}$$

$$\Rightarrow (\lambda+\mu) U_{1,12} + \mu U_{2,11} + (\lambda+2\mu) U_{2,22} = \rho \ddot{U}_2 \longrightarrow ②$$

Let $U_i = \phi_i(k x_2)\, e^{ik(x_1 - ct)}$

$$\Rightarrow U_{1,11} = -k^2 \phi_1(k x_2)\, e^{ik(x_1-ct)}$$

$$U_{1,22} = +k^2 \phi_1''(k x_2)\, e^{ik(x_1-ct)}$$

$$U_{2,12} = i k^2 \phi_1'(k x_2)\, e^{ik(x_1-ct)}$$

So equ ① implies

$$-(\lambda+2\mu) k^2 \phi_1(k x_2) + \mu k^2 \phi_1''(k x_2) + (\lambda+\mu) i k^2 \phi_2' = -\rho c^2 \phi_1 k^2$$

$$\Rightarrow \mu \phi_1'' + i(\lambda+\mu)\phi_2' - (\lambda+2\mu-\rho c^2)\phi_1 = 0 \longrightarrow ③$$

Similarly for 2nd equation

$$(\lambda+\mu) i k^2 \phi_1' + \mu (ik)^2 \phi_2 + (\lambda+2\mu) k^2 \phi_2'' = -\rho c^2 \phi_2 k^2$$

$$\Rightarrow (\lambda+2\mu)\phi_2'' + i(\lambda+\mu)\phi_1' - (\mu - \rho c^2)\phi_2 = 0 \longrightarrow ④$$

Now by Laplace Transform of equ ③ & ④

from ③ $\mu \{ s^2 \bar{\phi}_1(s) - s\phi_1(0) - \phi_1'(0) \} + i(\lambda+\mu)\{ s\bar{\phi}_2(s) - \phi_2(0) \} - \{ \lambda + 2\mu - \rho c^2 \} \bar{\phi}_1(s) = 0$

$$\Rightarrow \{ \mu s^2 - (\lambda+2\mu-\rho c^2) \} \bar{\phi}_1(s) + i(\lambda+\mu) s \bar{\phi}_2$$

$$= \mu\{ s\phi_1(0) - \phi_1'(0)\} + i(\lambda+\mu)\phi_2(0)$$

$$\Rightarrow \{\mu s^2 - (\lambda+2\mu-\rho c^2)\}\bar{\phi}_1(s) - i(\lambda+\mu)s\bar{\phi}_2(s) = A_1$$

Now from equ (4)

$$(\lambda+2\mu)\{s^2\bar{\phi}_2(s) - s\phi_2(0) - \phi_2'(0)\} + i(\lambda+\mu)\{s\bar{\phi}_1(s)$$
$$- \phi_1(0)\} - (\mu-\rho c^2)\bar{\phi}_2(s) = 0$$

$$\Rightarrow is(\lambda+\mu)\bar{\phi}_1(s) + \{(\lambda+2\mu)s^2 - (\mu-\rho c^2)\}\bar{\phi}_2(s) = A_2$$

⇒ Speed of Waves Under Gravity:-

$$G_i = \rho g u_{3,i}$$

Equation of motion in the presence of gravity can be written as

$$\sigma_{ij,j} + G_i = \rho \ddot{u}_i$$

where $G_i = \rho g u_{3,i}$

For isotropic medium

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$$

or $\sigma_{ij} = \lambda u_{k,k} \delta_{ij} + \mu(u_{i,j} + u_{j,i})$

$$\Rightarrow \sigma_{ij,j} = \lambda u_{k,ki} + \mu(u_{i,jj} + u_{j,ij})$$

Then equation of motion becomes

$$\lambda u_{k,ki} + \mu(u_{i,jj} + u_{j,ij}) + \rho g u_{3,i} = \rho \ddot{u}_i \longrightarrow (1)$$

Let $u_i = A e^{i(x_j n_j - ct)} p_i$

Then equation (1) becomes

$$-\lambda n_i n_k p_k + \mu(-n_j n_j p_i - n_j p_j n_i) + \rho g n_i p_3 = -\rho c^2 p_i$$

$$\Rightarrow \{(\lambda+\mu) n_j p_j - i\rho g p_3\} n_i + (\mu - \rho c^2) p_i = 0$$

For Longitudinal waves :- $n_i \parallel p_i \Rightarrow n_i = p_i$

$$\Rightarrow [\{(\lambda+\mu) - i\rho g p_3\} + (\mu - \rho c^2)] p_i = 0$$

$$\Rightarrow \lambda + 2\mu - \rho c^2 - i\rho g p_3 = 0$$

* If $p_3 = 0$ ($\Rightarrow$ in x-y Plane)

$$\lambda + 2\mu - \rho c^2 = 0 \rightarrow c = \sqrt{\frac{\lambda+2\mu}{\rho}}$$

* If $P_3 \neq 0$

$$\Rightarrow c = \sqrt{\frac{\lambda + 2\mu - i\rho g P_3}{\rho}}$$

$\Rightarrow$ There exis decaying waves

Let $c = \alpha \pm i\beta$

$$\Rightarrow u_i = A e^{i\xi\{x_j n_j - (\alpha \pm i\beta)t\}} P_i$$

$$\Rightarrow u_i = A e^{-\beta t} e^{i(x_j n_j - \alpha t)} P_i$$

For Transverse waves:- $n_i \perp P_i \Rightarrow n_i P_i = 0$

$$\Rightarrow \mu P_i - i\rho g P_3 n_i - \rho c^2 P_i = 0$$

$$\Rightarrow (\mu - \rho c^2) P_i - i\rho g P_3 n_i = 0$$

Since $n_i \perp P_i$

$\Rightarrow n_i$ and P_i are linearly-indipendent

$\Rightarrow \mu - \rho c^2 = 0$ And $i\rho g P_3 = 0$

$\Rightarrow c = \sqrt{\frac{\mu}{\rho}}$ $\Big|$ $\Rightarrow P_3 = 0$

$\Rightarrow$ In the case of Gravity Transverse waves only propagate in XY- Plane.

$\Rightarrow$ **Speed of Waves Under magnetic Field:-**

$$F_i = \mu_0 H_0^2 (U_{j,ji} - \mu_0 \epsilon . \ddot{U}_i)$$

Equation of motion

$$\sigma_{ij,j} + F_i = \rho \ddot{U}_i$$

$$\Rightarrow \lambda u_{k,ki} + \mu(u_{i,jj} + u_{j,ij}) + \mu_0 H_0^2 (u_{j,ij} - \mu_0 \epsilon_0 \ddot{u}_i) = \rho \ddot{u}_i$$

$$\Rightarrow \lambda u_{k,ki} + \mu u_{i,jj} + (\mu + \mu_0 H_0^2) u_{j,ij} = (\rho + \mu_0^2 \epsilon_0 H_0^2) \ddot{u}_i$$

Let $u_i = A e^{ik(x_j n_j - ct)} P_i$

$$\Rightarrow \lambda n_i n_k P_k + \mu P_i + (\mu + \mu_0 H_0^2) n_j P_j n_i$$
$$= (\rho + \mu_0^2 \epsilon_0 H_0^2) c^2 P_i$$

For Longitudinal Waves:- $n_i \parallel P_i \Rightarrow n_i = P_i$

$$\Rightarrow \{ \lambda + \mu + (\mu + \mu_0 H_0^2) - (\rho + \mu_0^2 \epsilon_0 H_0^2) c^2 \} P_i = 0$$

$P_i \neq 0$

$$\Rightarrow c_L = \sqrt{\frac{\lambda + 2\mu + \mu_0 H_0^2}{\rho + \epsilon_0 \mu_0^2 H_0^2}}$$

For Transverse Waves:- $n_i P_i = 0$

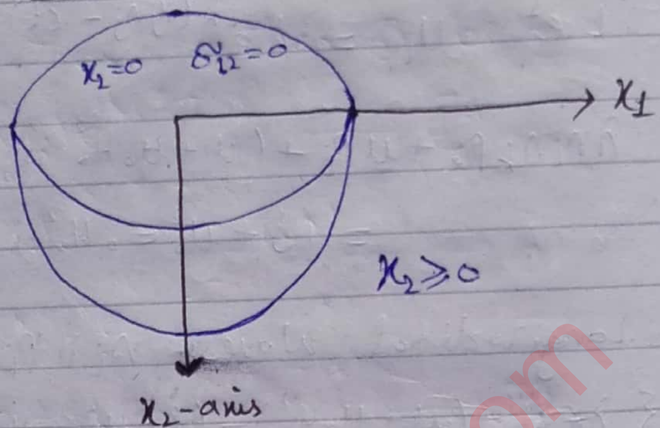
$$\Rightarrow \{ \mu - (\rho + \mu_0 \epsilon_0 H_0^2) c^2 \} P_i = 0$$

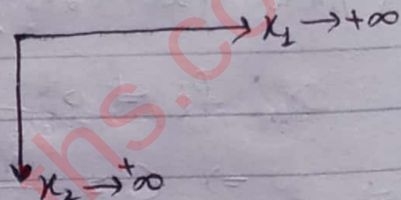
$$\Rightarrow c_T = \sqrt{\frac{\mu}{\rho + \mu_0^2 \epsilon_0 H_0^2}}$$

When $H_0 \longrightarrow \infty$

$$\Rightarrow c_T \longrightarrow 0$$

⇒ Rayleigh / Surface Waves in an Isotropic Medium (Without Rotation)





$U_3 = 0 \qquad U_{,3} = 0$

For an isotropic medium, suppose that material lies in $x_2 \geq 0$ and free surface is $x_2 = 0$

$\Rightarrow \sigma_{i2} = 0$ at $x_2 = 0$

We know that the equation of motion for isotropic ~~medium~~ material is

$$\sigma_{ij,j} = \rho \ddot{u}_i \quad , \text{where } i,j = 1,2 \longrightarrow (1)$$

We also know that for an isotropic material

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} \longrightarrow (2)$$

Since $\epsilon_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i}) \longrightarrow (i)$

Put $i,j = k$ in (i) we get

$$\epsilon_{kk} = \frac{1}{2}(U_{k,k} + U_{k,k})$$

$$\Rightarrow \epsilon_{kk} = U_{k,k} \longrightarrow (ii)$$

using (i) and (ii) in equ ② we get

$$\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + 2\mu \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$= \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i})$$

$$\Rightarrow \sigma_{ij,j} = \lambda u_{k,kj}\delta_{ij} + \mu(u_{i,jj} + u_{j,ij}) \longrightarrow ③$$

Comparing equation ① and ③ we get

$$\lambda u_{k,kj}\delta_{ij} + \mu(u_{i,jj} + u_{j,ij}) = \rho \ddot{u}_i \longrightarrow ④$$

Since $f_j\delta_{ij} = f_i$ so equ ④ becomes

$$\lambda u_{k,ki} + \mu(u_{i,jj} + u_{j,ij}) = \rho \ddot{u}_i \longrightarrow ⑤$$

Now we use the normal mode solution

Let

$$u_i = \phi_i(kx_2)\, e^{ik(x_1 - ct)} \longrightarrow ⑥$$

From equation ⑤

$$\lambda u_{k,ki} + \mu u_{i,jj} + \mu u_{j,ij} = \rho \ddot{u}_i$$

$$\lambda u_{k,ki} + \mu u_{j,ij} + \mu u_{i,jj} = \rho \ddot{u}_i$$

Put $j = k$ in 2nd term

$$\lambda u_{k,ki} + \mu u_{k,ik} + \mu u_{i,jj} = \rho \ddot{u}_i$$

$$\Rightarrow \lambda u_{k,ki} + \mu u_{k,ki} + \mu u_{i,jj} = \rho \ddot{u}_i$$

$$\Rightarrow (\lambda + \mu) u_{k,ki} + \mu u_{i,jj} = \rho \ddot{u}_i \longrightarrow ⑦$$

Put $i = 1$ in equ ⑦ we get

$$(\lambda + \mu) u_{k,k1} + \mu u_{1,jj} = \rho \ddot{u}_1 \longrightarrow ⑧$$

Put $j, k = 1, 2$ in equ ⑧ we get

$$(\lambda + \mu)(u_{1,11} + u_{2,21}) + \mu(u_{1,11} + u_{1,22}) = \rho \ddot{u}_1 \longrightarrow ⑨$$

Now from equ ⑥

$$u_i = \phi_i(kx_2)\, e^{ik(x_1 - ct)}$$

when $i=1$ then equation ⑥ becomes

$$u_1 = \phi_1(kx_2)\, e^{ik(x_1-ct)} \longrightarrow (10)$$

$$u_{1,1} = \phi_1(kx_2)\, e^{ik(x_1-ct)} (ik)$$

$$u_{1,11} = \phi_1(kx_2)\, e^{ik(x_1-ct)} (ik)^2$$

$$= -k^2 \phi_1(kx_2)\, e^{ik(x_1-ct)} \longrightarrow (iii)$$

when $i=2$, then equ ⑥ becomes

$$u_2 = \phi_2(kx_2)\, e^{ik(x_2-ct)}$$

$$u_{2,2} = \phi_2'(kx_2)(k)\, e^{ik(x_1-ct)}$$

$$u_{2,21} = \phi_2'(kx_2)\, k\, e^{ik(x_1-ct)} (ik)$$

$$\Rightarrow u_{2,21} = ik^2 \phi_2'(kx_2)\, e^{ik(x_1-ct)} \longrightarrow (iv)$$

from equation (10)

$$u_1 = \phi_1(kx_2)\, e^{ik(x_1-ct)}$$

$$u_{1,2} = \phi_1'(kx_2)\, k\, e^{ik(x_1-ct)}$$

$$u_{1,22} = \phi_1''(kx_2)\, k^2\, e^{ik(x_1-ct)}$$

$$\Rightarrow u_{1,22} = k^2 \phi_1''(kx_2)\, e^{ik(x_1-ct)} \longrightarrow (v)$$

from equation (10)

$$u_1 = \phi_1(kx_2)\, e^{ik(x_1-ct)}$$

Differentiate w.r.t 't'

$$\dot{u}_1 = \phi_1(kx_2)\, e^{ik(x_1-ct)}(-ick)$$

$$\ddot{u}_1 = \phi_1(kx_2)\, e^{ik(x_1-ct)}(-ick)^2$$

$$\Rightarrow \ddot{u}_1 = -c^2k^2\phi_1(kx_2)\, e^{ik(x_1-ct)} \longrightarrow (vi)$$

using (iii), (iv), (v) and (vi) in equation ⓐ we get

$$(\lambda+\mu)\left[-k^2\phi_1 e^{ik(x_1-ct)} + ik^2\phi_2' e^{ik(x_1-ct)}\right]$$

$$+\mu\left[-k^2\phi_1 e^{ik(x_1-ct)} + k^2\phi_1'' e^{ik(x_1-ct)}\right] = -c^2k^2\phi_1 \rho e^{ik(-}$$

$$\Rightarrow e^{ik(x_1-ct)}(\lambda+\mu)\left[-k^2\phi_1 + ik^2\phi_2'\right] + e^{ik(x_1-ct)}\mu$$

$$\left[-k^2\phi_1 + k^2\phi_1'\right] = e^{ik(x_1-ct)}\left[-c^2k^2\phi_1\rho\right]$$

$$\Rightarrow e^{ik(x_1-ct)}\left[(\lambda+\mu)(-k^2\phi_1 + ik^2\phi_2') + \mu(-k^2\phi_1 + k^2\phi_1')\right]$$
$$= e^{ik(x_1-ct)}\left[-c^2k^2\rho\phi_1\right]$$

$$\Rightarrow (\lambda+\mu)(-k^2\phi_1 + ik^2\phi_2') + \mu(-k^2\phi_1 + k^2\phi_1'') = -c^2k^2\rho\phi_1$$

$$\Rightarrow k^2\left[(\lambda+\mu)(-\phi_1 + i\phi_2') + \mu(-\phi_1 + \phi_1'')\right] = -c^2k^2\rho\phi_1$$

$$\Rightarrow (\lambda+\mu)(-\phi_1 + i\phi_2') + \mu(-\phi_1 + \phi_1'') = -c^2\rho\phi_1$$

$$\Rightarrow \lambda(-\phi_1 + i\phi_2') + \mu(-\phi_1 + i\phi_2') + \mu(-\phi_1 + \phi_1'') = -c^2\rho\phi_1$$

$$\Rightarrow -\lambda\phi_1 + i\lambda\phi_2' - \mu\phi_1 + i\mu\phi_2' - \mu\phi_1 + \mu\phi_1'' = -c^2\rho\,\phi_1$$

$$\Rightarrow \mu\phi_1'' + i(\lambda+\mu)\phi_2' - 2\mu\phi_1 - \lambda\phi_1 = -c^2\rho\phi_1$$

$$\Rightarrow \mu\phi_1'' + i(\lambda+\mu)\phi_2' - (2\mu+\lambda)\phi_1 = -c^2\rho\phi_1$$

$$\Rightarrow \phi_1'' + i(\lambda+\mu)\phi_2' - (2\mu+\lambda)\phi_1 + c^2\rho\phi_1 = 0$$

$$\mu\phi_1'' + i(\lambda+\mu)\phi_2' - (\lambda+2\mu-\rho c^2)\phi_1 = 0 \longrightarrow (A)$$

From equ (7)

$$(\lambda+\mu)u_{k,ki} + \mu u_{i,jj} = \rho\ddot{u}_2$$

Put $i=2$ in equ (7) we get

$$(\lambda+\mu)u_{k,k2} + \mu u_{2,jj} = \rho\ddot{u}_2 \longrightarrow (11)$$

Put $j,k = 1,2$ in equ (11) we get

$$(\lambda+\mu)[u_{1,12} + u_{2,22}] + \mu[u_{2,11} + u_{2,22}] = \rho\ddot{u}_2 \longrightarrow (12)$$

Now from equ (6)

$$u_i = \phi_i(kx_2)e^{ik(x_1-ct)}$$

$$\Rightarrow u_1 = \phi_1(kx_2)e^{ik(x_1-ct)}$$

$$u_{1,1} = \phi_1(kx_2)e^{ik(x_1-ct)}(ik)$$

$$u_{1,12} = \phi_1'(kx_2)ke^{ik(x_1-ct)}(ik)$$

$$\Rightarrow u_{1,12} = ik^2\phi_1'(kx_2)e^{ik(x_1-ct)} \longrightarrow (vii)$$

Put $i=2$ in equ (6)

$$u_2 = \phi_2(kx_2)e^{ik(x_1-ct)}$$

$$u_{2,2} = \phi_2'(kx_2)ke^{ik(x_1-ct)}$$

$$u_{2,22} = \phi_2''(kx_2)k^2e^{ik(x_1-ct)}$$

$$u_{2,22} = k^2\phi_2''(kx_2)e^{ik(x_1-ct)} \longrightarrow (viii)$$

Now $u_2 = \phi_2(kx_2)\, e^{ik(x_1 - ct)}$

$u_{2,1} = \phi_2(kx_2)\, e^{ik(x_1 - ct)} (ik)$

$u_{2,11} = \phi_2(kx_2)\, e^{ik(x_1 - ct)} (ik)^2$

$\Rightarrow u_{2,11} = -k^2 \phi_2(kx_2)\, e^{ik(x_1 - ct)} \longrightarrow (ix)$

using relation (vi), (vii)

$\ddot{u}_2 = -c^2 k^2 \phi_2(kx_2)\, e^{ik(x_1 - ct)} \longrightarrow (x)$

using relation (vii), (viii), (ix) and (x) in eqn (12)

$(\lambda+\mu)[u_{1,12} + u_{2,22}] + \mu[u_{2,11} + u_{2,22}] = \rho \ddot{u}_2$

$\Rightarrow (\lambda+\mu)[ik^2\phi_1' e^{ik(x_1 - ct)} + k^2 \phi_2'' e^{ik(x_1 - ct)}]$

$+ \mu[-k^2\phi_2 e^{ik(x_1 - ct)} + k^2 \phi_2'' e^{ik(x_1 - ct)}]$

$= \rho[-c^2 k^2 \phi_2 e^{ik(x_1 - ct)}]$

$\Rightarrow e^{ik(x_1 - ct)} [(\lambda+\mu)(ik^2\phi_1' + k^2\phi_2'') + \mu(-k^2\phi_2 + k^2\phi_2'')]$

$= e^{ik(x_1 - ct)}[-\rho c^2 k^2 \phi_2]$

$\Rightarrow (\lambda+\mu)(ik^2\phi_1' + k^2\phi_2'') + \mu(-k^2\phi_2 + k^2\phi_2'') = -\rho c^2 k^2 \phi_2$

$\Rightarrow \lambda(ik^2\phi_1' + k^2\phi_2'') + \mu(ik^2\phi_1' + k^2\phi_2'') + \mu(-k^2\phi_2 + k^2\phi_2'')$

$= -\rho c^2 k^2 \phi_2$

$\Rightarrow i\lambda k^2\phi_1' + \lambda k^2 \phi_2'' + i\mu k^2 \phi_1' + \mu k^2 \phi_2'' - \mu k^2 \phi_2$

$+ \mu k^2 \phi_2'' = -\rho c^2 k^2 \phi_2$

Dividing throughout by k^2

$\Longrightarrow$

$$\Rightarrow i\lambda\phi_1' + \lambda\phi_2'' + i\mu\phi_1' + \mu\phi_2'' - \mu\phi_2 + \mu\phi_2'' = -\rho c^2\phi_2$$

$$\Rightarrow i(\lambda+\mu)\phi_1' + \lambda\phi_2'' + 2\mu\phi_2'' - \mu\phi_2 = -\rho c^2\phi_2$$

$$\Rightarrow i(\lambda+\mu)\phi_1' + (\lambda+2\mu)\phi_2'' - \mu\phi_2 + \rho c^2\phi_2 = 0$$

$$\Rightarrow i(\lambda+\mu)\phi_1' + (\lambda+2\mu)\phi_2'' - (\mu-\rho c^2)\phi_2 = 0$$

$$\Rightarrow (\lambda+2\mu)\phi_2'' + i(\lambda+\mu)\phi_1' - (\mu-\rho c^2)\phi_2 = 0 \longrightarrow (B)$$

From equation (A)

$$\mu\phi_1'' + i(\lambda+\mu)\phi_2' - (\lambda+2\mu-\rho c^2)\phi_1 = 0 \longrightarrow (A)$$

Taking laplace transform on both sides of (A)

Since $\mathcal{L}(\phi_1'') = s^2\bar{\phi}_1(s) - s\phi_1(0) - \phi_1'(0) \longrightarrow (xi)$

$$\mathcal{L}(\phi_2') = s\bar{\phi}_2(s) - \phi_2(0) \longrightarrow (xii)$$

$$\mathcal{L}(\phi_1) = \bar{\phi}_1(s) \longrightarrow (xiii)$$

using relation (xi), (xii) and (xiii) in (A) we get

$$\mu[s^2\bar{\phi}_1(s) - s\phi_1(0) - \phi_1'(0)] + i(\lambda+\mu)[s\bar{\phi}_2(s) - \phi_2(0)] - (\lambda+2\mu-\rho c^2)\bar{\phi}_1(s) = 0$$

$$\Rightarrow \mu s^2\bar{\phi}_1(s) - \mu s\phi_1(0) - \mu\phi_1'(0) + is(\lambda+\mu)\bar{\phi}_2(s) - i(\lambda+\mu)\phi_2(0) - \lambda\bar{\phi}_1(s) - 2\mu\bar{\phi}_1(s) + \rho c^2\bar{\phi}_1(s) = 0$$

$$\Rightarrow \mu s^2\bar{\phi}_1(s) - \lambda\bar{\phi}_1(s) - 2\mu\bar{\phi}_1(s) + \rho c^2\bar{\phi}_1(s) + is(\lambda+\mu)\bar{\phi}_2(s) - \mu s\phi_1(0) - \mu\phi_1'(0) - i(\lambda+\mu)\phi_2(0) = 0$$

$$\Rightarrow [s^2-\lambda-2\mu+\rho c^2]\bar{\Phi}_1(s)+is(\lambda+\mu)\bar{\Phi}_1(s)-\mu s\phi_1(0)$$
$$-\mu\phi_1'(0)-i(\lambda+\mu)\phi_2(0)=0 \longrightarrow \text{(C)}$$

Now from equ (B)

$$(\lambda+2\mu)\phi_2''+i(\lambda+\mu)\phi_1'-(\mu-\rho c^2)\phi_2=0 \longrightarrow \text{(B)}$$

Taking Laplace transform on both sides of (B). Since

$$\mathcal{L}\{\phi_2''\}=s^2\bar{\Phi}_2(s)-s\phi_2(0)-\phi_2'(0) \longrightarrow (xiv)$$
$$\mathcal{L}\{\phi_1'\}=s\bar{\Phi}_1(s)-\phi_1(0) \longrightarrow (xv)$$
$$\mathcal{L}\{\phi_2\}=\bar{\Phi}_2(s) \longrightarrow (xvi)$$

using these values (xiv) $\rightarrow$ (xvi) in (B) we get

$$(\lambda+2\mu)[s^2\bar{\Phi}_2(s)-s\phi_2(0)-\phi_2'(0)]+i(\lambda+\mu)[s\bar{\Phi}_1(s)$$
$$-\phi_1(0)]+(\mu-\rho c^2)\bar{\Phi}_2(s)=0$$

$$\Rightarrow \lambda[s^2\bar{\Phi}_2(s)-s\phi_2(0)-\phi_2'(0)]+2\mu[s^2\bar{\Phi}_2(s)-s\phi_2(0)$$
$$-\phi_2'(0)]+i\lambda[s\bar{\Phi}_1(s)-\phi_1(0)]+i\mu[s\bar{\Phi}_1(s)-\phi_1(0)]$$
$$+\mu\bar{\Phi}_2(s)-\rho c^2\bar{\Phi}_2(s)=0$$

$$\Rightarrow \lambda s^2\bar{\Phi}_2(s)-\lambda s\phi_2(0)-\lambda\phi_2'(0)+2\mu s^2\bar{\Phi}_2(s)$$
$$-2\mu s\phi_2(0)-2\mu\phi_2'(0)+i\lambda s\bar{\Phi}_1(s)-i\lambda\phi_1(0)$$
$$+i\mu s\bar{\Phi}_1(s)-i\mu\phi_1(0)+\mu\bar{\Phi}_2(s)-\rho c^2\bar{\Phi}_2(s)=0$$

$$\Rightarrow \lambda s^2\bar{\Phi}_2(s)+2\mu s^2\bar{\Phi}_2(s)+\mu\bar{\Phi}_2(s)-\rho c^2\bar{\Phi}_2(s)+i\lambda s\bar{\Phi}_1(s)$$
$$+i\mu s\bar{\Phi}_1(s)-\lambda s\phi_2(0)-2\mu s\phi_2(0)-\lambda\phi_2'(0)$$
$$-2\mu\phi_2'(0)-i\lambda\phi_1(0)-i\mu\phi_1(0)=0$$

$$\Rightarrow [\lambda s^2 + 2\mu s^2 + \mu - \rho c^2]\bar{\Phi}_2(s) + [i\lambda s + i\mu s]\bar{\Phi}_1(s)$$
$$- [\lambda s + 2\mu s]\phi_2(0) - (\lambda + 2\mu)\phi_2'(0) - [i\lambda + i\mu]\phi_1(0) =$$

$$\Rightarrow [(\lambda+2\mu)s^2 + \mu - \rho c^2]\bar{\Phi}_2(s) + i(\lambda+\mu)s\bar{\Phi}_1(s) - (\lambda+2\mu)s\phi$$
$$- (\lambda+2\mu)\phi_2'(0) - i(\lambda+\mu)\phi_1(0) \longrightarrow (D)$$

From (C) put $\mu s^2 - \lambda - 2\mu + \rho c^2 = a_1$

$is(\lambda+\mu) = a_2$ and

$-\mu s\,\phi_1(0) - \mu\,\phi_1'(0) - i(\lambda+\mu)\,\phi_2(0) = -c_1$

we get

$$a_1\bar{\Phi}_1(s) + a_2\bar{\Phi}_2(s) - c_1 = 0 \longrightarrow (E)$$

From (D) put $(\lambda+2\mu)s^2 + \mu - \rho c^2 = b_2$

$i(\lambda+\mu)s = b_1$ and

$-(\lambda+2\mu)s\,\phi_2(0) - (\lambda+2\mu)\phi_2'(0) - i(\lambda+\mu) = -c$

we get

$$b_2\bar{\Phi}_2(s) + b_1\bar{\Phi}_1(s) - c_2 = 0 \longrightarrow (F)$$

Now from equation (E) and (E) we have

$$a_1\bar{\Phi}_1(s) + a_2\bar{\Phi}_2(s) = c_1$$

$$b_1\phi_1(s) + b_2\bar{\Phi}_2(s) = c_2$$

$$\Rightarrow \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\begin{bmatrix} \bar{\Phi}_1(s) \\ \bar{\Phi}_2(s) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Now we have to find $\bar{\Phi}_1(s)$ and $\bar{\Phi}_2(s)$

So by crammer Rule we have

$$\bar{\phi}_1(s) = \frac{\begin{vmatrix} c_1 & a_2 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} = \frac{c_1 b_2 - b_1 c_1}{a_1 b_2 - a_2 b_1}$$

And

$$\bar{\phi}_2(s) = \frac{\begin{vmatrix} a_1 & c_1 \\ b_1 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} = \frac{a_1 c_2 - b_1 c_1}{a_1 b_2 - a_2 b_1}$$

Here $a_1 b_2 - a_2 b_1$ is quadratic in s^2.

Let s_1^2 and s_2^2 be the roots of $a_1 b_2 - a_2 b_1$

$$\Rightarrow \quad a_1 b_2 - b_1 a_2 = (s^2 - s_1^2)(s^2 - s_2^2)$$

Thus we have

$$\bar{\phi}_1(s) = \frac{1}{(s^2 - s_1^2)(s^2 - s_2^2)}$$

By using Partial fraction method we have

$$\bar{\phi}_1(s) = \frac{A_1}{s - s_1} + \frac{A_2}{s + s_1} + \frac{A_3}{s - s_2} + \frac{A_4}{s + s_2} \longrightarrow (13)$$

Now by taking Laplace inverse transform on both sides.

$$\therefore \quad \mathcal{L}^{-1}[\bar{\phi}_1(s)] = \phi_1(kx_2) \longrightarrow (xvii)$$

$$\& \quad \mathcal{L}^{-1}\left[\frac{1}{s \mp a}\right] = e^{\pm a x_2} \longrightarrow (xviii)$$

using (xvii) & (xviii) in eqn (13) we get

$$\phi_1(kx_2) = A_1 e^{s_1 x_2} + A_2 e^{-s_1 x_2} + A_3 e^{s_2 x_2} + A_4 e^{-s_2 x_2} \longrightarrow (14)$$

For bounded solution $A_1 = A_3 = 0$

as $x_2 \to \infty$, $\phi_1(kx_2) \to 0$

So equ (14) becomes

$$\phi_1(kx_2) = A_2 e^{-s_1 x_2} + A_4 e^{-s_2 x_2} \longrightarrow (15)$$

Let $A_2 = A$ & $A_4 = B$

$$\Rightarrow \phi_1(kx_2) = Ae^{-s_1 x_2} + Be^{-s_2 x_2} \longrightarrow (16)$$

differentiate equ (16) twice w.r.t x_2 we get,

$$\phi_1' = -s_1 A e^{-s_1 x_2} - s_2 B e^{-s_2 x_2}$$

$$\phi_1'' = s_1^2 A e^{-s_1 x_2} + s_2^2 B e^{-s_2 x_2} \longrightarrow (17)$$

From equ (A)

$$\mu \phi_1'' + i(\lambda+\mu)\phi_2' - (\lambda + 2\mu - \rho c^2)\phi_1 = 0 \longrightarrow (A)$$

using values from (16) and (17) in (A) we get

$$\mu[s_1^2 A e^{-s_1 x_2} + s_2^2 B e^{-s_2 x_2}] + i(\lambda+\mu)\phi_2'$$
$$-(\lambda + 2\mu - \rho c^2)[Ae^{-s_1 x_2} + Be^{-s_2 x_2}] = 0$$

$$\Rightarrow i(\lambda+\mu)\phi_2' = (\lambda + 2\mu - \rho c^2)[Ae^{-s_1 x_2} + Be^{-s_2 x_2}] -$$
$$\mu[s_1^2 A e^{-s_1 x_2} + s_2^2 A e^{-s_2 x_2}]$$

$$\Rightarrow i(\lambda+\mu)\phi_2' = (\lambda + 2\mu - \rho c^2)Ae^{-s_1 x_2} + (\lambda + 2\mu - \rho c^2)Be^{-s_2 x_2}$$
$$-\mu s_1^2 A e^{-s_1 x_2} - \mu s_2^2 B e^{-s_2 x_2}$$

$$\Rightarrow i(\lambda+\mu)\phi_2' = (\lambda+2\mu-\rho c^2-\mu s_1^2)Ae^{-s_1x_2}$$
$$+(\lambda+2\mu-\rho c^2-\mu s_2^2)Be^{-s_2x_2}$$

Multiplying both sides by i

$$-(\lambda+\mu)\phi_2' = i(\lambda+2\mu-\rho c^2-\mu s_1^2)Ae^{-s_1x_2}$$
$$+i(\lambda+2\mu-\rho c^2-\mu s_2^2)Be^{-s_2x_2}$$

Multiplying both sides by -1 we get

$$(\lambda+\mu)\phi_2' = -i(\lambda+2\mu-\rho c^2-\mu s_1^2)Ae^{-s_1x_2}$$
$$-i(\lambda+2\mu-\rho c^2-\mu s_1^2)Be^{-s_2x_2}$$

$$\Rightarrow \phi_2' = \frac{-i}{\lambda+\mu}[\lambda+2\mu-\rho c^2-\mu s_1^2]Ae^{-s_1x_2}$$
$$-\frac{i}{\lambda+\mu}[\lambda+2\mu-\rho c^2-\mu s_2^2]Be^{-s_2x_2}$$

Integrating both sides w.r.t x_2 we get

$$\phi_2 = \frac{-iA}{\lambda+\mu}[\lambda+2\mu-\rho c^2-\mu s_1^2]\frac{e^{-s_1x_2}}{-s_1}$$
$$-\frac{iB}{\lambda+\mu}[\lambda+2\mu-\rho c^2-\mu s_2^2]\frac{e^{-s_2x_2}}{-s_2}$$

$$\Rightarrow \phi_2(kx_2) = \frac{iA}{(\lambda+\mu)s_1}[\lambda+2\mu-\rho c^2-\mu s_1^2]e^{-s_1x_2}$$
$$+\frac{iB}{(\lambda+\mu)s_2}[\lambda+2\mu-\rho c^2-\mu s_2^2]e^{-s_2x_2}$$

⇒ Thermal Effects On Surface:-

Field equations in the presence of thermal effect (Temperature difference) can be written as.

$$\sigma_{ij} = C_{ijkl} - \beta_{ij}(T-T_0)$$

with heat equation

$$\frac{\partial}{\partial x_i}\left\{K_{ij}\frac{\partial}{\partial x_j}(T-T_0)\right\} = \rho C_v \frac{\partial}{\partial t}(T-T_0)$$

β_{ij} is the coefficients of linear thermal expansion. This is conductivity of the material.

C_v is specific heat.

T_0 is the refrence (initial) temperature.

For isotropic.

$$K_{ij} = K\delta_{ij} \quad , \beta_{ij} = \beta\delta_{ij}$$

For transversely isotropic medium

$$K_{ij} = K_1\delta_{ij} + (K_2 - K_1)\delta_{i3}\delta_{j3}$$

$$\beta_{ij} = \beta_1\delta_{ij} + (\beta_2 - \beta_1)\delta_{i3}\delta_{j3}$$

Equation of motion for isotropic medium becomes

$$\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} - \beta\delta_{ij}(T-T_0)$$

$$\Rightarrow \sigma_{ij,j} = \lambda u_{k,ki} + \mu(u_{i,jj} + u_{j,ij}) - \beta T_{,i}$$

$$\boxed{\begin{aligned}&\beta\delta_{ij}(T-T_0)_{,j}\\ &= \beta(T-T_0)_{,i}\\ &= \beta(T_{,i})\end{aligned}}$$

Thus equation of motion

$$\sigma_{ij,j} = \rho\ddot{u}_i$$

becomes

$$\lambda u_{k,ki} + \mu(u_{i,jj} + u_{j,ij}) - \beta T_{,i} = \rho\ddot{u}_i \longrightarrow (1)$$

$$\frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial T}{\partial x_j}\right) = \rho C_v \frac{\partial T}{\partial t}$$

$$\left(K\delta_{ij}\frac{\partial T}{\partial x_j}\right)_{,i} = \rho C_v \frac{\partial T}{\partial t}$$

And heat equation becomes

$$K T_{,jj} = \rho C_v \frac{\partial T}{\partial t} \longrightarrow ②$$

$$u_i = \phi_i(Kx_2)\, e^{iK(x_1 - ct)} \quad ; i = 1,2$$

$$T = \phi_3(Kx_2)\, e^{iK(x_1 - ct)}$$

from equation ①

for $i=1$

$$\lambda u_{k,k1} + \mu(u_{1,jj} + u_{j,1j}) - \beta T_{,1} = \rho \ddot{u}_1$$

$$\Rightarrow \lambda(u_{1,11} + u_{2,21}) + \mu(u_{1,11} + u_{1,22} + u_{1,11} + u_{2,12}) - \beta T_{,1} = \rho \ddot{u}_1$$

$$\Rightarrow \lambda\{-K^2\phi_1(Kx_2) + iK\phi_2'(Kx_2)\} + \mu\{-2K^2\phi_1(Kx_2) + \phi_1'' + iK\phi_2'\} - \beta iK\phi_3 = \rho(-K^2c^2)\phi_1$$

$$\Rightarrow \mu\phi_1'' + (\lambda+\mu)iK\phi_2' - (\lambda K^2 + 2K^2\mu - \rho K^2c^2)\phi_1 - iK\beta\phi_3 = 0 \longrightarrow Ⓐ$$

for $i=2$

$$\lambda u_{k,k2} + \mu(u_{2,jj} + u_{j,2j}) - \beta T_{,2} = \rho \ddot{u}_2$$

$$\rightarrow \lambda(u_{1,12} + u_{2,22}) + \mu(u_{2,11} + u_{2,22} + u_{1,21} + u_{2,22}) - \beta T_{,2} = \rho \ddot{u}_2$$

$$\Rightarrow \lambda\{iK\phi_1'(Kx_2) + \phi_2''(Kx_2)\} + \mu\{-K^2\phi_2 + 2\phi_2'' + iK\phi_1'\} - \beta\phi_3' = \rho(-K^2c^2)\phi_2$$

$$\Rightarrow (\lambda + 2\mu)\phi_2'' + (\lambda+\mu)iK\phi_1' - K^2(\mu - \rho c^2)\phi_2 - \beta\phi_3' = 0 \longrightarrow Ⓑ$$

Now from equation ②

$$T_{,11} + T_{,22} = \frac{\delta C_v}{K} \frac{\delta T}{\delta t}$$

$$\Rightarrow -k^2 \Phi_3 + \phi_3'' = \frac{\delta C_v}{K} ikc\Phi_3$$

$$\Rightarrow \phi_3'' - \left(k^2 + \frac{ikc\delta C_v}{K}\right)\phi_3 = 0 \longrightarrow (C)$$

Taking Laplace Transform of equ (A)

$$\mu\{s^2\bar{\Phi}_1(s) - s\phi_1(0) - \phi_1'(0)\} + ik(\lambda+\mu)\{s\bar{\Phi}_2(s) - \phi_2(0)\}$$
$$- \{\lambda k^2 + 2k^2\mu - \delta k^2 c^2\}\bar{\Phi}_1 - ik\beta\bar{\Phi}_3 = 0$$

$$\Rightarrow (\mu s^2 - \lambda k^2 - 2k^2\mu + \delta k^2 c^2)\bar{\Phi}_1(s) + isk(\lambda+\mu)\bar{\Phi}_2(s)$$
$$- ik\beta\bar{\Phi}_3(s) = \mu s\phi_1(0) + \mu\phi_1'(0) + ik(\lambda+\mu)\phi_2(0) \longrightarrow (*)$$

Similarly taking Laplace Transform of (B)

$$(\lambda+2\mu)\{s^2\bar{\Phi}_2(s) - s\phi_2(0) - \phi_2'(0)\} + ik(\lambda+\mu)\{s\bar{\Phi}_1(s) - \phi_1(0)\}$$
$$- k^2(\mu - \delta c^2)\bar{\Phi}_2 - \beta\{s\bar{\Phi}_3(s) - \phi_3(0)\} = 0$$

$$\Rightarrow \{\lambda s^2 + 2\mu s^2 - \mu k^2 + \delta c^2 k^2\}\bar{\Phi}_2(s) + iks(\lambda+\mu)\bar{\Phi}_1(s)$$
$$- \beta s\bar{\Phi}_3(s) = (\lambda+2\mu)s\phi_2(0) + (\lambda+2\mu)\phi_2'(0)$$
$$+ ik(\lambda+\mu)\phi_1(0) + \beta\phi_3(0) = 0 \longrightarrow (**)$$

Now taking Laplace Transform of equation (C)

$$\Rightarrow s^2\bar{\Phi}_3(s) - s\phi_3(0) - \phi_3'(0) - \xi\bar{\Phi}_3(s) = 0$$

$$\Rightarrow (s^2 + \xi)\bar{\Phi}_3(s) = s\phi_3(0) + \phi_3'(0) \longrightarrow (***)$$

from ⊛ Put $(\mu s^2 - \lambda k^2 - 2k^2\mu + \rho k^2 c^2) = a_1$,
$isk(\lambda+\mu) = a_2$; $-ik\beta = a_3$ &
$\mu s\phi_1(0) + \mu\phi_1'(0) + ik(\lambda+\mu)\phi_2(0) = d_1$

from ⊛⊛ Put; $(\lambda s^2 + 2\mu s^2 - \mu k^2 + \rho c^2 k^2) = b_2$,
$isk(\lambda+\mu) = b_1$, $-\beta s = b_3$ and
$-(\lambda+2\mu)s + (\lambda+2\mu) - ik(\lambda+\mu) + \beta = d_2$

from ⊛⊛⊛ $s^2 + \xi = c_3$ and $s\phi_3(0) + \phi_3'(0) = d_3$

$$⊛ \Rightarrow \quad a_1\bar{\Phi}_1(s) + a_2\bar{\Phi}_2(s) + a_3\bar{\Phi}_3(s) = d_1 \longrightarrow (3)$$

$$⊛⊛ \Rightarrow \quad b_1\bar{\Phi}_1(s) + b_2\bar{\Phi}_2(s) + b_3\bar{\Phi}_3(s) = d_2 \longrightarrow (4)$$

$$⊛⊛⊛ \Rightarrow \quad c_3\bar{\Phi}_3(s) = d_3 \longrightarrow (5)$$

$$\Rightarrow \bar{\Phi}_1(s) = \frac{\begin{vmatrix} d_1 & a_2 & a_3 \\ d_2 & b_2 & b_3 \\ d_3 & 0 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix}} = \frac{\begin{vmatrix} \quad \end{vmatrix}}{c_3(a_1b_2 - b_1a_2)}$$

$$\bar{\Phi}_2(s) = \frac{\begin{vmatrix} a_1 & d_1 & a_3 \\ b_1 & d_2 & b_3 \\ 0 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix}} = \frac{\begin{vmatrix} \quad \end{vmatrix}}{c_3(a_1b_2 - b_1a_2)}$$

$$\bar{\Phi}_3(s) = \frac{\begin{vmatrix} a_1 & a_2 & d_1 \\ b_1 & b_2 & d_2 \\ 0 & 0 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix}} = \frac{\begin{vmatrix} \quad \end{vmatrix}}{c_3(a_1b_2 - b_1a_2)}$$

Here $c_3(a_1b_2 - b_1a_2)$ is quadratic in s^2

Let s_1^2 and s_2^2 be the roots of

$c_3(a_1b_2 - b_1a_2)$

$\Rightarrow c_3(a_1b_2 - b_1a_2) = c_3(s^2 - s_1^2)(s^2 - s_2^2)$

Thus we have

$$\bar{\Phi}_1(s) = \frac{1}{c_3 s(s^2 - s_1^2)(s^2 - s_2^2)}$$

$$\Rightarrow \bar{\Phi}_1(s) = \frac{A_1}{c_3} + \frac{A_2}{s - s_1} + \frac{A_3}{s - s_2} + \frac{A_4}{s + s_1} + \frac{A_5}{s + s_2}$$

Now by taking inverse Laplace Transform on both sides

$$\mathcal{L}^{-1}\{\bar{\Phi}_1(s)\} = \mathcal{L}^{-1}\{\quad\quad\}$$

$\Rightarrow \phi_1(kx_2) = A$

⇒ Reflection of P-waves as a P-wave :-

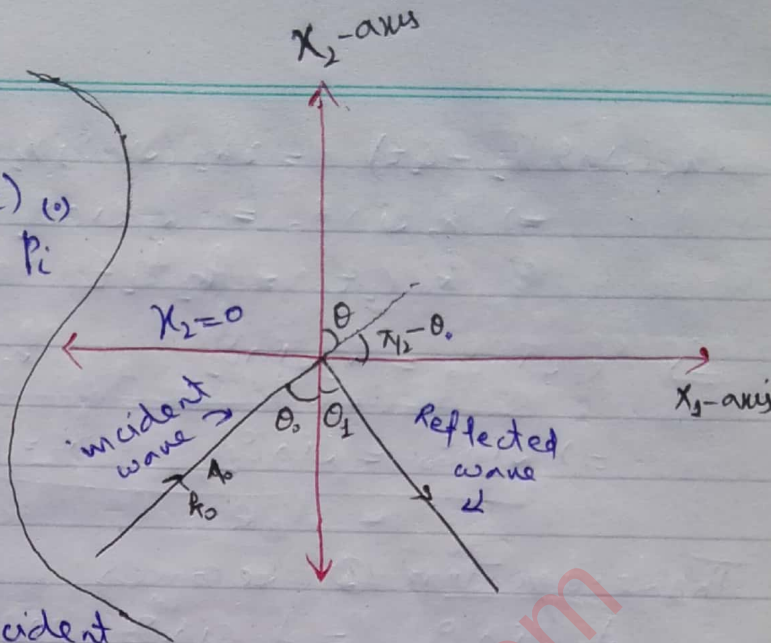
P-waves ⟶ longitudinal waves
SV- waves ⟶ Transverse waves

$$C_L = \sqrt{\frac{\lambda+\mu}{\rho}} \quad , \quad C_T = \sqrt{\frac{\mu}{\rho}}$$

with boundary conditions

1) Clamped Boundary Conditions:- In these boundary conditions displacement u_i vanish at boundary i.e $u_i = 0$ at $x_2 = 0$
2) Free Boundary Conditions:- In these boundary conditions stress temperature vanish at the boundary i.e. $\sigma_{i2} = 0$ at $x_2 = 0$

*** Now Consider Reflection of P-waves as a P-wave:-** Let θ_0 be the angle of incident, A_0 is the amplitude of incident wave and k_0 is the wave number. Similarly θ_1, A_1 and k_1 are angle of the reflected wave, amplitude of the reflected wave and wave number of the reflected wave respectively.



Let

$$u_i^{(0)} = A e^{ik_0(n_j^{(0)} x_j - c_L t)} p_i^{(0)}$$

$$u_i^{(1)} = A e^{ik_1(n_j^{(1)} x_j - c_L t)} p_i^{(1)}$$

be the displacement vector for incident and reflected waves.

Here
$$\underline{n}_i^{(0)} = (n_1^{(0)}, n_2^{(0)}, 0)$$
$$= \left(\cos\left(\tfrac{\pi}{2}-\theta_0\right), \sin\left(\tfrac{\pi}{2}-\theta_0\right), 0\right)$$
$$\Rightarrow \quad n_i^{(0)} = (\sin\theta_0, \cos\theta_0, 0)$$
$$\Rightarrow \quad p_i^{(0)} = (\sin\theta_0, \cos\theta_0, 0)$$
$\because$ we are considering P-waves $\Rightarrow n_i \parallel p_i$

Also
$$n_i^{(1)} = (\sin\theta_1, -\cos\theta_1, 0)$$
$$\Rightarrow \quad p_i^{(1)} = (\sin\theta_1, -\cos\theta_1, 0) \quad \because n_i \parallel p_i$$

Now for Clamped Boundary Conditions:-

i.e $u_i = 0$ at $x_2 = 0$

$$\Rightarrow \quad u_i^{(0)} + u_i^{(1)} = 0 \quad \text{at} \quad x_2 = 0$$

for $i=1$
$$u_1^{(0)} + u_1^{(1)} = 0 \quad \text{at} \quad x_2 = 0$$
$$\Rightarrow \quad A_0 p_1 e^{ik_0(n_1^{(0)} x_1 - c_L t)} + A_1 p_1 e^{ik_1(n_1^{(1)} x_1 - c_L t)} = 0$$
$$\Rightarrow \quad A_0 \sin\theta_0 e^{ik_0(\sin\theta_0 x_1 - c_L t)} + A_1 \sin\theta_1 e^{ik_1(\sin\theta_1 x_1 - c_L t)} = 0$$

by stress Law

$$ik_0(\sin\theta_0 x_1 - c_L t) = ik_1(\sin\theta_1 x_1 - c_L t)$$
$$\Rightarrow k_0 \sin\theta_0 = k_1 \sin\theta_1$$
$$\& \quad k_0 c_L t = k_1 c_L t$$
$$\Rightarrow k_0 \sin\theta_0 = k_1 \sin\theta_1 \quad \& \quad k_0 = k_1$$
$$\Rightarrow \boxed{\sin\theta_0 = \sin\theta_1 \quad \& \quad k_0 = k_1}$$

Then equation ① becomes
$$A_0 \sin\theta_0 = A_1 \sin\theta_1$$
$$\Rightarrow A_0 = A_1 \qquad \text{or} \qquad A_1 = A_0$$

Now for free Boundary Conditions
i.e $\sigma_{i2} = 0$ at $x_2 = 0$
$$\Rightarrow \sigma_{i2}^{(0)} + \sigma_{i2}^{(1)} = 0 \text{ at } x_2 = 0$$

for Isotropic medium
$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$$
$$\Rightarrow \sigma_{i2} = \lambda \epsilon_{kk} \delta_{i2} + 2\mu \epsilon_{i2}$$
$$\Rightarrow \sigma_{12} = 2\mu \epsilon_{12} + \mu(u_{1,2} + u_{2,1})$$
$$\Rightarrow \sigma_{12}^{(0)} = \mu(u_{1,2}^{(0)} + u_{2,1}^{(0)})$$
$$\sigma_{12}^{(1)} = \mu(u_{1,2}^{(1)} + u_{2,1}^{(1)})$$

for $i=1$
$$\sigma_{12}^{(0)} + \sigma_{12}^{(1)} = 0$$
$$\Rightarrow \{u_{1,2}^{(0)} + u_{2,1}^{(0)}\} + \{u_{1,2}^{(1)} + u_{2,1}^{(1)}\} = 0$$
$$\Rightarrow A_0 \{n_2^{(0)} p_1^{(0)} + n_1^{(0)} p_2^{(0)}\} e^{ik_0(n_1^{(0)} x_1 - c_L t)} + A_1 \{n_2^{(1)} p_1^{(1)} + n_1^{(1)} p_2^{(1)}\} e^{ik_1(n_1^{(1)} x_1 - c_L t)} = 0$$

By stress Law.

$$k_0 n_1^{(0)} = k_1 n_1^{(1)} \quad \text{and} \quad k_0 = k_1$$

$$\Rightarrow n_1^{(0)} = n_1^{(1)} \quad \text{and} \quad k_0 = k_1$$

$$\Rightarrow \sin\theta_0 = \sin\theta_1 \,, \quad k_0 = k_1$$

$$\Rightarrow \boxed{\theta_0 = \theta_1 \;\; \& \;\; k_0 = k_1} \qquad \Rightarrow A_0 - A_1 = 0$$

for $i=2$

$$\sigma_{i2} = \lambda \epsilon_{kk} \delta_{i2} + 2\mu \epsilon_{i2}$$

$$\Rightarrow \sigma_{22} = \lambda \epsilon_{kk} + 2\mu \epsilon_{22}$$

$$= \lambda \epsilon_{11} + (\lambda + 2\mu)\epsilon_{22}$$

$$\Rightarrow \sigma_{22} = \lambda u_{1,1} + (\lambda + 2\mu) u_{2,2}$$

Now

$$\sigma_{22}^{(0)} + \sigma_{22}^{(1)} = 0$$

$$\Rightarrow \left\{ \lambda u_{1,1}^{(0)} + (\lambda+2\mu) u_{2,2}^{(0)} \right\} + \left\{ \lambda u_{1,1}^{(1)} + (\lambda+2\mu) u_{2,2}^{(1)} \right\} = 0$$

$$\Rightarrow A_0 \left\{ \lambda n_1^{(0)} p_1^{(0)} + (\lambda+2\mu) n_2^{(0)} p_2^{(0)} \right\} + A_1 \left\{ \lambda n_1^{(1)} p_1^{(1)} + (\lambda+2\mu) (n_2^{(1)} p_2^{(1)}) \right\} = 0$$

$$\Rightarrow A_0 \left\{ \lambda \sin^2\theta_0 + (\lambda+2\mu)\cos^2\theta_0 \right\} + A_1 \left\{ \lambda \sin^2\theta_1 + (\lambda+2\mu)\cos^2\theta_1 \right\} = 0$$

$\because \theta_0 = \theta_1$

$$\Rightarrow \left.\begin{aligned} A_0 + A_1 &= 0 \\ A_0 - A_1 &= 0 \end{aligned}\right\} \Rightarrow A_1 = 0$$

Then we have seen in both cases the problem can not be solved by assuming only a reflected P-wave.

⇒ Reflection of S.V. waves as a P-Wave:-

Incident wave is SV-wave and reflected wave is P-wave. The displacement vector of the wave are given

$$u_i^{(0)} = A_0 e^{ik_0(n_j^{(0)} x_j - ct)} p_i^{(0)}$$

Incident wave:-

$$n_i^{(0)} \perp p_0 \qquad n^{(0)} = (\sin\theta, \cos\theta, 0)$$

$$p_0 = (-\cos\theta, \sin\theta, 0)$$

The incident wave is S.V. wave

$$u_i^{(0)} = A_0 e^{ik_0(n_j^{(0)} x_j - c_2 t)} p_i^{(0)}$$

$$\Rightarrow u_1^{(0)} = A_0 e^{ik_0(\sin\theta_0 x_1 + \cos\theta_0 x_2 - c_2 t)} (-\cos\theta_0)$$

$$u_2^{(0)} = A_0 e^{ik_0(x_1 \sin\theta_0 + x_2 \cos\theta_0 - c_2 t)} (\sin\theta_0)$$

Reflected wave:-

$$n^1 \parallel p^1$$

$$n^{(1)} = (\sin\theta, -\cos\theta, 0) = p^{(1)}$$

$$u_i^{(1)} = A_1 e^{ik_1(n_j^{(1)} x_j - c_1 t)} p_i^{(1)}$$

$$u_1^{(1)} = A_1 e^{ik_1(x_1 \sin\theta_1 - x_2 \cos\theta_2 - c_1 t)} (\sin\theta_1)$$

$$u_2^{(1)} = A_1 e^{ik_1(x_1 \sin\theta_1 - x_2 \cos\theta_2 - c_1 t)} (-\cos\theta_1)$$

For Free Boundary Condition:-

Boundry $x_2 = 0$ is free mean $\sigma_{i2} = 0$ where σ_{ij} is stress tensor

$$\sigma_{12} = \lambda \epsilon_{kk}\delta_{12} + 2\mu \epsilon_{12}$$

$$= (0) + 2\mu (u_{1,2} + u_{2,1})\cdot\frac{1}{2} \qquad \because \delta_{12} = 0$$

$$= \mu (u_{1,2} + u_{2,1})$$

$$= \mu (u^{(0)}_{1,2} + u^{(1)}_{1,2} + u^{(0)}_{2,1} + u^{(1)}_{2,1})$$

$$= \mu \{ -A_0 e^{ik_0(x_1 \sin\theta_0 - Gt)} ik_0 \cos\theta_0 \cos\theta_0 + A_1 e^{\cancel{ik_1(x_1 \sin\theta_1 - Gt)}} \sin\theta_1 \cos\theta_1 + A_0 e^{ik_0(x_1\sin\theta_0 - Gt)} ik_0 \sin\theta_0 \sin\theta_0 - A_1 e^{\cancel{ik_1(x_1\sin\theta_1 - Gt)}} \cos\theta_1 \sin\theta_1$$

$$= \mu\{ -A_0 e^{ik_0(x_1 \sin\theta_0 - Gt)} ik_0 (\cos^2\theta_0 - \sin^2\theta_0)\}$$

$\sigma_{12} = 0 \Rightarrow$

$$-\mu A_0 e^{ik_0(x_1\sin\theta_0 - Gt)} (\cos^2\theta_0 - \sin^2\theta_0) = 0$$

$$\Rightarrow A_0 \cos 2\theta_0 = 0$$

$\Rightarrow A_0 = 0 \Rightarrow$ Reflection does not exist.

⇒ Reflection of SV-Waves as SV-waves:-

The incident wave is SV and the reflected wave is SV. the displacement vector of the wave are given

$$u_i^{(0)} = A_0 e^{ik_0(x_j n_j^{(0)} - Gt)} p_i^{(0)}$$

$$u_i^{(1)} = A_1 e^{ik_1(x_j n_j^{(1)} - Gt)} p_i^{(1)}$$

$n^0 \perp p^0$

$n^0 = (\sin\theta_0, \cos\theta_0, 0)$, $p^0 = (-\cos\theta_0, \sin\theta_0, 0)$

$n^1 = (\sin\theta_1, -\cos\theta_1, 0) \Rightarrow p^1 = (\cos\theta_1, +\sin\theta_1, 0)$

For free Boundary Conditions:-

$\sigma_{12} = 0$, $\sigma_{22} = 0$

$$\sigma_{12} = \lambda \epsilon_{kk} \delta_{12} + 2\mu \epsilon_{12}$$

$$= \lambda \epsilon_{kk}(0) + 2\mu(u_{1,2} + u_{2,1})$$

$$= \mu(u^{(0)}_{1,2} + u^{(1)}_{1,2} + u^{(0)}_{2,1} + u^{(1)}_{2,1})$$

$$= \mu\{-A_0 e^{ik_0(x_1\sin\theta_0 - Gt)} ik_0 \cos\theta_0 \cos\theta_0 - A_1 e^{ik_1(x_1\sin\theta_1 - Gt)} ik_1(\cos\theta_1 \cos\theta_1) + A_0 e^{ik_0(x_1\sin\theta_0 - Gt)} ik_0 \sin\theta_0 \sin\theta_0 + A_1 e^{ik_1(x_1\sin\theta_1 - Gt)} ik_1 \sin\theta_1 \sin\theta_1$$

$$= \mu\{-A_0 e^{ik_0(x_1\sin\theta_0 - Gt)} ik_0(\cos^2\theta_0 - \sin^2\theta_0) - A_1 e^{ik_1(x_1\sin\theta_1 - Gt)} ik_1(\cos^2\theta_1 - \sin^2\theta_1)\}$$

$$= \mu\{-A_0 e^{ik_0(x_1\sin\theta_0 - Gt)} ik_0 \cos 2\theta_0 - A_1 e^{ik_1(x_1\sin\theta_1 - Gt)} ik_1 \cos 2\theta_1\}$$

Now $\sigma_{12} = 0 \Rightarrow$

$$-A_0 e^{ik_0(x_1\sin\theta_0 - Gt)} ik_0 \cos 2\theta_0 - A_1 e^{ik_1(x_1\sin\theta_1 - Gt)} ik_1 \cos 2\theta_1 =$$

To make exponents identical

$k_0 \sin\theta_0 = k_1 \sin\theta_1$

This is possible only if $\theta_0 = \theta_1$ & k

If it holds then

$$-A_0 \cos 2\theta_0 - A_1 \cos 2\theta_0 = 0$$

$$\Rightarrow \boxed{A_0 = -A_1}$$

Now

$$\sigma_{22} = \lambda(\epsilon_{11} + \epsilon_{22}) + 2\mu(u_{2,2})$$

$$= \lambda[u_{1,1} + u_{2,2}] + 2\mu(u_{2,2})$$

$$\Rightarrow \sigma_{22} = \lambda[u_{1,1}^{(0)} + u_{1,1}^{(1)} + u_{2,2}^{(0)} + u_{2,2}^{(1)}] + 2\mu(u_{2,2}^{(0)} + u_{2,2}^{(1)})$$

$$= \lambda[-A_0 e^{ik_0(x_1\sin\theta_0 - Gt)} ik_0 \sin\theta_0 \cos\theta_0 + A_1 e^{ik_1(x_1\sin\theta_1 - Gt)} ik_1(\sin\theta_1\cos\theta_1) + A_0 e^{ik_0(x_1\sin\theta_0 - Gt)} ik_0(\cos\theta_0\sin\theta_0) - A_1 e^{ik_1(x_1\sin\theta_1 - Gt)} ik_1 \cos\theta_1 \sin\theta_1] + 2\mu[A_0 e^{ik_0(x_1\sin\theta_0 - Gt)} ik_0 \cos\theta_0 \sin\theta_0 - A_1 e^{ik_1(x_1\sin\theta_1 - Gt)} \cos\theta_1 \sin\theta_1]$$

$$= \lambda(0) + 2\mu[A_0 e^{ik_0(x_1\sin\theta_0 - Gt)} ik_0 \sin\theta_0 \cos\theta_0) - A_1 e^{ik_1(x_1\sin\theta_1 - Gt)} \sin\theta_1 \cos\theta_1]$$

Now $\sigma_{22} = 0$ implies

$$2\mu A_0 \cos\theta_0 \sin\theta_0 - 2\mu A_1 \cos\theta_0 \sin\theta_0 = 0$$

$$\Rightarrow \boxed{A_0 = A_1}$$

Thus we have seen in both cases the problem can not be solved by assuming only a reflected SV - wave.

⇒ Reflection of P-waves as SV-wave + P-wave:-

Here we assume a reflected P-wave, there is a reflected SV-wave too.

The displacement vector $u_i^{(2)}$ is given

$$u_i^{(2)} = A_2 e^{ik_2(x_j n_j^{(2)} - c_2 t)} p_i^{(2)}$$

$$n^2 = (\sin\theta_2, -\cos\theta_2, 0)$$

$$p^2 = (\cos\theta_2, \sin\theta_2, 0) \quad \because n^2 \perp p^2$$

<u>Boundry is free:-</u>

Boundry $x_2 = 0$ is free mean $\sigma_{i2} = 0$, where σ_{ij} is stress tensor.

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} \quad , \text{for } j=2$$

$$\sigma_{12} = \lambda(0) + 2\mu\epsilon_{12}$$

$$= 2\mu\left(\frac{u_{1,2} + u_{2,1}}{2}\right)$$

$$\Rightarrow \sigma_{12}^{(2)} = \mu\left(u_{1,2}^{(2)} + u_{2,1}^{(2)}\right)$$

$$\Rightarrow \sigma_{12}^{(2)} = -\mu i k_2 A_2 \cos 2\theta_2 e^{ik_2(x_1 \sin\theta_0 - c_2 t)}$$

for $i=1$

$$\sigma_{12} = \mu\left(u_{1,2}^{(0)} + u_{1,2}^{(1)} + u_{1,2}^{(2)} + u_{2,1}^{(0)} + u_{2,1}^{(1)} + u_{2,1}^{(2)}\right)$$

$$\sigma_{12} = 0 \text{ at } x_2 = 0 \quad \forall x, t$$

$$\Rightarrow 0 = \mu\Big\{ A_0 e^{ik_0(x_1\sin\theta_0 - c_1 t)} ik_0 \sin\theta_0 \cos\theta_0 - A_1 e^{ik_1(x_1\sin\theta_1 - c_1 t)} ik_1(\sin\theta_1\cos\theta_1) - A_2 e^{ik_2(x_1\sin\theta_2 - c_2 t)} ik_2 \cos\theta_2 \cos\theta_2$$

$$+A_0 e^{ik_0(x_1\sin\theta_0 - c_L t)} ik_0 \sin\theta_0 \cos\theta_0 - A_1 e^{ik_1(x_1\sin\theta_1 - c_L t)}$$

$$ik_1(\sin\theta_1\cos\theta_1) + A_2 e^{ik_2(x_1\sin\theta_2 - c_T t)} ik_2(\sin\theta_2 \sin\theta_2)$$

$$\Rightarrow 0 = \mu_0 A_0 ik_0(\cos\theta_0\sin\theta_0 + \cos\theta_0\sin\theta_0) e^{ik_0(x_1\sin\theta_0 - c_L t)}$$
$$-\mu A_1 ik_1 \sin 2\theta_1 e^{ik_1(x_1\sin\theta_1 - c_L t)} - \mu ik_2 A_2(\cos^2\theta_2 - \sin^2\theta_2$$

$$\Rightarrow 0 = \mu_0 A_0 ik_0 \sin 2\theta_0 e^{ik_0(x_1\sin\theta_0 - c_L t)} - \mu A_1 ik_1 \sin 2\theta_1$$
$$e^{ik_1(x_1\sin\theta_1 - c_L t)} - \mu A_2 ik_2 \cos 2\theta_2 e^{ik_2(x_1\sin\theta_2 - c_T t)}$$

To make exponent identical

$$k_0\sin\theta_0 = k_1\sin\theta_1 = k_2\sin\theta_2$$

And $\quad k_0 c_L = k_1 c_L = k_2 c_T$

This hold if $k_0 = k_1$, $\theta_0 = \theta_1$

$$\Rightarrow k_0 c_L = k_2 c_T \Rightarrow k_2 = k_0 \frac{c_L}{c_T}$$

$$\Rightarrow k_2 = k_0 K \Rightarrow K = \frac{k_2}{k_0}$$

$$\Rightarrow A_0 k_0 \sin 2\theta_0 - A_1 k_0 \sin 2\theta_0 - A_2 k_2 \cos 2\theta_2 = 0 \longrightarrow (a)$$

for $i=2$

$$\sigma_{22} = \lambda \epsilon_{kk}(1) + 2\mu(\epsilon_{22})$$
$$= \lambda(\epsilon_{11} + \epsilon_{22} + \cancel{\epsilon_{33}}^{0}) + 2\mu(\epsilon_{22})$$
$$= \lambda(u_{1,1} + u_{2,2}) + 2\mu(u_{2,2})$$

$$\Rightarrow \sigma_{22} = \lambda(u^{(0)}_{1,1} + u^{(1)}_{1,1} + u^{(0)}_{2,2} + u^{(1)}_{2,2} + u^{(2)}_{1,1} + u^{(2)}_{2,2})$$
$$+ 2\mu(u^{(0)}_{2,2} + u^{(1)}_{2,2} + u^{(2)}_{2,2})$$

$\sigma_{22} = 0$ at $x_2 = 0 \;\forall\, x, t$

$$\Rightarrow 0 = \lambda\{A_0 i k_0 \sin\theta_0 e^{ik_0(x_1\sin\theta_0 - c_1 t)} \sin\theta_0 + A_1 i k_1 \sin\theta_1 e^{ik_1(x_1\sin\theta_1 - c_1 t)} \sin\theta_1 + A_2 i k_2 \sin\theta_2 e^{ik_2(x_1\sin\theta_2 - c_T t)} \cos\theta_2$$

$$+ A_0 i k_0 \cos\theta_0 e^{ik(x_1\sin\theta_0 - c_1 t)} \cos\theta_0 + A_1 i k_1 \cos\theta_1 e^{ik_1(x_1\sin\theta_1 - c_1 t)} \cos\theta_1 - A_2 i k_2 \cos\theta_2 e^{ik_2(x_1\sin\theta_2 - c_T t)} \sin\theta_2\}$$

$$+ 2\mu\{A_0 i k_0 \cos\theta_0 e^{ik_0(x_1\sin\theta_0 - c_1 t)} + A_1 i k_1 \cos\theta_1 e^{ik_1(x_1\sin\theta_1 - c_1 t)} \cos\theta_1 - A_2 i k_2 \cos\theta_2 e^{ik_2(x_1\sin\theta_2 - c_T t)} \sin\theta_2\}$$

$$\Rightarrow 0 = (\lambda + 2\mu\cos^2\theta_0) A_0 i k_0 e^{ik_0(x_1\sin\theta_0 - c_1 t)} + (\lambda + 2\mu\cos^2\theta_1)(A_1 i k_1 e^{ik_1(x_1\sin\theta_1 - c_1 t)} - \mu\sin 2\theta_2 A_2 i k_2 e^{ik_2(x_1\sin\theta_2 - c_T t)} \cancel{\sin\theta_2}$$

Therefore equation gives

$$0 = (\lambda + 2\mu\cos^2\theta_0) A_0 k_0 + (\lambda + 2\mu\cos^2\theta_1) A_1 k_1 - \mu A_2 k_2 \sin 2\theta_2 \longrightarrow (b)$$

from (A)

$$A_0 k_0 \sin 2\theta_0 - A_1 k_0 \sin 2\theta_0 - A_2 k_2 \cos 2\theta_2 = 0$$

$$\Rightarrow \frac{A_1}{A_0}\sin 2\theta_0 + \frac{A_2}{A_0}\frac{k_2}{k_0}\cos 2\theta_2 = \sin 2\theta_0$$

$$\Rightarrow \frac{A_1}{A_0}\sin 2\theta_0 + \frac{A_2}{A_0} K \cos 2\theta_2 = \sin 2\theta_0 \longrightarrow (c)$$

from ⓑ $A_1 k_0(\lambda+2\mu\cos^2\theta_0) - \mu A_2 k_2(\sin 2\theta_2)$

$$= -A_0 k_0(\lambda+2\mu\cos^2\theta_0)$$

$$\Rightarrow \frac{A_1}{A_0}(\lambda+2\mu\cos^2\theta_0) - \mu\frac{A_2}{A_0}\frac{k_2}{k_0}\sin 2\theta_2 = -(\lambda+2\mu\cos^2\theta_0)$$

$$\Rightarrow \frac{A_1}{A_0}(\lambda+2\mu\cos^2\theta_0) - \mu\frac{A_2}{A_0}K\sin 2\theta_2 = -(\lambda+2\mu\cos^2\theta_0)$$

$$\because \frac{k_2}{k_0} = \frac{c_1}{c_2} = K$$

$$\Rightarrow K = \sqrt{\frac{\lambda+2\mu}{\mu}} \Rightarrow K^2 = \frac{\lambda+2\mu}{\mu}$$

$$\Rightarrow K^2\mu - 2\mu = \lambda$$

Put in above equation

$$\frac{A_1}{A_0}(K^2\mu - 2\mu + 2\mu\cos^2\theta_0) - \mu\frac{A_2}{A_0}K\sin 2\theta_0 = -(K^2\mu - 2\mu + 2\mu\cos^2\theta_0)$$

$$\Rightarrow \frac{A_1}{A_0}(K^2\mu - 2\mu(1-\cos^2\theta_0)) - \mu\frac{A_2}{A_0}K\sin 2\theta_0 = -(K^2\mu - 2\mu\sin^2\theta_0)$$

$$\Rightarrow \frac{A_1}{A_0}(K^2\mu - 2\mu\sin^2\theta_0) - \mu K\sin 2\theta_2\frac{A_2}{A_0} = -(K^2\mu - 2\mu\sin^2\theta_0)$$

$$\because \sin^2\theta_2 = \frac{1}{K^2}\sin^2\theta_0$$

$$\Rightarrow \frac{A_1}{A_0}(K^2\mu - 2\mu\sin^2\theta_0) - \mu K\sin 2\theta_2\frac{A_2}{A_0} = -K^2\mu(1-2\sin^2\theta_2)$$

$$\Rightarrow (K^2 - 2\sin^2\theta_0)\frac{A_1}{A_0} - K\sin 2\theta_2\frac{A_2}{A_0} = -K^2\cos 2\theta \longrightarrow (d)$$

Now multiplying equation ⓒ by $\sin 2\theta_2$

× eqn (d) by $\cos 2\theta_2$ and add

$$\Rightarrow \sin 2\theta_0 \sin 2\theta_2 \frac{A_1}{A_0} + k\cos 2\theta_2 \sin 2\theta_0 \frac{A_2}{A_0} = \sin 2\theta_2 \sin 2\theta_0$$

$$(k^2 - 2\sin^2\theta_0)\cos 2\theta_2 \frac{A_1}{A_0} - k\cos 2\theta_2 \sin 2\theta_0 \frac{A_2}{A_0} = -k^2\cos^2 2\theta_2$$

$$\Rightarrow \frac{A_1}{A_0}\left[\sin 2\theta_0 \sin 2\theta_2 + (k^2 - 2\sin^2\theta_0)\cos 2\theta_2\right] = \sin 2\theta_2 \sin 2\theta_0 - k^2\cos^2 2\theta_2$$

$$\Rightarrow \boxed{\frac{A_1}{A_0} = \frac{\sin 2\theta_0 \sin 2\theta_2 - k^2\cos^2 2\theta_2}{\sin 2\theta_0 \sin 2\theta_2 + k^2\cos^2 2\theta_2}}$$

Put it in eqn (c)

$$\Rightarrow \frac{A_2}{A_0} = \frac{2k\cos 2\theta_2 \sin 2\theta_0}{\sin 2\theta_0 \sin 2\theta_2 + k^2\cos^2 2\theta_2}$$

⇒ Reflection of SV-waves as P-waves + SV-waves:-

Consider the reflection of an incident SV-wave propagation in x_1x_2-plane.
We assume that on reflection there is a P-wave and SV-wave.

<u>Incident wave:-</u>

$$u_i^{(0)} = A_0 e^{ik_0(x_j n_j^{(0)} - C_T t)} p_i^{(0)}$$

$$n_i^{(0)} = (\sin\theta_0, \cos\theta_0, 0) \quad , \quad p_i^{(0)} = (-\cos\theta_0, \sin\theta_0, 0)$$

$$\because n_i^{(0)} \perp p_i^{(0)}$$

<u>Reflected P-wave:-</u>

$$u_i^{(1)} = A_1 e^{ik_1(x_j n_j^{(1)} - C_L t)} p_i^{(1)}$$

$n_i^{(1)} = (\sin\theta_1, -\cos\theta_1, 0)$, $P_i^{(1)} = (\sin\theta_1, -\cos\theta_1, 0)$

Reflected S.V waves:

$$u_i^{(2)} = A_2 e^{ik_2(x_j n_j^{(2)} - c_T t)} P_i^{(2)}$$

$n_i^{(2)} = (\sin\theta_2, -\cos\theta_2, 0)$, $P_i^{(2)} = (\cos\theta_2, \sin\theta_2, 0)$

For free Boundary Conditions:-

$x_2 = 0$ is free mean $\sigma_{i2} = 0$

$\Rightarrow \sigma_{12} = 0$, $\sigma_{22} = 0$, $\sigma_{32} = 0$

$$\sigma_{12} = \lambda \epsilon_{kk} \delta_{12} + 2\mu \epsilon_{12}$$

$$= 2\mu\left(\frac{u_{1,2} + u_{2,1}}{2}\right)$$

$$= \mu(u_{1,2} + u_{2,1})$$

$$= \mu(u_{1,2}^{(0)} + u_{1,2}^{(1)} + u_{1,2}^{(2)} + u_{2,1}^{(0)} + u_{2,1}^{(1)} + u_{2,1}^{(2)})$$

$$\Rightarrow 0 = \mu\Big\{-A_0 e^{ik_0(x_1\sin\theta_0 - c_L t)} ik_0(\cos^2\theta_0 - \sin^2\theta_0) - A_1 ik_1 e^{ik_1(x_1\sin\theta_1 - c_L t)}(\sin\theta_1\cos\theta_1 + \cos\theta_1\sin\theta_1) - A_2 ik_2 e^{ik_2(x_1\sin\theta_2 - c_T t)}\cos 2\theta_2 \longrightarrow (1)$$

$$\sigma_{22} = \lambda \epsilon_{kk} \delta_{22} + 2\mu \epsilon_{22}$$

$$= \lambda(\epsilon_{11} + \epsilon_{22}) + 2\mu(u_{2,2})$$

$$= \lambda(u_{1,1} + u_{2,2}) + 2\mu(u_{2,2})$$

$\sigma_{22} = 0$ at $x_2 = 0$

$$\sigma_{22} = \sigma_{22}^{(0)} + \sigma_{22}^{(1)} + \sigma_{22}^{(2)} = 0$$

implies

$$0 = \lambda\{-A_0 e^{ik_0(x_1\sin\theta_0 - c_1 t)} ik_0 \sin\theta_0\cos\theta_0 + A_1 e^{ik_1(x_1\sin\theta_1 - c_1 t)} ik_1\sin^2\theta_1 + A_2 ik_2 e^{ik_2(x_1\sin\theta_2 - c_1 t)} \sin\theta_2\cos\theta_2 + A_0 ik_0 e^{ik_0(x_1\sin\theta_0 - c_1 t)} \cos\theta_0 \sin\theta_0 + A_1 ik_1 e^{ik_1(x_1\sin\theta_1 - c_1 t)} \cos^2\theta_1 - A_2 ik_2 e^{ik_2(x_1\sin\theta_2 - c_1 t)} \sin\theta_2\cos\theta_2\} + 2\mu\{A_0 ik_0 e^{ik_0(x_1\sin\theta_0 - c_1 t)} \cos\theta_0 \sin\theta_0 + A_1 ik_1 e^{ik_1(x_1\sin\theta_1 - c_1 t)} \cos^2\theta_1 + A_2 ik_2 e^{ik_2(x_1\sin\theta_2 - c_1 t)} \sin\theta_2\cos\theta_2\}$$

$$\Rightarrow 0 = \mu A_0 ik_0 \sin 2\theta_0 e^{ik_0(x_1\sin\theta_0 - c_1 t)} + (\lambda + 2\mu\cos^2\theta_1) e^{ik_1(x_1\sin\theta_1 - c_1 t)} - \mu A_2 ik_2 \sin 2\theta_2 e^{ik_2(x_1\sin\theta_2 - c_1 t)} \longrightarrow (2)$$

To satisfy these equation, exponent must be identical

$$\Rightarrow k_0 \sin\theta_0 = k_1 \sin\theta_1 = k_2 \sin\theta_2$$

$$\& \quad k_0 c_1 = k_1 c_2 = k_2 c_1$$

$$\Rightarrow \boxed{k_0 = k_2 \quad , \quad \theta_0 = \theta_2}$$

$$\Rightarrow k_1 = k_0 \frac{c_1}{c_2} \quad \Rightarrow k_1 = \frac{k_0}{K}$$

$$\Rightarrow \sin\theta_1 = K \sin\theta_0$$

$$K = \frac{c_2}{c_1} = \sqrt{\frac{\lambda + 2\mu}{\mu}}$$

$$\Rightarrow K^2 = \frac{\lambda + 2\mu}{\mu}$$

$$\Rightarrow \lambda = k^2\mu - 2\mu$$

equ ① $\Rightarrow -A_0 k_0 \cos 2\theta_0 - A_1 k_1 \sin 2\theta_1 - A_2 k_2 \cos 2\theta_2 = 0$

$$\Rightarrow A_1 k_1 \sin 2\theta_1 + A_2 k_2 \cos 2\theta_2 = -A_0 k_0 \cos 2\theta_0$$

$$k_2 = k_0 \quad , \quad \theta_2 = \theta_0$$

$$\Rightarrow A_1 k_1 \sin 2\theta_1 + A_2 k_0 \cos 2\theta_0 = -A_0 k_0 \cos 2\theta_0$$

$$\Rightarrow \frac{A_2}{A_0}\frac{k_0}{k_0}\cos 2\theta_0 + \frac{A_1}{A_0}\frac{k_1}{k_0}\sin 2\theta_1 = -\cos 2\theta_0$$

$$\Rightarrow \frac{A_2}{A_0}\cos 2\theta_0 + \frac{A_1}{A_0}\cdot\frac{1}{k}\sin 2\theta_1 = -\cos 2\theta_0 \longrightarrow ③$$

② $\Rightarrow \mu A_0 k_0 \sin 2\theta_0 + (\lambda + 2\mu\cos^2\theta_1)A_1 k_1 - \mu \sin 2\theta_2 A_2 k_2 = 0$

$$\Rightarrow (\lambda + 2\mu\cos^2\theta_1)A_1 k_1 - \mu\sin 2\theta_2 A_2 k_2 = -\mu A_0 k_0 \sin 2\theta_0$$

$$\because \lambda = k^2\mu - 2\mu$$

$$\Rightarrow (k^2\mu - 2\mu + 2\mu\cos^2\theta_1)A_1 k_1 - \mu\sin 2\theta_2 A_2 k_2 = -\mu A_0 k_0 \sin 2\theta_0$$

$$\Rightarrow (k^2\mu - 2\mu(1-\cos^2\theta_1))A_1 k_1 - \mu\sin 2\theta_2 A_2 k_2 = -\mu A_0 k_0 \sin 2\theta_0$$

$$\Rightarrow (k^2\mu - 2\mu\sin^2\theta_1)A_1 k_1 - \mu\sin 2\theta_2 A_2 k_2 = -\mu A_0 k_0 \sin 2\theta_0$$

$$\Rightarrow (k^2\mu - 2\mu\sin^2\theta_1)\frac{A_1}{A_0}\cdot\frac{k_1}{k_0} - \mu\sin 2\theta_2 \frac{A_2}{A_1}\frac{k_0}{k_0} = -\mu\sin 2\theta_0$$

$$\Rightarrow (k^2 - 2\sin^2\theta_1)\frac{A_1}{A_0} - \frac{A_2}{A_1}k\sin 2\theta_0 = -k\sin 2\theta_0$$

$$\because \sin\theta_1 = k\sin\theta_0$$

$$\Rightarrow (k^2-2k^2\sin^2\theta_0)\frac{A_1}{A_0} - \frac{A_2}{A_0}k\sin 2\theta_0 = -k\sin 2\theta_0$$

$$\Rightarrow k\frac{A_1}{A_0}\cos 2\theta_0 - \frac{A_2}{A_0}\sin 2\theta_0 = -\sin 2\theta_0 \longrightarrow (4)$$

Multiply equ (3) by $\sin 2\theta_0$ & equ (4) by $k\cos 2\theta_0$

$$\Rightarrow \frac{A_1}{A_0}\sin 2\theta_1 \sin 2\theta_0 + \frac{A_2}{A_0}k\cos 2\theta_0 \sin 2\theta_0 = -k\cos 2\theta_0 \sin 2\theta_0$$

$$\frac{A_1}{A_0}k^2\cos 2\theta_0 \cos 2\theta_0 - \frac{A_2}{A_0}\sin 2\theta_0 \cos 2\theta_0 k = -k\sin 2\theta_0 \cos 2\theta_0$$

$$\Rightarrow \frac{A_1}{A_0}\left(\sin 2\theta_0 \sin 2\theta_1 + k^2\cos^2 2\theta_0\right) = -2k\sin 2\theta_0 \cos 2\theta_0$$

$$\Rightarrow \boxed{\frac{A_1}{A_0} = \frac{-k\sin 4\theta_0}{\sin 2\theta_0 \sin 2\theta_1 + k^2\cos^2 2\theta_0}}$$

Put in equ (3)

$$\frac{-k\sin 4\theta_0 \sin 2\theta_1}{\sin 2\theta_0 \sin 2\theta_1 + k^2\cos^2 2\theta_0} + \frac{A_2}{A_0}k\cos 2\theta_0 = -k\cos 2\theta_0$$

$$\Rightarrow \frac{A_2}{A_0}k\cos 2\theta_0 = -k\cos 2\theta_0 + \frac{k\sin 4\theta_0 \sin 2\theta_1}{\sin 2\theta_0 \sin 2\theta_1 + k^2\cos^2 2\theta_0}$$

$$\Rightarrow \frac{A_2}{A_0} = -1 + \frac{k\sin 4\theta_0 \sin 2\theta_1}{\sin 2\theta_0 \sin 2\theta_1 + k^2\cos^2 2\theta_0} \cdot \frac{1}{k\cos 2\theta_0}$$

$$= -1 + \frac{2\sin 2\theta_0 \sin 2\theta_1}{\sin 2\theta_0 \sin 2\theta_1 + k^2\cos^2 2\theta_0}$$

$$\Rightarrow \frac{A_2}{A_0} = \frac{-\sin 2\theta_0 \sin 2\theta_1 - k^2\cos^2 2\theta_0 + 2\sin 2\theta_0 \sin 2\theta_1}{\sin 2\theta_0 \sin 2\theta_1 + k^2 \cos^2 2\theta_0}$$

$$\Rightarrow \boxed{\frac{A_2}{A_0} = \frac{\sin 2\theta_0 \sin 2\theta_1 - k^2\cos^2 2\theta_0}{\sin 2\theta_0 \sin 2\theta_1 + k^2\cos^2 2\theta_0}}$$

⇒ Reflection of P-wave as SV-wave :-

The incident wave is P-wave and reflected wave is SV-wave. The displacement vector $u_i^{(0)}$ for incident wave and displacement vector $u_i^{(1)}$ for reflected wave.

$$n_i^{(0)} = p_i^{(0)} = (\sin\theta_0, \cos\theta_0, 0) \quad \because n^{(0)} \parallel p^{(0)}$$

$$n_i^{(1)} = (\sin\theta_1, -\cos\theta_1, 0), \quad p_i^{(1)} = (\cos\theta_1, \sin\theta_1, 0)$$

$$\because p_i^{(1)} \perp n_i^{(1)}$$

$$u_i^{(0)} = A_0 e^{ik_0(x_1\sin\theta_0 + x_2\cos\theta_0 - c_1 t)} p_i^{(0)}$$

$$\Rightarrow u_1^{(0)} = A_0 e^{ik_0(x_1\sin\theta_0 + x_2\cos\theta_0 - c_1 t)} \sin\theta_0$$

$$u_1^{(1)} = A_1 e^{ik_1(x_1\sin\theta_1 - x_2\cos\theta_1 - c_T t)} \cos\theta_1$$

$$u_2^{(0)} = A_0 e^{ik_0(x_1\sin\theta_0 + x_2\cos\theta_0 - c_1 t)} \cos\theta_0$$

$$u_2^{(1)} = A_1 e^{ik_1(x_1\sin\theta_1 - x_2\cos\theta_1 - c_T t)} \sin\theta_1$$

Free Boundry

$$\sigma_{12} = \sigma_{12}^{(0)} + \sigma_{12}^{(1)}$$

$\sigma_{12} = 0 \quad \Rightarrow \quad \sigma_{12} = 2\mu(\epsilon_{12})$

$= 2\mu(u_{1,2} + u_{2,1})$

$= \mu(u_{1,2} + u_{2,1}) = 0$

$\Rightarrow 0 = \mu\{A_0 e^{ik_0(x_1 \sin\theta_0 - c_L t)} ik_0 \sin\theta_0 \cos\theta_0 + A_1 e^{ik_1(x_1 \sin\theta_1 - c_T t)}$

$ik_1 \cos\theta_1 \cos\theta_1 + A_0 e^{ik_0(x_1 \sin\theta_0 - c_L t)} ik_0 \cos\theta_0 \sin\theta_0 +$

$A_1 e^{ik_1(x_1 \sin\theta_1 - c_T t)} ik_1 \sin\theta_1 \sin\theta_1\}$

$0 = \mu\{A_0 e^{ik_0(x_1 \sin\theta_0 - c_L t)} ik_0 \sin 2\theta_0 + A_1 e^{ik_1(x_1 \sin\theta_1 - c_T t)} ik_1\}$

To make exponents identical

$k_0 \sin\theta_0 = k_1 \sin\theta_1$

$\Rightarrow \boxed{\theta_0 = \theta_1 \quad \Rightarrow k_0 = k_1}$

$\Rightarrow \quad A_0 e^{ik_0 c_L t} + A_1 e^{ik_0 c_T t} \longrightarrow (1)$

$\sigma_{22} = \lambda \epsilon_{kk} + 2\mu \epsilon_{22}$

$= \lambda(\epsilon_{11} + \epsilon_{22}) + 2\mu(\epsilon_{22})$

$= \lambda(u_{1,1} + u_{2,2}) + 2\mu(u_{2,2})$

$= \lambda[u_{1,1}^{(0)} + u_{1,1}^{(1)} + u_{2,2}^{(0)} + u_{2,2}^{(1)}] + 2\mu[u_{2,2}^{(0)} + u_{2,2}^{(1)}]$

$\sigma_{22} = 0$ implies

$0 = \lambda\{A_0 e^{ik_0(x_1 \sin\theta_0 - c_L t)} ik_0 \sin\theta_0 \sin\theta_0 + \cancel{A_1 e^{ik_1(x_1 \sin\theta_1 - c_T t)} ik_1 \sin\theta_1 \cos\theta_1}$

$+ A_0 e^{ik_0(x \sin\theta_0 - c_L t)} ik_0 \cos\theta_0 \cos\theta_0 + \cancel{A_1 e^{ik_1(x_1 \sin\theta_1 - c_T t)} ik_1 \sin\theta_1 \cos\theta_1}\}$

$$+2\mu\left\{A_0 e^{ik_0(x_1\sin\theta_0 - c_1 t)} ik_0\cos\theta_0\cos\theta_0 - A_1 e^{ik_1(x_1\sin\theta_1 - c_1 t)} \sin\theta_1\cos\theta_1\right\}$$

$$\Rightarrow 0 = (\lambda + 2\mu\cos^2\theta_0) ik_0 A_0 e^{ik_0(x_1\sin\theta_0 - c_1 t)} - \mu A_1 e^{ik_1(x_1\sin\theta_1 - c_1 t)} \sin 2\theta_1 \longrightarrow (2)$$

Multiply eqn (1) by μ and add in (2)

$$\mu A_0 e^{ik_0 c_1 t} + \mu A_1 \cancel{e^{ik_1 c_1 t}} = 0$$

$$(\lambda + 2\mu\cos^2\theta_0) A_0 e^{ik_0 c_1 t} - \mu A_1 \cancel{e^{ik_1 c_1 t}} = 0$$

$$\Rightarrow (\mu + \lambda + 2\mu\cos^2\theta_0) A_0 e^{ik_0 c_1 t} = 0$$

$$\Rightarrow A_0 = 0$$

$\Rightarrow$ Reflection does not exist.

$\Rightarrow$ Transversly Isotropic Material

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{c_{11}-c_{12}}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{13} \\ \epsilon_{12} \end{bmatrix}$$

$\Rightarrow$ Reflection of P-waves as P-waves:-

$$n^0 \parallel p^0 \quad \text{and} \quad n^1 \parallel p^1$$

$$n^0 = (\sin\theta_0, \cos\theta_0, 0) = p^0$$

$$n^1 = (\sin\theta_1, -\cos\theta_1, 0) = p^1$$

Free Boundry:- Boundry $x_2 = 0$ is free mean $\sigma_{2} = 0$, where σ_{ij} is stress tensor.

$$\sigma_{12} = \left(\frac{C_1 - C_2}{2}\right) 2 \in_{12}$$

$$= (C_1 - C_2)\in_{12} = C_1 - C_2 \left(\frac{U_{1,2} + U_{2,1}}{2}\right)$$

$$\Rightarrow \sigma_{12} = \left(\frac{C_1 - C_2}{2}\right)\left[U^{(0)}_{1,2} + U^{(1)}_{1,2} + U^{(0)}_{2,1} + U^{(1)}_{2,1}\right]$$

$$= \left[\frac{C_1 - C_2}{2}\right]\Big[A_0 i k_0 \cos\theta_0 e^{ik_0(x_1 \sin\theta_0 - c_2 t)} \sin\theta_0 - A_1 i k_1 \cos\theta_1 e^{ik_1(x_1 \sin\theta_1 - c_2 t)} \sin\theta_1 + A_0 i k_0 e^{ik_0(x_1 \sin\theta_0 - c_2 t)} \sin\theta_0 \cos\theta_0 - A_1 i k_1 \sin\theta_1 e^{ik_1(x_1 \sin\theta_1 - c_2 t)} \cos\theta_1\Big]$$

$$= \left[\frac{C_1 - C_2}{2}\right]\left[A_0 i k_0 2\sin\theta_0 \cos\theta_0 e^{ik_0(x_1 \sin\theta_0 - c_2 t)} - A_1 i k_1 2 \sin\theta_1 \cos\theta_1 e^{ik_1(x_1 \sin\theta_1 - c_2 t)}\right]$$

To make exponents identical

$k_0 \sin\theta_0 = k_1 \sin\theta_1$. This is possible only if

$k_0 = k_1$, $\theta_0 = \theta_1$

$\sigma_{12} = 0 \Rightarrow \boxed{A_0 - A_1 = 0}$

Now solve σ_{22}

$$\sigma_{22} = C_{12}\in_{11} + C_{22}\in_{22} + \cancel{C_{23}\in_{33}}^{\,0}$$

$$= C_{12}\in_{11} + C_{22}\in_{22}$$

$$= C_{12}\left(U^{(0)}_{1,1} + U^{(1)}_{1,1}\right) + C_{22}\left(U^{(0)}_{2,2} + U^{(1)}_{2,2}\right)$$

$\sigma_{22} = 0$ implies

$$0 = C_{12}[A_0 i k_0 \sin\theta_0 e^{ik_0(x_1 \sin\theta_0 - C_t t)} \sin\theta_0 + A_1 i k_1 \sin\theta_1 e^{ik_1(x_1 \sin\theta_1 - C_t t)} \sin\theta_1] + C_{22}[A_0 i k_0 \cos^2\theta_0 e^{ik_0(x_1 \sin\theta_0 - C_t t)} + A_1 i k_1 \cos^2\theta_1 e^{ik_1(x_1 \sin\theta_1 - C_t t)}]$$

$$\Rightarrow (C_{12}\sin^2\theta_0 + C_{22}\cos^2\theta_0) i k_0 e^{ik_0(x_1 \sin\theta_0 - C_t t)} A_0 + (C_{22}\cos^2\theta_1 + C_{12}\sin^2\theta_1) i k_1 e^{ik_1(x_1 \sin\theta_1 - C_t t)} A_1 = 0$$

$\because \theta_0 = \theta_1$, $k_0 = k_1$

$$\left.\begin{aligned} \Rightarrow \quad A_0 + A_1 &= 0 \\ \& \quad A_0 - A_1 &= 0 \end{aligned}\right\} \Rightarrow A_0 = 0$$

Hence no reflecting P-wave.

Reflection of SV-wave as SV-wave :-

For Transversly isotropic medium

Incident wave SV:-

$$u_i^{(0)} = A_0 e^{ik_0(x_j n_j^{(0)} - C_t t)} p_i^{(0)}$$

$n^0 \perp p^0$; $n^0 = (\sin\theta_0, \cos\theta_0, 0)$, $p^0 = (-\cos\theta_0, \sin\theta_0, 0)$

Reflecting wave SV:-

$$u_i^{(1)} = A_1 e^{ik_1(x_j n_j^{(0)} - C_t t)} p_i^{(1)}$$

$n^1 \perp p^1$; $n^1 = (\sin\theta_1, -\cos\theta_1, 0)$, $p^1 = (\cos\theta_1, \sin\theta_1, 0)$

Free Boundry:-

$$\sigma_{12} = \left(\frac{C_{11} - C_{12}}{2}\right) 2\epsilon_{12} = C_{11} - C_{12}\left(\frac{u_{1,2} + u_{2,1}}{2}\right)$$

$$= \left(\frac{C_{11} - C_{12}}{2}\right)[u^0_{1,2} + u^1_{1,2} + u^0_{2,1} + u^1_{2,1})$$

$$\Rightarrow \sigma'_{12} = \left[\frac{C_{11}-C_{12}}{2}\right]\left[-A_0 e^{ik_0(x_1\sin\theta_0 - Gt)} ik_0 \cos 2\theta_0 - A_1 e^{ik_1(x_1\sin\theta_1 - Gt)} ik_1 \cos 2\theta_1\right]$$

$$\Rightarrow \left[\frac{C_{11}-C_{12}}{2}\right]\left(A_0 e^{ik_0(x_1\sin\theta_0 - Gt)} ik_0 \cos 2\theta_0 - A_1 e^{ik_1(x_1\sin\theta_1 - Gt)} ik_1 \cos 2\theta_1\right) = 0$$

To make exponents identical.

$k_0 \sin\theta_0 = k_1 \sin\theta_1$ This is possible only if $\theta_0 = \theta_1$ and $k_0 = k_1$ if holds this

Then $\quad -A_0 \cos 2\theta_0 - A_1 \cos 2\theta_0 = 0$

$$\Rightarrow \boxed{A_0 = -A_1}$$

Now

$$\sigma'_{22} = C_{12}\epsilon_{11} + C_{22}\epsilon_{22} + \cancel{C_{23}\epsilon_{33}}^{\,0}$$

$$= C_{12}\left(u^{(0)}_{1,1} + u^{(1)}_{1,1}\right) + C_{22}\left(u^{(0)}_{2,2} + u^{(1)}_{2,2}\right)$$

$$\Rightarrow \sigma'_{22} = C_{12}\left[-A_0 e^{ik_0(x_1\sin\theta_0 - Gt)} ik_0 \sin\theta_0 \cos\theta_0 + A_1 ik_1 e^{ik_1(x_1\sin\theta_1 - Gt)} \sin\theta_1 \cos\theta_1\right] + C_{22}\left[A_0 e^{ik_0(x_1\sin\theta_0 - Gt)} ik_0 \sin\theta_0 \cos\theta_0 - A_1 e^{ik_1(x_1\sin\theta_1 - Gt)} ik_1 \cos\theta_1 \sin\theta_1\right]$$

$\sigma'_{22} = 0 \;;\; \theta_0 = \theta_1 ,\; k_0 = k_1$ implies

$$-(C_{12}-C_{22})A_0 e^{ik_0(x_1\sin\theta_0 - Gt)} ik_0 \sin\theta_0 \cos\theta_0 + (C_{12}-C_{22}) A_1 ik_1 e^{ik_1(x_1\sin\theta_1 - Gt)} \sin\theta_1 \cos\theta_1 = 0$$

$$\Rightarrow -(C_{12}-C_{22})A_0 + (C_{12}-C_{22})A_1 = 0$$

$$\Rightarrow -(C_{12}-C_{22})A_0 = -(C_{12}-C_{22})A_1$$

$$\Rightarrow \quad A_0 = A_1 \longrightarrow ②$$

from ① & ②

$$\left.\begin{aligned} A_0 + A_1 &= 0 \\ A_0 - A_1 &= 0 \end{aligned}\right\} \Longrightarrow A_0 = 0$$

$\Rightarrow$ SV $\longrightarrow$ SV does not exist.

$\Rightarrow$ Reflection of P-wave as SV-wave :-

$$u_i^{(0)} = A_0 e^{ik_0(x_1 \sin\theta_0 + x_2 \cos\theta_0 - c_1 t)} P_i^{(0)}$$

$$n_i^0 = P_i^{(0)} = (\sin\theta_0 , \cos\theta_0 , 0)$$

$$u_i^{(1)} = A_1 e^{ik_1(x_1 \sin\theta_1 - x_2 \cos\theta_1 - c_2 t)} P_i^{(1)}$$

$$n^1 \perp P^1 ; \quad n_i^1 = (\sin\theta_1, -\cos\theta_1, 0) , \quad P_i^1 = (\cos\theta_1, \sin\theta_1, 0)$$

Free Boundry:- $\sigma_{i2} = 0$ at $x_2 = 0$

$$\Rightarrow \sigma_{12} = 0 \quad \& \quad \sigma_{22} = 0$$

$$\sigma_{12} = \left(\frac{c_1 - c_2}{2}\right) 2 \epsilon_{12} = (c_1 - c_2)\left[\frac{u_{1,2} + u_{2,1}}{2}\right]$$

$$= \left(\frac{c_1 - c_2}{2}\right)\left[u_{1,2}^{(0)} + u_{1,2}^{(1)} + u_{2,1}^{(0)} + u_{2,1}^{(1)}\right]$$

$$= \left[\frac{c_1 - c_2}{2}\right]\left[A_0 e^{ik_0(x_1 \sin\theta_0 - c_1 t)} ik_0 \sin\theta_0 \cos\theta_0 - A_1 e^{ik_1 (x_1 \sin\theta_1 - c_2 t)} ik_1 \cos^2\theta_1 + A_0 e^{ik_0(x_1 \sin\theta_0 - c_1 t)} ik_0 \cos\theta_0 \sin\theta_0 \right.$$

$$\left. + A_1 e^{ik_1(x_1 \sin\theta_1 - c_2 t)} ik_1 \sin^2\theta_1 \right]$$

$$= \left[\frac{c_1 - c_2}{2}\right]\left[A_0 e^{ik_0(x_1 \sin\theta_0 - c_1 t)} ik_0 \sin 2\theta_0 - \right.$$

$$A_1 e^{ik_1(x_1\sin\theta_1 - Gt)} ik_1 (\cos^2\theta_1 - \sin^2\theta_1)$$

$\sigma_{12} = 0$ implies

$$\Rightarrow \left[\frac{C_{11}-C_{12}}{2}\right]\left[A_0 e^{ik_0(x_1\sin\theta_0 - Gt)} ik_0 \sin 2\theta_0 - A_1 e^{ik_1(x_1\sin\theta_1 - Gt)} ik_1 \cos 2\theta_1\right] = 0$$

To make exponent identical

$k_0 \sin\theta_0 = k_1 \sin\theta_1$ & $k_0 G = k_1 G$

This is possible if

$k_0 = k_1$, $\theta_0 = \theta_1$ If this holds then

$$G = G$$

$$\Rightarrow \left[\frac{C_{11}-C_{12}}{2}\right]\left[A_0 e^{ik_0(x_1\sin\theta_0 - Gt)} ik_0 \sin 2\theta_0 - A_1 e^{ik_0(x_1\sin\theta_0 - Gt)} ik_0 \cos 2\theta_0\right] = 0$$

$$\Rightarrow A_0 \sin 2\theta_0 - A_1 \cos 2\theta_0 = 0$$

$$\Rightarrow A_0 = A_1 \frac{\cos 2\theta_0}{\sin 2\theta_0} \longrightarrow (1)$$

Now

$$\sigma_{22} = C_{12}\epsilon_{11} + C_{22}\epsilon_{22} + \cancel{C_{23}\epsilon_{33}}^{\,0}$$

$$= C_{12}(U^{(0)}_{1,1} + U^{(1)}_{1,1}) + C_{22}(U^{(0)}_{2,2} + U^{(1)}_{2,2})$$

$$\Rightarrow \sigma_{22} = C_{12}\left[A_0 e^{ik_0(x_1\sin\theta_0 - Gt)} ik_0 \sin^2\theta_0 + A_1 ik_1 e^{ik_1(x_1\sin\theta_1 - Gt)} \sin\theta_1\cos\theta_1\right] + C_{22}\left[A_0 e^{ik_0(x_1\sin\theta_0 - Gt)} ik_0\cos^2\theta_0 - A_1 ik_1 e^{ik_1(x_1\sin\theta_1 - Gt)} \cos\theta_1\sin\theta_1\right]$$

$$= (C_{12}\sin^2\theta_0 + C_{22}\cos^2\theta_0) A_0 e^{ik_0(x_1\sin\theta_0 - Gt)} ik_0$$

$$+ (C_{12}\sin\theta_1\cos\theta_1 - C_{22}\cos\theta_1\sin\theta_1)A_1 e^{ik_1(x_1\sin\theta_1 - C_1t)} ik_1$$

To make exponent identical

$k_0 \sin\theta_0 = k_1 \sin\theta_1$ this is possible only if

$k_0 = k_1$, $\theta_0 = \theta_1$ if this holds then

$k_0 C_1 = k_0 C \Rightarrow C_1 = C$

$\sigma_{22} = 0$ implies

$$(C_{12}\sin^2\theta_0 + C_{22}\cos^2\theta_0) A_0 ik_0 e^{\cancel{ik_0(x_1\sin\theta_0 - C_1t)}} +$$

$$(C_{12}\sin\theta_0\cos\theta_0 - C_{22}\cos\theta_0\sin\theta_0)A_1 e^{\cancel{ik_0(x_1\sin\theta_0 - C_1t)}} ik_0 = 0$$

$$\Rightarrow (C_{12}\sin^2\theta_0 + C_{22}\cos^2\theta_0)A_0 + (C_{12}\sin\theta_0\cos\theta_0 - C_{22}\cos\theta_0\sin\theta_0)A_1 = 0$$

$$\Rightarrow (C_{12}\sin^2\theta_0 + C_{22}\cos^2\theta_0)A_1 \frac{\cos 2\theta_0}{\sin 2\theta_0} + (C_{12} - C_{22})\sin 2\theta_0 A_1 = 0$$

MUHAMMAD TAHIR

(FA15-RMT-007)

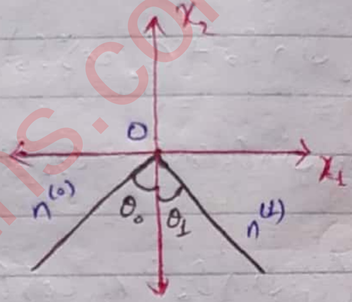
M.S. MATH

COMSATS UNIVERSITY

⇒ Reflection of SH-wave as SH-wave :-

A transverse wave (shear or secondry wave) is a wave in which P_i is perpendicular to n_i, where (P_i is polarazation vector and n_i is propagation vector)

Consider the reflection of SH-wave in x_1x_2-plane, by definition of SH-wave, the displacement component will be in x_3-direction only.



In the figure,
θ_0 is angle of incident,
θ_1 is angle of reflection,
$n^{(0)}$ is unit propagation vector along incident ray, $n^{(1)}$ is unit propagation vector along reflected ray.
Incident ray is as follows

$$u_3^{(0)} = A_0 e^{ik_0(n_j^{(0)}x_j - Ct)} P_3 \longrightarrow ①$$

where $n_i^{(0)} = (n_1^{(0)}, n_2^{(0)}, n_3^{(0)})$, $\bar{x} = (x_1, x_2, 0)$, $P_i = (0,0,1)$

$$\Rightarrow n_i^{(0)} = (\sin\theta_0, -\cos\theta_0, 0)$$

So equ ① becomes

$$u_3^{(0)} = A_0 e^{ik_0(x_1\sin\theta_0 - x_2\cos\theta_0 - Ct)}$$

Similarly for reflected wave

$$u_3^{(1)} = A_1 e^{ik_1(x_1\sin\theta_1 - x_2\cos\theta_1 - Ct)}$$

For Clamped Boundary Conditions:-

For clamped boundry conditions we have,

$u_3^{(0)} + u_3^{(1)} = 0$ at $x_2 = 0$ for at x_1, t

$$\Rightarrow A_0 e^{ik_0(x_1 \sin\theta_0 - Gt)} + A_1 e^{ik_1(x_1 \sin\theta_1 - Gt)} = 0 \longrightarrow ②$$

A_0, k_0 & θ_0 are given, we choose A_1, k_1 and θ_1 in such a way that equ ② is satisfied

To make exponent identical

$k_0 \sin\theta_0 = k_1 \sin\theta_1$ This is possible only if $k_0 = k_1$ and $\theta_0 = \theta_1$. If this holds then

$$A_0 + A_1 = 0 \quad \Rightarrow \quad A_0 = -A_1$$

<u>For Free Boundary</u>:- Boundary $x_2 = 0$ is free mean $\sigma_{i2} = 0$, where σ_{ij} is stress tensor.

By hooke's law for isotropic material

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} \quad , \text{ for } j = 2$$

$$\sigma_{i2} = \lambda \epsilon_{kk} \delta_{i2} + 2\mu \epsilon_{i2}, \text{ where } \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$\sigma_{12} = 0 \quad \because u_1, u_2 = 0$$

$$\sigma_{22} = \lambda(u_{1,1} + u_{2,2} + u_{3,3}) + \mu(u_{1,2} + u_{2,1})$$

$$= \lambda u_{3,3} \qquad \because u_1, u_2 = 0$$

$$\Rightarrow \sigma_{22} = \lambda \frac{\partial u_3}{\partial x_3} = 0 \qquad \because x_3 \notin u_3$$

And

$$\sigma_{32} = \mu(u_{3,2} + u_{2,3})$$

$$= \mu(u_{3,2})$$

So the only non-zero component for σ_{i2} is σ_{32}

This implies $\sigma_{32}^{(0)} = \mu(U_{3,2}^{(0)})$, $\sigma_{32}^{(1)} = \mu(U_{3,2}^{(1)})$

$$\Rightarrow \sigma_{32}^{(0)} = \mu A_0 i k_0 \cos\theta_0 e^{ik_0(x_1 \sin\theta_0 + x_2\cos\theta_0 - Gt)}$$

$$\sigma_{32}^{(1)} = -\mu A_1 i k_1 \cos\theta_1 e^{ik_1(x_1\sin\theta_1 - x_2\cos\theta_1 - Gt)}$$

If boundry $x_2 = 0$ is free then

$$\sigma_{32}^{(0)} + \sigma_{32}^{(1)} = 0 \quad \text{at } x_2 = 0 \;\; \forall x_1, t$$

$$\Rightarrow \mu A_0 i k_0 \cos\theta_0 e^{ik_0(x_1\sin\theta_0 - Gt)} - \mu A_1 i k_1 \cos\theta_1 e^{ik_1(x_1\sin\theta_1 - Gt)} = 0 \longrightarrow (3)$$

To make exponent identical
$k_0 \sin\theta_0 = k_1 \sin\theta_1$, This is possible only if $k_0 = k_1$, $\theta_0 = \theta_1$; if This holds then

$$A_0 - A_1 = 0 \Rightarrow A_0 = A_1$$

$\Rightarrow$ In a reflected SH-wave if angle of reflection is equal to the angle of incident then its wave number and amplitude remains the same.

Clamped Boundary:-

For clamped boundary condition, we have

$$u_1^{(0)} + u_1^{(1)} + u_1^{(2)} = 0 \quad \text{at } x_2 = 0 \;\; \forall x_1, t$$

and

$$u_2^{(0)} + u_2^{(1)} + u_2^{(2)} = 0 \quad \text{at } x_2 \;\; \forall x_1, t$$

Putting values in equations ( ) & ( ) we have

$$A_0 e^{ik_0(x_1\sin\theta_0 - C_1 t)} \sin\theta_0 + A_1 e^{ik_1(x_1\sin\theta_1 - C_1 t)} \sin\theta_1 +$$

$$A_2 e^{ik_2(x_1\sin\theta_2 - C_T t)} \cos\theta_2 = 0$$

$$\Rightarrow A_0 e^{ik_0(x_1\sin\theta_0 - C_1 t)} \cos\theta_0 - A_1 e^{ik_1(x_1\sin\theta_1 - C_1 t)} +$$

$$A_2 e^{ik_2(x_1\sin\theta_2 - C_T t)} \sin\theta_2 = 0$$

To satisfy these equations exponents must be identical, so

$$k_0 \sin\theta_0 = k_1 \sin\theta_1 = k_2 \sin\theta_2$$

and $k_0 C_1 = k_1 C_1 = k_2 C_T$

This holds if

$$k_0 = k_1 \quad , \quad \theta_0 = \theta_1$$

$$\Rightarrow k_2 = k_0 \frac{C_1}{C_T} = k_0 k, \quad \text{where } k = \frac{C_1}{C_T} > 1$$

So we have

$$\frac{A_1}{A_0}\sin\theta_0 + \frac{A_2}{A_0}\cos\theta_2 = -\sin\theta_0 \longrightarrow (a)$$

$$-\frac{A_1}{A_0}\cos\theta_0 + \frac{A_2}{A_0}\sin\theta_2 = -\cos\theta_0 \longrightarrow (b)$$

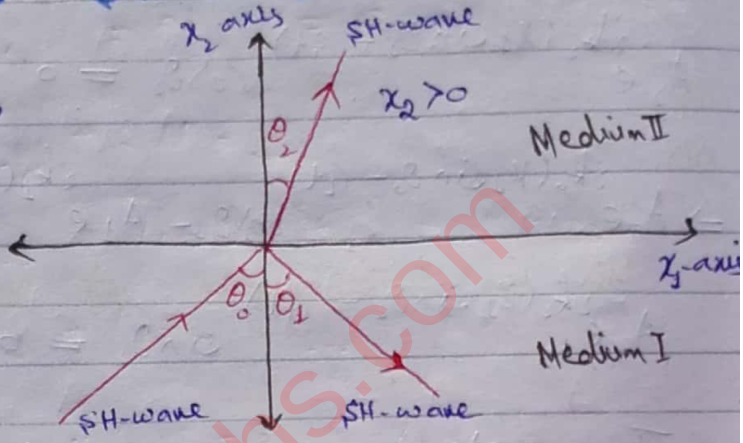
by solving equ (a) & (b), we have

$$\frac{A_1}{A_0} = \frac{\cos\theta_0 \cos\theta_2 - \sin\theta_0 \sin\theta_2}{\cos\theta_0 \cos\theta_2 + \sin\theta_0 \sin\theta_2} = \frac{\cos(\theta_2 + \theta_0)}{\cos(\theta_2 - \theta_0)}$$

$$\frac{A_2}{A_0} = \frac{-\sin 2\theta_0}{\cos\theta_0 \cos\theta_2 + \sin\theta_0 \sin\theta_2} = \frac{-\sin 2\theta_0}{\cos(\theta_2 - \theta_0)}$$

⇒ Transmission:-

⇒ Reflection and Transmission of SH-wave as SH-wave:-



If a space $x_2>0$ is filled with another medium (may liquid or solid) then waves transmit across $x_2=0$.

Let an SH-wave incident at $x_2=0$ i.e. at the interface of two mediums. Both reflected and transmitted waves are considered SH-waves.

Let incident wave makes an angle θ_0 with vertical then

$$n_i^{(0)} = (\sin\theta_0, \cos\theta_0, 0)$$

Similarly reflected and transmitted waves makes angle θ_1 and θ_2 respecti—

$$\Rightarrow \quad n_i^{(1)} = (\sin\theta_1, -\cos\theta_1, 0)$$

$$\text{q.}\ n_i^{(2)} = (\sin\theta_2, \cos\theta_2, 0)$$

Since all waves are SH-waves.

$$\Rightarrow \quad p_i^{(0)} = p_i^{(1)} = p_i^{(2)} = (0, 0, 1)$$

Let $u_i^{(0)} = A_0 e^{ik_0(x_j n_j - c_1 t)} p_i$

$$\Rightarrow \quad u_3^{(0)} = A_0 e^{ik_0(x_1\sin\theta_0 + x_2\cos\theta_0 - c_1 t)} \qquad \because p_3 = 1$$

Similarly

$$u_3^{(1)} = A_1 e^{ik_1(x_1 \sin\theta_1 - x_2 \cos\theta_1 - Gt)}$$

& $$u_3^{(2)} = A_2 e^{ik_2(x_1 \sin\theta_2 + x_2 \cos\theta_2 - G^\beta t)}$$

We shall use the following boundary conditions at the interface $x_2 = 0$

(i) u_i is continuous at $x_2 = 0$

(ii) σ_{i2} is continuous at $x_2 = 0$

1) u_i is continuous at $x_2 = 0$

$$\Rightarrow u_3^{(2)} = u_3^{(0)} + u_3^{(1)} \quad \text{at } x_2 = 0$$

$$\Rightarrow A_2 e^{ik_2(x_1 \sin\theta_2 - G^\beta t)} = A_0 e^{ik_0(x_1 \sin\theta_0 - Gt)} + A_1 e^{ik_1(x_1 \sin\theta_1 - Gt)} \longrightarrow (1)$$

By Snell's law implies that

$$k_2 \sin\theta_2 = k_0 \sin\theta_0 = k_1 \sin\theta_1$$

& $$k_2 G^\beta = k_0 G = k_1 G$$

$$\Rightarrow \boxed{k_0 = k_1} \quad \text{and} \quad \boxed{k_2 = k_0 \frac{G}{G^\beta}}$$

$$\Rightarrow \boxed{\theta_0 = \theta_1} \quad \text{and} \quad \sin\theta_2 = \frac{k_0}{k_2} \sin\theta_0$$

$$\Rightarrow \boxed{\sin\theta_2 = \frac{G^\beta}{G} \sin\theta_0}$$

Thus equation (1) implies

$$A_2 = A_0 + A_1 \longrightarrow (*_1)$$

2) σ_{i2} is continuous at $x_2 = 0$

$$\Rightarrow \sigma_{i2}^{(2)} = \sigma_{i2}^{(0)} + \sigma_{i2}^{(1)}$$

Since $\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$

$$\Rightarrow \sigma_{12} = 2\mu \epsilon_{12} = 2\mu \cdot \frac{1}{2}(u_{1,2} + u_{2,1})$$

$$= \mu(u_{1,2} + u_{2,1})$$

$$\Rightarrow \sigma_{12}^{(0)} = \sigma_{12}^{(1)} = \sigma_{12}^{(2)} = 0 \qquad (\because u_1, u_2 = 0$$

Now

$$\sigma_{22} = \lambda \epsilon_{kk} + 2\mu \epsilon_{22}$$

$$= \lambda \epsilon_{11} + (\lambda + 2\mu)\epsilon_{22} + \lambda \epsilon_{33}$$

$$= \lambda u_{1,1} + (\lambda + 2\mu) u_{2,2} + \lambda u_{3,3}$$

$$\Rightarrow \sigma_{22} = 0 \qquad \because u_1, u_2 = 0 \text{ \& } x_3 \notin u_3$$

$$\Rightarrow \sigma_{22}^{(0)} = \sigma_{22}^{(1)} = \sigma_{22}^{(2)} = 0$$

Now

$$\sigma_{23} = 0 + 2\mu \epsilon_{23}$$

$$= \mu(u_{2,3} + u_{3,2})$$

$$\Rightarrow \sigma_{23} = \mu u_{3,2}$$

Thus $\sigma_{23}^{(2)} = \sigma_{23}^{(0)} + \sigma_{23}^{(1)}$ at $x_2 = 0$

$$\Rightarrow \mu^{\beta} u_{3,2}^{(2)} = \mu \{ u_{3,2}^{(0)} + u_{3,2}^{(1)} \} \text{ at } x_2 = 0$$

$$\Rightarrow \mu^{\beta} \cos\theta_2 (ik_2 A_2) e^{ik_2(x_1 \sin\theta_2 - C t)} = \mu \{ \cos\theta_0 (ik_0 A_0) e^{ik_0(x_1 \sin\theta_0 - Ct)}$$

$$+ (-\cos\theta_1)(ik_1 A_1) e^{ik_1(x_1 \sin\theta_1 - Ct)} \}$$

$$\Rightarrow k_2 \mu^{\beta} \cos\theta_2 A_2 = k_0 \mu \cos\theta_0 A_0 - k_1 \mu \cos\theta_1 A_1$$

Put $\theta_1 = \theta_0$ & $k_1 = k_0$

$$\Rightarrow \quad k_2 u^{\beta} \cos\theta_2 A_2 = k_0 u \cos\theta_0 (A_0 - A_1) \longrightarrow (*_2)$$

Solving $(*_1)$ and $(*_2)$

$$(*_2) \Rightarrow \quad \frac{k_2 u^{\beta} \cos\theta_2 A_2}{k_0 u \cos\theta_0} = A_0 - A_1 \longrightarrow (a)$$

Adding $(*_1)$ and (a) implies

$$\left(1 + \frac{k_2 u^{\beta} \cos\theta_2}{k_0 u \cos\theta_0}\right) A_2 = 2A_0$$

$$\Rightarrow \quad A_2 = \left(\frac{2k_0 u \cos\theta_0}{k_0 u \cos\theta_0 + k_2 u^{\beta} \cos\theta_2}\right) A_0 \longrightarrow (b)$$

Put the value of A_2 in (a) implies

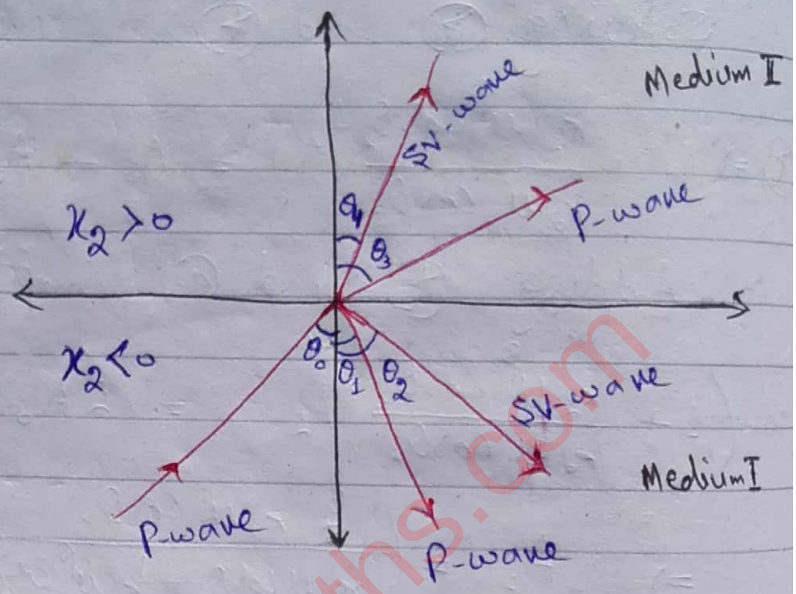
$$\frac{k_2 u^{\beta} \cos\theta_2}{k_0 u \cos\theta_0} \cdot \frac{2(k_0 u \cos\theta_0)}{k_0 u \cos\theta_0 + k_2 u^{\beta} \cos\theta_2} \cdot A_0 = A_0 - A_1$$

$$\Rightarrow \quad A_1 = \left(1 - \frac{2k_2 u^{\beta} \cos\theta_2}{k_0 u \cos\theta_0 + k_2 u^{\beta} \cos\theta_2}\right) A_0$$

$$= \left(\frac{k_0 u \cos\theta_0 + k_2 u^{\beta} \cos\theta_2 - 2k_2 u^{\beta} \cos\theta_2}{k_0 u \cos\theta_0 + k_2 u^{\beta} \cos\theta_2}\right) A_0$$

$$\Rightarrow A_1 = \left(\frac{k_0 u \cos\theta_0 - k_2 u^{\beta} \cos\theta_2}{k_0 u \cos\theta_0 + k_2 u^{\beta} \cos\theta_2}\right) A_0 \longrightarrow (c)$$

⇒ Reflection and Transmission of P-wave as P-wave and SV-wave:



★ Boundary Conditions:- u_i and σ_{i2} are continuous at $x_2 = 0$

$$n_i^{(0)} = (\sin\theta_0, \cos\theta_0, 0) = P_i^{(0)}$$

$$n_i^{(1)} = (\sin\theta_1, -\cos\theta_1, 0) = P_i^{(1)}$$

$$n_i^{(2)} = (\sin\theta_2, -\cos\theta_2, 0), \; P_i^{(2)} = (\cos\theta_2, \sin\theta_2, 0)$$

$$n_i^{(3)} = (\sin\theta_3, \cos\theta_3, 0) = P_i^{(3)}$$

$$n_i^{(4)} = (\sin\theta_4, \cos\theta_4, 0), \; P_i^{(4)} = (\cos\theta_4, -\sin\theta_4, 0)$$

This implies

$$U_i^{(0)} = A_0 e^{ik_0(x_1\sin\theta_0 + x_2\cos\theta_0 - c_1 t)} (\sin\theta_0, \cos\theta_0, 0)$$

$$U_i^{(1)} = A_1 e^{ik_1(x_1\sin\theta_1 - x_2\cos\theta_1 - c_1 t)} (\sin\theta_1, -\cos\theta_1, 0)$$

$$U_i^{(2)} = A_2 e^{ik_2(x_1\sin\theta_2 - x_2\cos\theta_2 - c_1 t)} (\cos\theta_2, \sin\theta_2, 0)$$

$$u_i^{(3)} = A_3 e^{ik_3(x_1 \sin\theta_3 + x_2 \cos\theta_3 - c_t^{\beta} t)} (\sin\theta_3, \cos\theta_3, 0)$$

$$u_i^{(4)} = A_4 e^{ik_4(x_1 \sin\theta_4 + x_2 \cos\theta_4 - c_l^{\beta} t)} (\cos\theta_4, -\sin\theta_4, 0)$$

1) U_i is continuous at $x_2 = 0$

$$\Rightarrow u_i^{(0)} + u_i^{(1)} + u_i^{(2)} = u_i^{(3)} + u_i^{(4)}$$

for $i=1$

$$\Rightarrow u_1^{(0)} + u_1^{(1)} + u_1^{(2)} = u_1^{(3)} + u_1^{(4)}$$

$$\Rightarrow A_0 e^{ik_0(x_1 \sin\theta_0 - c_l t)} \sin\theta_0 + A_1 e^{ik_1(x_1 \sin\theta_1 - c_l t)} \sin\theta_1$$
$$+ A_2 e^{ik_2(x_1 \sin\theta_2 - c_t t)} \cos\theta_2 = A_3 e^{ik_3(x_1 \sin\theta_3 - c_t^{\beta} t)} \sin\theta_3$$
$$+ A_4 e^{ik_4(x_1 \sin\theta_4 - c_l^{\beta} t)} \cos\theta_4 \longrightarrow (A)$$

By Snell's law

$$k_0 \sin\theta_0 = k_1 \sin\theta_1 = k_2 \sin\theta_2 = k_3 \sin\theta_3 = k_4 \sin\theta_4$$

And $\underbrace{k_0 c_l = k_1 c_l} = k_2 c_t = k_3 c_t^{\beta} = k_4 c_l^{\beta}$

$$\Rightarrow \boxed{k_1 = k_0} \Rightarrow \boxed{\theta_1 = \theta_0}$$

$$\Rightarrow \boxed{k_2 = \frac{c_l}{c_t} k_0}, \quad \boxed{k_3 = \frac{c_l}{c_t^{\beta}} k_0}$$

$$\boxed{k_4 = \frac{c_l}{c_l^{\beta}} k_0}$$

$$\sin\theta_2 = \frac{k_0}{k_2} \sin\theta_0 \Rightarrow \boxed{\frac{c_t}{c_l} \sin\theta_0 = \sin\theta_2}$$

$$\sin\theta_3 = \frac{k_0}{k_3} \sin\theta_0 \Rightarrow \boxed{\sin\theta_3 = \frac{c_t^{\beta}}{c_l} \sin\theta_0}$$

$$\sin\theta_4 = \frac{k_0}{k_4^{p}}\sin\theta_0 \Rightarrow \boxed{\sin\theta_4 = \frac{C_T^{p}}{C_L}\sin\theta_0}$$

so (A) implies

$$A_0\sin\theta_0 + A_1\sin\theta_1 + A_2\cos\theta_2 - A_3\cos\theta_3 - A_4\cos\theta_4 = 0 \longrightarrow (1)$$

for $i=2$

$$u_2^{(0)} + u_2^{(1)} + u_2^{(2)} = u_2^{(3)} + u_2^{(4)}$$

$$\Rightarrow A_0 e^{ik_0(x_1\sin\theta_0 - C_L t)}\cos\theta_0 - A_1 e^{ik_1(x_1\sin\theta_1 - C_L t)}\cos\theta_1$$

$$+ A_2 e^{ik_2(x_1\sin\theta_2 - C_T t)}\sin\theta_2 = A_3 e^{ik_3(x_1\sin\theta_3 - C_L^{p} t)}\cos\theta_3$$

$$- A_4 e^{ik_4(x_1\sin\theta_4 - C_T^{p} t)}\sin\theta_4$$

By snell's law

$$A_0\cos\theta_0 - A_1\cos\theta_1 + A_2\sin\theta_2 - A_3\cos\theta_3 + A_4\sin\theta_4 = 0 \longrightarrow (2)$$

2) σ_{i2} is continuous at $x_2 = 0$

$$\Rightarrow \sigma_{i2}^{(0)} + \sigma_{i2}^{(1)} + \sigma_{i2}^{(2)} = \sigma_{i2}^{(3)} + \sigma_{i2}^{(4)} \longrightarrow (*)$$

Since $\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij}$

$$\Rightarrow \sigma_{i2} = \lambda\epsilon_{kk}\delta_{i2} + 2\mu\epsilon_{i2}$$

for $i=1$

$$\sigma_{12} = 2\mu\epsilon_{12} = \mu(u_{1,2} + u_{2,1})$$

$$\Rightarrow \mu\left\{\left(u_{1,2}^{(0)} + u_{2,1}^{(0)}\right) + \left(u_{1,2}^{(1)} + u_{2,1}^{(1)}\right) + \left(u_{1,2}^{(2)} + u_{2,1}^{(2)}\right)\right\}$$

$$= \mu^{p}\left\{\left(u_{1,2}^{(3)} + u_{2,1}^{(3)}\right) + \left(u_{1,2}^{(4)} + u_{2,1}^{(4)}\right)\right\}$$

$$\Rightarrow \mu\left\{A_0k_0(\sin\theta_0\cos\theta_0 + \sin\theta_0\cos\theta_0) - A_1k_1(\sin\theta_1\cos\theta_1 + \sin\theta_1\cos\theta_1)\right.$$

$$+A_2 k_2(-\cos^2\theta_2+\sin^2\theta_2) = \mu^\beta\{A_3 k_3(\sin\theta_3\cos\theta_3+\sin\theta_3\cos\theta_3)$$
$$+A_4 k_4(\cos^2\theta_4-\sin^2\theta_4)\}$$

$$\Rightarrow \mu[A_0 k_0 \sin 2\theta_0 - A_1 k_1 \sin 2\theta_1 - A_2 k_2 \cos 2\theta_2] =$$
$$\mu^\beta[A_3 k_3(\sin 2\theta_3)+A_4 k_4 \cos 2\theta_4)]$$

$$\Rightarrow A_0 k_0 \sin 2\theta_0 - A_1 k_1 \sin 2\theta_1 - A_2 k_2 \cos 2\theta_2 = \frac{\mu^\beta}{\mu}[A_3 k_3 \sin 2\theta_3$$
$$+A_4 k_4 \cos 2\theta_4] \longrightarrow ③$$

For $i=2$

$$\sigma_{22} = \lambda\epsilon_{kk}+2\mu\epsilon_{22}$$
$$= \lambda(\epsilon_{11}+\epsilon_{22})+2\mu\,\epsilon_{22}$$
$$\sigma_{22} = \lambda(u_{1,1}+u_{2,2})+2\mu(u_{2,2})$$

So ⊛ implies

$$\lambda(u_{1,1}^{(0)}+u_{2,2}^{(0)}+u_{1,1}^{(1)}+u_{2,2}^{(1)}+u_{1,1}^{(2)}+u_{2,2}^{(2)})$$
$$+2\mu(u_{2,2}^{(0)}+u_{2,2}^{(1)}+u_{2,2}^{(2)}) = \lambda^\beta(u_{1,1}^{(3)}+u_{2,2}^{(3)}+$$
$$u_{1,1}^{(4)}+u_{2,2}^{(4)})+2\mu^\beta(u_{2,2}^{(3)}+u_{2,2}^{(4)})$$

$$\Rightarrow \lambda\{A_0 k_0\sin^2\theta_0+A_0 k_0\cos^2\theta_0+A_1 k_1\sin^2\theta_1+A_1 k_1\cos^2\theta_1+$$
$$A_2 k_2\cos\theta_2\cancel{\sin\theta_2}-A_2 k_2\cancel{\cos\theta_2}\sin\theta_2\}+2\mu\{A_0 k_0\cos^2\theta_0$$
$$+A_1 k_1\cos^2\theta_1+A_2 k_2\sin\theta_2\cos\theta_2\} = \lambda^\beta\{A_3 k_3\sin^2\theta_3+$$
$$A_3 k_3\cos^2\theta_3+A_4 k_4\sin\theta_4\cancel{\cos\theta_4}-A_4 k_4\cancel{\sin\theta_4}\cos\theta_4\}$$
$$+2\mu^\beta\{A_3 k_3\cos^2\theta_3-A_4 k_4\sin\theta_4\cos\theta_4\}$$

$$\Rightarrow \lambda(A_0k_0 + A_1k_1) + 2\mu(A_0k_0\cos^2\theta_0 + A_1k_1\cos^2\theta_1 + A_2k_2\cos\theta_2\sin\theta_2) = \lambda^\beta(A_3k_3) + 2\mu^\beta(A_3k_3\cos^2\theta_3 - A_4k_4\sin\theta_4\cos\theta_4)$$

$$\Rightarrow (\lambda + 2\mu\cos^2\theta_0)A_0k_0 + (\lambda + 2\mu\cos^2\theta_1)A_1k_1 - \mu\sin 2\theta_2 A_2k_2 - (\lambda^\beta + 2\mu^\beta\cos^2\theta_3)A_3k_3 + \mu^\beta\sin 2\theta_4 A_4k_4 = 0 \longrightarrow (4)$$

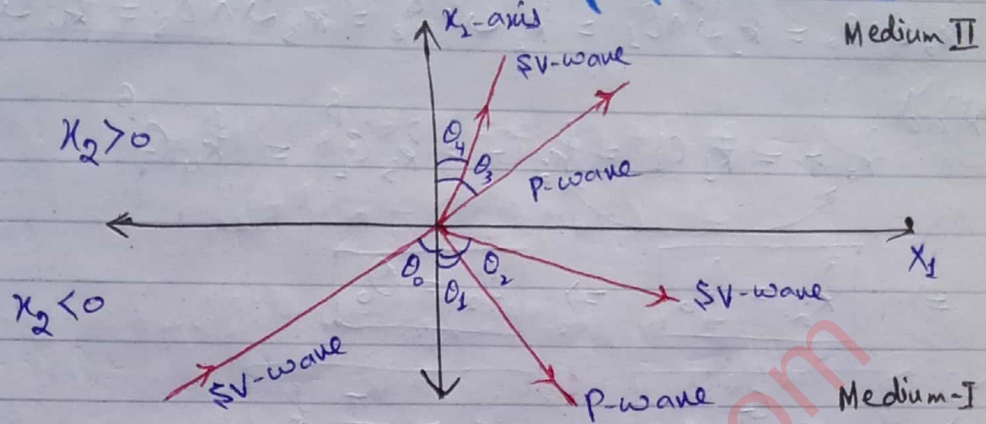
from equation (1) $\rightarrow$ (4)

$$\begin{bmatrix} \sin\theta_1 & \cos\theta_2 & -\sin\theta_3 & -\cos\theta_4 \\ -\cos\theta_1 & \sin\theta_2 & -\cos\theta_3 & \sin\theta_4 \\ -k_1\sin 2\theta_1 & -k_2\cos 2\theta_4 & -\frac{\mu^\beta}{\mu}k_3\sin 2\theta_3 & -\frac{\mu^\beta}{\mu}k_4\cos 2\theta_4 \\ k_1(\lambda+2\mu\cos^2\theta_1) & -k_2\mu\sin 2\theta_2 & -k_3(\lambda^\beta+2\mu^\beta\cos^2\theta_3) & k_4\mu^\beta\sin 2\theta_4 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} -A_0\sin\theta_0 \\ -A_0\cos\theta_0 \\ -A_0k_0\sin 2\theta_0 \\ -(\lambda+2\mu\cos^2\theta_0)A_0k_0 \end{bmatrix}$$

Solve above system of equations (matrix) and find the values of the unknowns (A_1, A_2, A_3, A_4)

This is our solution.

⇒ Reflection And Transmission of SV-wave as P-wave & SV-wave.



★ Boundary Conditions:- u_i and σ_{i2} are continuous at $x_2 = 0$

$$u_i^{(0)} = A_0 e^{ik_0(x_1 \sin\theta_0 + x_2 \cos\theta_0 - c_1 t)} (-\cos\theta_0, \sin\theta_0, 0)$$

$$u_i^{(1)} = A_1 e^{ik_1(x_1 \sin\theta_1 - x_2 \cos\theta_1 - c_1 t)} (\sin\theta_1, -\cos\theta_1, 0)$$

$$u_i^{(2)} = A_2 e^{ik_2(x_1 \sin\theta_2 - x_2 \cos\theta_2 - c_1 t)} (\cos\theta_2, \sin\theta_2, 0)$$

$$u_i^{(3)} = A_3 e^{ik_3(x_1 \sin\theta_3 + x_2 \cos\theta_3 - c_1^{\beta} t)} (\sin\theta_3, \cos\theta_3, 0)$$

$$u_i^{(4)} = A_4 e^{ik_4(x_1 \sin\theta_4 + x_2 \cos\theta_4 - c_1^{\beta} t)} (-\cos\theta_4, \sin\theta_4, 0)$$

1) u_i is continuous at $x_2 = 0$

$$\Rightarrow u_i^{(0)} + u_i^{(1)} + u_i^{(2)} = u_i^{(3)} + u_i^{(4)}$$

for $i=1$

$$u_1^{(0)} + u_1^{(1)} + u_1^{(2)} = u_1^{(3)} + u_1^{(4)}$$

$$\Rightarrow -A_0 e^{ik_0(x_1 \sin\theta_0 - c_1 t)} \cos\theta_0 + A_1 e^{ik_1(x_1 \sin\theta_1 - c_1 t)} \sin\theta_1$$
$$+ A_2 e^{ik_2(x_1 \sin\theta_2 - c_1 t)} \cos\theta_2 = A_3 e^{ik_3(x_1 \sin\theta_3 - c_1^{\beta} t)} \sin\theta_3$$
$$- A_4 e^{ik_4(x_1 \sin\theta_4 - c_1^{\beta} t)} \cos\theta_4 \rightarrow (A)$$

By Snell's law

$$k_0 \sin\theta_0 = k_1 \sin\theta_1 = k_2 \sin\theta_2 = k_3 \sin\theta_3 = k_4 \sin\theta_4$$

And

$$k_0 C_T = k_1 C_L = k_2 C_T = k_3 C_L^{\beta} = k_4 C_T^{\beta}$$

$$\Rightarrow \boxed{k_2 = k_0} \quad \Rightarrow \boxed{\theta_2 = \theta_0}$$

$$\Rightarrow \boxed{k_1 = \frac{C_T}{C_L} k_0} \qquad \boxed{k_3 = \frac{C_T}{C_L^{\beta}} k_0}$$

$$\boxed{k_4 = \frac{C_T}{C_T^{\beta}} k_0}$$

$$\sin\theta_1 = \frac{k_0}{k_1} \sin\theta_0 \Rightarrow \boxed{\sin\theta_1 = \frac{C_L}{C_T} \sin\theta_0}$$

$$\sin\theta_3 = \frac{k_0}{k_3} \sin\theta_0 \Rightarrow \boxed{\sin\theta_3 = \frac{C_L^{\beta}}{C_T} \sin\theta_0}$$

$$\sin\theta_4 = \frac{k_0}{k_4} \sin\theta_0 \Rightarrow \boxed{\sin\theta_4 = \frac{C_T^{\beta}}{C_T} \sin\theta_0}$$

So equation (A) implies

$$-A_0 \cos\theta_0 + A_1 \sin\theta_1 + A_2 \cos\theta_2 = A_3 \sin\theta_3 - A_4 \cos\theta_4$$

$$\Rightarrow -A_0 \cos\theta_0 + A_1 \sin\theta_1 + A_2 \cos\theta_2 - A_3 \sin\theta_3 + A_4 \cos\theta_4 = 0$$

$\longrightarrow$ (4)

for $i=2$ $\quad U_2^{(0)} + U_2^{(1)} + U_2^{(2)} = U_2^{(3)} + U_2^{(4)}$

$$\Rightarrow A_0 \sin\theta_0 - A_1 \cos\theta_1 + A_2 \sin\theta_2 = A_3 \cos\theta_3 + A_4 \sin\theta_4$$

$$\Rightarrow A_0 \sin\theta_0 - A_1 \cos\theta_1 + A_2 \sin\theta_2 - A_3 \cos\theta_3 - A_4 \sin\theta_4 = 0 \longrightarrow (2)$$

(2) σ_{i2} is continuous at $x_2 = 0$

$$\Rightarrow \sigma_{i2}^{(0)} + \sigma_{i2}^{(1)} + \sigma_{i2}^{(2)} = \sigma_{i2}^{(3)} + \sigma_{i2}^{(4)} \longrightarrow (*)$$

Since $\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$

$$\Rightarrow \sigma_{i2} = \lambda \epsilon_{kk} \delta_{i2} + 2\mu \epsilon_{i2}$$

for $i=1$

$$\sigma_{12} = 2\mu \epsilon_{12} = \mu(U_{1,2} + U_{2,1})$$

$$\Rightarrow \mu\left\{(U_{1,2}^{(0)} + U_{2,1}^{(0)}) + (U_{1,2}^{(1)} + U_{2,1}^{(1)}) + (U_{1,2}^{(2)} + U_{2,1}^{(2)})\right\} = \mu^{\beta}\left\{(U_{1,2}^{(3)} + U_{2,1}^{(3)}) + (U_{1,2}^{(4)} + U_{2,1}^{(4)})\right\}$$

$$\Rightarrow \mu\left\{A_0 k_0(-\cos^2\theta_0 + \sin^2\theta_0) + A_1 k_1(-\sin\theta_1 \cos\theta_1 - \sin\theta_1 \cos\theta_1) + A_2 k_2(-\cos^2\theta_2 + \sin^2\theta_2)\right\} = \mu^{\beta}\left\{A_3 k_3(\sin\theta_3 \cos\theta_3 + \sin\theta_3 \cos\theta_3) + A_4 k_4(-\cos^2\theta_4 + \sin^2\theta_4)\right\}$$

$$\Rightarrow \mu\left\{-A_0 k_0 \cos 2\theta_0 - A_1 k_1 \sin 2\theta_1 - A_2 k_2 \cos 2\theta_2\right\} = \mu^{\beta}\left\{A_3 k_3 \sin 2\theta_3 - A_4 k_4 \cos 2\theta_4\right\}$$

$$\Rightarrow -A_0 k_0 \mu \cos 2\theta_0 - A_1 k_1 \mu \sin 2\theta_1 - A_2 k_2 \mu \cos 2\theta_2 - A_3 k_3 \mu^\beta \sin 2\theta_3 + A_4 k_4 \mu^\beta \cos 2\theta_4 = 0 \quad \longrightarrow (3)$$

for $i=2$

$$\sigma_{22} = \lambda \epsilon_{kk} + 2\mu \epsilon_{22}$$
$$= \lambda(\epsilon_{11} + \epsilon_{22}) + 2\mu \epsilon_{22}$$
$$= \lambda(u_{1,1} + u_{2,2}) + 2\mu(u_{2,2})$$

So (*) implies

$$\lambda\left(u_{1,1}^{(0)} + u_{2,2}^{(0)} + u_{1,1}^{(1)} + u_{2,2}^{(1)} + u_{1,1}^{(2)} + u_{2,2}^{(2)}\right) + 2\mu\left(u_{2,2}^{(0)} + u_{2,2}^{(1)} + u_{2,2}^{(2)}\right) = \lambda^\beta\left(u_{1,1}^{(3)} + u_{2,2}^{(3)} + u_{1,1}^{(4)} + u_{2,2}^{(4)}\right) + 2\mu^\beta\left(u_{2,2}^{(3)} + u_{2,2}^{(4)}\right)$$

$$\Rightarrow \lambda\left\{A_0 k_0 \cancel{\sin\theta_0 \cos\theta_0} + A_0 k_0 \cancel{\sin\theta_0 \cos\theta_0} + A_1 k_1 \sin^2\theta_1 + A_1 k_1 \cos^2\theta_1 + A_2 k_2 \cancel{\sin\theta_2 \cos\theta_2} - A_2 k_2 \cancel{\sin\theta_2 \cos\theta_2}\right\}$$
$$+ 2\mu\left\{A_0 k_0 \sin\theta_0 \cos\theta_0 + A_1 k_1 \cos^2\theta_1 - A_2 k_2 \sin\theta_2 \cos\theta_2\right\}$$
$$= \lambda^\beta\left\{A_3 k_3 \sin^2\theta_3 + A_3 k_3 \cos^2\theta_3 - A_4 k_4 \cancel{\sin\theta_4 \cos\theta_4} + A_4 k_4 \sin\theta_4 \cancel{\cos\theta_4}\right\} + 2\mu^\beta\left\{A_3 k_3 \cos^2\theta_3 + A_4 k_4 \sin\theta_4 \cos\theta_4\right\}$$

$$\Rightarrow \lambda\{A_1 k_1\} + A_0 k_0 \mu \sin 2\theta_0 + A_1 k_1 2\mu \cos^2\theta_1 - A_2 k_2 \mu \sin 2\theta_2 = \lambda^\beta A_3 k_3 + A_3 k_3 2\mu^\beta \cos^2\theta_3 + A_4 k_4 \mu^\beta \sin 2\theta_4$$

$$\Rightarrow A_0 k_0 \mu \sin 2\theta_0 + A_1 k_1 (\lambda + 2\mu \cos^2\theta_1) - A_2 k_2 \mu \sin 2\theta_2$$
$$- A_3 k_3 (\lambda^\beta + 2\mu^\beta \cos^2\theta_3) - A_4 k_4 \mu^\beta \sin 2\theta_4 \longrightarrow (4)$$

from equation (1) $\longrightarrow$ (4)

$$\begin{bmatrix} \sin\theta_1 & \cos\theta_2 & -\sin\theta_3 & \cos\theta_4 \\ -\cos\theta_1 & \sin\theta_2 & -\cos\theta_3 & -\sin\theta_4 \\ -k_1\mu\sin 2\theta_1 & -k_2\mu\cos 2\theta_2 & -k_3\mu^\beta\sin 2\theta_3 & k_4\mu^\beta\cos 2\theta_4 \\ k_1(\lambda+2\mu\cos^2\theta_1) & -k_2\mu\sin 2\theta_2 & -k_3(\lambda^\beta+2\mu^\beta\cos^2\theta_3) & -k_4\mu^\beta\sin 2\theta_4 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}$$

$$= \begin{bmatrix} A_0 \cos\theta_0 \\ -A_0 \sin\theta_0 \\ A_0 k_0 \mu \cos 2\theta_0 \\ -A_0 k_0 \mu \sin 2\theta_0 \end{bmatrix}$$

By solving this matrix (system of equations) we will find the four unknowns (A_1, A_2, A_3, A_4)

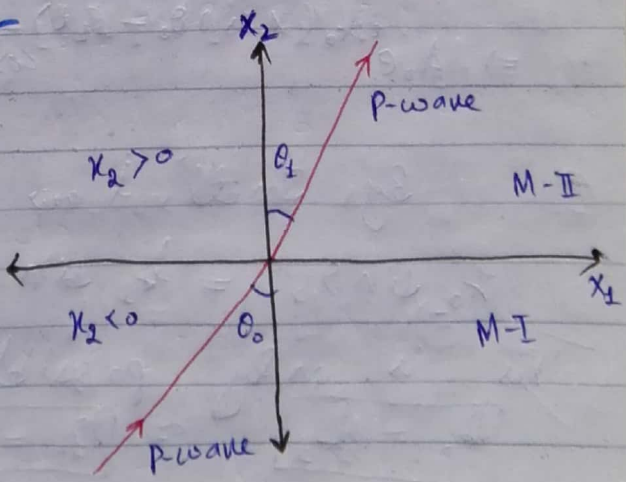
* Refraction (Transmission) of P-wave as P-wave :-

$n_i^{(0)} = (\sin\theta_0, \cos\theta_0, 0)$

$P_i^{(0)} = (\sin\theta_0, \cos\theta_0, 0)$

$n_i^{(1)} = (\sin\theta_1, \cos\theta_1, 0)$

$P_i^{(1)} = (\sin\theta_1, \cos\theta_1, 0)$



$$u_i^{(0)} = A_0 e^{ik_0(x_1\sin\theta_0 + x_2\cos\theta_0 - c_L t)} (\sin\theta_0, \cos\theta_0, 0)$$

$$u_i^{(1)} = A_1 e^{ik_1(x_1\sin\theta_1 + x_2\cos\theta_1 - c_L^\beta t)} (\sin\theta_1, \cos\theta_1, 0)$$

1) u_i is continuous at $x_2 = 0$

$$\Rightarrow u_i^{(0)} = u_i^{(1)}$$

for $i=1$

$$A_0 e^{ik_0(x_1\sin\theta_0 - c_L t)} \sin\theta_0 = A_1 e^{ik_1(x_1\sin\theta_1 - c_L^\beta t)} \sin\theta_1$$

By Snell's law

$$k_0 \sin\theta_0 = k_1 \sin\theta_1 \quad \& \quad k_0 c_L = k_1 c_L^\beta$$

$$\Rightarrow \boxed{k_1 = \frac{c_L}{c_L^\beta} k_0}$$

$$\Rightarrow \sin\theta_1 = \frac{k_0}{k_1}\sin\theta_0 \Rightarrow \boxed{\sin\theta_1 = \frac{c_L^\beta}{c_L}\sin\theta_0}$$

$$\Rightarrow A_0 = A_1 \longrightarrow (1)$$

for $i=2$

$$u_2^{(0)} = u_2^{(1)}$$

$$\Rightarrow A_0 e^{ik_0(x_1\sin\theta_0 - c_L t)} \cos\theta_0 = A_1 e^{ik_1(x_1\sin\theta_1 - c_L^\beta t)} \cos\theta_1$$

By Snell's law

$$k_0 \sin\theta_0 = k_1 \sin\theta_1$$

This is only possible if $\boxed{k_0 = k_1}$, & $\boxed{\theta_0 = \theta_1}$ if this holds

then

$$c_L = c_L^\beta$$

$$\Rightarrow A_0 \cos\theta_0 = A_1 \cos\theta_0$$

$$\Rightarrow A_0 = A_1 \longrightarrow ②$$

② σ_{i2} is continuous at $x_2 = 0$

$$\Rightarrow \sigma_{i2}^{(0)} = \sigma_{i2}^{(1)} \longrightarrow (*)$$

where $\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$

$$\Rightarrow \sigma_{i2} = \lambda \epsilon_{kk} \delta_{i2} + 2\mu \epsilon_{i2}$$

for i=1

$$\sigma_{12} = 2\mu \epsilon_{12} = \mu(U_{1,2} + U_{2,1})$$

So

$$\mu\{U_{1,2}^{(0)} + U_{2,1}^{(0)}\} = \mu^{\beta}\{U_{1,2}^{(1)} + U_{2,1}^{(1)}\}$$

$$\Rightarrow \mu\{A_0 k_0 (\sin\theta_0 \cos\theta_0 + \sin\theta_0 \cos\theta_0)\} = \mu^{\beta}\{A_1 k_1 (\sin\theta_1 \cos\theta_1 + \sin\theta_1 \cos\theta_1)\}$$

$$\Rightarrow A_0 k_0 \mu \sin 2\theta_0 = A_1 k_1 \mu^{\beta} \sin 2\theta_1 \longrightarrow ③$$

for i=2

$$\sigma_{22} = \lambda \epsilon_{kk} + 2\mu \epsilon_{22}$$

$$= \lambda(\epsilon_{11} + \epsilon_{22}) + 2\mu \epsilon_{22}$$

$$= \lambda(U_{1,1} + U_{2,2}) + 2\mu(U_{2,2})$$

$$\Rightarrow \lambda(U_{1,1}^{(0)} + U_{2,2}^{(0)}) + 2\mu(U_{2,2}^{(0)}) = \lambda^{\beta}(U_{1,1}^{(1)} + U_{2,2}^{(1)}) + 2\mu^{\beta} U_{2,2}^{(1)}$$

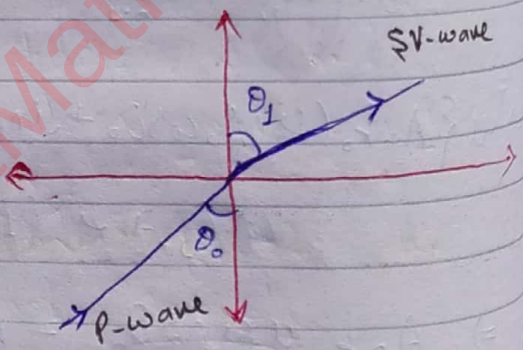
$$\Rightarrow \lambda\{A_0 k_0 (\sin^2\theta_0 + \cos^2\theta_0)\} + 2\mu A_0 k_0 \cos^2\theta_0 =$$

$$\lambda^{\beta}\{A_1 k_1 (\sin^2\theta_1 + \cos^2\theta_1)\} + 2\mu^{\beta}\{A_1 k_1 \cos^2\theta_1\}$$

$$\Rightarrow A_0 k_0 \lambda + 2A_0 k_0 \mu \cos^2\theta_0 = A_1 k_1 \lambda^{\beta} + 2A_1 k_1 \mu^{\beta} \cos^2\theta_1 \longrightarrow (4)$$

There are four equations and one unknown. So there does not exist any unique solution. Hence Refraction of P-wave as a P-wave not exist.

⇒ Transmission of P-wave as SV-wave :-



$$u_i^{(0)} = A_0 e^{ik_0(x_1 \sin\theta_0 + x_2 \cos\theta_0 - c_L t)} (\sin\theta_0, \cos\theta_0, 0)$$

$$u_i^{(1)} = A_1 e^{ik_1(x_1 \sin\theta_1 + x_2 \cos\theta_1 - c_T^{\beta} t)} (-\cos\theta_1, \sin\theta_1, 0)$$

(1) u_i is continuous at $x_2 = 0$

$$\Rightarrow u_i^{(0)} = u_i^{(1)}$$

for $i=1$

$$u_1^{(0)} = u_1^{(1)}$$

$$\Rightarrow A_0 e^{ik_0(x_1 \sin\theta_0 - c_L t)} \sin\theta_0 = -A_1 e^{ik_1(x_1 \sin\theta_1 - c_T^{\beta} t)} \cos\theta_1 \longrightarrow (A)$$

To make exponent identical

$k_0 \sin\theta_0 = k_1 \sin\theta_1$ And $k_0 c_1 = k_1 c_1^{\beta}$

This is only possible if

$\boxed{k_0 = k_1}$ & $\boxed{\theta_0 = \theta_1} \Rightarrow c_1 = c_1^{\beta}$

So equ Ⓐ implies

$A_0 \sin\theta_0 = -A_1 \cos\theta_1$

$\Rightarrow A_0 \sin\theta_0 + A_1 \cos\theta_1 = 0 \longrightarrow (1)$

for $i=2$

$u_2^{(0)} = u_2^{(1)}$

$\Rightarrow A_0 \cos\theta_0 = A_1 \sin\theta_1$

$\Rightarrow A_0 \cos\theta_0 - A_1 \sin\theta_1 = 0 \longrightarrow (2)$

(2) σ_{i2} is continuous at $x_2 = 0$

$\Rightarrow \sigma_{i2}^{(0)} = \sigma_{i2}^{(1)}$

Since $\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$

for $i=1$

$\sigma_{12}^{(0)} = \sigma_{12}^{(1)}$

$\Rightarrow \mu \{ u_{1,2}^{(0)} + u_{2,1}^{(0)} \} = \mu^{\beta} \{ u_{1,2}^{(1)} + u_{2,1}^{(1)} \}$

$\Rightarrow \mu \{ A_0 k_0 (\sin\theta_0 \cos\theta_0 + \sin\theta_0 \cos\theta_0) \} = \mu^{\beta} \{ A_1 k_1 (-\cos^2\theta_1 + \sin^2\theta_1) \}$

$\Rightarrow \mu A_0 k_0 \sin 2\theta_0 = -\mu^{\beta} A_1 k_1 \cos 2\theta_1$

$\Rightarrow A_0 k_0 \mu \sin 2\theta_0 + A_1 k_1 \mu^{\beta} \cos 2\theta_1 = 0 \longrightarrow (3)$

for i=2 $\quad \sigma_{22}^{(0)} = \sigma_{22}^{(1)}$

$$\Rightarrow \lambda(U_{1,1}^{(0)} + U_{2,2}^{(0)}) + 2\mu(U_{2,2}^{(0)}) = \lambda^\beta(U_{1,1}^{(1)} + U_{2,2}^{(1)}) + 2\mu^\beta U_{2,2}^{(1)}$$

$$\Rightarrow \lambda\{A_0k_0(\sin^2\theta_0 + \cos^2\theta_0)\} + 2\mu A_0 k_0 \cos^2\theta_0 =$$

$$\lambda^\beta\{A_1k_1(-\sin\theta_1\cancel{\cos\theta_1} + \sin\theta_1\cancel{\cos\theta_1})\} + 2\mu^\beta A_1k_1 \sin\theta_1\cos\theta_1$$

$$\Rightarrow \lambda A_0k_0 + A_0k_0 2\mu\cos^2\theta_0 - A_1k_1\mu^\beta \sin2\theta_1 = 0$$

$$\Rightarrow A_0k_0(\lambda + 2\mu\cos^2\theta_0) - A_1k_1\mu^\beta\sin2\theta_1 = 0 \longrightarrow (4)$$

There are four equations an one unknown, so there does not exist any unique solution.

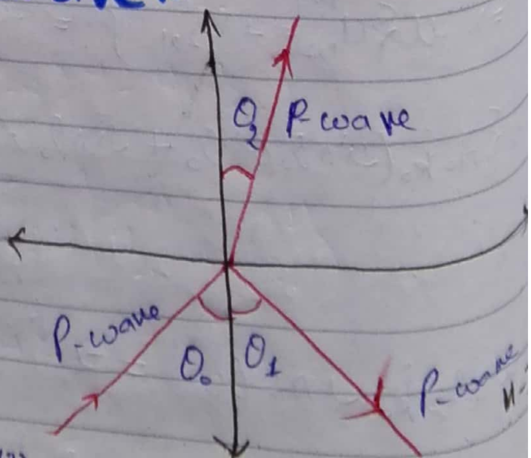
Hence Refraction of P-wave as SV-wave do not possible.

⇒ Reflection & Transmission of P-wave as P-wave :-

$n_i^{(0)} = (\sin\theta_0, \cos\theta_0, 0)$
$= P_i^{(0)}$

$n_i^{(1)} = (\sin\theta_1, -\cos\theta_1, 0)$
$= P_i^{(1)}$

$n_i^{(2)} = (\sin\theta_2, \cos\theta_2, 0)$
$= P_i^{(2)}$



$$\Rightarrow u_i^{(0)} = A_0 e^{ik_0(x_1\sin\theta_0 + x_2\cos\theta_0 - c_L t)}(\sin\theta_0, \cos\theta_0, 0)$$

$$u_i^{(1)} = A_1 e^{ik_1(x_1\sin\theta_1 - x_2\cos\theta_1 - c_L t)}(\sin\theta_1, -\cos\theta_1, 0)$$

$$u_i^{(2)} = A_2 e^{ik_2(x_1\sin\theta_2 + x_2\cos\theta_2 - c_L^{\beta} t)}(\sin\theta_2, \cos\theta_2, 0)$$

① u_i is continuous at $x_2 = 0$

$$\Rightarrow u_i^{(0)} + u_i^{(1)} = u_i^{(2)}$$

for $i=1$

$$u_1^{(0)} + u_1^{(1)} = u_1^{(2)}$$

$$\Rightarrow A_0 e^{ik_0(x_1\sin\theta_0 - c_L t)}\sin\theta_0 + A_1 e^{ik_1(x_1\sin\theta_1 - c_L t)}\sin\theta_1$$

$$= A_2 e^{ik_2(x_1\sin\theta_2 - c_L^{\beta} t)}\sin\theta_2$$

By Snell's law

$$k_0\sin\theta_0 = k_1\sin\theta_1 = k_2\sin\theta_2$$

And $k_0 c_L = k_1 c_L = k_2 c_L^{\beta}$

$$\Rightarrow \boxed{k_1 = k_0} \quad \Rightarrow \boxed{k_2 = \frac{c_L}{c_L^{\beta}} k_0}$$

$$\Rightarrow \boxed{\theta_1 = \theta_0}$$

$$\sin\theta_2 = \frac{k_0}{k_2}\sin\theta_0 \Rightarrow \boxed{\sin\theta_2 = \frac{c_L^{\beta}}{c_L}\sin\theta_0}$$

$$\Rightarrow A_0\sin\theta_0 + A_1\sin\theta_1 = A_2\sin\theta_2$$

$$\Rightarrow A_0\sin\theta_0 + A_1\sin\theta_1 - A_2\sin\theta_2 = 0 \quad ①$$

for i=2

$$u_2^{(0)} + u_2^{(1)} = u_2^{(2)}$$

$$\Rightarrow A_0 \cos\theta_0 - A_1 \cos\theta_1 = A_2 \cos\theta_2$$

$$\Rightarrow A_0 \cos\theta_0 - A_1 \cos\theta_1 - A_2 \cos\theta_2 = 0 \longrightarrow (2)$$

(2) σ_{i2} is continuous at $x_2 = 0$

$$\Rightarrow \sigma_{i2}^{(0)} + \sigma_{i2}^{(1)} = \sigma_{i2}^{(2)} \longrightarrow (*)$$

since $\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$

$$\Rightarrow \sigma_{i2} = \lambda \epsilon_{kk} \delta_{i2} + 2\mu \epsilon_{i2}$$

for i=1

$$\sigma_{12} = 2\mu \epsilon_{12} = \mu(u_{1,2} + u_{2,1})$$

So (*) implies

$$\mu(u_{1,2}^{(0)} + u_{2,1}^{(0)} + u_{1,2}^{(1)} + u_{2,1}^{(1)}) = \mu^{\beta}(u_{1,2}^{(2)} + u_{2,1}^{(2)})$$

$$\Rightarrow \mu\{A_0 k_0(\sin\theta_0\cos\theta_0 + \sin\theta_0\cos\theta_0) + A_1 k_1(-\sin\theta_1\cos\theta_1 - \sin\theta_1\cos\theta_1)\} = \mu^{\beta}\{A_2 k_2(\sin\theta_2\cos\theta_2 + \sin\theta_2\cos\theta_2)\}$$

$$\Rightarrow A_0 k_0 \mu \sin 2\theta_0 - A_1 k_1 \mu \sin 2\theta_1 - A_2 k_2 \mu^{\beta} \sin 2\theta_2 = 0 \longrightarrow (3)$$

for i=2

$$\sigma_{22} = \lambda \epsilon_{kk} + 2\mu \epsilon_{22}$$

$$= \lambda(\epsilon_{11} + \epsilon_{22}) + 2\mu \epsilon_{22}$$

$$= \lambda(u_{1,1} + u_{2,2}) + 2\mu(u_{2,2})$$

So (*) becomes

$$\lambda(u_{1,1}^{(0)} + u_{2,2}^{(0)} + u_{1,1}^{(1)} + u_{2,2}^{(1)}) + 2\mu(u_{2,2}^{(0)} + u_{2,2}^{(1)})$$

$$= \lambda^{\beta}(u_{1,1}^{(2)} + u_{2,2}^{(2)}) + 2\mu^{\beta}(u_{2,2}^{(2)})$$

$$\Rightarrow \lambda\left\{A_0 k_0(\sin^2\theta_0+\cos^2\theta_0)+A_1 k_1(\sin^2\theta_1+\cos^2\theta_1)\right\}$$
$$+2\mu\left\{A_0 k_0\cos^2\theta_0 + A_1 k_1\cos^2\theta_1\right\} = \lambda^\beta\left\{A_2 k_2(\sin^2\theta_2+\cos^2\theta_2)\right\} + 2\mu^\beta A_2 k_2 \cos^2\theta_2$$

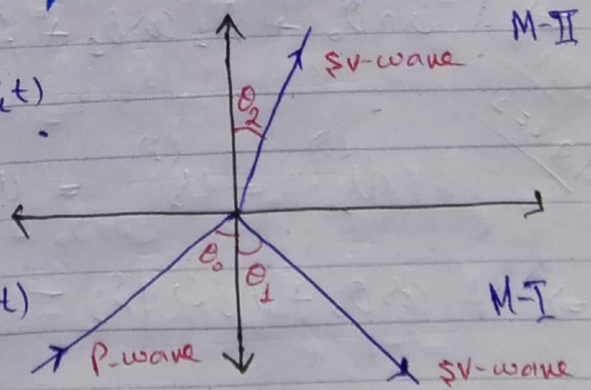
$$\Rightarrow \lambda\left\{A_0 k_0 + A_1 k_1\right\}+2\mu\left\{A_0 k_0\cos^2\theta_0 + A_1 k_1 \cos^2\theta_1\right\}$$
$$+\lambda^\beta\left\{A_2 k_2\right\}+2\mu^\beta A_2 k_2\cos^2\theta_2$$

$$\Rightarrow A_0 k_0(\lambda+2\mu\cos^2\theta_0)+A_1 k_1(\lambda+2\mu\cos^2\theta_1)$$
$$-A_2 k_2(\lambda^\beta+2\mu^\beta\cos^2\theta_2)=0 \longrightarrow (4)$$

There are four equations and two unknowns, so there does not exist any unique solution.

Hence Refraction and Reflection of P-wave as a P-wave does not exist.

⇒ Transmission & Reflection of P-wave as SV-wave :-



$$u_i^{(0)} = A_0 e^{ik_0(x_1\sin\theta_0 + x_2\cos\theta_0 - c_1 t)}$$
$$(\sin\theta_0,\ \cos\theta_0,\ 0)$$

$$u_i^{(1)} = A_1 e^{ik_1(x_1\sin\theta_1 - x_2\cos\theta_1 - c_2 t)}$$
$$(\cos\theta_1,\ \sin\theta_1,\ 0)$$

$$u_i^{(2)} = A_2 e^{ik_2(x_1\sin\theta_2 + x_2\cos\theta_2 - c_2^\beta t)} \quad (-\cos\theta_2,\ \sin\theta_2,\ 0)$$

1) u_i is continuous at $x_2 = 0$

$$\Rightarrow u_i^{(0)} + u_i^{(1)} = u_i^{(2)}$$

for $i=1$

$$u_1^{(0)} + u_1^{(1)} = u_1^{(2)}$$

$$\Rightarrow A_0 e^{ik_0(x_1 \sin\theta_0 - Ct)} \sin\theta_0 + A_1 e^{ik_1(x_1\sin\theta_1 - Ct)} \cos\theta_1$$

$$= -A_2 e^{ik_2(x_1 \sin\theta_2 - C^\beta t)} \cos\theta_2$$

By Snell's law

$$k_0 \sin\theta_0 = k_1 \sin\theta_1 = k_2 \sin\theta_2$$

And $k_0 C = k_1 C = k_2 C^\beta$

This is possible if

$$\boxed{k_1 = k_0} \ \& \ \boxed{\theta_1 = \theta_0} \Rightarrow C = C$$

$$\Rightarrow \boxed{k_2 = \frac{C}{C^\beta} k_0}$$

And $\sin\theta_2 = \frac{k_0}{k_2}\sin\theta_0 \Rightarrow \boxed{\sin\theta_2 = \frac{C^\beta}{C}\sin\theta_0}$

$$\Rightarrow A_0 \sin\theta_0 + A_1 \cos\theta_1 + A_2 \cos\theta_2 = 0 \longrightarrow (1)$$

for $i=2$

$$u_2^{(0)} + u_2^{(1)} = u_2^{(2)}$$

$$\Rightarrow A_0 \cos\theta_0 + A_1 \sin\theta_1 - A_2 \sin\theta_2 = 0 \longrightarrow (2)$$

(2) σ_{i2} is continuous at $x_2 = 0$

$$\Rightarrow \sigma_{i2}^{(0)} + \sigma_{i2}^{(1)} = \sigma_{i2}^{(2)} \longrightarrow (*)$$

since $\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$

$$\Rightarrow \sigma_{i2} = \lambda \epsilon_{kk} \delta_{i2} + 2\mu \epsilon_{i2}$$

for i=1

$$\sigma_{12} = 2\mu \epsilon_{12} = \mu(u_{1,2} + u_{2,1})$$

so ⊛ implies

$$\mu\{u_{1,2}^{(0)} + u_{2,1}^{(0)} + u_{1,2}^{(1)} + u_{2,1}^{(1)}\} = \mu^{\beta}\{u_{1,2}^{(2)} + u_{2,1}^{(2)}\}$$

$$\Rightarrow \mu\{A_0 k_0(\sin\theta_0\cos\theta_0 + \sin\theta_0\cos\theta_0) + A_1 k_1(-\cos^2\theta_1 + \sin^2\theta_1)\}$$
$$= \mu^{\beta}\{A_2k_2(-\cos^2\theta_2 + \sin^2\theta_2)\}$$

$$\Rightarrow A_0 k_0 \mu \sin 2\theta_0 - A_1 k_1 \mu \cos 2\theta_1 + A_2 k_2 \cos 2\theta_2 = 0 \longrightarrow (3)$$

for i=2

$$\sigma_{22} = \lambda(u_{1,1} + u_{2,2}) + 2\mu(u_{2,2})$$

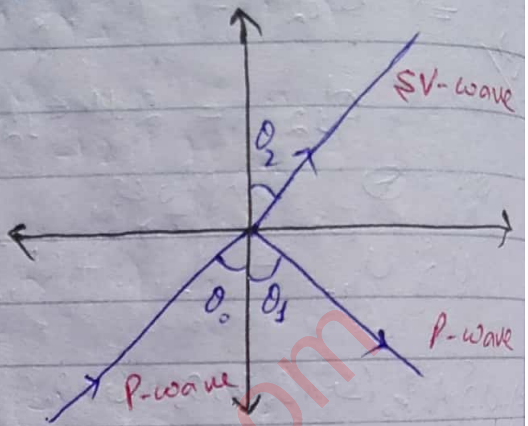
$$\Rightarrow \lambda\{u_{1,1}^{(0)} + u_{2,2}^{(0)} + u_{1,1}^{(1)} + u_{2,2}^{(1)}\} + 2\mu\{u_{2,2}^{(0)} + u_{2,2}^{(1)}\}$$
$$= \lambda^{\beta}\{u_{1,1}^{(2)} + u_{2,2}^{(2)}\} + 2\mu^{\beta}(u_{2,2}^{(2)})$$

$$\Rightarrow \lambda\{A_0 k_0(\sin^2\theta_0 + \cos^2\theta_0) + A_1 k_1(\sin\cancel{\theta_1}\cos\theta_1 - \sin\cancel{\theta_1}\cos\theta_1)\}$$
$$+ 2\mu\{A_0 k_0 \cos^2\theta_0 - A_1 k_1 \sin\theta_1\cos\theta_1\} = \lambda^{\beta}\{A_2 k_2(-\sin\theta_2$$
$$\cos\theta_2 + \sin\theta_2\cancel{\cos\theta_2})\} + 2\mu^{\beta}(A_2 k_2 \sin\theta_2 \cos\theta_2)$$

$$\Rightarrow \lambda A_0 k_0 + 2\mu A_0 k_0 \cos^2\theta_0 - \mu A_1 k_1 \sin 2\theta_1 = \mu^{\beta} A_2 k_2 \sin 2\theta_2$$

$$\Rightarrow A_0 k_0(\lambda + 2\mu\cos^2\theta_0) - A_1 k_1 \mu \sin 2\theta_1 - A_2 k_2 \mu^{\beta} \sin 2\theta_2 = 0 \longrightarrow (4)$$

⇒ Reflection of P-wave as P-wave with refrated SV-wave also:-



$$n_i^{(0)} = (\sin\theta_0, \cos\theta_0, 0)$$
$$= p_i^{(0)}$$

$$n_i^{(1)} = (\sin\theta_1, -\cos\theta_1, 0)$$
$$= p_i^{(1)}$$

$$n_i^{(2)} = (\sin\theta_2, \cos\theta_2, 0)$$

$$p_i^{(2)} = (-\cos\theta_2, \sin\theta_2, 0)$$

$$\Rightarrow u_i^{(0)} = A_0 e^{ik_0(x_1\sin\theta_0 + x_2\cos\theta_2 - c_1 t)} (\sin\theta_0, \cos\theta_0, 0)$$

$$u_i^{(1)} = A_1 e^{ik_1(x_1\sin\theta_1 - x_2\cos\theta_1 - c_1 t)} (\sin\theta_1, -\cos\theta_1, 0)$$

$$u_i^{(2)} = A_2 e^{ik_2(x_1\sin\theta_2 + x_2\cos\theta_2 - c_2 t)} (-\cos\theta_2, \sin\theta_2, 0)$$

① u_i is continuous at $x_2 = 0$

$$\Rightarrow u_i^{(0)} + u_i^{(1)} = u_i^{(2)}$$

for $i=1$

$$u_1^{(0)} + u_1^{(1)} = u_1^{(2)}$$

$$\Rightarrow A_0 e^{ik_0(x_1\sin\theta_0 - c_1 t)} \sin\theta_0 + A_1 e^{ik_1(x_1\sin\theta_1 - c_1 t)} \sin\theta_1$$
$$= -A_2 e^{ik_2(x_1\sin\theta_2 - c_2 t)} \cos\theta_2$$

By Snell's law

$$k_0 \sin\theta_0 = k_1 \sin\theta_1 = k_2 \sin\theta_2$$

And $k_0 C_1 = k_1 C_1 = k_2 C^\beta$

$\Rightarrow \boxed{k_0 = k_1} \Rightarrow \boxed{\theta_1 = \theta_0}$

$\Rightarrow \boxed{k_2 = \frac{C_1}{C^\beta} k_0}$

$\Rightarrow \sin\theta_2 = \frac{k_0}{k_2} \sin\theta_0 \quad \Rightarrow \boxed{\sin\theta_2 = \frac{C^\beta}{C_1} \sin\theta_0}$

$\Rightarrow A_0 \sin\theta_0 + A_1 \sin\theta_1 + A_2 \cos\theta_2 = 0 \longrightarrow ①$

for $i=2$

$$u_2^{(0)} + u_2^{(1)} = u_2^{(2)}$$

$\Rightarrow A_0 \cos\theta_0 - A_1 \cos\theta_1 = A_2 \sin\theta_2$

$\Rightarrow A_0 \cos\theta_0 - A_1 \cos\theta_1 - A_2 \sin\theta_2 = 0 \longrightarrow ②$

② σ_{i2} is continuous at $x_2 = 0$

$\Rightarrow \sigma_{i2}^{(0)} + \sigma_{i2}^{(1)} = \sigma_{i2}^{(2)} \longrightarrow (*)$

And $\sigma_{i2} = \lambda \epsilon_{kk} \delta_{i2} + 2\mu \epsilon_{i2}$

for $i=1$

$$\sigma_{12} = \mu(u_{1,2} + u_{2,1})$$

$\Rightarrow \mu\{u_{1,2}^{(0)} + u_{2,1}^{(0)} + u_{1,2}^{(1)} + u_{2,1}^{(1)}\} = \mu^\beta\{u_{1,2}^{(2)} + u_{2,1}^{(2)}\}$

$\Rightarrow \mu\{A_0 k_0(\sin\theta_0\cos\theta_0 + \sin\theta_0\cos\theta_0) + A_1 k_1(-\sin\theta_1\cos\theta_1 - \sin\theta_1\cos\theta_1)\} = \mu^\beta\{A_2 k_2(-\cos^2\theta_2 + \sin^2\theta_2)\}$

$\Rightarrow \mu A_0 k_0 \sin 2\theta_0 - \mu A_1 k_1 \sin 2\theta_1 = -\mu^\beta A_2 k_2 \cos 2\theta_2$

$$\Rightarrow A_0 k_0 \mu \sin 2\theta_0 - A_1 k_1 \mu \sin 2\theta_1 + A_2 k_2 \mu^\beta \cos 2\theta_2 = 0 \longrightarrow (3)$$

for $i=2$

$$\sigma'_{22} = \lambda(u_{1,1} + u_{2,2}) + 2\mu(u_{2,2})$$

$$\Rightarrow \lambda\{u_{1,1}^{(0)} + u_{2,2}^{(0)} + u_{1,1}^{(1)} + u_{2,2}^{(1)}\} + 2\mu\{u_{2,2}^{(0)} + u_{2,2}^{(1)}\}$$
$$= \lambda^\beta\{u_{1,1}^{(2)} + u_{2,2}^{(2)}\} + 2\mu^\beta\{u_{2,2}^{(2)}\}$$

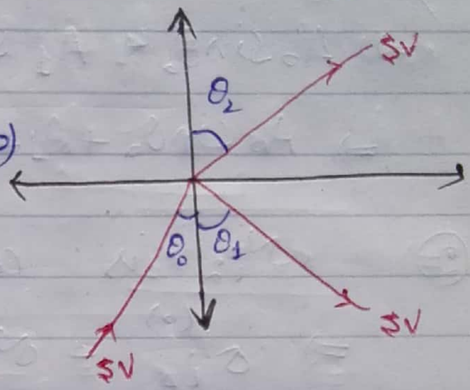
$$\Rightarrow \lambda\{A_0 k_0(\sin^2\theta_0 + \cos^2\theta_0) + A_1 k_1(\sin^2\theta_1 + \cos^2\theta_1)\}$$
$$+ 2\mu\{A_0 k_0 \cos^2\theta_0 + A_1 k_1 \cos^2\theta_1\} = \lambda^\beta\{A_2 k_2(-\sin\theta_2 \cos\theta_2$$
$$+ \sin\theta_2 \cos\theta_2)\} + 2\mu^\beta\{A_2 k_2 \sin\theta_2 \cos\theta_2\}$$

$$\Rightarrow \lambda\{A_0 k_0 + A_1 k_1\} + 2\mu A_0 k_0 \cos^2\theta_0 + 2\mu A_1 k_1 \cos^2\theta_1$$
$$= \mu^\beta A_2 k_2 \sin 2\theta_2$$

$$\Rightarrow A_0 k_0(\lambda + 2\mu\cos^2\theta_0) + A_1 k_1(\lambda + 2\mu\cos^2\theta_1)$$
$$- \mu^\beta A_2 k_2 \sin 2\theta_2 = 0 \longrightarrow (4)$$

Solution not exist

⇒ Reflection And Transmission of SV-wave as SV-wave:-



$$u_i^{(0)} = A_0 e^{ik_0(x_1 \sin\theta_0 + x_2 \cos\theta_0 - Gt)} \quad (-\cos\theta_0, \sin\theta_0, 0)$$

$$u_i^{(1)} = A_1 e^{ik_1(x_1 \sin\theta_1 - x_2 \cos\theta_1 - Gt)} \quad (\cos\theta_1, \sin\theta_1, 0)$$

$$u_i^{(2)} = A_2 e^{ik_2(x_1 \sin\theta_2 + x_2 \cos\theta_2 - Gt^{\beta})} \quad (-\cos\theta_2, \sin\theta_2, 0)$$

① u_i is continuous at $x_2 = 0$

$$\Rightarrow u_i^{(0)} + u_i^{(1)} = u_i^{(2)}$$

for $i=1$

$$u_1^{(0)} + u_1^{(1)} = u_1^{(2)}$$

$$\Rightarrow -A_0 e^{ik_0(x_1 \sin\theta_0 - Gt)} \cos\theta_0 + A_1 e^{ik_1(x_1 \sin\theta_1 - Gt)} \cos\theta_1$$

$$= -A_2 e^{ik_2(x_1 \sin\theta_2 - Gt^{\beta})} \cos\theta_2$$

By Snell's law

$$k_0 \sin\theta_0 = k_1 \sin\theta_1 = k_2 \sin\theta_2$$

And $k_0 G = k_1 G = k_2 G^{\beta}$

$$\Rightarrow \boxed{k_0 = k_1} \quad \& \quad \boxed{\theta_0 = \theta_1}$$

$$\Rightarrow \boxed{k_2 = \frac{G}{G^{\beta}} k_0}$$

$$\Rightarrow \sin\theta_2 = \frac{k_0}{k_2} \sin\theta_0 \quad \Rightarrow \boxed{\sin\theta_2 = \frac{G^{\beta}}{G} \sin\theta_0}$$

$$\Rightarrow -A_0 \cos\theta_0 + A_1 \cos\theta_1 + A_2 \cos\theta_2 = 0 \longrightarrow ①$$

for $i=2$

$$u_2^{(0)} + u_2^{(1)} = u_2^{(2)}$$

$$\Rightarrow A_0 \sin\theta_0 + A_1 \sin\theta_1 = A_2 \sin\theta_2$$

$$\Rightarrow A_0 \sin\theta_0 + A_1 \sin\theta_1 - A_2 \sin\theta_2 = 0 \longrightarrow (2)$$

(2) σ_{i2} is continuous at $x_2 = 0$

$$\Rightarrow \sigma_{i2}^{(0)} + \sigma_{i2}^{(1)} = \sigma_{i2}^{(2)} ;$$

where $\sigma_{i2} = \lambda \epsilon_{kk} \delta_{i2} + 2\mu \epsilon_{i2}$

for $i=1$

$$\sigma_{12} = \mu(u_{1,2} + u_{2,1})$$

$$\Rightarrow \mu\{u_{1,2}^{(0)} + u_{2,1}^{(0)} + u_{1,2}^{(1)} + u_{2,1}^{(1)}\} = \mu^{\beta}\{u_{1,2}^{(2)} + u_{2,1}^{(2)}\}$$

$$\Rightarrow \mu\{A_0 k_0(-\cos^2\theta_0 + \sin^2\theta_0) + A_1 k_1(-\cos^2\theta_1 + \sin^2\theta_1)\}$$

$$= \mu^{\beta}\{A_2 k_2(-\cos^2\theta_2 + \sin^2\theta_2)\}$$

$$\Rightarrow A_0 k_0 \mu \cos 2\theta_0 - A_1 k_1 \mu \cos 2\theta_1 + A_2 k_2 \mu^{\beta} \cos 2\theta_2 = 0$$

for $i=2$

$$\sigma_{22} = \lambda(u_{1,1} + u_{2,2}) + 2\mu(u_{2,2})$$

$$\Rightarrow \lambda\{u_{1,1}^{(0)} + u_{2,2}^{(0)} + u_{1,1}^{(1)} + u_{2,2}^{(1)}\} + 2\mu\{u_{2,2}^{(0)} + u_{2,2}^{(1)}\}$$

$$= \lambda^{\beta}(u_{1,1}^{(2)} + u_{2,2}^{(2)}\} + 2\mu^{\beta} u_{2,2}^{(2)}$$

$$\Rightarrow \lambda\{A_0 k_0(-\cos\theta_0 \cancel{\sin\theta_0} + \cos\theta_0 \cancel{\sin\theta_0}) + A_1 k_1(\cos\theta_1 \cancel{\sin\theta_1}$$

$$- \cos\theta_1 \cancel{\sin\theta_1})\} + 2\mu\{A_0 k_0 \cos\theta_0 \sin\theta_0 - A_1 k_1 \cos\theta_1$$

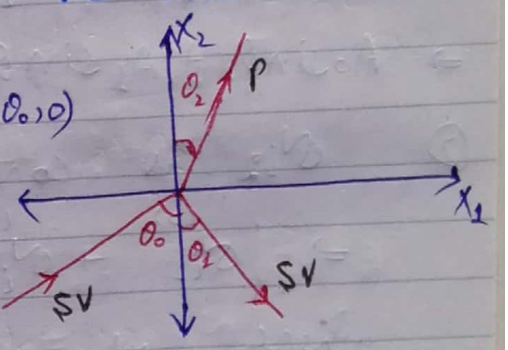
$$= \lambda^{\beta}\{A_2 k_2(-\sin\theta_2 \cancel{\cos\theta_2} + \sin\theta_2 \cancel{\cos\theta_2})\} + 2\mu^{\beta}\{A_2 k_2 \sin\theta_2$$

$$\cos\theta_2\}$$

$$\Rightarrow \mu A_0 k_0 \sin 2\theta_0 - \mu A_1 k_1 \sin 2\theta_1 = \mu^{\beta} A_2 k_2 \sin 2\theta_2$$

$$\Rightarrow A_0 k_0 \mu \sin 2\theta_0 - A_1 k_1 \mu \sin 2\theta_1 - A_2 k_2 \mu^{\beta} \sin 2\theta_2 = 0 \quad \text{(4)}$$

⇒ Reflection of SV-wave as SV-wave with refracted P-wave also:-



$$U_i^{(0)} = A_0 e^{ik_0(x_1 \sin\theta_0 + x_2 \cos\theta_0 - c_T t)} \quad (-\cos\theta_0, \sin\theta_0, 0)$$

$$U_i^{(1)} = A_1 e^{ik_1(x_1 \sin\theta_1 - x_2 \cos\theta_1 - c_T t)} \quad (\cos\theta_1, \sin\theta_1, 0)$$

$$U_i^{(2)} = A_2 e^{ik_2(x_1 \sin\theta_2 + x_2 \cos\theta_2 - c_L^{\beta} t)} \quad (\sin\theta_2, \cos\theta_2, 0)$$

① U_i is continuous at $x_2 = 0$

$$\Rightarrow U_i^{(0)} + U_i^{(1)} = U_i^{(2)}$$

For $i=1$

$$-A_0 e^{ik_0(x_1 \sin\theta_0 - c_T t)} \cos\theta_0 + A_1 e^{ik_1(x_1 \sin\theta_1 - c_T t)} \cos\theta_1$$

$$= A_2 e^{ik_2(x_1 \sin\theta_2 - c_L^{\beta} t)} \sin\theta_2$$

By Snell's law

$k_0 \sin\theta_0 = k_1 \sin\theta_1 = k_2 \sin\theta_2$

And $k_0 G = k_1 G = k_2 G^{\beta}$

$\Rightarrow \boxed{k_0 = k_1}$ And $\boxed{\theta_1 = \theta_0}$

$\Rightarrow \boxed{k_2 = \frac{G}{G^{\beta}} k_0}$

And $\sin\theta_2 = \frac{k_0}{k_2}\sin\theta_0 \Rightarrow \boxed{\sin\theta_2 = \frac{G^{\beta}}{G}\sin\theta_0}$

$\Rightarrow -A_0\cos\theta_0 + A_1\cos\theta_1 - A_2\sin\theta_2 = 0 \longrightarrow (1)$

for i=2

$$u_2^{(0)} + u_2^{(1)} = u_2^{(2)}$$

$\Rightarrow A_0\sin\theta_2 + A_1\sin\theta_1 - A_2\cos\theta_2 = 0 \longrightarrow (2)$

(2) σ_{i2} is continuous at $x_2 = 0$

$$\Rightarrow \sigma_{i2}^{(0)} + \sigma_{i2}^{(1)} = \sigma_{i2}^{(2)}$$

where $\sigma_{i2} = \lambda\epsilon_{kk}\delta_{i2} + 2\mu\epsilon_{i2}$

for i=1

$$\sigma_{12} = \mu(u_{1,2} + u_{2,1})$$

$\Rightarrow \mu\{u_{1,2}^{(0)} + u_{2,1}^{(0)} + u_{1,2}^{(1)} + u_{2,1}^{(1)}\} = \mu^{\beta}\{u_{1,2}^{(2)} + u_{2,1}^{(2)}\}$

$\Rightarrow \mu\{A_0k_0(-\cos^2\theta_0 + \sin^2\theta_0) + A_1k_1(-\cos^2\theta_1 + \sin^2\theta_1)\} = \mu^{\beta}\{A_2k_2(\sin\theta_2\cos\theta_2 + \sin\theta_2\cos\theta_2)\}$

$\Rightarrow -\mu A_0k_0\cos 2\theta_0 - \mu A_1k_1\cos 2\theta_1 = \mu^{\beta}A_2k_2\sin 2\theta_2$

$$\Rightarrow -A_0 k_0 \mu \cos 2\theta_0 - A_1 k_1 \mu \cos 2\theta_1 - A_2 k_2 \mu^\beta \sin 2\theta_2 = 0 \longrightarrow (3)$$

for $i=2$

$$\sigma_{22} = \lambda(u_{1,1} + u_{2,2}) + 2\mu(u_{2,2})$$

$$\Rightarrow \lambda\{u_{1,1}^{(0)} + u_{2,2}^{(0)} + u_{1,1}^{(1)} + u_{2,2}^{(1)}\} + 2\mu\{u_{2,2}^{(0)} + u_{2,2}^{(1)}\}$$

$$= \lambda^\beta\{u_{1,1}^{(2)} + u_{2,2}^{(2)}\} + 2\mu^\beta\{u_{2,2}^{(2)}\}$$

$$\Rightarrow \lambda\{A_0 k_0(-\sin\theta_0 \cancel{\cos\theta_0} + \sin\theta_0 \cos\theta_0) + A_1 k_1(\sin\theta_1 \cancel{\cos\theta_1} - \sin\theta_1 \cancel{\cos\theta_1})\} + 2\mu\{A_0 k_0 \sin\theta_0 \cos\theta_0 - A_1 k_1 \sin\theta_1 \cos\theta_1)\}$$

$$= \lambda^\beta\{A_2 k_2 \sin^2\theta_2 + A_2 k_2 \cos^2\theta_2\} + 2\mu^\beta\{A_2 k_2 \cos^2\theta_2\}$$

$$\Rightarrow \mu A_0 k_0 \sin 2\theta_0 - \mu A_1 k_1 \sin 2\theta_1 = A_2 k_2 \lambda^\beta + 2\mu^\beta A_2 k_2 \cos^2\theta_2$$

$$\Rightarrow A_0 k_0 \mu \sin 2\theta_0 - A_1 k_1 \mu \sin 2\theta_1 - (\lambda^\beta + 2\mu^\beta \cos^2\theta_2) A_2 k_2 = 0 \longrightarrow (4)$$

Solution not exist

⇒ Equation of Motion for Homogeneous Transversely Isotropic Medium:-

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(C_{22}-C_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix}$$

Here axis of symmetric is x_1 - axis.
We have 5-indipendent elastic constants.
For Fibre Reinforced Material

$$\sigma_{ij} = \lambda \epsilon_{kk}\delta_{ij} + 2\mu_T \epsilon_{ij} + \alpha(a_k a_m \epsilon_{km}\delta_{ij} + \epsilon_{kk} a_i a_j) + 2(\mu_L - \mu_T)(a_i a_k \epsilon_{kj} + a_j a_k \epsilon_{ki}) + \beta(a_k a_m \epsilon_{km}\delta_{ij})$$

Equation of motion is

$$\sigma_{ij,j} = \rho \ddot{u}_i$$

$$\Rightarrow \left.\begin{aligned} \sigma_{11,1} + \sigma_{12,2} &= \rho \ddot{u}_1 \\ \sigma_{21,1} + \sigma_{22,2} &= \rho \ddot{u}_2 \\ \sigma_{31,1} + \sigma_{32,2} &= \rho \ddot{u}_3 \end{aligned}\right\} \longrightarrow ①$$

Now $\sigma_{11} = C_{11}\epsilon_{11} + C_{12}\epsilon_{22}$

$$\Rightarrow \sigma_{11} = C_{11}\cdot\frac{1}{2}(u_{1,1} + u_{1,1}) + \frac{1}{2}C_{12}(u_{2,2} + u_{2,2})$$

$$= C_{11}\, u_{1,1} + \frac{1}{2}C_{12}(u_{2,2} + u_{2,2})$$

$$\Rightarrow \sigma_{11,1} = C_{11} U_{1,11} + C_{12} U_{2,21} \longrightarrow (a)$$

$$* \quad \sigma_{12} = 2 C_{66} \epsilon_{12} = 2 C_{66} \cdot \frac{1}{2} (U_{1,2} + U_{2,1})$$

$$= C_{66} (U_{1,2} + U_{2,1})$$

$$\Rightarrow \sigma_{12,2} = C_{66} (U_{1,22} + U_{2,12}) \longrightarrow (b)$$

$$\sigma_{21} = 2 C_{66} \epsilon_{12}$$

$$= C_{66} (U_{1,2} + U_{2,1})$$

$$\Rightarrow \sigma_{21,1} = C_{66} (U_{1,21} + U_{2,11}) \longrightarrow (c)$$

$$\sigma_{22} = C_{12} \epsilon_{11} + C_{22} \epsilon_{22}$$

$$= C_{12} U_{1,1} + C_{22} U_{2,2}$$

$$\Rightarrow \sigma_{22,2} = C_{12} U_{1,12} + C_{22} U_{2,22} \longrightarrow (d)$$

$$\sigma_{31} = 2 C_{66} \epsilon_{13}$$

$$= C_{66} (U_{1,3} + U_{3,1})$$

$$= C_{66} U_{3,1} \qquad \because U_{1,3} = 0$$

$$\Rightarrow \sigma_{31,1} = C_{66} U_{3,11} \longrightarrow (e)$$

$$\sigma_{32} = \frac{1}{2} (C_{22} - C_{23}) 2 \epsilon_{23}$$

$$= \frac{1}{2} (C_{22} - C_{23}) (U_{2,3} + U_{3,2})$$

$$= \frac{1}{2} (C_{22} - C_{23}) U_{3,2} \qquad \because U_{2,3} = 0$$

$$\Rightarrow \sigma_{32,2} = \frac{1}{2} (C_{22} - C_{23}) U_{3,22} \longrightarrow (f)$$

using ⓐ → ⓟ equ ① becomes

$$\left.\begin{aligned} C_{11} U_{1,11} + C_{12} U_{2,21} + C_{66} U_{1,22} + C_{66} U_{2,21} &= \rho \ddot{U}_1 \\ C_{22} U_{2,22} + C_{12} U_{1,12} + C_{66} U_{2,11} + C_{66} U_{1,12} &= \rho \ddot{U}_2 \\ C_{66} U_{3,11} + \tfrac{1}{2}(C_{22} - C_{23}) U_{3,22} &= \rho \ddot{U}_3 \end{aligned}\right\} \rightarrow ②$$

Now for fibre Reinforced material
Let $a_i = (0,0,1)$

$$\sigma_{11} = \lambda \epsilon_{kk} \delta_{11} + 2\mu_T \epsilon_{11}$$
$$= \lambda \epsilon_{kk} + 2\mu_T \epsilon_{11}$$
$$= \lambda \epsilon_{11} + \lambda \epsilon_{22} + 2\mu_T \epsilon_{11}$$
$$= \lambda U_{1,1} + \lambda U_{2,2} + 2\mu_T U_{1,1}$$

$$\Rightarrow \sigma_{11,1} = (\lambda + 2\mu_T) U_{1,11} + \lambda U_{2,21} \longrightarrow ⓐ$$

$$\sigma_{12} = 2\mu_T \epsilon_{12} = \mu_T (U_{1,2} + U_{2,1})$$

$$\Rightarrow \sigma_{12,2} = \mu_T U_{1,22} + \mu_T U_{2,12} \longrightarrow ⓑ$$

$$\Rightarrow (\lambda + 2\mu_T) U_{1,11} + (\lambda + \mu_T) U_{2,21} + \mu_T U_{1,22} = \rho \ddot{U}_1$$

Comparing equation 1 of ② & ③, we have

$$\boxed{\lambda + 2\mu_T = C_{11}} \qquad \boxed{C_{66} = \mu_T}$$

$$C_{12} + C_{66} = \lambda + \mu_T$$

$$\Rightarrow \boxed{C_{12} = \lambda}$$

similarly we can find others speed of waves

$$\sigma_{21} = 2\mu_T \epsilon_{21} = \mu_T(u_{2,1} + u_{1,2})$$

$$\Rightarrow \sigma_{21,1} = \mu_T u_{2,11} + \mu_T u_{1,21} \longrightarrow (i)$$

$$\sigma_{22} = \lambda \epsilon_{kk} + 2\mu_T \epsilon_{22}$$

$$= \lambda u_{1,1} + \lambda u_{2,2} + 2\mu_T u_{2,2}$$

$$\Rightarrow \sigma_{22,2} = (\lambda + 2\mu_T)_{2,22} + \lambda u_{1,12} \longrightarrow (j)$$

$$\Rightarrow \mu_T u_{2,11} + (\lambda + \mu_T) u_{1,12} + (\lambda + 2\mu_T) u_{2,22} = \rho \ddot{u}_2 \longrightarrow (4)$$

Comparing equation 2 of (2) & (4) we have

$$\boxed{C_{66} = \mu_T} \qquad C_{12} + C_{66} = \lambda + \mu_T$$

$$\Rightarrow \boxed{C_{12} = \lambda} \qquad \because C_{66} = \mu_T$$

$$\boxed{C_{22} = \lambda + 2\mu_T}$$

$$\sigma_{31} = 2\mu_T \epsilon_{31} + 2(\mu_L - \mu_T)\epsilon_{31}$$

$$= \mu_T(u_{3,1} + u_{1,3}) + (\mu_L - \mu_T)(u_{3,1} + u_{1,3})$$

$$\Rightarrow \sigma_{31,1} = \mu_T u_{3,11} + (\mu_L - \mu_T) u_{3,11}$$

$$\Rightarrow \sigma_{31,1} = \mu_L u_{3,11} \longrightarrow (k)$$

$$\sigma_{32} = 2\mu_T \epsilon_{32} + 2(\mu_L - \mu_T)\epsilon_{32}$$

$$= \mu_T(u_{3,2} + u_{2,3}) + (\mu_L - \mu_T)(u_{3,2} + u_{2,3})$$

$$\Rightarrow \sigma_{32,2} = \mu_T u_{3,22} + (\mu_L - \mu_T) u_{3,22}$$

$$\Rightarrow \sigma_{32,2} = \mu_L u_{3,22} \longrightarrow (l)$$

$$\Rightarrow \mu_L U_{3,11} + \mu_L U_{3,22} = \rho \ddot{U}_3 \longrightarrow (5)$$

Comparing equation 3 of (2) & (5) we have

$$\boxed{C_{66} = \mu_L} \quad \& \quad \frac{1}{2}(C_{22} - C_{23}) = \mu_L$$

$$\because C_{22} = \lambda + 2\mu_T$$

$$\Rightarrow \frac{1}{2}(\lambda + 2\mu_T - C_{23}) = \mu_L$$

$$\Rightarrow \frac{1}{2}\lambda + \mu_T - \mu_L = \frac{1}{2} C_{23}$$

$$\Rightarrow \frac{1}{2}\lambda = \frac{1}{2} C_{23}$$

$$\boxed{\because C_{66} = \mu_T \;\&\; C_{66} = \mu_L \Rightarrow \mu_T = \mu_L}$$

$$\Rightarrow \boxed{C_{23} = \lambda}$$