General form of PDE's. 01-02-2016

$$f(x, y, \ldots, u, u_x, u_y, \ldots, u_{xx}, u_{xy}, \ldots) = 0 \longrightarrow (1)$$

$$\sum_{i,j=1}^{n} A_{ij} U_{x_i} U_{x_j} + \sum_{i=1}^{n} b_i U_{x_i} + F(x_1, x_2, \ldots) = G(x_1, x_2, \ldots, x_n) \longrightarrow (2)$$

$$A_{ij} = A_{ji}$$

both the forms (1) & (2) are identical.

where "f" is any general function of independent variables $x, y, \ldots$ "u" is the unknown function. $U_x, U_y, \ldots, U_{xx}, U_{yy}$ are the partial derivatives & the subscripts on the dependent variables denote the derivative.

$U_{xy} = \frac{\partial^2 u}{\partial y \partial x}$ LHS is performed left to right & vice-versa.

Laplace Eq,

$$U_{xx} + U_{yy} = 0$$

1D wave Eq $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

1-D Heat Eq, $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$

Solution should be differentiable in the domain of given D.E.

Where
$$u = u(x, y, z, \cdots)$$
which satisfies (i) identically is a suitable domain "D" of n-dimensional space $\mathbb{R}^n$ in the independent variables $x, y, \ldots,$ Such functions (if they exist) are called the solution of (1).

Example:-
$$U_{xx} - U_{yy} = 0$$
has solutions
$$u = (x+y)^3 \quad , \quad u = \sin(x-y)$$
Polynomial Continuous   Again Continuous

Unique solution are obtained when the D.Eq is a subject to some conditions.

The order of PDE is the highest derivative of PDE, appearing in the D.Eq.

Example:-
$$U_{xxy} + x\,U_{yy} + 8u = f_y \rightarrow 3^{rd} \text{ order}$$
$$U_{xx} + 2x\,U_{xy} + U_{yy} = e^y \rightarrow 2^{nd} \text{ order}.$$

A PDE is said to be Linear if

1):- Linear in unknown function.
2):- All of it's partial derivatives with co-efficients depending only on the independent variables.

If dependent variable becomes exponential then PDE is non-linear.

A PDE is Quasi-Linear if it is linear in the highest derivative of the unknown function.

If forces terms are non-linear but rest of the PDE is linear. is said to be almost linear.
Force term is called input.
Zero force term implies homogenity.

$$U_{xx} - U_{yy} = 0$$

$$u = c_1(x+y)^3 + c_2 \sin(x-y)$$

$$U_{xx} = c_1 6(x+y) - c_2 \sin(x-y)$$

$$U_{yy} = c_1 6(x+y) - c_2 \sin(x-y)$$

$$\Rightarrow U_{xx} - U_{yy} = \cancel{c_1 6(x+y)} - \cancel{c_1 6(x+y)} - \cancel{c_2 \sin(x-y)} + \cancel{c_2 \sin(x-y)} = 0$$

Superposition principle always satisfies linear homogeneous PDE.

Example :- $u_x - u_y = 0$

$\xi = x+y, \quad \eta = x-y$

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$

$u_y = u_\xi \xi_y + u_\eta \eta_y = u_\xi - u_\eta$

So, $u_x - u_y = 2u_\eta = 0 \Rightarrow 2u = f(\xi)$

OR simply $\Rightarrow u = f(\xi) = f(x+y)$

thus "u" can have solution

$(x+y)^n$, $\cos n(x+y)$, $\sin n(x+y)$

$\exp(n(x+y))$, $n = 1, 2, 3, ---$

Note: First lecture on page #12

Happy Birthday

Adv. PDE 02-02-2016

Dr. Mahmood-ul-Hassan.

Operator.

An operator is a mathematical Rule which when applied to a function produces it to an other function.

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$Lu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Linear:-

$$L[c_1 u_1 + c_2 u_2] = c_1 L[u_1] + c_2 L[u_2]$$

$$L\left[\sum_{i=1}^{n} c_i u_i\right] = \sum_{i=1}^{n} c_i L[u_i]$$

Associative:-

$$L+M = M+L$$

$$LM = ML \leftarrow \text{(not hold)}$$

Example

$$L = \frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y}$$

$$M = \frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial x}$$

$$(L+M)+N = L+(M+N)$$

also holds

$$L(MN) = (LM)N$$

$$L(M+N) = LM + LN$$

Result: If u_1 & u_2 are solutions of a linear inhomogeneous Eq. then

$u_1 - u_2$ is a solution of corresponding homogeneous solution.

Boundary & Initial conditions.

Conditions are given on discrete points in ODE.

But in PDE on Curves.

DNR

Drichlet - Neumann - Robon's.

Solution of a BVP.

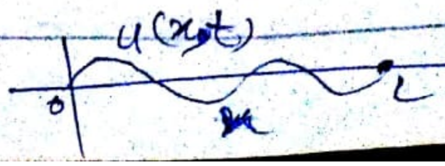
By a solution to a BVP on an open region D we mean a function u that satisfies the Differential Eq. on D. and is continuous on $D \cup \partial D$ ←Boundary. and satisfies the boundary condition on ∂D.

The B.C's are linear if they aspect a linear relationship b/w u & its partial derivative on ∂D.

For second order PDE linear B.C's can take one of the following three forms.

(i):- The B.C's specify the values of unknown function u on the boundary this type of the B.C is called Dirichlet B.C's.

Example: If $u(x,t)$ is the displacement of a vibrating spring and its ends are fixed at $x=0$ & $x=L$



$u(0,t)=0$

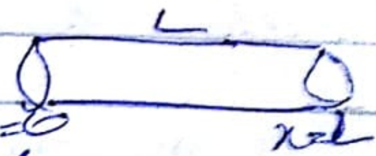
$u(L,t)=0$

(2) The B.C's specify the value of the derivative of u in the direction normal to boundary written as.

$$\frac{\partial u}{\partial n}$$

This type is called Neumann conditions.

Example: Suppose $u(x,t)$ is the temperature in a rod of length 'L' if the rod is perfectly insulated at $x=0$ and $x=L$ the heat flux at these points is zero.



$$\frac{\partial u(0,t)}{\partial x}=0 \quad \& \quad \frac{\partial u(L,t)}{\partial x}=0$$

(3) The B.Cs specify a linear relationship b/w u & its normal derivative on the

boundary these are reffered as mixed B.C's or robin B.C's.

General form

$$\left[\alpha u + \beta \frac{\partial u}{\partial n}\right]_{\partial D} = f(x)\Big|_{\partial D}$$

α, β are constants.

Example : Suppose in Example 2 poor insulated is used at the ends of the rod then B.C's take the form

$$u(0,t) + \frac{\partial u(0,t)}{\partial x} = 0$$

$$u(L,t) + \frac{\partial u(L,t)}{\partial x} = 0$$

Note:- Similar to superposition principle for the solution to linear PDE's, we have the superposition principle for Linear B.C's.

If u_1 & u_2 are solution of Linear homogeneous PDE with the linear B.C's

$$\left[\alpha u_1(\bar{x}) + \beta \frac{\partial u_1(\bar{x})}{\partial \bar{n}}\right]\Bigg|_{\partial D} = f(\bar{x})\Big|_{\partial D}$$

$$\left[\alpha u_2(\bar{x}) + \beta \frac{\partial u_2(\bar{x})}{\partial \bar{n}}\right]\Bigg|_{\partial D} = g(\bar{x})\Big|_{\partial D}$$

where α, β are constants.

Then

$w = u_1 + u_2$ is a solution of PDE that satisfy

$$\left[\alpha w(\bar{x}) + \beta \frac{\partial w(\bar{x})}{\partial \bar{n}}\right]\Bigg|_{\partial D} = \left[f(\bar{x}) + g(\bar{x})\right]_{\partial D}$$

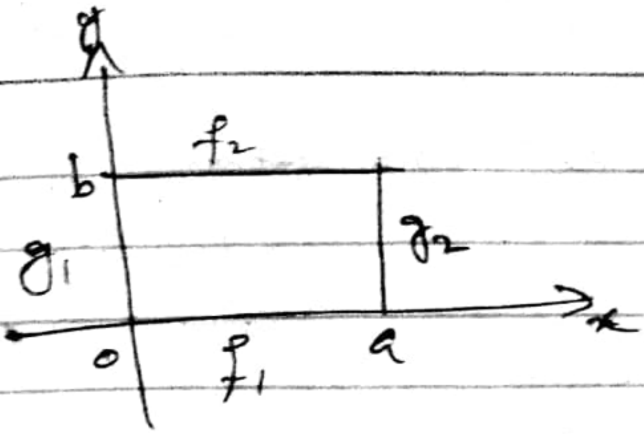
Expl Consider

$\nabla^2 u = 0$ in rectangle

with the following linear B.C.

$$\left.\begin{aligned} u(x, 0) &= f_1(x) \\ u(x, b) &= f_2(x) \end{aligned}\right\} \longrightarrow (1)$$

$$\left.\begin{aligned} u(0, y) &= g_1(y) \\ u(a, y) &= g_2(y) \end{aligned}\right\} \longrightarrow (2)$$



We strick the problem into two parts along x-axis & along y-axis.

x-axis ⇒

$$\nabla^2 u_1 = 0$$

$$u_1(x,0) = f_1(x)$$

$$u_1(x,b) = f_2(x)$$

$$u_1(0,y) = 0$$

$$u_1(a,y) = 0$$

y-axis.

$$\nabla^2 u_2 = 0$$

$$u_2(x,0) = 0$$

$$u_2(x,b) = 0$$

$$u_2(0,y) = g_1(y)$$

$$u_2(a,y) = g_2(y)$$

Obviously if we solve u_1 & u_2 then $u_1 + u_2$ is a solution of Laplace Eq. which satisfies ① & ②.

Note: Neumann B.C's usually donot specify a unique solution of BVP.

Because
⇒ derivative of constant is zero.

An Eq. written in the form

$$f(x,y,z,p,q)=0 \longrightarrow ①$$

is called first order PDE.
where
$$p=\frac{\partial z}{\partial x},\ q=\frac{\partial z}{\partial y}$$

x, y are independent.
z is dependent variable.

$$x^2+y^2+(z-c)^2=a^2 \longrightarrow ②$$

with a & c are arbitrary constants.

$$2x+2(z-c)p=0$$

$$2y+2(z-c)q=0$$

$\Rightarrow$ $x+(z-c)p=0 \quad \Rightarrow z-c=\frac{-x}{p}$

$y+(z-c)q=0 \quad \Rightarrow z-c=-y/p$

Now compare these Eq's.

$$\Rightarrow \frac{x}{p} = y/q$$

$$yp - xq = 0 \longrightarrow (3)$$

that is set of all spheres with centres on the z-axis is characterized by Eq. (3).

Example.

$$x^2 + y^2 = (z-c)^2 \tan^2\alpha$$

circular cone.

$$\Rightarrow 2x = 2(z-c)\tan^2\alpha\, p$$

$$2y = 2(z-c)\tan^2\alpha\, q$$

$$\Rightarrow x = (z-c)\tan^2\alpha\, p$$

$$y = (z-c)\tan^2\alpha\, q$$

$$\Rightarrow \frac{x}{p} = y/q$$

$$\Rightarrow yp - xq = 0 \longrightarrow (3)$$

Note: It means that more then one geometrical entities can be discribed by the same Eq.

All surfaces of revolution with this property (the spheres & The cone ... what have in common that they are surfaces of revolution which have line oz as axis of symmetry) are characterized by Eq. of the form

(i) $z = f(x^2+y^2)$ ⇒ Same as Ⓑ

(ii) $z = f(x+it) + g(x-it)$
⇒ Laplace Eq.

Sol: (i). $z = f(x^2+y^2) \longrightarrow$ Ⓐ

Diff Ⓐ w.r.t. x

$$\frac{\partial z}{\partial x} = f'(x^2+y^2) \cdot 2x$$

$$\Rightarrow \frac{p}{x} = 2f'(x^2+y^2) \longrightarrow (i)$$

where $p = \frac{\partial z}{\partial x}$

Diff Ⓐ w.r.t y.

$$\frac{\partial z}{\partial y} = f'(x^2+y^2) \cdot 2y$$

$$\Rightarrow \frac{q}{y} = 2f'(x^2+y^2) \longrightarrow (ii)$$

comparing (i) & (ii).

$$\frac{q}{y} = \frac{p}{x}$$

$$\Rightarrow \boxed{py - qx = 0}$$

Solution (ii) $\quad z = f(x+it) + g(x-it) \rightarrow Ⓑ$

Two times Diff. (B) w.r.t x.

$$\frac{\partial z}{\partial x} = f'(x+it) + g'(x-it)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+it) + g''(x-it) \longrightarrow (i)$$

Two times Diff. (B) w.r.t "t".

$$\frac{\partial z}{\partial t} = f'(x+it) \cdot i + g'(x-it)(-i)$$

$$\frac{\partial^2 z}{\partial t^2} = f''(x+it)\, i^2 + g''(x-it)(i^2)$$

$$\frac{\partial^2 z}{\partial t^2} = -f''(x+it) - g''(x-it) \rightarrow (ii)$$

By adding (ii) & (i).

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$$

Sankara Rao

Linear PDE. 08-02-2016

Any Eq. of the form

$$Pp + Qq = R \longrightarrow (1)$$

where P, Q & R are given functions of x, y & z which don't envolve p, q. and derivatives. (1) is known as first order PDEs. also Lagrange Eq.

Result:- The General solution of linear PDE (1) is $F(u, v) = 0$. where F is an arbitrary function & $u(x,y,z) = c_1$ & $v(x,y,z) = c_2$ forms a solution of the auxillary Lagrange Eq.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

General PDE

$$Au_x + Bu_y + Cu = D$$

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

Example:-

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x+y)z$$

By comparing:- with $Pp+Qq=R$

$$P = x^2, \; Q = y^2, \; R = (x+y)z$$

aux. Eq.

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} \longrightarrow (1)$$

Now

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

$$\int x^{-2} dx = \int y^{-2} dy$$

$$-x^{-1} = -y^{-1} + c_1$$

$$\Rightarrow \frac{1}{y} - \frac{1}{x} = c_1$$

Let $u = y^{-1} - x^{-1}$

$$\frac{dx - dy}{x^2 - y^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$$

$$\frac{dx - dy}{\cancel{(x+y)}(x-y)} = \frac{dz}{\cancel{(x+y)}z}$$

not satisfy

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{a \pm c}{b \pm d} = \frac{c}{d}$$

$$\frac{d(x-y)}{x-y} = \frac{dz}{z}$$

Integrating.

$$\ln(x-y) = \ln z + \ln A$$

$$x - y = zA$$

$$\frac{x-y}{z} = A = V$$

$$F(u, V) = 0$$

If $u_i(x_1, x_2, \ldots x_n, z) = c_i$

$i = 1, 2, \ldots, n.$

If u_i are independent solutions of

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \cdots = \frac{dx_n}{P_n} = \frac{dz}{R}$$

then the relation

$$\phi(u_1, u_2, \ldots, u_n) = 0,$$

In which ϕ is arbitrary.

is a general solution of linear PDE.

$$P_1 \frac{\partial z}{\partial x_1} + P_2 \frac{\partial z}{\partial x_2} + \cdots + P_n \frac{\partial z}{\partial x_n} = R$$

If $u = u(x, y, z)$ which satisfy

$$(y-z)\frac{\partial u}{\partial x} + (z-x)\frac{\partial u}{\partial y} + (x-y)\frac{\partial u}{\partial z} = 0$$

Show that u contains x, y and z only in the combination $x+y+z$ and $x^2+y^2+z^2$

$P_1 = y-z$, $P_2 = z-x$, $P_3 = x-y$, $R = 0$

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{du}{0}$$

By adding. $\frac{dx+dy+dz}{\cancel{y}-\cancel{z}+\cancel{z}-\cancel{x}+\cancel{x}-\cancel{y}} = \frac{dx+dy+dz}{0} = \frac{dz}{x-y}$

$\Rightarrow x+y+z = c_1$

$$\frac{x\,dx + y\,dy + z\,dz}{x(y-z)+y(z-x)+z(x-y)} = \frac{dy}{z-x}$$

$$\frac{x\,dx+y\,dy+z\,dz}{\cancel{xy}-\cancel{xz}+\cancel{yz}-\cancel{yx}+\cancel{xz}-\cancel{yz}} = \frac{dy}{z-x}$$

$$\Rightarrow x^2+y^2+z^2 = c_2$$

Now $\dfrac{dz}{x-y} = \dfrac{du}{0}$

$$0 = du$$

$$\Rightarrow u = c_3$$

$$\phi(x+y+z,\ x^2+y^2+z^2,\ u) = 0$$

$$\Rightarrow u = f(x+y+z,\ x^2+y^2+z^2)$$

Integral surfaces passing through the given curves

$$u(x(t), y(t), z(t)) = 0,\quad V(x(t), y(t), z(t)) = 0$$

Find the integral surface of the linear PDE

$$x(y^2+z)p - y(z+x^2)q = (x^2-y^2)z \longrightarrow (*)$$

containing the straight line

$$x+y=0 \; , \; z=1$$

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(z+x^2)} = \frac{dz}{(x^2-y^2)z}$$

$$\frac{x\,dx + y\,dy}{\cancel{x^2y^2} + x^2z - y^2z \cancel{- y^2x^2}} = \frac{dz}{x^2z - y^2z}$$

$$x\,dx + y\,dy = dz$$

$$\frac{x^2+y^2}{2} = z + c_1$$

$x+y=0$

$x=-y$

$y=t$

$x=-t$

$z=1$

$$\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{\cancel{y^2}\cancel{+y^2}\cancel{-z}\cancel{-x^2}\cancel{+x^2}\cancel{-y^2}} = \frac{dz}{(x^2-y^2)z}$$

$$\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$$

$$\ln x+\ln y+\ln z=\ln c_1$$

$$\ln(xyz)=\ln c_1$$

$$\boxed{xyz=c_1}$$

$$\frac{x^2}{2}+\frac{y^2}{2}-z=c_2$$

$$x+y=0$$

$$\Rightarrow x=-y \qquad \text{let}, \; y=t$$

$$x=-t$$

$$z=1$$

$$(-t)(t)(1) = c_1$$

$$-t^2 = c_1 \longrightarrow (1)$$

$$\frac{t^2}{2} + \frac{t^2}{2} - 1 = c_2$$

$$t^2 - 1 = c_2$$

$$-c_1 - 1 = c_2$$

$$c_1 + c_2 + 1 = 0$$

$$xyz + \frac{x^2}{2} + \frac{y^2}{2} - z + 1 = 0 \longrightarrow (A)$$

Put $\boxed{x = -y}$ $\boxed{z = 1}$

$$(-y)(y)(1) + \frac{y^2}{2} + \frac{y^2}{2} \cancel{-1} \cancel{+1} = 0$$

$$-y^2 + y^2 = 0$$

$$0 = 0$$

Diff (A) w.r.t x.

$$xyp + yz + x - p = 0$$

$$p(xy - 1) = -yz - x$$

$$p = \frac{yz + x}{1 - xy}$$

Diff (A) w.r.t y.

$$xyq + xz + y - q = 0$$

$$q(xy-1) = -y - xz$$

$$q = \frac{(xz+y)}{(1-xy)}$$

Put p & q in (A).

$$x(y^2+z)\frac{yz+x}{1-xy} - y(z+x^2)\frac{xz+y}{1-xy} = (x^2-y^2)z$$

$$(y^2+z)(xyz+x^2) - (z+x^2)(xyz+y^2) = (x^2-y^2)z(1-xy)$$

$$y^2(xyz+\cancel{x^2}) + \cancel{xyz^2} + x^2z - \cancel{xyz^2} - zy^2 - x^2(xyz+\cancel{y^2}) = (x^2-y^2)z(1-xy)$$

$$xy^3z \quad \cancel{+xyz^2} + (x^2-y^2)z - x^3yz = (x^2-y^2)z(1-xy)$$

$$xyz(y^2-x^2) + (x^2-y^2)z = \text{''}$$

$$(y^2-x^2)(xy-1)z = (y^2-x^2)z(xy-1)$$

The General 2^{nd} order Linear PDE in one independent variable $u = u(x,y)$ can be written in the form:-

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$$

where A, B, C are functions of x, y independent variables & don't vanish simultaneously.

D, E & F are also function of x & y.

Classifications

(i):- If Discreminant

$B^2 - AC < 0$ elliptic

(ii) $B^2 - AC = 0$ Parabola.

(iii) $B^2 - AC > 0$ Hyperbola.

$$u_{xx} + u_{yy} = 0$$

$A = 1$, $B = 0$, $C = 1$

$B^2 - AC = -1 < 0$ Elliptic

$$u_{xx} = \alpha^2 u_t$$

$A=1,\ B=0,\ C=0$

$B^2 - AC = 0 \Rightarrow$ parabolic

correct

$$u_{xx} = \alpha^2 u_{yy}$$

$$u_{xx} - \alpha^2 u_{yy} = 0$$

$A=1,\ B=0,\ C=-\alpha^2$

$B^2 - AC = +\alpha^2 > 0$ Hyperbolic.

$$u_{xx} = \frac{1}{c^2} u_{tt}$$

$$u_{xx} - \frac{1}{c^2} u_{tt} = 0$$

$A=1,\ B=0,\ C=-\frac{1}{c^2}$

$B^2 - AC = \frac{1}{c^2} > 0$ Hyperbolic.

Reduction of 2^{nd} order PDE to the canonical form

Consider

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + H(x, y, u, u_x, u_y) = 0 \quad \rightarrow (1)$$

where A, B, C are functions of x, y and u have contineous first and 2^{nd} order partial derivatives let us transform

$\xi = \xi(x,y)$

$\eta = \eta(x,y)$

$$J(\xi,\eta) = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$$

in the region under consideration.

(ξ,η)

$$\bar{A}u_{\xi\xi} + 2\bar{B}u_{\xi\eta} + \bar{C}u_{\eta\eta} + \bar{H}(\xi,\eta,u,u_\xi,u_\eta) = 0 \longrightarrow (2)$$

where $\bar{A} = A(\xi_x)^2 + 2B\xi_{xy} + C(\xi_y)^2$

$\bar{C} = A(\eta_x)^2 + 2B\eta_{xy} + C(\eta_y)^2$

$\bar{B} = A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x)$
$+ C\xi_y\eta_y$

The resulting Eq. 2 is in the same form to the orignal Eq. 1.

The nature of the Eq remains invariant under such a transformation if the Jacobian does not vanish.

$$\boxed{\bar{B}^2 - \bar{A}\bar{C} = J^2 (B^2 - AC)}$$

Consider

$A u_{xx} + 2B u_{xy} + C u_{yy} + H(x, y, u, u_x, u_y) = 0$

We suppose first that none of A.B.C is zero.

Let (ξ, η) be new variables such that

$$\bar{A} = 0$$

$$\& \quad \bar{C} = 0$$

then

$$\left.\begin{aligned} & A(\xi_x)^2 + 2B\,\xi_x \xi_y + C(\xi_y)^2 = 0 \\ \& \quad & A(\eta_x)^2 + 2B\,\eta_x \eta_y + C(\eta_y)^2 = 0 \end{aligned}\right\} \rightarrow 3$$

These two Eq. are the same type. Therefore we can write

$$A(\phi_x)^2 + 2B\,\phi_{xy} + C(\phi_y)^2 = 0$$

$\div A$

$$(\phi_x)^2 + 2\frac{B}{A}\,\phi_{xy} + \frac{C}{A}(\phi_y)^2 = 0 \longrightarrow **$$

which is quadratic in (ϕ_x/ϕ_y).

$$\left(\frac{\phi_x}{\phi_y}\right)^2 + \frac{2B}{A}\left(\frac{\phi_x}{\phi_y}\right) + \frac{C}{A} = 0$$

$$\frac{\phi_x}{\phi_y} = \frac{-\frac{2B}{A} \pm \sqrt{4\frac{B^2}{A^2} - 4\frac{C}{A}}}{2} = \frac{1}{A}\left[-B \pm \sqrt{B^2-AC}\right]$$

$$A\phi_x = \left[B \pm \sqrt{B^2-AC}\right]\phi_y$$

$$A\phi_x + \left[B \pm \sqrt{B^2-AC}\right]\phi_y = 0$$

$$\left[A\phi_x + \left(B + \sqrt{B^2-AC}\right)\phi_y\right]\left[A\phi_x + \left(B - \sqrt{B^2-AC}\right)\phi_y\right] = 0$$

$$\Rightarrow \begin{cases} A\phi_x + \left(B + \sqrt{B^2-AC}\right)\phi_y = 0 \longrightarrow (4) \\ A\phi_x + \left(B - \sqrt{B^2-AC}\right)\phi_y = 0 \longrightarrow (5) \end{cases}$$

Characteristics Eq.s / Lagrange Eq.

Note that the solution of (4) & (5) is the sol. of (**).

To solve (4) & (5) the corresponding auxillary Eq-s will be

$$\frac{dx}{A} = \frac{dy}{B+\sqrt{B^2-AC}}$$

&

$$\frac{dx}{A} = \frac{dy}{B-\sqrt{B^2-AC}}$$

Solutions of (4) & (5) called the characteristics family of curves in xy-plane along which $\xi =$ constant & $\eta =$ constant.

$$\left.\begin{aligned} \phi_1(x,y) &= C_1 \\ \phi_2(x,y) &= C_2 \end{aligned}\right\} \text{characteristics curves.}$$

Hence the transformation will be

$$\xi = \xi(x,y) = \phi_1(x,y)$$

$$\eta = \eta(x,y) = \phi_2(x,y)$$

Example:-

$$y^2 u_{xx} - x^2 u_{yy} = 0 \longrightarrow (1)$$

$$A = y^2, \quad B = 0, \quad C = -x^2$$

$$B^2 - AC = x^2 y^2 > 0 \quad \text{Hyperbolic.}$$

$$x \neq 0 \neq y$$

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

$$\frac{dy}{dx} = \frac{x}{y}, \quad \frac{dy}{dx} = -\frac{x}{y}$$

$$y\,dy = x\,dx \qquad \Big| \qquad \frac{y^2}{2} + \frac{x^2}{2} = c_2$$

$$\frac{y^2}{2} - \frac{x^2}{2} = c_1$$

Let

$$\xi = \frac{y^2}{2} - \frac{x^2}{2}$$

$$\eta = \frac{y^2}{2} + \frac{x^2}{2}$$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$= -x u_\xi + x u_\eta$$

$$u_{xx} = -u_\xi - x[u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x]$$

$$+1\cdot u_\eta + x\{u_{\eta\eta}\eta_x + u_{\eta\xi}\xi_x\}$$

$$= -u_\xi - x[u_{\xi\xi}(-x) + x\, u_{\xi\eta}]$$

$$+ u_\eta + x\{u_{\eta\eta}(x) + (-x)\, u_{\eta\xi}\}$$

$$= u_\eta - u_\xi + x^2[u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}]$$

Answer will be $\Rightarrow$ $\boxed{u_{\xi\eta} = \frac{\eta}{2(\xi^2-\eta^2)} u_\xi - \frac{\xi}{2(\xi^2-\eta^2)} u_\eta}$

$$u_y = u_\xi \xi_y + u_\eta \eta_y = y\, u_\xi + y\, u_\eta$$

$$u_{yy} = u_\xi + u_\eta + y^2[u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}]$$

Put these values in (1)

$$y^2[u_\eta - u_\xi + x^2[u_{\xi\xi} - 2u_{\eta\xi} + u_{\eta\eta}]] - x^2[u_\xi + u_\eta + y^2[u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}]] = 0$$

$$\Rightarrow y^2u_\eta - y^2u_\xi + x^2y^2(4u_{\xi\eta}) - x^2u_\xi - x^2u_\eta = 0$$

$$\Rightarrow u_{\xi\eta} = \frac{(y^2-x^2)\,u_\eta}{4x^2y^2} - \frac{(y^2+x^2)\,u_\xi}{4x^2y^2}$$

$2\xi = y^2 - x^2$ & $2\eta = y^2 + x^2$

$4\xi^2 = y^4 + x^4 - 2x^2y^2$; $4\eta^2 = y^4 + x^4 + 2x^2y^2$

$$\Rightarrow 4(\eta^2 - \xi^2) = 4x^2y^2$$

$$\Rightarrow u_{\xi\eta} = \frac{2\xi\,u_\eta}{4(\eta^2-\xi^2)} - \frac{2\eta}{4(\eta^2-\xi^2)}u_\xi$$

$$u_{\xi\eta} = \frac{\xi}{2(\eta^2-\xi^2)}u_\eta - \frac{\eta}{2(\eta^2-\xi^2)}u_\xi$$

15-02-2016

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

For parabola

$$\sqrt{B^2 - AC} = 0$$

Example:-

$$\frac{\partial^2 u}{\partial x^2} - 2x\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial u}{\partial y} = 0$$

$A = 1$, $B = -x$, $C = x^2$

$$B^2 - AC = x^2 - x^2 = 0$$

$\Rightarrow$ Parabolic Eq.

Then

$$\frac{dy}{dx} = \frac{-x}{1} = -x$$

$$y = -\frac{x^2}{2} + c$$

$$y + \frac{x^2}{2} = c = \xi(x,y)$$

Let $\eta = x$ (condition: Jacobian should not be zero.)

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} x & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$= x u_\xi + u_\eta$$

$$u_{xx} = u_\xi + x[u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x]$$

$$+ u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x$$

$$= u_\xi + x(u_{\xi\xi}(x) + u_{\xi\eta}(1))$$

$$+ u_{\eta\xi}(x) + u_{\eta\eta}(1)$$

$$= u_\xi + x^2 u_{\xi\xi} + 2x u_{\xi\eta} + u_{\eta\eta}$$

$$\frac{\partial u}{\partial y} = u_\xi \xi_y + u_\eta \eta_y$$

$$= u_\xi$$

$$\frac{\partial^2 u}{\partial x \partial y} = u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x$$

$$= x u_{\xi\xi} + u_{\xi\eta}$$

$$u_{yy} = u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y$$

$$\boxed{u_{yy} = u_{\xi\xi}}$$

$$x^2 \cancel{u_{\xi\xi}} + \cancel{2xu_{\xi\eta}} + u_{\eta\eta} + u_{\xi}$$
$$-\cancel{2x^2u_{\xi\xi}} - \cancel{2xu_{\xi\eta}} + \cancel{x^2u_{\xi\xi}} - 2u_{\xi} = 0$$

$$\boxed{u_{\eta\eta} - u_{\xi} = 0}$$

Example:-

$$x^2u_{xx} + 2u_{xy}(xy) + y^2u_{yy} = 0 \rightarrow (1)$$

x result will be $\boxed{\cancel{x}u_{\eta\eta} = 0}$

$A = x^2$, $B = xy$, $C = y^2$

$B^2 - AC = x^2y^2 - x^2y^2 = 0$ (Parabolic PDE)

$$\frac{dy}{dx} = \frac{+xy}{x^2} = +y/x$$

$$\frac{1}{y}\,dy = +\frac{1}{x}\,dx$$

$$\Rightarrow \ln y = +\ln x + \ln c$$

$$y/x = c$$

$\Rightarrow \xi = y/x$ let $\eta = x$

$$u_x = -\frac{y}{x^2}u_{\xi} + u_{\eta}$$

$u_{xx} = y[u_{\xi\xi}\, y + u_{\xi\eta}] + u_{\eta\xi}\, y + u_{\eta\eta}$

$\boxed{u_{xx} = y^2 u_{\xi\xi} + 2y\, u_{\xi\eta} + u_{\eta\eta}}$

$u_{xy} = u_\xi + y[u_{\xi\xi}\, x + u_{\xi\eta}(0)] + u_{\eta\xi}\, x + 0$

$\boxed{u_{xy} = u_\xi + xy\, u_{\xi\xi} + x\, u_{\xi\eta}}$

$u_y = x\, u_\xi$

$\boxed{u_{yy} = x^2\, u_{\xi\xi}}$

$x^2\{y^2 u_{\xi\xi} + 2y\, u_{\xi\eta} + u_{\eta\eta}\} + 2xy[u_\xi + xy\, u_{\xi\xi} + x\, u_{\eta\xi}]$
$+ y^2 x^2 u_{\xi\xi} = 0$

$4x^2y^2 u_{\xi\xi} + 4y(x^2) u_{\eta\xi} + x^2 u_{\eta\eta} + 2xy\, u_\xi = 0$

$4\xi^2 u_{\xi\xi} + 4\xi\eta\, u_{\xi\eta} + \eta^2 u_{\eta\eta} + 2\xi\eta\, u_\xi = 0$

$\boxed{u_{xx} = \frac{2y}{x^3} u_\xi + \frac{y^2}{x^4} u_{\xi\xi} - \frac{2y}{x^2} u_{\xi\eta}}$

$\boxed{u_{xy} = -\frac{1}{x^2} u_\xi - \frac{y}{x^3} u_{\xi\xi} + \frac{1}{x} u_{\eta\xi}}$

$u_y = \frac{1}{x} u_\xi$

$u_{yy} = \frac{1}{x^2} u_{\xi\xi}$

Put values in (1).

$2\frac{y}{x}\cancel{u_\xi} + \cancel{\frac{y^2}{x^2} u_{\xi\xi}} - \cancel{2y\, u_{\xi\eta}} + x^2 u_{\eta\eta} - \cancel{2\frac{y}{x} u_\xi}$
$-2\frac{y^2}{x^2}\cancel{u_{\xi\xi}} + \cancel{2y\, u_{\eta\xi}} + \frac{y^2}{x^2}\cancel{u_{\xi\eta}} = 0$

$\Rightarrow x^2 u_{\eta\eta} = 0$

$\Rightarrow u_{\eta\eta} = 0 \qquad \because x^2 \neq 0$

$\Rightarrow u_\eta = f(\xi)$

$u = \int f(\xi)\, d\eta$

$= f(\xi)\cdot\eta + g(\xi)$

If $B^2 - AC < 0$

$\Rightarrow \phi_1 \pm i\phi_2$

Then take $\phi_1 = \xi, \quad \phi_2 = \eta.$

Example:-

$u_{xx} + x^2 u_{yy} = 0$

$A = 1 \; , B = 0 \; , C = x^2$

$B^2 - AC = 0 - x^2 < 0$

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

$$\Rightarrow \frac{dy}{dx} = \pm i x$$

Then $y = \pm i \frac{x^2}{2} + c$

$y \pm i \frac{x^2}{2} = c$

$u = \phi_1 \pm i\phi_2$

$\phi_1 = \xi = y \ , \ \phi_2 = \eta = \frac{x^2}{2}$

$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ x & 0 \end{vmatrix} = -x \neq 0$

$u_x = u_\xi \, \xi_x + u_\eta \, \eta_x$

$= x u_\eta$

$u_{xx} = u_\eta + x[u_{\eta\xi} \cancel{\xi_x} + u_{\eta\eta}\eta_x]$

$= u_\eta + x^2 u_{\eta\eta}$

$u_y = u_\xi \xi_y + u_\eta \cancel{\eta_y}^{0} = u_\xi$

$u_{yy} = u_{\xi\xi} \xi_y + u_{\xi\eta} \cancel{\eta_y}^{0}$

$= u_{\xi\xi}$

$u_{xx} + x^2 u_{yy} = 0$

$x^2 u_{\eta\eta} + u_\eta + x^2 u_{\xi\xi} = 0$

$\because \frac{x^2}{2} = \eta$

$\boxed{2\eta u_{\eta\eta} + u_\eta + 2\eta u_{\xi\xi} = 0}$

Example:- Reduced into canonical form & find the General solution of wave Eq. with the Cauchy data.

$$u_{xx} = \frac{1}{c^2} u_{tt}$$

Cauchy data ⇒ $u(x,0) = f(x)$

$u_t(x,0) = g(x)$

conditions on curves not on a point.

Solution:-

$$u_{xx} - \frac{1}{c^2} u_{tt} = 0 \longrightarrow (A)$$

$A = 1,\ B = 0,\ C = -\frac{1}{c^2}$

$B^2 - AC = 0 + \frac{1}{c^2} > 0$

$$\frac{dt}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

$$\frac{dt}{dx} = \pm\sqrt{\frac{1}{c^2}} = \pm\frac{1}{c}$$

$\pm c\,dt = dx$

$x = \pm ct + c_1$

$x \pm ct = c_1$

then

$\pm ct + x = c_1$

$\xi = x + ct$

$\eta = x - ct$

$u_x = u_\xi + u_\eta$

$u_{xx} = u_{\xi\xi} + 2u_{\eta\xi} + u_{\eta\eta}$

$u_t = cu_\xi - cu_\eta$

$u_{tt} = c^2 u_{\xi\xi} - 2c^2 u_{\eta\xi} + c^2 u_{\eta\eta}$

Ⓐ $\Rightarrow \cancel{u_{\xi\xi}} + 2u_{\eta\xi} + \cancel{u_{\eta\eta}} - \cancel{u_{\xi\xi}} + 2u_{\eta\xi} - \cancel{u_{\eta\eta}} = 0$

$4u_{\eta\xi} = 0$

Final Result is $u_{\xi\eta} = 0$

$$u_\xi = f(\xi)$$

$$u = \int f(\xi)\, d\xi + \psi(\eta) \quad (1)$$

$$u = \phi(\xi) + \psi(\eta) \quad (2)$$

$$= \phi(x+ct) + \psi(x-ct) \rightarrow (B)$$

This wave travel to left.

Using conditions.

$u(x,0) = f(x)$

$\Rightarrow f(x) = \phi(x) + \psi(x) \longrightarrow (1)$

&

$u_t = c\phi'(x+ct) - c\psi'(x-ct)$

using $u_t(x,0) = g(x)$

$g(x) = c\phi'(x) - c\psi'(x) \rightarrow (2)$

From ① Diff. w.r.t x

$$f'(x) = \phi'(x) + \psi'(x)$$

$$\phi'(x) = f'(x) - \psi'(x)$$

Put in ②

$$g(x) = c\,f'(x) - c\psi'(x) - c\psi'(x)$$

$$\boxed{\psi'(x) = \frac{f'(x)}{2} - \frac{g(x)}{2c}}$$

Now put $\psi'(x) = f'(x) - \phi'(x)$ in Eq. ②.

$$g(x) = c\phi'(x) - c\,f'(x) + c\phi'(x)$$

$$\boxed{\frac{g(x)}{2c} + \frac{f'(x)}{2} = \phi'(x)}$$

On integrating.

$$\phi(x) = \frac{1}{2c}\int_0^x g(z)\,dz + \frac{1}{2}f(x) + K_1$$

& $$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(z)\,dz + K_2$$

By adding.

$$\phi(x) + \psi(x) = f(x) + K_1 + K_2$$

$\because f(x) = \phi(x) + \psi(x)$

So,

$$\cancel{f(x)} = \cancel{f(x)} + K_1 + K_2$$

$$\Rightarrow K_1 = -K_2$$

Let $K_1 = K \Rightarrow K_2 = -K$

$$\Rightarrow \phi(x+ct) = \frac{1}{2} f(x+ct) + \frac{1}{2c} \int_0^{x+ct} g(z)dz + K$$

$$\& \ \psi(x-ct) = \frac{1}{2} f(x-ct) - \frac{1}{2c} \int_0^{x-ct} g(z)dz - K$$

Put these values in (B)

$$u(x,t) = \frac{1}{2} f(x+ct) + \frac{1}{2c} \int_0^{x+ct} g(z)dz + \cancel{K} + \frac{1}{2} f(x-ct) - \frac{1}{2c} \int_0^{x-ct} g(z)dz - \cancel{K}$$

$$u(x,t) = \frac{1}{2} \left[f(x+ct) + f(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(z)dz$$

which is known as D' Alembert solution of PDE.

22-02-2016

$$u(x,t) = \frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(z)\, dz$$

$$= \frac{1}{2} \sin(x+ct) + \frac{1}{2} \sin(x-ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} \cos z\, dz$$

Question $\Rightarrow$ $u_{tt} = c^2 u_{xx}$ $\quad u(x,0) = \sin x$, $u_t(x,0) = \cos x$

$$= \frac{1}{2}(2 \sin x \cos ct) + \frac{1}{2c} \left[\sin z\right]_{x-ct}^{x+ct}$$

$$= \sin x \cos ct + \frac{1}{2c}\left(\sin(x+ct) - \sin(x-ct)\right)$$

$$= \sin x \cos ct + \frac{1}{c} \cos x \sin ct$$

Example:-

$$u_x + \cos x\, u_y + u = xy \longrightarrow (1)$$

characteristics Eq. $\quad Pp + Qq = R$

$$\frac{dx}{1} = \frac{dy}{\cos x} = \frac{du}{xy - u}$$

$$\frac{dx}{1} = \frac{dy}{\cos x}$$

$$\int \cos x\, dx = \int dy$$

$$\sin x = y + c$$

$$\varphi = \sin x - y = c$$

Let $\eta = x$

$$\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} \cos x & -1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0$$

Transforms Eqs

$\xi = \sin x - y$

$\eta = x$

$u_x = u_\xi \xi_x + u_\eta \eta_x$

$\Rightarrow u_x = \cos x \, u_\xi + u_\eta$

$u_y = u_\xi \xi_y + u_\eta \eta_y$

$u_y = -u_\xi$

Put in Eq ①

$\cancel{\cos x \, u_\xi} + u_\eta - \cancel{\cos x \, u_\xi} + u = xy$

$\because y = \sin\eta - \xi; \quad \eta = x$

$$u_\eta + u = \eta(\sin\eta - \xi) \rightarrow (*)$$

$$I.F = e^{\int d\eta} = e^\eta$$

multiplying I.F with Eq.(*)

$$e^\eta u_\eta + e^\eta u = e^\eta \eta(\sin\eta - \xi)$$

$$\int \frac{\partial}{\partial\eta}(e^\eta u)\, d\eta = \int e^\eta \eta(\sin\eta - \xi)\, d\eta$$

$$e^\eta u = \underbrace{\int \eta e^\eta \sin\eta\, d\eta}_{I_1} - \xi \underbrace{\int e^\eta \eta\, d\eta}_{I_2} + f(\xi)$$

$$I_1 = \int \underbrace{e^\eta \eta}_{I} \underbrace{\sin\eta}_{II} d\eta$$

$$= -e^\eta \eta \cos\eta + \int (e^\eta + \eta e^\eta)\cos\eta\, d\eta$$

$$= -e^\eta \eta \cos\eta + \int \underbrace{e^\eta}_{II} \underbrace{\cos\eta}_{I} d\eta + \int \underbrace{e^\eta \eta}_{I} \underbrace{\cos\eta}_{II} d\eta$$

$$= -e^\eta \eta \cos\eta + e^\eta \cos\eta + \cancel{\int e^\eta \sin\eta\, d\eta} + \eta e^\eta \sin\eta - \int (\cancel{e^\eta} + \eta e^\eta)\sin\eta\, d\eta$$

$$I_1 = -e^\eta \eta \cos\eta + \eta e^\eta \sin\eta + e^\eta \cos\eta - I_1$$

$$I_1 = -\frac{1}{2} e^\eta \eta \cos\eta + \frac{1}{2}\eta e^\eta \sin\eta + \frac{1}{2} e^\eta \cos\eta$$

$$I_2 = \int \underbrace{e^\eta}_{II} \underbrace{\eta}_{I} d\eta = e^\eta \eta - \int e^\eta d\eta$$

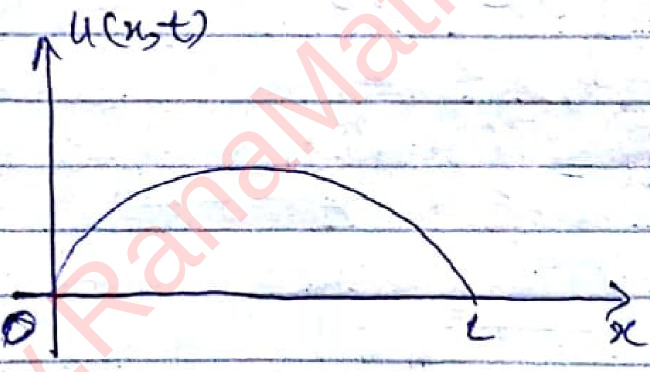
$$= e^\eta \eta - e^\eta$$

$$e^{\eta}u = -\frac{1}{2}e^{\eta}\eta\cos\eta + \frac{1}{2}\eta e^{\eta}\sin\eta + \frac{1}{2}e^{\eta}\cos\eta - \xi\eta e^{\eta} + \xi e^{\eta} + f(\xi)$$

$$u = -\xi(\eta - 1) + \frac{1}{2}\eta(\sin\eta - \cos\eta) + \frac{1}{2}\cos\eta + e^{-\eta}f(\xi)$$

Wind is not a wave.

Derivation of the Wave Eq. of the vibrating string.



Let us consider a strech string of length L fixed at both ends, we want to determine the Eq. of motion which characterises the position u(x,t) of the string in time t after initial disturbance. In order to obtain a simple Eq. we make the following assumptions.

1):- The string is flexible and elastic → (tension will be tengential)

2):- The tension in the string is uniform throughout it's length.

3): The weight of the string is small compare with the tension in the string.

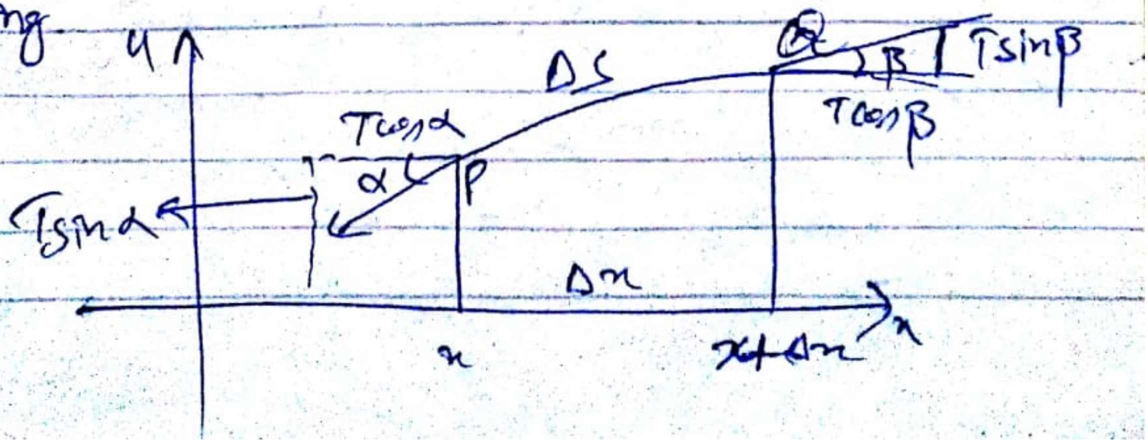
(4) The wave disterbance is small. i.e the displacement in the string is very small compare with length of the string.

(5) The slope of the displaced string at any point is small compare with unity. (α is very small).

(6) There is only (pure) transverse vibrations.

(7) The Gravity affect neglegible.

Consider a small portion of string.



Consider an element Δs in a displaced string.

The forces acting on the string are Tension T acting at point P & Q in the direction of tangent to the element Δs. The resulting force acting vertically on the mass $(m = \rho \Delta s)$ of element Δs.

Resulting force $= T\sin\beta - T\sin\alpha$

$= ma$

$= \rho \Delta s \, u_{tt}$

where ρ is the line density.

$$\Rightarrow \frac{T(\sin\beta - \sin\alpha)}{\Delta s} = \rho u_{tt}$$

If angles (α, β) are very small then

$\sin\alpha \approx \tan\alpha$

$\sin\beta \approx \tan\beta$

So,

$$\frac{T(\tan\beta - \tan\alpha)}{\Delta S} = \rho u_{tt}$$

$$\frac{T\left[u_x\Big|_{at\ x+\Delta x} - u_x\Big|_{at\ x}\right]}{\Delta S} = \rho u_{tt}$$

Since slope of string is small

then $\Delta S \approx \Delta x$

$$\frac{T\left[u_x\Big|_{at\ x+\Delta x} - u_x\Big|_{at\ x}\right]}{\Delta x} = \rho u_{tt}$$

when $\Delta x \to 0$

$$T u_{xx} = \rho u_{tt}$$

$$u_{tt} = \frac{T}{\rho} u_{xx} = c^2 u_{xx}$$

which is called one dimensional wave Eq.

Note. If there is any external force F per unit length acting on the string then new Eq. will be

$$u_{tt} = c^2 u_{xx} + \frac{F}{\rho}$$

24-02-2016

Drive the wave Eqn for vibrating membrain.

① The membrain is flexible & elastic.

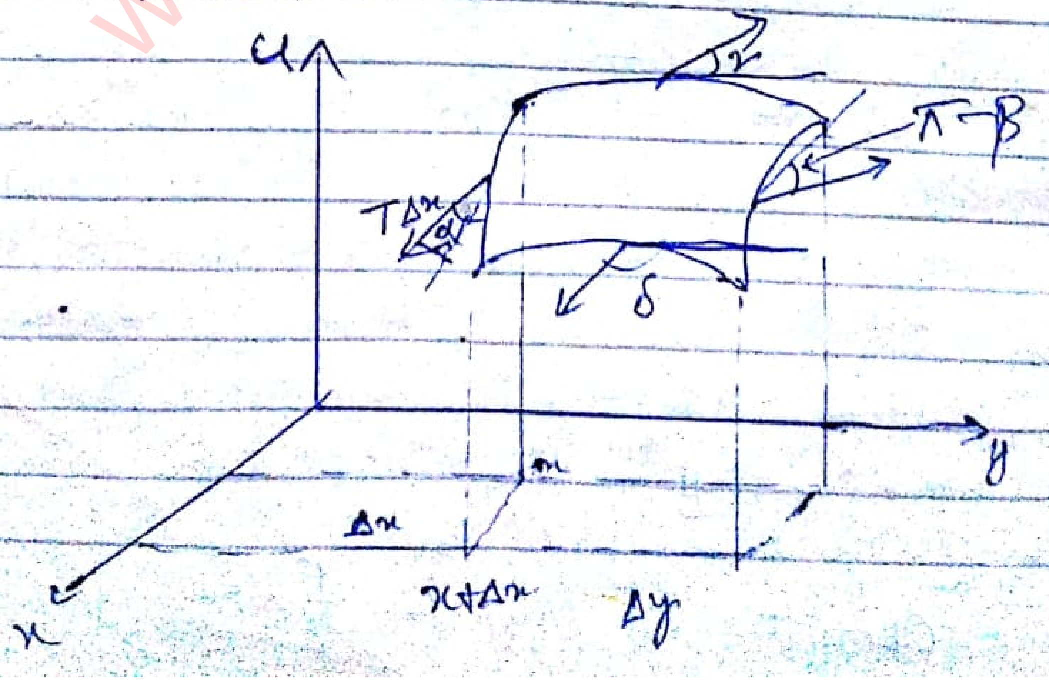
② The tension is constant throughout it's length.

③ The weight is small as compare to it's tension.

④ The deflection is small as compare to the

⑤ The slope of the displaced membrain at any point is small as compare with unity.

⑥. Only transverse vibrations.



$$\Delta A = \Delta x \cdot \Delta y$$

let T be the tension in the membrane per unit length. The forces acting on membrane vertically are

$$-T\Delta x \sin\alpha + T\Delta x \sin\beta - T\Delta y \sin\gamma$$
$$+ T\Delta y \sin\delta = \underbrace{\rho\Delta A}_{\text{mass}}\, u_{tt}$$

angles are very small.

so, $\sin\alpha \approx \tan\alpha$

$$+T\Delta x\left[\tan\beta - \tan\alpha\right]$$
$$+T\Delta y\left[\tan\delta - \tan\gamma\right] = \rho\Delta x\,\Delta y\, u_{tt}$$

$$\tan\alpha = u_y(x, y)$$
$$\tan\beta = u_y(x_1, y+\Delta y)$$
$$\tan\delta = u_x(x+\Delta x, y_1)$$
$$\tan\gamma = u_x(x, y_1)$$

$$\Rightarrow T\Delta x\left[u_y(x_1, y+\Delta y) - u_y(x_1, y)\right]$$

$$+ T\Delta y\left[u_x(x+\Delta x, y_1) - u_x(x, y_1)\right] = \rho\,\Delta x\,\Delta y\,u_{tt}$$

$$T\left[\frac{u_y(x_1, y+\Delta y) - u_y(x_1, y)}{\Delta y}\right]$$

$$+ T\left[\frac{u_x(x+\Delta x, y_1) - u_x(x, y_1)}{\Delta x}\right] = \rho\, u_{tt}$$

$$T\{u_{yy} + u_{xx}\} = \rho\, u_{tt}$$

$$u_{xx} + u_{yy} = \frac{1}{c^2} u_{tt}$$

$$\text{where } \frac{1}{c^2} = \frac{\rho}{T}$$

Eq. of Continuity $\Rightarrow c^2 = \frac{T}{\rho}$

Objective:-

Derive a model. Eq. for trafic flow on a highway without exits and with one interence and one leave.

Assumptions:-

(1). we asume that the highway is infinite.

2) We define the car density $\rho(x,t)$

$$\rho(x,t) = \frac{\text{no. of the cars on } [x, x+\Delta x]}{\Delta x}$$

where
Δx must be large compare to the car length.
3):- We assume that there are no accident on the highway. (If any then negligible).

④ They holds car conservation law.
The rate at which the no. of cars on the segment $[a,b]$ is changing equals the rate at which they are entering, the rate at which they are leaving.

Car flux is

$$q(x,t) = \rho(x,t)\, u(x,t)$$

where u is car speed at x.

The number N of cars in segment at any time 't'.

$$n(t,a,b) = \int_a^b f(x,t)\,dx$$

The rate of change in the quantity is

$$\frac{dn}{dt} = \int_a^b \frac{\partial f(x,t)}{\partial t}\,dx$$

This rate must be equal to

$$= q(a,t) - q(b,t)$$

$$= -\int_a^b \frac{\partial q(x,t)}{\partial x}\,dx$$

$$\Rightarrow \int_a^b \left[\frac{\partial f}{\partial t} + \frac{\partial q}{\partial x}\right] dx = 0$$

$$\int_a^b \left[\frac{\partial f}{\partial t} + \frac{\partial (fu)}{\partial x}\right] dx = 0$$

$$\Rightarrow \frac{\partial f}{\partial t} + \frac{\partial (fu)}{\partial x} = 0$$

$$\frac{\partial f}{\partial t} + \operatorname{div} \vec{J} = 0$$

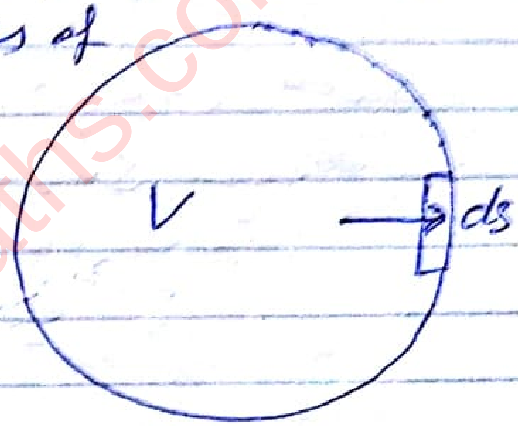
f is ~~charge~~ density

$\vec{J} \longrightarrow$ current density

Laplace & Pioson Eq.

Derive the Laplace & Piosson Eq. at .....

According to Gauss flux theorem in electrostatics "the Total flux of the electrostatic intensity $\vec{E}$ across the closed surface S is equal to four times of the total charge enclosed within the surface S.



$$\int_S \vec{E} \cdot \vec{ds} = 4\pi \int_V \rho \, dv$$

where ρ is the volume density of the charge, S is surface that bounds volume V.

$$\int_V \operatorname{div} \vec{E} \, dv = 4\pi \int_V \rho \, dv$$

$$\Rightarrow \int_V [\operatorname{div} \vec{E} - 4\pi \rho] \, dv = 0$$

$$\Rightarrow \operatorname{div} E = 4\pi \rho$$

which is Piosson Eq

and

$$\vec{E} = -\nabla\phi$$

$\nabla \rightarrow$ operator

where ϕ is potential.

then

$$-\nabla^2\phi = 4\pi\rho \rightarrow \text{Poisson Eqn}$$

when there is no charge (free surface)...

$$\nabla^2\Phi = 0$$

03-03-2016

Method of Separation of variables:-

$$AU_{xx}+2BU_{xy}+CU_{yy}+DU_x+EU_y + FU=0$$

$$U(x,y)= X(x)\, Y(y)$$

$$U_{xx}+U_{yy}+U_{zz}=0 \longrightarrow (1)$$

Let $U(x,y,z)= X(x)\,Y(y)\,Z(z)$

$$U_x = X' Y Z$$

$$U_{xx} = X'' Y Z$$

$$U_{yy} = X Y'' Z$$

$$U_{zz} = X Y Z''$$

Then

$$(1) \Rightarrow X'' Y Z + X Y'' Z + X Y Z'' = 0$$

$\div$ by XYZ assuming that $X\neq 0, Y\neq 0, Z\neq 0$

$$\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}=0$$

$$\frac{X''}{X}+\frac{Y''}{Y}=-\frac{Z''}{Z}$$

How it is possible R.H.S is independent of x & y but L.H.S is dependent only when R.H.S is constant.

So, let

Seperation constant

$$\frac{X''}{X} + \frac{Y''}{Y} = -\frac{Z''}{Z} = -\lambda^2$$

$$\Rightarrow \quad \frac{Z''}{Z} = \lambda^2$$

$$\& \quad \frac{X''}{X} + \frac{Y''}{Y} = -\lambda^2$$

$$Z'' - \lambda^2 Z = 0$$

$$D^2 - \lambda^2 = 0$$

$$D = \pm\lambda$$

$$Z = Ae^{-\lambda z} + Be^{+\lambda z} = \begin{cases} e^{-\lambda z} \\ e^{\lambda z} \end{cases}$$

Now

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda^2$$

$$\Rightarrow \quad \frac{X''}{X} = -\lambda^2 - \frac{Y''}{Y}$$

Now this is possible only

$$\frac{X''}{X} = -\lambda^2 - \frac{Y''}{Y} = -\mu^2$$

$$\Rightarrow \quad \frac{X''}{X} = -\mu^2$$

$\Rightarrow \quad D^2 = \pm i\mu$

$$X = \begin{cases} \sin \mu x \\ \cos \mu x \end{cases}$$

Now

$$-\frac{Y''}{Y} - \lambda^2 = -\mu^2$$

$$\frac{Y''}{Y} = \mu^2 - \lambda^2$$

case I $\mu^2 > \lambda^2$, case II $\mu^2 < \lambda^2$

$$D^2 = \mu^2 - \lambda^2$$

$$D = \pm\sqrt{\mu^2 - \lambda^2}$$

$$\Rightarrow \quad Y = \begin{cases} e^{\sqrt{\mu^2-\lambda^2}\, y} \\ e^{-\sqrt{\mu^2-\lambda^2}\, y} \end{cases}$$

Result will be after putting values of x, y & z in $u(x, y, z) = X(x)\, Y(y)\, Z(z)$

Example:-

$$x^2 u_{xx} + x u_x + 9 u_y = 0$$

like Cauchy Eq.

$$x^n \frac{d^n y}{dx^n} + x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + x y' + y = 0$$

Let $u = XY$

then

$$x^2 X'' Y + x X' Y + 9 X Y' = 0$$

$\div$ ing by XY

$$x^2 \frac{X''}{X} + x \frac{X'}{X} + 9 \frac{Y'}{Y} = 0$$

$$x^2 \frac{X''}{X} + x \frac{X'}{X} = -9 \frac{Y'}{Y} = -\mu^2$$

$$9 \frac{Y'}{Y} = \mu^2$$

$$\Rightarrow D = \frac{\mu^2}{9}$$

Solution will be

$$Y = e^{\frac{\mu^2}{9} y}$$

$$x^2\frac{X''}{X}+x\frac{X'}{X}=-\mu^2 \longrightarrow (A)$$

Put $x=e^t$

Now $X(x)=X(t)$

$$X'=\frac{dX}{dt}\frac{dt}{dx}$$

$$=\frac{dX}{dt}\cdot\frac{1}{e^t}=\frac{1}{x}\frac{dX}{dt}$$

$$xX'=\dot{X} \Rightarrow \frac{d}{dx}(xX')=\frac{d}{dt}(\dot{X})\cdot\frac{dt}{dx}$$

$$X'+xX''=\ddot{X}\cdot\frac{1}{x}$$

$$xX'+x^2X''=\ddot{X}$$

$$x^2X''=\ddot{X}-\dot{X}$$

$$\text{Eq. (A)} \Rightarrow \frac{\ddot{X}}{X}-\cancel{\frac{\dot{X}}{X}}+\cancel{\frac{\dot{X}}{X}}=-\mu^2$$

$$\frac{\ddot{X}}{X}=-\mu^2$$

$$D^2=-\mu^2$$

$$D=\pm i\mu$$

$$X=\begin{cases}\cos\mu t\\ \sin\mu t\end{cases} \Rightarrow X(x)=\begin{cases}\cos\mu(\ln x)\\ \sin\mu(\ln x)\end{cases}$$

"Orthogonal & orthonormal set of functions."

A finite or infinite seq or set of real valued function $u_n(x)$ $(n = 1, 2, \dots)$ Defined in $[a, b]$ is said to form an orthogonal set on $[a, b]$ if

inner product $\Rightarrow \displaystyle\int_a^b u_n(x)\, u_m(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases}$

where $L > 0$

If $L = 1$ then set of function $u_n(x)$ is said to be orthonormal set.

$$\int_a^b u_n(x)\, u_m(x)\, dx = \delta_{mn}$$

also

We can make an orthogonal set, an orthonormal set if we take

$$\frac{u_n(x)}{\sqrt{L}}$$

$$\Rightarrow \frac{1}{L}\int_a^b u_n(x)\, u_m(x)\, dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

Some Results.

① $\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$

same for $\cos mx \cos nx$.

② $\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0$

for any choice of m & n.

③ $\int_{0}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi}{L} x\,dx$

$= \begin{cases} 0 & \text{if } m \neq n \\ L/2 & \text{if } m = n \end{cases}$

Same for $\cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L}$

But $\int_{0}^{L} \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\,dx = 0$

Remark on Eigenvalues & Eigenfunc:

Suppose we are given a set of Eq. involving a parameter λ.

and we required to find the value of λ for which the given set of Eq. have a non-trivial solution. Such type of solutions are called Eigenvalues solution.

For which the eigenvalue the function is called Eigen function.

Spectrum:-

The set of all eigenvalues of a problem is called a spectrum.

Example:- Consider

$$5x + (1-\lambda)y = 0$$

$$(1-\lambda)x + 5y = 0$$

non-trivial solution

if $\text{Det}[A - \lambda I] = 0$

$AX = 0$

$A = \begin{bmatrix} 5 & 1-\lambda \\ 1-\lambda & 5 \end{bmatrix}$

$X = \begin{bmatrix} x \\ y \end{bmatrix}$

So,
$$\begin{vmatrix} 5 & 1-\lambda \\ 1-\lambda & 5 \end{vmatrix} = 0$$

$$25 - (1-\lambda)^2 = 0$$

$$(1-\lambda)^2 = (25)^2$$

$$1-\lambda = \pm 5$$

$1-\lambda = 5$ | $1-\lambda = -5$

$\boxed{-4 = \lambda}$ | $\boxed{6 = \lambda}$

These are Eigen values of the problem

$$\text{Spectrum} = \{-4, 6\}$$

when $\lambda = -4$

$$5x + 5y = 0$$
$$5x + 5y = 0$$
$$\Rightarrow \boxed{x = -y} \leftarrow \text{infinite solutions}$$

when $\lambda = 6$

$$\left.\begin{aligned} 5x - 5y &= 0 \\ -5x + 5y &= 0 \end{aligned}\right\} \Rightarrow \boxed{x = y}$$

Question:

$$U_{xx} = \frac{1}{c^2} U_{tt}$$

$$U_{xx} - \frac{1}{c^2} U_{tt} = 0$$

let

$$U(x,t) = X(x)\,T(t)$$

then

$$X''T - \frac{1}{c^2} X T'' = 0$$

$$\frac{X''}{X} - \frac{1}{c^2}\frac{T''}{T} = 0$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \alpha^2$$

$$\Rightarrow \frac{X''}{X} = \alpha^2$$

$$D^2 = \alpha^2$$

$$D = \pm \alpha$$

$$\Rightarrow X(x) = \begin{cases} e^{\alpha x} \\ e^{-\alpha x} \end{cases}$$

$$\frac{1}{c^2}\frac{T''}{T} = \alpha^2$$

$$D^2 = c^2\alpha^2$$

$$D = \pm c\alpha$$

$$T(t) = \begin{cases} e^{c\alpha t} \\ e^{-c\alpha t} \end{cases}$$

Put in $U(x,t)$.

Question:-

$$u_{xx}+u_{yy}+u_{xy}=0$$

Let $u(x,y)=X(x)\,Y(y)$

$u_x = X'Y$ $\quad u_{xy}=X'Y'$

Then

$$X''Y+XY''+X'Y'=0$$

Dividing by XY.

$$\frac{X''}{X}+\frac{Y''}{Y}+\frac{X'Y'}{XY}=0$$

Diff. w.r.t x.

$$\left(\frac{X''}{X}\right)'+0+\frac{Y'}{Y}\left(\frac{X'}{X}\right)'=0$$

$$\left(\frac{X''}{X}\right)'=-\frac{Y'}{Y}\left(\frac{X'}{X}\right)'$$

$$\Rightarrow \left(\frac{X''}{X}\right)'\Big/\left(\frac{X'}{X}\right)'=-\frac{Y'}{Y}=-\lambda^2$$

$$\Rightarrow \frac{\left(\frac{X''}{X}\right)'}{(X'/X)'}=-\lambda^2, \quad -\frac{Y'}{Y}=-\lambda^2$$

Solution will be.

$$Y' = \lambda^2 Y \Rightarrow \boxed{Y(y) = c_1 e^{\lambda^2 y}}$$

Now

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{X'}{X} \cdot \frac{Y'}{Y} = 0 \longrightarrow (*)$$

Put $\boxed{\frac{Y'}{Y} = \lambda^2}$

& $Y' - \lambda^2 Y = 0$

$$\Rightarrow Y'' - \lambda^2 Y' = 0 \Rightarrow Y'' - \lambda^4 Y = 0$$

$$\Rightarrow \boxed{\frac{Y''}{Y} = \lambda^4}$$

$$(*) \Rightarrow \frac{X''}{X} + \lambda^4 + \frac{X'}{X}\lambda^2 = 0$$

$$m^2 + m\lambda^2 + \lambda^4 = 0$$

$$m = \frac{-\lambda^2 \pm \sqrt{\lambda^4 - 4\lambda^4}}{2} = \frac{-\lambda^2 \pm \sqrt{3}\,\lambda^2 i}{2}$$

07-03-2016

Show that the eigen functions of the problem form an orthogonal set and then make it orthonormal.

$$\frac{d^2u}{dx^2} + \lambda u = 0$$

$$u(0) = 0, \quad u(l) = 0$$

$$D^2 = -\lambda$$

$$D = \pm i\sqrt{\lambda}$$

$$u = A \sin\sqrt{\lambda}\, x + B \cos\sqrt{\lambda}\, x$$

Since $u(0) = 0$

$\Rightarrow B = 0$ then $u = A \sin\sqrt{\lambda}\, x$

when $u(l) = 0$

$\Rightarrow A \sin\sqrt{\lambda}\, l = 0$

we take $A \neq 0$

then

$$\sin\sqrt{\lambda}\, l = 0$$

$$\Rightarrow \lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n = 0, \pm 1, \ldots$$

$$U_n = A_n \sin \frac{n\pi}{l} x$$

$$\int_0^l U_n U_m \, dx = \int_0^l A_n A_m \sin \frac{n\pi}{l} x \sin \frac{m\pi}{l} x \, dx$$

$$= \begin{cases} 0 & \text{if } m \neq n \\ \frac{l}{2} A_n^2 & \text{if } m = n \end{cases}$$

So, U_n is orthogonal set.

Now

$$U_n = \frac{\sin \frac{n\pi}{l} x}{\sqrt{l/2}}$$ will be orthonormal.

Example:- Solve String Problem.

$$U_{xx} = \frac{1}{c^2} U_{tt} \qquad 0 < x < l$$

$$t > 0$$

$$\left. \begin{array}{l} U(0, t) = 0 \\ U(l, t) = 0 \end{array} \right\} t \geq 0$$

$$U(x, 0) = \phi(x)$$

$$\dot{U}(x, 0) = 0 \qquad 0 \leq x \leq l$$

Sol:- Let $u = XT$

$$X''T = \frac{1}{c^2} X\ddot{T}$$

$\div XT$

$$\frac{X''}{X} = \frac{1}{c^2}\cdot\frac{\ddot{T}}{T} = \lambda$$ Let

$$\frac{X''}{X} = \lambda$$

$$D^2 = \lambda$$

$$D = \pm\sqrt{\lambda}$$

Case I if $\lambda = 0$

$$\frac{X''}{X} = 0$$

$$D^2 = 0$$

$u(0,t) = 0$ $\Rightarrow X = A + Bx$

$$X(0) = 0$$

$$\Rightarrow A = 0$$

then $X = Bx$

using $X(l) = 0$

$$\Rightarrow lB = 0$$

$$\Rightarrow B = 0$$ Trivial solution

other case $\Rightarrow$ If $\lambda = \mu^2$

$$\frac{X''}{X} = \mu^2$$

$$D^2 = \mu^2$$

$$\Rightarrow D = \pm \mu$$

$$X = Ae^{\mu x} + Be^{-\mu x}$$

using $X(0) = 0$

$$\Rightarrow \boxed{0 = A+B} \Rightarrow \boxed{A = -B}$$

then using $X(l) = 0$

$$\Rightarrow Ae^{\mu l} + Be^{-\mu l} = 0$$

$$-Be^{\mu l} + Be^{-\mu l} = 0$$

$$-B\left[\frac{e^{\mu l} - e^{-\mu l}}{2}\right] = 0$$

always $\Rightarrow \frac{e^{\mu l} - e^{-\mu l}}{2} \neq 0$ thus $B = 0$

$$\Rightarrow A = 0$$

Again Trivial solution.

now take, $\lambda = -\mu^2$

$$\Rightarrow X = A\sin\mu x + B\cos\mu x$$

using $X(0)=0$

$\Rightarrow \boxed{B=0}$

Now using $X(l)=0$ we get

$\mu = \frac{n\pi}{l}$

$X_n = A_n \sin\left(\frac{n\pi}{l}\right)x$

Now

$\frac{\ddot{T}}{T} = -c^2\mu^2$

$D^2 = -c^2\mu^2$

$D = \pm ic\mu$

$T = F \sin c\mu t + D\cos(\mu c t)$

$\dot{T} = c\mu F \cos c\mu t - \mu c D \sin(\mu c t)$.

using $\dot{T}(0)=0$

$\Rightarrow \boxed{F=0}$

Then $T = D\cos(\mu c t)$.

$u = A_n \sin \frac{n\pi}{l} x \cos(c\mu t)$.

$u = \sum_{1}^{\infty} A_n \sin \frac{n\pi}{l} x \cos(c\mu t) \longrightarrow (A)$

where $\boxed{n \neq 0}$.

Now Using

$$u(x,0) = \phi(x) \quad \text{in } (A)$$

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right)$$

X ing $\sin\frac{m\pi}{l}x$

$$\phi(x)\sin\frac{m\pi}{l}x = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{l}x \sin\frac{m\pi}{l}x$$

$$\int_0^l \phi(x)\sin\frac{m\pi}{l}x\,dx = \int_0^l \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{l}x \sin\frac{m\pi}{l}x\,dx$$

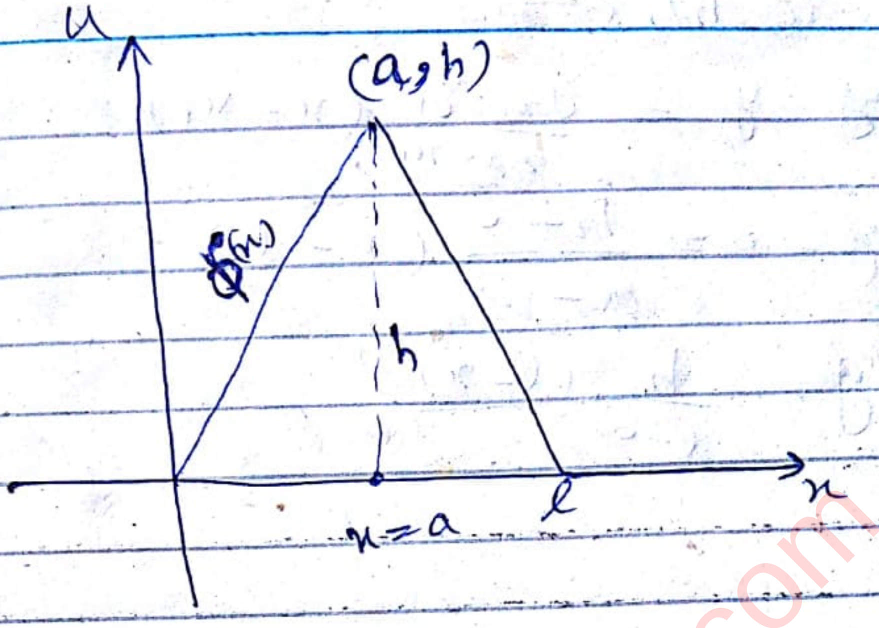
$$= \sum_{n=1}^{\infty} A_n\, \delta_{mn}\left(\frac{l}{2}\right)$$

$$= \frac{l}{2}A_m$$

Put A_m in (A)

$$u = \sum_{n=1}^{\infty}\left\{\frac{2}{l}\int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx\right\}\sin\frac{n\pi}{l}x \cos(\ldots t)$$

$u_t(x,0) = 0$



Suppose the string is raise to a height h at x = a and then released. The string will oscillate.

$$u(x,0) = \begin{cases} \frac{hx}{a} & 0 \le x \le a \\ \frac{h(l-x)}{l-a} & a \le x \le l \end{cases}$$

$$A_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x \, dx$$

For $0 \le x \le a$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 0 = \frac{h-0}{a-0}(x-0)$$

$$y = \frac{h}{a} x$$

$(l, 0)$, (a, h), for $a \le x \le l$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 0 = \frac{h - 0}{a - l}(x - l)$$

$$y = \frac{h}{l - a}(l - x).$$

Theorem:- 10-03-2015

Let $f(x)$ be piecewise smooth in $[-\pi, \pi]$. $f(x)$ is periodic upto period 2π, then the fourier series converges uniformly to "f" in every closed interval containing no discontinuity.

Theorem:- (Uniqueness Theorem)

There exist at most one solution of the wave eq,

$$u_{xx} = \frac{1}{c^2} u_{tt} \quad 0 < x < l$$
$$t > 0$$

I.C's

$$\left.\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}\right\} \quad 0 \le x \le l$$

B.C's $u(0,t) = 0$
$u(l,t) = 0$

where u is twice continuously differentiable func. w.r.t both x & t.

Proof:- Suppose there are 2 solutions u_1 & u_2.

$$V = u_1 - u_2 \longrightarrow (1)$$

then

$$u_{1tt} = c^2 u_{1xx}$$

Subtract and $u_{2tt} = c^2 u_{2xx}$

use ① $\Rightarrow V_{tt} = c^2 V_{xx}$ ⟶ ②

then $V(0,t) = 0$, $t \geq 0$ $\rightarrow V(x,0) = f(x) - f(x)$

$V(l,t) = 0$

$V(x,0) = 0$

$V_t(x,0) = 0$

energy $\rightarrow I(t) = \frac{1}{2} \int_0^l \left[c^2 V_x^2 + V_t^2 \right] dx$ ⟶ ③

which physically represents the total energy of the vibrating string at any time "t".

$$\Rightarrow \frac{dI}{dt} = \frac{1}{2} \int_0^l \left[c^2\, 2 V_x V_{xt} + 2 V_t V_{tt} \right] dx$$

$$= c^2 \int_0^l V_x V_{xt}\, dx + \int_0^l V_t V_{tt}\, dx$$

Integrating by parts.

$$= c^2 \left\{ V_x(l,t) V_t(l,t) - V_x(0,t) V_t(0,t) \right\} - c^2 \int_0^l V_{xx} V_t\, dx$$

$+ (\;''\;)$

using the given conditions we have

$$= -c^2\int_0^l V_{xx}V_t\,dx + \int_0^l V_t V_{tt}\,dx$$

from ②

$$= \int_0^l \left(-c^2 V_{xx} + V_{tt}\right)V_t\,dx$$

$$= 0 \Rightarrow \frac{dI}{dt} = 0$$

$$\Rightarrow I(t) = \text{constant} = K$$

$$I(0) = \frac{1}{2}\int_0^l \left[c^2 V_x^2(x,0) + V_t^2(x,0)\right]dx$$

$$= 0 \longrightarrow ⑤$$

$I(t) = K$
$I(0) = 0$
$\Rightarrow K = 0$

from ③, ④ & ⑤
$I(t) = 0$ So "V" is zero

hence $u_1 = u_2$.

$V_t^2 = 0$
$\Rightarrow V = K$
I.C $\Rightarrow \boxed{V=0}$

Example:– Heat-conduction Problem

$$u_{xx} = \alpha^2 u_t$$

$u(0,t) = 0$, $u(l,t) = T_0$, $u(x,0) = \phi(x)$

This problem is non-Homogeneous.

Let $w_{xx} = 0$

$w_x = c_1$

$w = c_1 x + c_2$

$w(0) = 0 \Rightarrow c_2 = 0$

$w(l) = T_0 \Rightarrow c_1 = \frac{T_0}{l}$

So, $w(x) = \frac{T_0}{l} x$

Let $V(x,t) = U(x,t) - W(x)$

$$U(x,t) = V(x,t) + \frac{T_0 x}{l}$$

$$U_{xx} = V_{xx} + 0$$

$$U_t = V_t$$

$$V_{xx} = \alpha^2 V_t$$

$$V(0,t) = 0$$

$$V(l,t) = 0$$

$$V(x,0) = \phi(x) - \frac{T_0 x}{l}$$

The result will be for $V(x,t)$

$$V(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{2}{l} \int_0^l \left[\phi(\tau) - \frac{T_0 \tau}{l} \right] \sin \frac{n\pi}{l} \tau \, d\tau \right\} e^{-\frac{n^2\pi^2}{l^2\alpha^2} t} \sin \frac{n\pi x}{l}$$

$$V(x,t) = T(t) X(x)$$

$$V_{xx} = T X''$$

$$V_t = \dot{T} X$$

$$V_{xx} = \alpha^2 V_t \Rightarrow T X'' = \alpha^2 \dot{T} X$$

$$\Rightarrow \quad \frac{X''}{X} = \frac{\dot{T}}{T}\alpha^2 = -\mu^2$$

$$\alpha^2 \frac{\dot{T}}{T} = -\mu^2$$

$$\frac{\dot{T}}{T} = -\frac{\mu^2}{\alpha^2}$$

$$D = -\frac{\mu^2}{\alpha^2}$$

$$T(t) = A e^{-\frac{\mu^2 t}{\alpha^2}}$$

$$\frac{X''}{X} = -\mu^2$$

$$D^2 = -\mu^2$$

$$D = \pm i\mu$$

$$X(x) = E \sin \mu x + F \cos \mu x$$

$$X(0,t) = 0 = 0 + F(1)$$

$$\Rightarrow \boxed{F = 0}$$

$$X(x) = E \sin \mu x$$

$$X(l,t) = 0 = E \sin(l\mu)$$

$$\sin(l\mu) = 0 \quad \because E \neq 0$$

$$\Rightarrow \mu = \frac{n\pi}{l}, \quad n = 1, 2, \ldots$$

$$V(x,t) = A e^{-\frac{\mu^2 t}{\alpha^2}} \sin\left(\frac{n\pi}{l} x\right)$$

$$V(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} e^{-\frac{\mu^2 t}{\alpha^2}}$$

Inner product

$$\int_0^l \sin\frac{m\pi x}{l} V(x,0)\, dx = \int_0^l \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l} e^{-\frac{\mu^2 t}{\alpha^2}} dx$$

$$\int_0^l \sin\frac{m\pi x}{l}\left(\phi(x) - \frac{T_0 x}{l}\right) dx = \int_0^l \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l}\, dx$$

$$'' = \frac{l}{2} A_m \qquad m=n$$

$$\Rightarrow A_n = \frac{2}{l}\int_0^l \sin\frac{n\pi x}{l}\left(\phi(x) - \frac{T_0 x}{l}\right) dx$$

$$\text{So, } V(x,t) = \frac{2}{l} \sum_{n=1}^{\infty} \int_0^l \sin\frac{n\pi x}{l}\left(\phi(x) - \frac{T_0 x}{l}\right) dx \; \sin\frac{n\pi}{l} x \, e^{-\frac{\mu^2 t}{\alpha^2}}$$

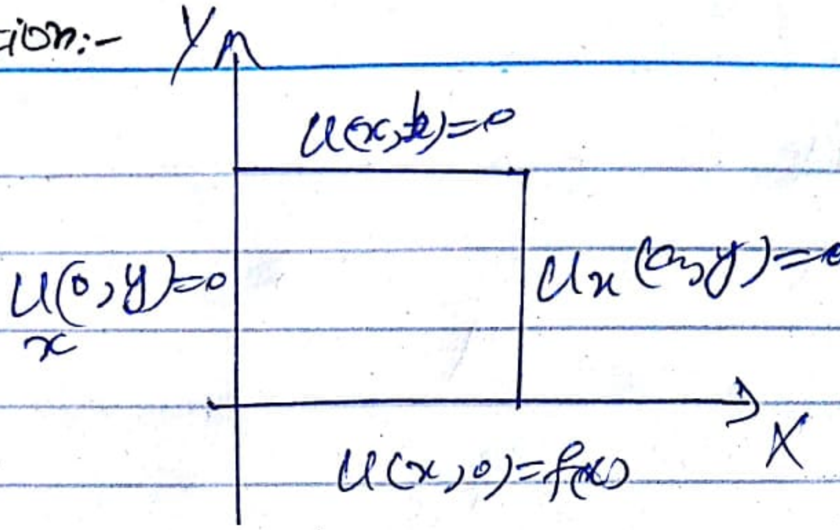
So, $u(x,t) = V(x,t) + \frac{T_0 x}{l}$

$$\Rightarrow u(x,t) = \frac{T_0 x}{l} + \frac{2}{l} \sum_{n=1}^{\infty} \int_0^l \sin\frac{n\pi}{l} x \left(\phi(x) - \frac{T_0 x}{l}\right) dx \; \sin\frac{n\pi}{l} x \, e^{-\frac{n^2\pi^2}{l^2\alpha^2} t}$$

Symmetric $\Rightarrow f(x) = f(-x)$

Anti-symmetric $\Rightarrow f(x) = -f(-x)$

Question:-



The governing Eq. will be

$$\nabla^2 u = 0$$

$$u(x,0) = f(x)$$

Example:- (A complete non-Homog.) 14 03-2016

$$u_{tt} = c^2 u_{xx} + F(x)$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

$$u(0,t) = A, \quad u(l,t) = B$$

Assume that

$$V(x,t) = u(x,t) - W(x)$$

$$u_{xx} = V_{xx} + W_{xx}$$

& $$u_{tt} = V_{tt}$$

then
$$V_{tt} = c^2(V_{xx} + W_{xx}) + F(x)$$

$$= c^2 V_{xx} + \left(c^2 W_{xx} + F(x)\right)$$

Let $c^2 W_{xx} + F(x) = 0$

then
$$u(x,0) = f(x)$$

$$\Rightarrow u(x,0) = f(x) = V(x,0) + W(x)$$

$$V(x,0) = f(x) - W(x)$$

$$u_t(x,0) = g(x) = V_t(x,0)$$

$$u(0,t) = A = V(0,t) + W(0)$$

$$\Rightarrow V(0,t) = A - W(0)$$

Hyberman ⇒ chap#8

Similarly

$$B = V(l,t) + w(l)$$

$$\Rightarrow V(l,t) = B - w(l).$$

Assume that

$$c^2 w_{xx} + f(x) = 0$$

$$w(0) = A$$

$$w(l) = B$$

then

$$V_{tt} = c^2 V_{xx}$$

$$V(x,0) = f(x) - w(x)$$

$$V_t(x,0) = g(x)$$

$$V(0,t) = 0$$

$$V(l,t) = 0$$

Solution will be of Eq. $w_{xx} = -\frac{1}{c^2}F(x)$

$$w(x) = A + \frac{(B-A)x}{l} + \frac{x}{l}\int_0^l \left[\frac{1}{c^2}\int_0^\eta F(\xi)\,d\xi\right]d\eta$$

$$- \int_0^x \left[\frac{1}{c^2}\int_0^\eta F(\xi)\,d\xi\right]d\eta$$

Example:-

$$u_{tt} = c^2 u_{xx} + x$$

$$u(x,0) = 0 \ , \ u_t(x,0) = 0$$

$$u(0,t) = 0 = u(l,t)$$

Let $V(x,t) = u(x,t) - w(x)$

$u = V + w$

$$c^2 w_{xx} + x = 0$$

$$w_{xx} = -\frac{x}{c^2}$$

Integrating

$$w_x = -\frac{x^2}{2c^2} + A_1$$

$$w = -\frac{x^3}{6c^2} + A_1 x + B_1$$

using $w(0) = 0$

$$\Rightarrow \boxed{B_1 = 0}$$

using $w(l) = 0$

$$\Rightarrow 0 = \frac{-l^3}{6c^2} + A_1 l$$

$$\boxed{A_1 = \frac{l^2}{6c^2}}$$

So, $w = -\frac{x^3}{6c^2} + \frac{l^2}{6c^2} x = \frac{x}{6c^2}(l^2 - x^2)$

$$V(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi c}{l} t \ \sin\frac{n\pi x}{l}$$

$$V(x,0) \Rightarrow -w(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$$

$$v(x,t) = -\frac{2}{\ell}\sum_{n=1}^{\infty}\int_0^{\ell}\frac{x}{6c^2}(\ell^2-x^2)\sin\frac{n\pi}{\ell}x\,dx\,\cos\frac{n\pi}{\ell}ct\,\sin\frac{n\pi}{\ell}x$$

$$a_n = -\frac{2}{\ell}\int_0^{\ell} w(x)\,\sin\frac{n\pi x}{\ell}\,dx$$

So, $u(x,t) = \frac{x}{6c^2}(\ell^2-x^2) - \frac{2}{\ell}\sum_{n=1}^{\infty}\int_0^{\ell}\frac{x}{6c^2}(\ell^2-x^2)\sin\frac{n\pi}{\ell}x\,dx\,\cos\frac{n\pi}{\ell}ct\,\sin\frac{n\pi}{\ell}x$

Related to Question in perivous lecture.

A Laplace Eq.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$$u(0,y) = g_1(y)$$
$$u(L,y) = g_2(y)$$
$$u(x,0) = f_1(x)$$
$$u(x,H) = f_2(x)$$

$\nabla^2 u = 0$

$u = f_2$

$u = g_2$

$u = f_1$

$u = g_2$

$0 < x < L$

$0 < y < H$

$$\frac{\partial^2 u_4}{\partial x^2}+\frac{\partial^2 u_4}{\partial y^2}=0$$

$$u_4(0,y) = g_1(y)$$
$$u_4(L,y) = 0$$
$$u_4(x,0) = 0$$
$$u_4(x,H) = 0$$

$$u_4 = X(x)\, Y(y)$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2\phi}{dy^2} = \lambda \text{ or } -\lambda$$

If the seperation constant is "$-\lambda$" then $X(x)$ oscillates & $Y(y)$ is composed of exponension function. This seems doubt full since $Y(0)=0$ & $Y(H)=0$ exponentials in Y are not expected to work.

If we take "$+\lambda$" then $X(x)$ is exponential & $Y(y)$ oscillates.

Example: 24-03-2016

Let us consider the problem of a finite string within external force acting on it, if ends are fixed then we have

$$u_{tt} - c^2 u_{xx} = h(x,t) \longrightarrow (1)$$

$$0 < x < l$$

$$t > 0$$

$$u(x,0) = f(x)$$

$$u_t(x,0) = g(x)$$

$$u(0,t) = 0$$

$$u(l,t) = 0$$

We assume the solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\frac{n\pi}{l}x \longrightarrow (2)$$

where $u_n(t)$ are to be determine.

$$u_{tt} - c^2 u_{xx} = 0$$

$$c^2\frac{d^2X}{dx^2} + \lambda^2 X = 0$$

$$\Rightarrow D^2 = -\frac{\lambda^2}{c^2} \Rightarrow D = \pm\frac{\lambda}{c}i$$

We get $U_n(x) = A\sin\frac{\lambda}{c}x + B\cos\frac{\lambda}{c}x$

$$U_n(x) = \sin\frac{n\pi}{l}x \qquad \boxed{B=0} \quad \& \quad \frac{\lambda}{c} = \frac{n\pi}{l}$$

Also assuming,

$$h(x,t) = \sum_{n=1}^{\infty} h_n(t) \sin \frac{n\pi}{l} x.$$

Since ② is solution of ①.

from ① $\Rightarrow$ $u_{tt} = \sum_{n=1}^{\infty} u_{n_{tt}}(t) \sin \frac{n\pi}{l} x$

& $u_{xx} = -\sum_{n=1}^{\infty} u_n(t) \left(\frac{n\pi}{l}\right)^2 \sin \frac{n\pi}{l} x$

$$c^2 \sum_{n=1}^{\infty} u_n(t) \left(\frac{n\pi}{l}\right)^2 \sin \frac{n\pi}{l} x + \sum_{n=1}^{\infty} u_{n_{tt}} \sin \frac{n\pi}{l} x$$
$$= \sum_{n=1}^{\infty} h_n(t) \sin \frac{n\pi}{l} x$$

using $\lambda_n = \frac{n\pi}{l} c$

$$\sum_{n=1}^{\infty} \left\{ u_n''(t) + \lambda_n^2 u_n(t) \right\} \sin \frac{n\pi}{l} x = \sum_{n=1}^{\infty} h_n(t) \sin \frac{n\pi}{l} x$$

xing $\sin \frac{n\pi}{l} x$

$$\sum_{n=1}^{\infty} \left[u_n''(t) + \lambda_n^2 u_n(t) \right] \int_0^l \sin^2 \frac{n\pi}{l} x \, dx = \sum_{n=1}^{\infty} h_n(t) \int_0^l \sin^2 \frac{n\pi}{l} x \, dx$$
$$= h_n(t) \, l/2$$

$$\Rightarrow h_n(t) = u_n''(t) + \lambda_n^2 u_n(t)$$

$$u_n(t) = u_{n_c}(t) + u_{n_p}(t)$$

$$D^2 = -\lambda_n^2$$

$$D = \pm i\lambda_n$$

$$(u_n(t))_c = A\cos\lambda_n t + B\sin\lambda_n t$$
$$= A y_1 + B y_2$$

$$(u_n(t))_p = V_1(t)\cos\lambda_n t + V_2(t)\sin\lambda_n t = V_1(t) y_1 + V_2(t) y_2$$

R.H.S of previous Eq.

$$V_{1n}'(t) = \frac{-y_2 R}{W} = -\frac{\sin\lambda_n t\, h_n(t)}{\lambda_n} \quad (\because \lambda_n \neq 0)$$

where W → Wronksian determinant

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos\lambda_n t & \sin\lambda_n t \\ -\lambda_n \sin\lambda_n t & \lambda_n \cos\lambda_n t \end{vmatrix} = \lambda_n$$

$$\Rightarrow V_{1n}(t) = -\frac{1}{\lambda_n}\int h_n(t)\sin\lambda_n t\, dt$$

$$V_{2n}'(t) = \frac{y_1 R}{W} = \frac{\cos\lambda_n t\, h_n(t)}{\lambda_n}$$

$$\Rightarrow V_{2n}(t) = \frac{1}{\lambda_n}\int \cos\lambda_n t\, h_n(t)\, dt$$

$$\Rightarrow (U_n(t))_p = -\frac{1}{\lambda_n}\left[\int h_n(t)\sin\lambda_n t\,dt\right]\overset{y_1}{\cos(\lambda_n t)} + \frac{1}{\lambda_n}\left[\int h_n(t)\cos\lambda_n t\,dt\right]\overset{y_2}{\sin(\lambda_n t)}$$

$$= \frac{1}{\lambda_n}\left[\left(\int_0^t h_n(\xi)\cos\lambda_n\xi\,d\xi\right)\sin\lambda_n t - \int_0^t h_n(\xi)\sin\lambda_n\xi\,d\xi\,\cos\lambda_n t\right]$$

$$= \frac{1}{\lambda_n}\int_0^t h_n(\xi)\{\cos\lambda_n\xi\sin\lambda_n t - \sin\lambda_n\xi\cos\lambda_n t\}\,d\xi$$

$$= \frac{1}{\lambda_n}\int_0^t h_n(\xi)\sin\lambda_n(t-\xi)\,d\xi$$

So,

$$② \Rightarrow \therefore U(x,t) = \sum_{n=1}^{\infty}\left[\left[A_n\cos\lambda_n t + B_n\sin\lambda_n t\right] + \frac{1}{\lambda_n}\int_0^t h_n(\xi)\sin\lambda_n(t-\xi)\,d\xi\right]\sin\frac{n\pi}{l}x$$

Using $U(x,0) = f(x)$.

$$f(x) = \sum_{n=1}^{\infty} A_n\sin\frac{n\pi}{l}x$$

③ By inner product.

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x\,dx$$

$$B_n = \frac{2}{l\lambda_n}\int_0^l g(x)\sin\frac{n\pi}{l}x\,dx$$

Example

Now we will consider the I-BVP with the time dependent boundary condition,

$$u_{tt} - c^2 u_{xx} = h(x,t)$$

$$u(x,0) = f(x)$$

$$u_t(x,0) = g(x)$$

$$u(0,t) = p(t)$$

$$u(l,t) = q(t)$$

Solution.

We assume the solution in the form.

$$V(x,t) = u(x,t) - S(x,t)$$

$$u(x,t) = V(x,t) + S(x,t)$$

$$V_{tt} - c^2 V_{xx} = h + c^2 S_{xx} - S_{tt}$$

$$u(x,0) = f(x)$$

$$\Rightarrow V(x,0) = f(x) - S(x,0)$$

$$u_t(x,0) = g(x)$$

$$\Rightarrow V_t(x,0) = g(x) - S_t(x,0)$$

$u(0,t) = p(t)$

$\Rightarrow v(0,t) = p(t) - S(0,t)$

$u(l,t) = q(t)$

$\Rightarrow v(l,t) = q(t) - S(l,t)$

For homogeneous :-

let $h + c^2 S_{xx} - S_{tt} = 0$

take $S(0,t) = p(t)$ ← using $v(0,t) = 0$ & $v(l,t) = 0$

& $S(l,t) = q(t)$

Thus

$$S(x,t) = p(t) + \frac{x}{l}[q(t) - p(t)]$$

This satisfy the upper conditions.

let $S_{xx} = 0$ implies

Now to find $v(x,t)$.

assume

$$v_{tt} - c^2 v_{xx} = h - S_{tt} = H(x,t)$$

$$v(x,0) = f(x) - S(x,0) = F(x)$$

$$v_t(x,0) = g(x) - S_t(x,0) = G(x)$$

$$v(0,t) = 0 \,, \quad v(l,t) = 0$$

Now this problem can be treated same as the previous example.

DYS Example:-

$$U_{tt} - U_{xx} = h \quad \text{constant}$$

$$u(x,0) = x(1-x)$$

$$u_t(x,0) = 0$$

$$u(0,t) = t$$

$$u(1,t) = \sin t$$

Bar Problem:- 28-03-2016

$$u_t = K u_{xx}$$

$$u(0,t) = A$$

$$u(l,t) = B$$

$$u(x,0) = f(x)$$

at Equilibrium Temp. $\rightarrow u_E''(x) = 0$

$$u_E(0) = A$$

$$u_E(l) = B$$

$$u_E(x) = A + \frac{(B-A)x}{l}$$

let $V(x,t) = u(x,t) - u_E(x) \longrightarrow (*)$

$$V_t(x,t) = K V_{xx}$$

$$V(0,t) = 0$$

$$V(l,t) = 0$$

$$V(x,0) = f(x) - u_E(x)$$

Solution will be

$$V(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{l} x \, e^{-K\left(\frac{n\pi}{l}\right)t}$$

Where $a_n = \frac{2}{l} \int_0^l \left[f(x) - u_E(x)\right] \sin \frac{n\pi}{l} x \, dx$

as $t \to \infty$ then $u \longrightarrow u_E$ (using $(*)$)

Example:-

$$u_t = K u_{xx} + Q(x,t) \longrightarrow (1)$$

$$u(0,t) = A(t), \quad u(l,t) = B(t), \quad u(x,0) = f(x)$$

Let

$$V(x,t) = u(x,t) - r(x,t)$$

$$r(x,t) = A(t) + \frac{x}{l}\left(B(t) - A(t)\right)$$

$$r(0,t) = A(t)$$

$$r(l,t) = B(t)$$

$$V_t = K V_{xx} + \bar{Q}(x,t) \longrightarrow (2)$$

$$\bar{Q}(x,t) = Q(x,t) - r_t + K r_{xx}$$

$$V(0,t) = 0$$

$$V(l,t) = 0$$

$$V(x,0) = f(x) - r(x,0)$$

Let

$$= f(x) - \left[A(0) + \frac{x}{l}\left(B(0) - A(0)\right)\right] = g(x)$$

The related homogeneous problem is

$$u_t = K u_{xx}$$

$$u(0,t) = 0$$

$$u(l,t) = 0$$

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

$\phi(0) = 0$

$\phi(l) = 0$ $\qquad \lambda_n = \left(\frac{n\pi}{l}\right)^2, \; n = 1, 2, \ldots$

$$\phi_n(x) = \sin\frac{n\pi x}{l}$$

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\, \phi_n(x)$$

$$v(x,0) = g(x)$$

$$g(x) = \sum_{n=1}^{\infty} a_n(0)\, \phi_n(x)$$

$$a_n(0) = \frac{\int_0^l g(x)\,\phi_n(x)\,dx}{\int_0^l \phi_n^2(x)\,dx}$$

$$v_t = \sum_{n=1}^{\infty} \frac{d}{dt} a_n(t)\, \phi_n(x)$$

$$v_{xx} = \sum_{n=1}^{\infty} a_n(t) \frac{d^2\phi_n}{dx^2}$$

$$(2) \Rightarrow \sum_{n=1}^{\infty} \left[\frac{d}{dt} a_n(t) + K\lambda_n a_n\right]\phi_n$$

$$= \bar{Q}$$

$$\frac{d}{dt} a_n(t) + K\lambda_n a_n = \frac{\int_0^l \phi_n \bar{Q}\, dx}{\int_0^l \phi_n^2\, dx} = \bar{q}_n(t)$$

$$\bar{Q}(x,t) = \sum_{n=1}^{\infty} \bar{q}_n(t)\, \phi_n(x)$$

$$\frac{d}{dt} a_n(t) + K\lambda_n a_n = \bar{q}_n(t)$$

A linear Eq.

Solution will be;

$$e^{\lambda_n K t}\left(\frac{d a_n}{dt} + \lambda_n K a_n\right) = \bar{q}_n e^{\lambda_n K t}$$

$$a_n(t)\, e^{\lambda_n K t} - a_n(0) = \int_0^t \bar{q}_n(\tau)\, e^{\lambda_n K \tau}\, d\tau$$

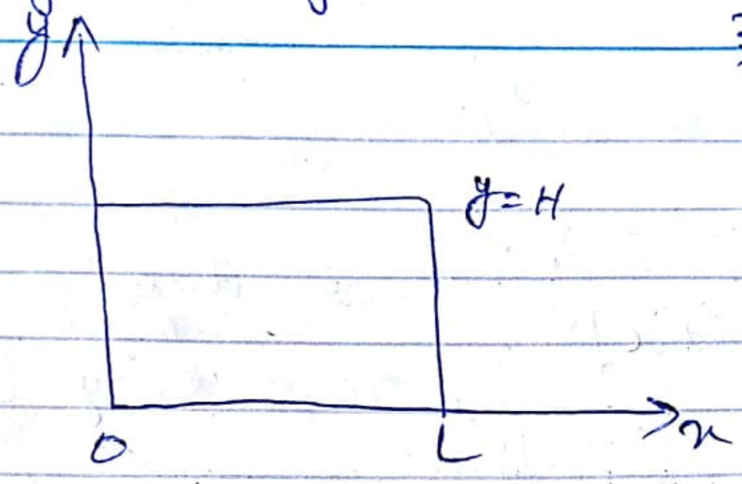
$$a_n(t) = e^{-\lambda_n t K}\left[\frac{\int_0^l g(x)\phi_n(x)\,dx}{\int_0^l \phi_n^2(x)\,dx} + \int_0^t \bar{q}_n(\tau)\, e^{\lambda_n K \tau}\, d\tau\right]$$

So,

$$V(x,t) = \sum_{n=1}^{\infty} e^{-\lambda_n K t}\left[\frac{\int_0^l g(x)\phi_n(x)\,dx}{\int_0^l \phi_n^2(x)\,dx} + \int_0^t \bar{q}_n \tau\, e^{\lambda_n K \tau}\, d\tau\right]\phi_n(x)$$

Vibrating Rectangular Membrane

31-03-2016



$u(x, y, t)$

$$u_{tt} = c^2 [u_{xx} + u_{yy}]$$

All the four sides are fixed

Subject to

$$\left.\begin{array}{l} u(0, y, t) = 0 \\ u(x, 0, t) = 0 \\ u(L, y, t) = 0 \\ u(x, H, t) = 0 \end{array}\right\} \text{B.Cs}$$

$$\left.\begin{array}{l} u(x, y, 0) = \alpha(x, y) \\ u_t(x, y, 0) = \beta(x, y) \end{array}\right\} \text{I.Cs}$$

Let $u(x, y, t) = h(t)\phi(x, y)$

$$h_{tt}\,\phi(x, y) = c^2 [h\,\phi_{xx} + h\,\phi_{yy}]$$

$\div$ by $c^2 h\phi$

$$\frac{h_{tt}}{c^2 h} = \frac{\phi_{xx} + \phi_{yy}}{\phi} = -\lambda$$

$$\Rightarrow h_{tt} = -\lambda c^2 h$$

$$D^2 = -c^2\lambda$$

$$D = \pm i c\sqrt{\lambda}$$

oscilations with frequency

$$h(t) = \begin{cases} \sin\sqrt{\lambda}\, c t \\ \cos\sqrt{\lambda}\, c t \end{cases}$$

"λ" is natural frequency of the vibrating membrane & "c" is frequency of oscillation.

Now

$$\phi_{xx} + \phi_{yy} = -\lambda\phi \longrightarrow (1)$$

$$\left.\begin{aligned} \phi(0,y) = 0 = \phi(x,0) \\ \phi(L,y) = 0 = \phi(x,H) \end{aligned}\right\} \longrightarrow (2)$$

We call (1) & (2) two dimension eigenvalue problem. It is always linear.

Let $\phi(x,y) = f(x)\, g(y)$

$(1) \Rightarrow f_{xx}\, g(y) + g_{yy} f = -\lambda f g$

$$\Rightarrow \frac{f_{xx}}{f} = -\lambda - \frac{g_{yy}}{g} = -\mu$$

$$\Rightarrow f_{xx} = -\mu f \longrightarrow (*)$$

$$f(0) = 0$$
$$f(L) = 0$$

and

$$g_{yy} = -(\lambda - \mu) g$$

$$g(0) = 0$$

$$g(H) = 0$$

(*) ⇒ which is sturm Liouville problem in x

$$f(x) = a \sin\sqrt{\mu}\, x + b \cos\sqrt{\mu}\, x$$

$0 = b.1, \; 0 = b$

$$f(0) = 0 \Rightarrow b = 0$$

$$f(x) = a \sin\sqrt{\mu}\, x$$

$$f(L) = 0 \Rightarrow a \sin\sqrt{\mu}\, L = 0, \; a \neq 0$$

$$\Rightarrow \sqrt{\mu}\, L = n\pi$$

since $a \neq 0$ for trivial sol.

Eigenvalues $\Rightarrow \mu_n = \dfrac{n^2\pi^2}{L^2}$ $\quad n = 1, 2, \ldots$

$$\Rightarrow f_n(x) = a_n \sin \frac{n\pi}{L} x$$

Eigenfunction

Now;

$$g_{yy} = -(\lambda-\mu_n) g$$

$$D^2 = -(\lambda-\mu_n)$$ assume $\lambda > \mu$

$$D = \pm \sqrt{\lambda-\mu_n}\, i$$

$$g(y) = c \sin\left(\sqrt{\lambda-\mu_n}\, y\right) + d \cos\left(\sqrt{\lambda-\mu_n}\, y\right)$$

$$g(0) = 0 \Rightarrow d = 0$$

$$g(H) = 0 \Rightarrow c \sin \sqrt{\lambda-\mu_n}\, H = 0$$

$$\Rightarrow \sqrt{\lambda-\mu_n}\, H = m\pi$$ $c \neq 0$

$m = 1, 2, \ldots$

$$\lambda - \mu_n = \frac{m^2\pi^2}{H^2}$$

$$\lambda = \mu_n + \frac{m^2\pi^2}{H^2}$$

$$\lambda_{mn} = \mu_n + \left(\frac{m\pi}{H}\right)^2$$ or $\lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$.

$n = 1, 2, \ldots$

$m = 1, 2, \ldots$

there are infinite number of eigenvalues for any "n".

So, $g_{mn}(y) = c_m \sin\left(\frac{m\pi}{H}\right)y$

There fore

$$\boxed{\phi_{mn}(x, y) = (\text{constant}) \sin\frac{n\pi}{L}x \sin\frac{m\pi}{H}y}$$

are two dimensional eigen functions corresponding to eigen value λ_{mn}.

Hence

$$u(x,y,t) = \begin{cases} \sin\frac{n\pi}{L}x \sin\frac{m\pi}{H}y \sin c\sqrt{\lambda_{mn}}t \\ \sin\frac{n\pi x}{L} \sin\frac{m\pi}{H}y \cos c\sqrt{\lambda_{mn}}t \end{cases}$$

are two doublee infinite values

As with the vibrating string each of this

The principle of superposition implies that we should consider a linear combination of all possible

$$U(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{n\pi}{L}x \sin\frac{m\pi}{H}y \sin c\sqrt{\lambda_{mn}}\,t$$
$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \sin\frac{n\pi x}{L} \sin\frac{m\pi}{H}y \cos c\sqrt{\lambda_{mn}}\,t$$

The two family of coefficient A_{mn} & B_{mn} will be determine from initial conditions.

$$U(x,y,0) = \alpha(x,y)$$

$$\Rightarrow \alpha(x,y) = \sum_{m=1}^{\infty}\left\{\sum_{n=1}^{\infty} B_{mn} \sin\frac{n\pi}{L}x\right\} \sin\frac{m\pi}{H}y$$

s.t.

$$\int_0^H \alpha(x,y) \sin\frac{l\pi}{H}y\,dy$$
$$= \sum_{m=1}^{\infty}\left\{\sum_{n=1}^{\infty} B_{mn} \sin\frac{n\pi}{L}x\right\} \int_0^H \sin\frac{m\pi}{H}y \sin\frac{l\pi}{H}y\,dy$$
$$= \sum_{m=1}^{\infty}\left\{\sum_{n=1}^{\infty} B_{mn} \sin\frac{n\pi}{L}x\right) \frac{H}{2}\delta_{ml}$$
$$= \sum_{n=1}^{\infty} B_{ln} \sin\frac{n\pi}{L}x \; \frac{H}{2}$$

$\Rightarrow$

$$B_{mn} = \frac{2}{L}\int_0^L \left[\frac{2}{H}\int_0^H \alpha(x,y) \sin\frac{m\pi}{H}y\,dy\right] \sin\frac{n\pi}{L}x\,dx$$
$$= \frac{4}{L\cdot H}\int_0^L\int_0^H \alpha(x,y) \sin\frac{m\pi}{H}y \sin\frac{n\pi}{L}x\,dy\,dx$$

Similarly,

$$A_{mn} = \frac{4}{L \cdot H} \cdot \frac{1}{c\sqrt{\lambda_{mn}}} \int_0^L \left[\int_0^H \beta(x,y) \sin\frac{m\pi}{H} y \, dy \right] \sin\frac{n\pi}{L} x \, dx$$

$$A_{mn} = \frac{4}{L \cdot H \cdot c \cdot \sqrt{\lambda_{mn}}} \int_0^L \int_0^H \beta(x,y) \sin\frac{m\pi y}{H} \sin\frac{n\pi}{L} x \, dy \, dx.$$

We have shown that all 3 independent variables seperate form a PDE. The results are then 3 ODE's to a which are eigenvalue problem.

In general, for a PDE in 'n' variables that completely seperat there will be n-ODE's, n-1 will be one dimensional eigenvalue problem.

Ex. 7.3 Haberman

DYS $u_{tt} = c^2(u_{xx} + u_{yy})$

$$u(x,0,t) = u(x,b,t) = 0$$

$$u(0,y,t) = u(a,y,t) = 0$$

$$u(x,y,0) = \phi(x)$$

$$u_t(x,y,0) = 0$$

Solution will be

$$u(x,y,t) = \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\int_0^a \int_0^b \phi(x) \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \, dy \, dx \right] \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \cos c\sqrt{\lambda_{mn}}\, t$$

7.4 Question ⇒

Answer will be

$$u(x,y,t)=\frac{288}{\pi^3}\sum_{n=1}^{\infty}\sum_{\substack{m=1,3,5\ldots \\ m\neq 0,2,4,\ldots}}^{\infty}\left(\frac{1}{m}\right)^3 \sin\left(\frac{n\pi x}{3}\right)\sin\left(\frac{m\pi y}{6}\right) e^{-9\lambda_{mn}t}$$

$$u_t=9(u_{xx}+u_{yy})$$

$$u(x,0,t)=0=u(x,6,t)$$

$$u(0,y,t)=0=u(3,y,t)$$

$$u(x,y,0)=-\sin\frac{n\pi x}{3}(y(y-6))$$

& $u_t(x,y,0)=0$

Question: $u_{tt}=c^2(u_{xx}+u_{yy})$

$$u(x,0,t)=0=u(x,\pi,t)$$

$$u(0,y,t)=0=u(\pi,y,t)$$

$$u(x,y,0)=xy(\pi-x)(\pi-y)$$

$$u_t(x,y,0)=0$$

The solution will be

$$u(x,y,t)=\frac{64}{\pi^2}\sum_{n=1,3,5\ldots}^{\infty}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{m^3n^3}\sin mx\sin my\cos c\sqrt{\lambda_{mn}}\,t$$

where $m\neq 0,2,4,\ldots$ & $n\neq 0,2,4,\ldots$

Bessel's Eq. of order n.

$$r^2 F''(r) + r F'(r) + (r^2 - n^2) F(r) = 0 \qquad (111)$$

07-04-2016

Vibrating circular membrane and the Bessel function.

Introduction:–

The verticle displacement u satisfies the 2-Dimensional wave Eq.

$$u_{tt} = c^2 \nabla^2 u \longrightarrow (1)$$

The geometry suggest that $u = u(r, \theta, t)$ (polar coordinates).

B.C $\quad u(a, \theta, t) = 0$ (On circular boundary)

I.C. $\quad u(r, \theta, 0) = \alpha(r, \theta)$

$$u_t(r, \theta, 0) = \beta(r, \theta)$$

&

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Let

$$u = \phi(r, \theta)\, H(t) \quad \text{(Separation of variables)}$$

$$H(t) = \begin{cases} \sin\sqrt{\lambda}\, c\, t \\ \cos\sqrt{\lambda}\, c\, t \end{cases}$$

where λ is the seperation const.

The natural frequencies of the

Viberations are $\sqrt{\lambda}\, c$, $\{$if $\lambda > 0$ and $\phi(r, \theta)$ satisfies the 2-D eigenvalue problem.

$$\nabla^2\phi + \lambda\phi = 0$$

$$\phi H_{tt} = Hc^2\nabla^2\phi$$

$$\frac{H_{tt}}{Hc^2} = \frac{\nabla^2\phi}{\phi} = \lambda$$

with $\phi(a, \theta) = 0$

Let $\phi = f(r)\, g(\theta)$, $0 < r < a$, $-\pi < \theta < \pi$

$$\left(\frac{1}{r}\right)\frac{\partial}{\partial r}\left(r\frac{\partial(fg)}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2(fg)}{\partial\theta^2}\right) + \lambda fg = 0$$

$$\frac{g}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{f}{r^2}\left(\frac{\partial^2 g}{\partial\theta^2}\right) + \lambda fg = 0$$

x'ing by $\frac{r^2}{gf}$

$$\frac{r}{f}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{g}\frac{\partial^2 g}{\partial\theta^2} + \lambda r^2 = 0$$

$$-\frac{1}{g}\frac{\partial^2 g}{\partial\theta^2} = \frac{r}{f}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \lambda r^2 = \mu$$

$$\frac{d^2 g}{d\theta^2} = -\mu g$$

$$g(\theta) = A\sin\sqrt{\mu}\,\theta + B\cos\sqrt{\mu}\,\theta$$

$$r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (r^2\lambda - \mu)f = 0$$

Two of these Eq.s must be eigenvalues problem the only B.V.C is

$$f(a) = 0$$
$$-\pi < \theta < \pi$$
$$0 < r < a$$

So, There should be B.C on both ends.

The periodic nature of solution in θ implies that

$$g(-\pi) = g(\pi).$$

&

$$\frac{dg(-\pi)}{d\theta} = \frac{dg(\pi)}{d\theta}$$

Polar coordinates are singular at $r = 0$. a singularity condition must be introduced there. Since the displacement must be finite

$$|f(0)| < \infty$$

Eigen value Problem:-

$$\frac{d^2 g}{d\theta^2} = -\mu g$$

$\Rightarrow g(\theta) = A \sin\sqrt{\mu}\,\theta + B \cos\sqrt{\mu}\,\theta$

using $g(-\pi) = g(\pi)$

$A \sin(\sqrt{\mu}(-\pi)) + B \cos\sqrt{\mu}(-\pi) = A \sin\sqrt{\mu}\pi + B\cos\sqrt{\mu}\pi$

$A \sin\sqrt{\mu}\,\pi = 0$

if $B = 0$ then $A \neq 0$

$\sin\sqrt{\mu}\,\pi = 0$

$\Rightarrow \mu_m = m^2, \quad m = 1, 2, \ldots$

$$\frac{dg}{d\theta} = A\sqrt{\mu}\cos\sqrt{\mu}\,\theta - B\sqrt{\mu}\sin\sqrt{\mu}\,\theta$$

Using $\frac{dg(-\pi)}{d\theta} = \frac{dg(\pi)}{d\theta}$

$\Rightarrow A\sqrt{\mu}\cos\sqrt{\mu}(-\pi) - B\sqrt{\mu}\sin\sqrt{\mu}(-\pi)$

$= A\sqrt{\mu}\cos\sqrt{\mu}\pi - B\sqrt{\mu}\sin\sqrt{\mu}\pi$

$B\sqrt{\mu}\sin\sqrt{\mu}\,\pi = 0$

If $A = 0$ & $B \neq 0$

the $\mu_m = m^2, \quad m = 0, 1, 2, \ldots$

Eigen Value Problem:- 11-04-2016

$$g(\theta) = A \sin m\theta + B \cos m\theta$$

m is the no. of crests in θ direction. For each integral value of 'm' we have

$$r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - m^2) f = 0 \longrightarrow (i)$$

$$f(a) = 0 \quad \& \quad |f(0)| < \infty$$

$$r^2\frac{d^2f}{dr^2} + r\frac{df}{dr} + (\lambda r^2 - m^2) f = 0 \longrightarrow (ii)$$

Eq. (ii) contains two parameters 'λ' & 'm'. 'λ' is yet to be determined. It would be quite traditous to solve (ii) numerically for various values of λ instead we might notice that simple scalling transformation

$\boxed{\frac{df}{dr} = \frac{df}{dz}\cdot\frac{dz}{dr}}$ $z = \sqrt{\lambda}\, r$ in Eq. (ii) we have

$$z^2\frac{d^2f}{dz^2} + z\frac{df}{dz} + (z^2 - m^2) f = 0 \longrightarrow (iii)$$

(iii) is known as Bessel Eq. of order m.

Standard form of Bessel's differential Eq. is $\frac{d^2f}{dz^2} + a(z)\frac{df}{dz} + b(z)f = 0$

If $a(z)$ & $b(z)$ & all their derivatives are finite at $z = z_0$ then z_0 is called ordinary point otherwise z_0 is singular point.

For Bessell diff. Eq. (iii) $z=0$ is a singular point & all other points are ordinary points. All solutions of Bessell diff. Eq. are well behaved at every finite point except possibly at $z=0$. The only difficulty can occur in the nbhd of $z=0$. We will investigate the expected behaviour of solution of Bessell diff. Eq. at $z=0$, If z is very close to zero then $z^2 f \ll m^2 f$ so the Eq (iii) becomes

$$z^2\frac{d^2f}{dr^2} + z\frac{df}{dr} - m^2 f \approx 0 \Rightarrow \text{Cauchy PDE}$$

Put $z = e^t \Rightarrow \ddot{Z} - \cancel{\dot{Z}} + \cancel{\dot{Z}} - m^2 = 0$

$$\Rightarrow Z = Ae^{mt} + Be^{-mt}$$

$$Z = A e^{m \ln z} + B e^{-m \ln z}$$

$$= A e^{(\ln z^m)} + B e^{(\ln z)^{-m}}$$

$$= A Z^m + B Z^{-m}, \quad m > 0$$

If $m = 0$

$D = 0$

$Z = a + bt = a + b \ln z \quad m = 0$

So, $f \simeq \begin{cases} a + b \ln z & m = 0 \\ A z^m + B z^{-m} & m > 0 \end{cases}$

We see that the independent solution can be chosen that one solution is well behaved and other is ill-behaved at $z = 0$.

$$f = C_1 J_m(z) + C_2 Y_m(z).$$

$J_m(z)$ is called first kind of Bessel function & $Y_m(z)$ is called second kind of Bessel function.

We simply note that this satisfy asymptotic property for small z.

$$J_m(z) \sim \begin{cases} 1 \text{ (constant)} & m=0 \\ \dfrac{1}{2^m m!} z^m & , m>0 \end{cases}$$

$$Y_m(z) \sim \begin{cases} \dfrac{2}{\pi} \ln z & m=0 \\ \dfrac{-2^m (m-1)! \, z^{-m}}{\pi} & m>0 \end{cases}$$

We see that J_m is bounded where as Y_m is not bounded at $z \to 0$.

Eigen Value Problem involving Bessel functions.

We determine the Eigen value of the singlar SL problem

$$\frac{d}{dr}\left(r \frac{df}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right) f = 0$$

$$f(a) = 0$$

$$|f(0)| < \infty$$

$$r f'' + f' + \left(\lambda r - \frac{m^2}{r}\right) f = 0$$

$$r^2 f'' + r f' + (\lambda r^2 - m^2) f = 0$$ 119

Similar to previous

$\Rightarrow$

$$f = c_1 J_m(\sqrt{\lambda}\, r) + c_2 Y_m(\sqrt{\lambda}\, r)$$

$$f(a) = 0$$

$$\Rightarrow \; 0 = c_1 J_m(\sqrt{\lambda}\, a) + c_2 Y_m(\sqrt{\lambda}\, a)$$

also to justify

$|f(0)| < \infty$ we take $c_2 = 0$

$$f(r) = c_1 J_m(\sqrt{\lambda}\, r)$$

$$J_m(\sqrt{\lambda}\, a) = 0$$

$$\Rightarrow \sqrt{\lambda}\, a = e$$

There is an infinite no. of zeros of each Bessel function J_m

Let

Z_{mn} designate the nth zero of $J_m(z)$.

$$\sqrt{\lambda}\, a = Z_{mn}$$

$$\Rightarrow \lambda_{mn} = \left(\frac{Z_{mn}}{a}\right)^2$$

For each n there is infinite no. of eigenvalues. Thus solution

$$u(r,\theta,t) = \left\{ J_m(\sqrt{\lambda_{mn}}\, r) \right\} \left\{ \begin{matrix} \cos m\theta \\ \sin m\theta \end{matrix} \right\} \left\{ \begin{matrix} \cos c\sqrt{\lambda_{mn}}\, t \\ \sin c\sqrt{\lambda_{mn}}\, t \end{matrix} \right\} \quad (\star)$$

Assume that

$$\frac{\partial u(r,\theta,0)}{\partial t} = \beta(r,\theta) = 0$$

Thus the $\sin c\sqrt{\lambda_{mn}}\, t$ terms in $(\star)$ is not necessary to include in this solution.

Thus we can write

$$= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}}\, r) \cos m\theta \cos c\sqrt{\lambda_{mn}}\, t$$

$$+ \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} B_{mn} J_m(\sqrt{\lambda_{mn}}\, r) \sin m\theta \cos c\sqrt{\lambda_{mn}}\, t$$

Use the I.C.

$$u(r,\theta,0) = \alpha(r,\theta)$$

$$\Rightarrow \alpha(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}}\, r) \cos m\theta + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} B_{mn} J_m(\sqrt{\lambda_{mn}}\, r) \sin m\theta$$

It is somewhat easier to determine all coefficient using the 2-D orthogonality property.

Let

$$\phi_\lambda(r,\theta) = J_m(\sqrt{\lambda_{mn}}\, r)\begin{cases} \cos(\theta m) \\ \sin(m\theta). \end{cases}$$

Thus

$$\alpha(r,\theta) = \sum_{\lambda} A_\lambda \phi_\lambda(r,\theta).$$

where the $\sum_{\lambda}$ stands for a summation over all eigenvalues (actually two double sums including both $\cos m\theta$ & $\sin m\theta$)

which implies.

$$A_\lambda = \frac{\iint \alpha(r,\theta)\, \phi_\lambda(r,\theta)\, dA}{\iint \phi_\lambda^2(r,\theta)\, dA}$$

where $dA = r\,dr\,d\theta$.

where r is not a weight function its a geometric function.

14-04-2016

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=\frac{1}{c^2}u_{tt}$$

$$\Rightarrow \frac{1}{\cancel{r}}\cancel{r}\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=\frac{1}{c^2}u_{tt} \longrightarrow (1)$$

If we looking for a periodic function in t.

$$u=F(r)\,e^{i\omega t}$$

Then $u'=F'e^{i\omega t}$

$$u''=F''e^{i\omega t}$$

$$u_{tt}=-\omega^2 F e^{i\omega t}$$

$$(1)\Rightarrow \left[F''(r)+\frac{1}{r}F'(r)\right]\cancel{e^{i\omega t}}=-\frac{\omega^2}{c^2}F\cancel{e^{i\omega t}}$$

$$F''(r)+\frac{1}{r}F'(r)=-\frac{\omega^2}{c^2}F$$

$$r^2F''(r)+rF'(r)+\frac{\omega^2}{c^2}r^2F=0$$

Bessel Eq. of zero order.

$$F(r)=A\,J_0\left(\frac{\omega r}{c}\right)+B\,Y_0\left(\frac{\omega r}{c}\right)$$

If $A=c_1+c_2$

$B=i(c_1-c_2)$

$$\Rightarrow F(r)=c_1\left[J_0\left(\frac{\omega r}{c}\right)+i\,Y_0\left(\frac{\omega r}{c}\right)\right]$$
$$+c_2\left[J_0\left(\frac{\omega r}{c}\right)-i\,Y_0\left(\frac{\omega r}{c}\right)\right]$$

رانا شکور احمد
MS(Mathematics)

$$= C_1 H_0^{(1)}\left(\frac{wr}{c}\right) + C_2 H_0^{(2)}\left(\frac{wr}{c}\right).$$

$$\left.\begin{aligned} H_0^{(1)} &= \sqrt{\frac{2}{\pi x}}\, e^{i(x-\frac{\pi}{4})} \\ H_0^{(2)} &= \sqrt{\frac{2}{\pi x}}\, e^{-i(x-\pi/4)} \end{aligned}\right\} \text{Ankel function.}$$

PYS.
Question

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{1}{c^2}u_{tt}$$

Solution will be

$$u = \left(\frac{wr}{c}\right)^{-1/2}\left[A\, J_{1/2}\left(\frac{wr}{c}\right) + B\, Y_{1/2}\left(\frac{wr}{c}\right)\right]$$

Fourier Transform: 25-04-2016

Let $f(x)$ be a func. defined on $(-\infty, \infty)$ & it is peicewise continuous, differentiable at each finite interval and it is absolutely integrable on $(-\infty, \infty)$

i.e $\int_{-\infty}^{\infty} f(x)\, dx$ exists.

Thus, fourier transform of $f(x)$ is

$$F[f(x), \cancel{\alpha}] = F[\alpha]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

$$f(x) = F^{-1}[F(\alpha)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha)\, e^{-i\alpha x}\, d\alpha$$

Fourier Sine Trans.

$$f_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \alpha x\, dx$$

$$= F_s[f(x)] \text{ or } S[f(x)]$$

$$F^{-1}[f_s(\alpha)] = f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(\alpha) \sin \alpha x\, d\alpha$$

Similarly for Fourier Cosine Transformation

$$\bar{f}_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \alpha x \, dx$$

$$= F_c[f(x)]$$

&

$$F_c^{-1}[F_c(\alpha)] = f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\alpha) \cos \alpha x \, d\alpha$$

Question:-

$f(x) = e^{-x^2/2}$ find fourier Trans. of this Gaussian function.

$$F[f(x), \alpha] = F(\alpha)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{i\alpha x} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2i\alpha x)} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x - i\alpha)^2/2} \cdot e^{-\alpha^2/2} \, dx$$

$$= \frac{e^{-\alpha^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} \, dt$$

Put $x - i\alpha = t$ $\qquad \int_{-\infty}^{\infty} e^{-\alpha t^2} dt$

$dx = dt$

$$= \frac{e^{-\alpha^2/2}}{\cancel{\sqrt{2\pi}}} \cdot \cancel{\sqrt{2\pi}} \qquad = \sqrt{\frac{\pi}{\alpha}}$$

$$= e^{-\alpha^2/2}$$

The Gausian itself is a fourier function

<u>Question:</u> Find the fourier trans. of function

$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$$

& evaluate.

$$\int_{-\infty}^{\infty} \frac{\sin\alpha a \cos\alpha x}{\alpha} d\alpha$$

$$\int_{0}^{\infty} \frac{\sin\alpha a}{a} d\alpha$$

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x) e^{i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left. \frac{e^{i\alpha x}}{i\alpha} \right|_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}\, i\alpha} \left[e^{i\alpha a} - e^{-i\alpha a} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{\alpha} \left[\frac{e^{i\alpha a} - e^{-i\alpha a}}{2i} \right]$$

$$= \begin{cases} \frac{1}{\alpha} \sqrt{\frac{2}{\pi}} \sin(\alpha a). & \text{if } \alpha > 0 \\ \sqrt{\frac{2}{\pi}}\, a & \text{if } \alpha = 0 \end{cases}$$

Now

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\alpha x} F(\alpha)\, d\alpha \qquad \alpha > 0$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \cdot \frac{e^{-i\alpha x}}{\alpha} \sin \alpha a \, d\alpha$$

$$= \begin{cases} 1 & |x| \le a \\ 0 & |x| > a \end{cases}$$

$$\Rightarrow \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \alpha a}{\alpha} [\cos \alpha x - i \sin \alpha x] \, d\alpha$$

$$= \begin{cases} 1 & \text{if } |x| \le a \\ 0 & \text{if } |x| > a \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin \alpha a \cos \alpha x}{\alpha} d\alpha = \begin{cases} \pi & |x| \le a \\ 0 & |x| > a \end{cases}$$

if $x = 0$

$$\int_{-\infty}^{\infty} \frac{\sin \alpha a}{\alpha} d\alpha = \pi$$

Even function. then we can write

$$2\int_0^\infty \frac{\sin\alpha a}{\alpha}\,d\alpha = \pi \quad \text{if } \alpha > 0$$

$$\int_0^\infty \frac{\sin\alpha a}{\alpha}\,d\alpha = \frac{\pi}{2}$$

Question:- Find Fourier cosine & Fourier sine

$f(x) = e^{-bx}$ and evaluate

(i) $\displaystyle\int_0^\infty \frac{\cos\alpha x}{\alpha^2+b^2}\,d\alpha$

(ii) $\displaystyle\int_0^\infty \frac{\alpha\sin\alpha x}{\alpha^2+b^2}\,d\alpha$

Sol:-

$$f_c(\alpha) = \sqrt{\frac{2}{\pi}}\underbrace{\int_0^\infty f(x)\cos\alpha x\,dx}_{I_1}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \underset{II}{e^{-bx}}\,\underset{I}{\cos\alpha x}\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\cos\alpha x\,\frac{e^{-bx}}{-b}\Big|_0^\infty - \frac{\alpha}{b}\int_0^\infty \underset{II}{e^{-bx}}\,\underset{I}{\sin\alpha x}\,dx\right] \quad I_2$$

$$= \sqrt{\frac{2}{\pi}}\left\{\left[0+\frac{1}{b}\right] - \frac{\alpha}{b}I_2\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1}{b} - \frac{\alpha}{b} I_2\right]$$

$$\Rightarrow I_1 = \frac{1}{b} - \frac{\alpha}{b} I_2 \longrightarrow (*)$$

and let

$$I_2 = \int_0^\infty e^{-bx} \sin\alpha x \, dx$$

$$I_2 = \left. -\frac{e^{-bx}}{b} \sin\alpha x \right|_0^\infty + \frac{\alpha}{b} \underbrace{\int_0^\infty e^{-bx} \cos\alpha x \, dx}_{I_1}$$

$$I_2 = \frac{\alpha}{b} I_1$$

$$(*) \Rightarrow I_1 = \frac{1}{b} - \frac{\alpha^2}{b^2} I_1$$

$$I_1\left[1 + \frac{\alpha^2}{b^2}\right] = \frac{1}{b}$$

$$I_1\left[\frac{b^2+\alpha^2}{b^2}\right] = \frac{1}{b}$$

$$I_1 = \frac{b}{\alpha^2+b^2}$$

$$\Rightarrow I_2 = \frac{\alpha}{\cancel{b}}\left(\frac{\cancel{b}}{\alpha^2+b^2}\right) = \frac{\alpha}{\alpha^2+b^2}$$

So,

$$f_c(\alpha) = \sqrt{\frac{2}{\pi}} \cdot \frac{b}{\alpha^2+b^2}$$

&

$$f_s(\alpha) = \sqrt{\frac{2}{\pi}} \cdot \frac{\alpha}{\alpha^2+b^2}$$

then

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_c(\alpha) \cos \alpha x \, d\alpha$$

$$e^{-bx} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{b}{\alpha^2+b^2} \cos \alpha x \, d\alpha$$

$$\Rightarrow \int_0^\infty \frac{\cos \alpha x}{\alpha^2+b^2} d\alpha = \frac{\pi}{2b} e^{-bx}$$

Similarly,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_s(\alpha) \sin \alpha x \, d\alpha$$

$$e^{-bx} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{\alpha}{\alpha^2+b^2}\right) \sin \alpha x \, d\alpha$$

$$\Rightarrow \int_0^\infty \frac{\alpha \sin \alpha x}{\alpha^2+b^2} d\alpha = \frac{\pi}{2} e^{-bx}$$

Transform of Derivatives:- 28-04-2016

$C\left[\frac{df}{dx}\right] = ?$ "C" is for cosine.

$$C\left[\frac{df}{dx}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{df}{dx} \cos\omega x \, dx$$ Taking $\frac{2}{\pi}$ here

$f(x)$ is bounded.

$$= \sqrt{\frac{2}{\pi}}\left[f(x)\cos\omega x\Big|_0^\infty + \int_0^\infty f(x)\,\omega \sin\omega x\, dx\right]$$

We assume that $f(x)$ is bounded ($f(x) \to 0$ when $x \to \infty$)

$$= \sqrt{\frac{2}{\pi}}\left[-f(0) + \omega \int_0^\infty f(x) \sin\omega x\, dx\right]$$

$$C\left[\frac{df}{dx}\right] = \sqrt{\frac{2}{\pi}}\left[-f(0)\right] + \omega\, S\left[f(x)\right] \longrightarrow (1)$$

Now $$S\left[\frac{df}{dx}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{df}{dx} \sin\omega x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[f(x)\sin\omega x\Big|_0^\infty - \omega\int_0^\infty f(x)\cos x\omega\, dx\right]$$

$$= -\omega\, C\left[f(x)\right] \longrightarrow (2)$$

Note:- If a PDE contains first derivatives w.r.t the potential ~~Partial~~ variable to be tranformed then the fourier sine or cosine transform will never work. Since sine and cosine transform of first derivatives always involve the other type of semi infinite transform.

(ii) $C\left[\frac{d^2f}{dx^2}\right]$

$$C\left[\frac{d^2f}{dx^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{df(x)}{dx}\cos wx\Big|_0^\infty + \int_0^\infty \frac{df(x)}{dx} w \sin wx\, dx\right]$$

Assumption on $f(x)$ & $\frac{df(x)}{dx}$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{df(0)}{dx}\right] + \frac{\sqrt{2}}{\sqrt{\pi}} w \int_0^\infty \frac{df}{dx} \sin wx\, dx$$

From ①

$$= -\frac{\sqrt{2}}{\sqrt{\pi}}\left[\frac{df(0)}{dx}\right] - w^2 C[f(x)]$$

$$S\left[\frac{d^2f}{dx^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{df(x)}{dx}\sin wx\Big|_0^\infty - \int_0^\infty \frac{df(x)}{dx} w \cos wx\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}}[0] - \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{df(x)}{dx} w \cos wx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\, w f(0) - w^2 S[f(x)]$$

Note:- If $f(0)$ is given use fourier sine transform and If $\frac{df(0)}{dx}$ given use fourier cosine transform.

(iii) $F_c\left[e^{-at^2}\right] = ?$

$$\Rightarrow I(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-at^2} \cos\alpha t \, dt$$

$$I = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-at^2} \cos\alpha t \, dt$$

$$\frac{dI}{d\alpha} = -\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-at^2} t \sin\alpha t \, dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d(e^{-at^2})}{2a} \sin\alpha t \, dt$$

$$= \frac{1}{\sqrt{2\pi}a} \int_0^\infty d(e^{-at^2}) \sin\alpha t \, dt$$

Integration by parts gives

$$= \frac{-\alpha}{\sqrt{2\pi}a} \int_0^\infty e^{-at^2} \cos\alpha t \, dt$$

$$= \frac{-\alpha}{2a} I$$

$$\frac{dI}{d\alpha} + \frac{\alpha}{2a} I = 0$$

$$\int \frac{dI}{I} = -\int \frac{\alpha}{2a} d\alpha$$

$$\ln I = -\frac{\alpha^2}{4a} + \ln A$$

$$I = Ae^{-\alpha^2/4a}$$

at $\alpha = 0$

$$I(0) = A$$

$$I = I(0)\, e^{-\alpha^2/4a}$$

$$I(0) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-at^2}\, dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\sqrt{\frac{\pi}{a}}\right] \frac{1}{2}$$

$$\int_{-\infty}^{\infty} \underbrace{e^{-at^2}}_{\text{Even func}} dt = \sqrt{\frac{\pi}{a}}$$

$$= 2\int_0^\infty e^{-at^2}\, dt$$

$$\boxed{I(0) = \frac{1}{\sqrt{2a}}}$$

$$\boxed{I = \frac{1}{\sqrt{2a}}\, e^{-\alpha^2/4a}} \longrightarrow \text{Answer.}$$

Heat Eq. on a Semi infinite interval:-

$$u_t = K u_{xx} \longrightarrow (1)$$

B.C's $\quad u(0,t) = g(t) \longrightarrow (2)$

I.C. $\quad u(x,0) = f(x) \longrightarrow (3)$

The B.C. at $x = 0$ is non-Homo.
So, we can't apply separation of variables condition is on u
So using
Sine fourier transform

Let $\bar{U}_s(\omega,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\sin\omega x\,dx$

Taking fourier sine transform of (1), we have

$$\frac{d}{dt}(\bar{U}_s) = K\left[\sqrt{\frac{2}{\pi}}\,\omega\, u(0,t) - \omega^2 \bar{U}_s(x,t)\right]$$

$$= K\left[\sqrt{\frac{2}{\pi}}\,\omega\, g(t) - \omega^2 \bar{U}_s\right]$$

Taking fourier sine transform of I.C gives,

$$\bar{U}_s(\omega,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\omega x\,dx$$

$$\frac{d}{dt}(\bar{U}_s) + \omega^2 K\bar{U}_s = \sqrt{\frac{2}{\pi}}\,K\omega\, g(t)$$

$$\frac{d}{dt}\left(e^{\omega^2 Kt}\bar{U}_s\right) = \sqrt{\frac{2}{\pi}}\,K\omega\, g(t)\, e^{\omega^2 Kt}$$

$$e^{\omega^2 Kt}\,\bar{U}_s = \sqrt{\frac{2}{\pi}}\,K\omega\int g(t)\, e^{\omega^2 Kt}\,dt$$

$$e^{\omega^2 Kt}\,\bar{U}_s(x,t) = \sqrt{\frac{2}{\pi}}\,K\omega\int g(t)\, e^{\omega^2 Kt}\,dt + A(\omega)$$

$$\bar{U}_s = \frac{\sqrt{2}\,K\omega\, e^{-\omega^2 Kt}}{\sqrt{\pi}}\int e^{\omega^2 Kt} g(t)\,dt + A(\omega)\, e^{-\omega^2 Kt}$$

Special case

If $g(t) = u_0$

$f(x) = 0$

05-05-2016

$$\Rightarrow \bar{u}_{s_t} = k\left[\sqrt{\tfrac{2}{\pi}}\,\omega\, g(t) - \omega^2 \bar{u}_s\right]$$

$$= k\left[\sqrt{\tfrac{2}{\pi}}\,\omega\, u_0 - \omega^2 \bar{u}_s\right]$$

$$\frac{d\bar{u}_s}{dt} + k\omega^2 \bar{u}_s = \sqrt{\frac{2}{\pi}}\, k\omega u_0$$

$$I.F = e^{\int k\omega^2 dt} = e^{\omega^2 k t}$$

$$\bar{u}_s e^{k\omega^2 t} = \int \sqrt{\frac{2}{\pi}}\, k\omega u_0\, e^{\omega^2 t k}\, dt + A(\omega)$$

$$\bar{u}_s e^{k\omega^2 t} = \sqrt{\frac{2}{\pi}}\, \cancel{k}\omega u_0 \frac{e^{\omega^2 t k}}{\cancel{k}\omega^2} + A(\omega)$$

Applying $\bar{u}_s(\omega, 0) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx$

$$\Rightarrow \bar{u}_s(\omega, 0) = 0 \qquad \because f(x) = 0$$

$$0 = \sqrt{\frac{2}{\pi}}\,\frac{u_0}{\omega} + A(\omega)$$

$$\Rightarrow A(\omega) = -\sqrt{\frac{2}{\pi}}\,\frac{u_0}{\omega}$$

So,

$$\bar{u}_s e^{\omega^2 t k} = \sqrt{\frac{2}{\pi}}\cdot\frac{u_0}{\omega} e^{\omega^2 t k} - \sqrt{\frac{2}{\pi}}\cdot\frac{u_0}{\omega}$$

$$\bar{u}_s(\omega, t) = \sqrt{\frac{2}{\pi}}\cdot\frac{u_0}{\omega} - \sqrt{\frac{2}{\pi}}\cdot\frac{u_0}{\omega} e^{-\omega^2 t k}$$

By Inverse Fourier sine Transform.

$$F_s^{-1}\left[\bar{u}_s(\omega,t)\right] = F_s^{-1}\left[\frac{\sqrt{2}}{\sqrt{\pi}}\frac{u_0}{\omega}\left(1-e^{-\omega^2 tk}\right)\right]$$

$$u(x,t) = \frac{2}{\pi}\cdot u_0 \int_0^\infty \frac{\left(1-e^{-\omega^2 tk}\right)}{\omega}\sin\omega x \, d\omega$$

To solve $\int_0^\infty e^{-\omega^2 tk} \frac{\sin\omega x}{\omega} d\omega$

$\operatorname{erf}(x) = \frac{2}{\pi}\int_0^x e^{-u^2} du$

also

$\int_0^\infty e^{-\alpha^2} \frac{\sin(2\alpha x)}{\alpha} d\alpha = \frac{\pi}{2}\operatorname{erf}(x)$

& $\int_0^\infty \frac{\sin\alpha x}{\alpha} d\alpha = \frac{\pi}{2}$

Put $\omega^2 kt = \alpha^2$

$\cancel{2}\omega kt \, d\omega = \cancel{2}\alpha \, d\alpha$

$d\omega = \frac{\alpha \, d\alpha}{\omega kt}$

Put $\omega = \alpha/\sqrt{kt}$

$\Rightarrow d\omega = \frac{\cancel{\alpha}\, d\alpha \sqrt{kt}}{\cancel{\alpha} kt}$

$d\omega = \frac{d\alpha}{\sqrt{kt}}$

$$\Rightarrow \int_0^\infty e^{-\omega^2 kt} \frac{\sin\omega x}{\omega} d\omega = \int_0^\infty \frac{\sin\left(\frac{\alpha x}{\sqrt{kt}}\right)}{\alpha/\cancel{\sqrt{kt}}} e^{-\alpha^2} \frac{d\alpha}{\cancel{\sqrt{kt}}}$$

$$= \int_0^\infty \frac{\sin\left(2\alpha\left(\frac{x}{2\sqrt{kt}}\right)\right)}{\alpha} e^{-\alpha^2} d\alpha$$

$$= \frac{\pi}{2}\operatorname{erf}\left(\frac{x}{2\sqrt{kt}}\right)$$

Haberman 477 examples

Santara

Thus

$$u(x,t) = \frac{2}{\pi} u_0 \left[\frac{\pi}{2} - \frac{\pi}{2} \operatorname{erf}\left(\frac{x}{2\sqrt{Kt}}\right) \right]$$

$$= u_0 \left[1 - \operatorname{erf}\left(\frac{x}{2\sqrt{Kt}}\right) \right]$$

$$= u_0 \operatorname{erf}_{comp.}\left(\frac{x}{2\sqrt{Kt}}\right)$$

if $g(t) = 0$ $\quad u_t = K u_{xx}$; $u(0,t) = 0$

$u(0,t) = 0$, $u(a,0) = f(x)$

Statement & Illustration of Theorem for Eigenvalue problem:-

$$\nabla^2 \phi + \lambda \phi = 0 \longrightarrow (1) \longleftarrow \text{Helmohtz Eq.}$$

For sepration of variables Heat & weve Eq i.e $u(x,y,z,t) = \phi(x,y,z)\, h(t)$ we get eigen value problem for spatial cordinates.

With $a\phi + b \nabla \phi \cdot \hat{n} = 0 \longrightarrow (2)$

Where $\hat{n}$ is normal to the surface

$$\nabla \cdot (p \nabla \phi) + q \phi + \lambda \sigma \phi = 0 \quad \text{SL-Prob.}$$

(1) & (3) $\Rightarrow$ $p = 1$, $q = 0$, $\sigma = 1$

Properties:-

(1):- All eigenvalues are real.

e.g $\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$

$m, n = 1, 2, 3, \ldots$

2):- There exist infinite no. of eigen values. There is a smallest eigenvalue but no larger value.

Smallest $\lambda_{1,1}$

3):- Corresponding to an eigenvalue there may be many eigenfunction (unlike Regular SL problems).

Example:- $\lambda_{nm} = \frac{\pi^2}{4H}\left[n^2 + 4m^2\right]$

$$\phi_{nm} = \sin\frac{n\pi}{2H}x \; \sin\frac{m\pi}{H}y$$

$$\lambda_{41} = \lambda_{22}$$

but

$$\phi_{41} = \sin\frac{2\pi}{H}x \; \sin\frac{\pi}{H}y$$

$$\phi_{22} = \sin\frac{\pi}{H}x \; \sin\frac{2\pi}{H}y$$

$\Rightarrow \phi_{41} \neq \phi_{22}$

$m=1,\ n=2 \Rightarrow \lambda_{41} = \lambda_{22}$

$$\lambda_{(2n)m} = \lambda_{(2m)n}$$

Put $n=3$, $m=5$

$$\lambda_{65} = \frac{\pi^2}{4H}\left[36 + 4(25)\right]$$

$$\lambda_{10\,3} = \frac{\pi}{4H}\left[100 + 36\right]$$

$$\Rightarrow \lambda_{65} = \lambda_{10\,3}$$

4:- The eigenfunction form a complete set meaning that any piecewise smooth func $f(x,y)$ / $\phi(x,y)$ can be represented as generalized fourier series of the eigen-function.

$$f(x,y) \sim \sum_{\lambda} a_\lambda \phi_\lambda(x,y)$$

~~Ex~~

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}$$

(5) The eigenfunctions belonging to different eigen values are orthogonal relative to the weight func over the entire region.

$$\iint_R \phi_{\lambda_1} \phi_{\lambda_2} \, dx\, dy = 0$$

if $\lambda_1 \neq \lambda_2$

We may assume that (5) is valid even $\lambda_1 = \lambda_2$ as long as ϕ_{λ_1} and ϕ_{λ_2} are different (or independent).

6):- An eigenvalue λ can be related to eigenfunction by the Rayleigh ---- quotient.

$$\lambda = \frac{-\oint \phi \nabla \phi \cdot \hat{n} ds + \iint_R |\nabla \phi|^2 dx dy}{\iint_R \phi^2 dx dy}$$

Consider

$$\nabla^2\phi + \lambda\phi = 0$$

xing by ϕ from left.

$$\Rightarrow \quad \phi\nabla^2\phi + \lambda\phi^2 = 0$$

$$\Rightarrow \quad \iint_R \phi\nabla^2\phi\, dx\,dy + \lambda\iint_R \phi^2\, dx\,dy = 0$$

$$\Rightarrow \quad \lambda = -\frac{\iint_R \phi\, \nabla^2\phi\, dx\,dy}{\iint_R \phi^2\, dx\,dy}$$

$$\nabla\cdot(fg) = f\,\nabla\cdot g + g\cdot\nabla f$$

if $\quad f = \phi$

$\quad\quad g = \nabla\phi$

$$\nabla\cdot(\phi\nabla\phi) = \phi(\nabla\cdot\nabla\phi) + \nabla\phi\cdot(\nabla\phi)$$

$$\nabla\cdot(\phi\nabla\phi) = \phi\nabla^2\phi + |\nabla\phi|^2$$

$$\Rightarrow \phi\nabla^2\phi = \nabla\cdot(\phi\nabla\phi) - |\nabla\phi|^2$$

Put in λ.

$$\lambda = \frac{-\iint_R [\nabla\cdot(\phi\nabla\phi) - |\nabla\phi|^2]\,dxdy}{\iint_R \phi^2\,dxdy}$$

$$= \frac{-\iint_R \nabla\cdot(\phi\nabla\phi)\,dxdy + \iint_R |\nabla\phi|^2\,dxdy}{\iint_R \phi^2\,dxdy}$$

$$\lambda = \frac{-\oint \phi\nabla\phi\cdot\hat{n}\,ds + \iint_R |\nabla\phi|^2\,dxdy}{\iint_R \phi^2\,dxdy}$$

using Gauss Div. theorem.

$$\iint_R \nabla\cdot\vec{F}\,dxdy = \oint \vec{F}\cdot\hat{n}\,ds$$

Example:– we consider any region in which the B.C is $\phi = 0$ on the entire boundary.

then

$$\cancel{\oint \phi\nabla\phi\,\hat{n}\cdot ds} \rightarrow 0 \leftarrow \text{on boundary.}$$

$$\iint_R \phi^2\,dxdy \neq 0 \leftarrow \text{on region.}$$

If $\lambda = 0$ then

$$\iint_R |\nabla\phi|^2 \, dx\, dy = 0$$

$$\nabla\phi = \frac{\partial}{\partial x}\phi\,\hat{i} + \frac{\partial\phi}{\partial y}\hat{j}$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = 0 \quad \& \quad \frac{\partial\phi}{\partial y} = 0$$

$\Rightarrow$ ϕ is equal to zero everywhere.

Therefore ϕ is zero "

but ϕ not an eigenfunction

hence $\lambda > 0$ is not an eigenfunct.

Selfadjoint

Green's Formula 09-05-2016

As

(i) $\nabla^2 u = \nabla\cdot(\nabla u)$

(ii) $\nabla\cdot(a\vec{B}) = a\,\nabla\cdot\vec{B} + \nabla a\cdot\bar{B}$

$$\nabla\cdot(u\cdot\nabla v) = u\cdot\nabla^2 v + \nabla u\cdot\nabla v \quad \to (1)$$

$$\nabla\cdot(v\,\nabla u) = v\,\nabla^2 u + \nabla v\cdot\nabla u \quad \to (2)$$

$$\iint_R (u\nabla^2 v - v\nabla^2 u)\,dx\,dy = \iint_R \left(\nabla\cdot(u\nabla v - v\nabla u)\right)dx\,dy$$

$$= \oint (u\nabla v - v\nabla u)\cdot\hat{n}\,ds$$

If u & v are functions such that

$$u\nabla v - v\nabla u = 0$$

then these are selfadjoint operators.

Then

$$\iint_R (u\nabla^2 v - v\nabla^2 u)\,dx\,dy = 0$$

Green Identity

Dirac Delta function.

$$\delta_\epsilon(t) = \begin{cases} \frac{1}{2\epsilon} & \text{if } |t| < \epsilon \\ 0 & \text{if } |t| > \epsilon \end{cases}$$

$$\int_{-\infty}^{\infty} \delta_\epsilon(t)\, dt = \int_{-\epsilon}^{\epsilon} \delta_\epsilon(t)\, dt = 1$$

$$\int_{-\infty}^{\infty} f(t)\, \delta_\epsilon(t)\, dt = \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} f(t)\, dt = f(\xi)$$

$$-\epsilon < \xi < \epsilon$$

Def.

$$\delta(t) = \lim_{\epsilon \to 0} \delta_\epsilon(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases} \longrightarrow (1)$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \longrightarrow (2)$$

This limiting $\delta(t)$ define by (1) & (2)

Properties:—

② $\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$

condition: f(t) should be const

③ $\int_{-\infty}^{\infty} \delta(t-a) f(t)\,dt = f(a)$

④ $\delta(-t) = \delta(t)$

⑤ $\delta(at) = \frac{1}{a}\delta(t)$, $a > 0$

⑥ $\int_{-\infty}^{\infty} f(t)\,\delta'(t)\,dt = -f'(0)$

⑦ $\int_{-\infty}^{\infty} \delta'(t-a) f(t)\,dt = -f'(a)$

Consider Differential Eq.

$L\,u(x) = f(x) \longrightarrow (1)$

where L is ordinary linear operator. $f(x)$ is known function while $u(x)$ is unknown function.

$L^{-1} f(x) = u(x) = L^{-1}(L\,u) = \int G(x,\xi) f(\xi)\,d\xi \rightarrow (2)$

Kernal

Greens function.

apply L-operator.

$$\Rightarrow f(x) = \int L\, G(x,\xi) f(\xi)\, d\xi \rightarrow ③$$

$$L\, G(x, \xi) = \delta(x-\xi) \longrightarrow ④$$

where $\delta(x-\xi)$ is called dirac delta.

The solution of the Eq. ④ is called singularity sol of ①.

consider

$$\int_{-\infty}^{\infty} \delta'(t)\, \phi(x)\, dx = -\phi'(0)$$

for every test func $\phi(x)$

we define a heaviside unit step function.

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$\phi(x)$ is a function of a real variable x. which possess derivative of all orders & vanishes outside of a finite interval such functions are called test function.

test func.

$$\int_{-\infty}^{\infty} \underset{II}{H'(x)}\,\underset{I}{\phi(x)}\,dx = H(x)\,\cancel{\phi(x)}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H(x)\,\phi'(x)\,dx$$

$$= -\int_0^{\infty} 1\cdot\phi'(x)\,dx = -\phi(x)\Big|_0^{\infty}$$

$$= \phi(0).$$

$$= \int \delta(x-0)\,\phi(x)\,dx$$

$$\Rightarrow \quad H'(x) = \delta(x-0)$$

$$\int_{-\infty}^{\infty} H'(x-\xi)\,\phi(x)\,d\xi = \int_{-\infty}^{\infty} \delta(x-\xi)\,\phi(x)\,dx$$

$$= \phi(\xi)$$

$$\frac{d}{dx}\left\{(x-\xi)\,H(x-\xi)\right\} = (x-\xi)\,H'(x-\xi) + H(x-\xi)$$

integrationg.

$$\int H'(x-\xi)(x-\xi)\,dx = 0$$

$$(x-\xi)\,H(x-\xi) = \int_{-\infty}^{\infty} H(x-\xi)\,d\xi$$

Example: $\dfrac{d^2u}{dx^2} = f(x)$ Green func by Heaviside.

$u(0) = 0 = u(1)$, $LG(x,\xi) = \delta(x-\xi)$

$$\Rightarrow \frac{d^2 G(x,\xi)}{dx^2} = \delta(x-\xi)$$

$$\frac{dG}{dx} = \int \delta(x-\xi)\,dx = \int H'(x-\xi)\,dx$$

$$\frac{dG}{dx} = H(x-\xi) + A(\xi).$$

$$\Rightarrow G = \int H(x-\xi)\,dx + xA(\xi) + B(\xi)$$

$$G(x,\xi) = (x-\xi)\,H(x-\xi) + xA(\xi) + B(\xi).$$

where A & B can be determine by boundary conditions.

$$u(x) = \int_0^x (x-\xi)\,H(x-\xi)\,f(\xi)\,d\xi + x\int_{-\infty}^{\infty} A(\xi) f(\xi)\,d\xi$$

$$+ \int_{-\infty}^{\infty} B(\xi) f(\xi)\,d\xi$$

$$u(0) = 0 \Rightarrow 0 = \int_{-\infty}^{\infty} B(\xi) f(\xi)\,d\xi$$

$$\Rightarrow \boxed{B(\xi) = 0}$$

Now

$u(1) = 0$

$$0 = \int_0^1 (1-\xi)H(1-\xi)\,f(\xi)\,d\xi + \int_{-\infty}^{\infty} A(\xi) f(\xi)\,d\xi$$

$$A(\xi) = (\xi - 1)\,\cancel{H(1-\xi)} \quad \text{if } 0 < \xi < 1$$

$$= 0 \quad \text{if } 0 > \xi > 1$$

$$\Rightarrow \boxed{A(\xi) = \xi - 1}$$

$$G(x,\xi) = (x-\xi)\, H(x-\xi) - x(1-\xi)$$

$$0 \le \xi \le 1$$

$$\Rightarrow G(0,\xi) = 0 = G(1,\xi)$$

12-05-2016

~~Establish G.F using~~ variation of parameters.

① $Ly = \{p(x)y'\}' + q(x)y = -f(x) \quad a<x<b$

② $\alpha y(a) + \beta y'(a) = 0$

③ $\alpha_1 y(b) + \beta_1 y'(b) = 0$

If $\lambda = 0$ is not an eigenvalue of the associated S.L system

$y(x) = u_1 y_1 + u_2 y_2$

Here y_1 & y_2 are solutions of the homogeneous Eq.

To be specific we determine y_1 & y_2 up to a constant multiply by requiring that y_1 satisfied condition ② & y_2 satisfied condition ③.

① $\Rightarrow$ $py'' + p'y' + qy = -f(x)$

$$y'' + \frac{p'}{p}y' + \frac{q}{p}y = -\frac{f(x)}{p}$$

$$c_1' = -\frac{y_2 R}{w}$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$c_1' = \frac{+y_2 f(x)}{pw}$$

& $$c_2' = -\frac{y_1 f(x)}{pw}$$

we determine c_1 & c_2 uniquely and obtained

$$y(x) = y_2(x)\int_a^x \frac{y_1(\xi) f(\xi)}{P(\xi) w(\xi)} d\xi - y_1(x)\int_x^b \frac{y_2(\xi) f(\xi)}{P(\xi) w(\xi)} d\xi$$

Let $$G(x,\xi) = \begin{cases} -\dfrac{y_2(x) y_1(\xi)}{P(\xi) w(\xi)} & a \le \xi \le x \\[2ex] -\dfrac{y_1(x) y_2(\xi)}{P(\xi) w(\xi)} & x \le \xi \le b \end{cases}$$

$$\Rightarrow y = \int_a^b G(x,\xi) f(x)\, dx$$

Property $G(x,\xi) = G(\xi, x)$

Example:-

$$y'' = -f$$

$$y(0) = 0 = y'(l)$$

$D^2 = 0$

$\Rightarrow$

$$y = A + Bx$$

$$y_1 = A$$

$$y_2 = Bx$$

$y_1(0) = 0$

$\Rightarrow A = 0$

$y'(l) = 0 \Rightarrow B = 0$

$\Rightarrow y = 0$ $\lambda = 0$ ~~there~~ is not an eigenvalue.

Assume that

$y_1 = 1$ $\qquad p(x) = 1$

$y_2 = x$ $\qquad w(\xi) = 1$

$$G(x,\xi) = \begin{cases} -\dfrac{y_2(x)\, y_1(\xi)}{p(\xi)\, w(\xi)} = -x \\ -\dfrac{y_1(x)\, y_2(\xi)}{p(\xi)\, w(\xi)} = -\xi \end{cases}$$

Haberman Chap# 9.

$$W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1$$

$$(py')' + (q + \lambda R)y = 0$$

<u>To obtain Fourier representation of Green's function:</u>

Let $\{\lambda_n\}_{n\geq 1}$ be the eigenvalue and $\{\phi_n\}_{n\geq 1}$ a set of normalized eigenfunctions for the Eq.

$$(py')' + qy + \delta(x)\lambda y = 0$$

where $\delta = 1$ <u>weight func.</u>

then the Green func. is written as

$$G(x, \xi) = \sum_{n=1}^{\infty} \frac{\phi_n(\xi)\,\phi_n(x)}{\lambda_n \delta}$$

This series is absolutely uniform & convergent b/w interval $[a, b]$.

Example:- $\lambda = 0$ not an eigenvalue.

$$\phi_n'' + \lambda \phi_n = 0 \qquad u'' = -f(x)$$

$$u(0) = 0 = u'(l)$$

$$D^2 = -\lambda$$

$$D = \pm\sqrt{\lambda}\, i$$

$$\phi_n(0) = 0 = \phi_n'(l)$$

$$\phi_n = A\cos\sqrt{\lambda}\, x + B\sin\sqrt{\lambda}\, x$$

$$\phi_n(0) = 0 \Rightarrow A = 0$$

$$\phi_n'(l) = 0 \Rightarrow 0 = \sqrt{\lambda} B\cos\sqrt{\lambda}\, l$$

$$B \neq 0 \text{ but } \cos\sqrt{\lambda}\, l = 0$$

$$\sqrt{\lambda} = \frac{(2n-1)\pi}{2l}$$

$$\therefore \phi_n(x) = B_n \sin\sqrt{\lambda}\, x$$

To normalize

$$\int_0^l \phi_n^2\, dx = \int_0^l B_n^2 \sin^2\sqrt{\lambda}\, x\, dx$$

$$= \frac{B_n^2}{2}\left(x - \frac{\sin 2\sqrt{\lambda}\, x}{2\sqrt{\lambda}} \Big|_0^l \right)$$

$$= \frac{B_n^2}{2}(l)$$

Let $B_n = \sqrt{\frac{2}{l}}$

then

$$\int_0^l \phi_n^2 \, dx = 1$$

So, $\phi_n = \sqrt{\frac{2}{l}} \sin\sqrt{\lambda} x$

$$G(x, \xi) = \sum_{n=1}^{\infty} \frac{\sqrt{2/l} \, \sin\sqrt{\lambda} x \, \sqrt{\frac{2}{l}} \, \sin\sqrt{\lambda} \xi}{\left(\frac{(2n-1)\pi}{2l}\right)^2}$$

$$G(x, \xi) = \sum_{n=1}^{\infty} \left(\sqrt{\frac{2}{l}}\right)^2 \frac{\sin\sqrt{\lambda} x \, \sin\sqrt{\lambda} \xi}{\frac{((2n-1)\pi)^2}{(2l)^2}}$$

$$= \sum_{n=1}^{\infty} 8l \frac{\sin\sqrt{\lambda} x \, \sin\sqrt{\lambda} \xi}{((2n-1)\pi)^2}$$

P#383

Properties:-

$$(p(x)y')' + q(x)y + \lambda\, \delta(x)\, y = +f(x)$$

$$R_0(y) = \alpha\, y(a) + \beta\, y'(a) = 0$$

$$R_1(y) = \alpha\, y(b) + \beta\, y'(b) = 0$$

$$y = \int_a^b G(x,\xi) f(\xi)\, d\xi.$$

① :- The func. $G(x,\xi)$ will satisfies

$$L(G) = 0 \quad \text{if } x \neq \xi$$

$$[a,\xi) \cup (\xi, b]$$

②. G satisfies the same B.C's.

$$R_0(G) = 0$$

$$R_1(G) = 0$$

③. The func. G is continuous for $a \le \xi$ & $x \le b$

④. The Partial derivatives are continuous except $x=\xi$

$$\left(\frac{\partial G}{\partial x}\right)(\xi+0,\xi)-\left(\frac{\partial G}{\partial x}\right)(\xi-0,\xi)=\frac{-1}{P(\xi)}$$

⑤ The func. G is symmetric

$$G(x,\xi)=G(\xi,x) \text{ for } a\le \xi, x\le b$$

Question:- $u''+\lambda u=-f(x)$

$$u''+\lambda u=0$$

$$u(0)=0=u(1)$$

Ist step
no eigenvalue at $\lambda=0$

$$\lambda=0 \Rightarrow u''=0$$

$$u=A+Bx$$

Use property ① $\Rightarrow$
$G(0)=0=G(1)$

$$G''+\lambda G=0$$

$$G=A+Bx \quad x\neq\xi$$

16-05-16

$$u'' + \lambda u = 0$$

$$u(0) = 0, \quad u(1) = 0$$

For $\lambda = 0$

$$u'' = 0$$

$$m^2 = 0$$

$$u = A + Bx$$

$u(0) = 0 \Rightarrow A = 0$

$u(1) = 0 \Rightarrow B = 0$

$\lambda = 0$ is not an eigenvalue.

Step 2.

$LG = 0$, at $x \neq \xi$

$$\Rightarrow G''(x, \xi) = 0$$

$$G(x, \xi) = \begin{cases} A_1 + B_1 x & 0 \leq x < \xi \\ a_1 + b_1 x & \xi < x \leq 1 \end{cases}$$

Step 3. $G(0, \xi) = 0$

$G(1, \xi) = 0$

Applying $G(0,\xi)=0 \Rightarrow A_1=0$

$G(1,\xi)=0 \Rightarrow a_1+b_1=0$

$\Rightarrow a_1=-b_1$

$$G(x,\xi)=\begin{cases} B_1 x & 0\le x<\xi \\ -b_1+b_1 x & \xi<x\le 1 \end{cases}$$

Step # 4. Since G is continuous on $[0,1]$.

Therefore G is conti at $x=\xi$

$B_1\xi = b_1(\xi-1)$

$B_1 = \dfrac{b_1(\xi-1)}{\xi}$

$$\therefore G(x,\xi)=\begin{cases} \dfrac{b_1(\xi-1)}{\xi}x & 0\le x<\xi \\ b_1(x-1) & \xi<x\le 1 \end{cases}$$

Step 5. $G'(\xi+0,\xi)-G'(\xi-0,\xi)=-\dfrac{1}{P(\xi)}$

Derivative w.r.t. x

$b_1\xi - b_1(\xi-1) = -\xi$

$\boxed{b_1=-\xi}$

$$G(x,\xi)=\begin{cases} -(\xi-1)x & 0\le x<\xi \\ -\xi(x-1) & \xi<x\le 1 \end{cases}$$

$y''=-f$

$y(0)=0$

$y'(\ell)=0$