"VECTOR SPASES"

FIELD:-

A set of F with two binary operation "+" and "$\circ$" is called field iff

(i) $(F, +)$ is abelian group.

(ii) $(F - \{0\}, \circ)$ is abelian group.

(iii) Distribution law holds in F i.e

$$a \circ (b + c) = a \circ b + a \circ c \quad \forall\, a, b, c \in F$$

$$(a + b) \circ c = a \circ c + b \circ c$$

where 0 is identity element of F w.r.t "+".

Note:-

A field contains atleast two elements.

Example:-

The set $Z_5 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ is a field under addition and multiplication of residue classes.

Proof:-

(1) For $(Z_5, +)$ the Cayley's table is

$+$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{4}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{4}$	$\bar{4}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$

$\Rightarrow (Z_5, +)$ is an abelian group

(2) For $(Z_5 - \{0\}, \cdot)$ Caylay's table is

$\cdot$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{2}$	$\bar{2}$	$\bar{4}$	$\bar{1}$	$\bar{3}$
$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{4}$	$\bar{2}$
$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$

$\Rightarrow (Z_5 - \{0\}, \cdot)$ is an abelian group

(3) $\forall\ \bar{a}, \bar{b}, \bar{c} \in Z_5$

$\bar{a} \cdot (\bar{b} + \bar{c}) = \bar{a} \cdot \bar{b} + \bar{a} \cdot \bar{c}$

and $(\bar{a} + \bar{b}) \cdot \bar{c} = \bar{a} \cdot \bar{c} + \bar{b} \cdot \bar{c}$

i-e distributive laws holds in Z_5

$\Rightarrow Z_5$ is a field.

Notation :-

If F is a field with binary operations "+" and "." then we denote it as $(F, +, \cdot)$

Examples :-

(1) $(R, +, \cdot)$ is a field.

(2) $(Q, +, \cdot)$ " " "

(3) $(\mathbb{C}, +, \cdot)$ " " "

VECTOR SPACE :-

Let F be a field and V be a non-empty set with binary operation of addition "+" an for every $a \in F$ and $v \in V$

$$av \in V \quad \text{or} \quad va \in V$$

then V is called a vector space over F if

(1) $(V, +)$ is an abelian group.

(2) $a(v_1 + v_2) = av_1 + av_2 \quad \forall\, a \in F$ and $v_1, v_2 \in V$

(3) $(a+b)v = av + bv \quad \forall\, a, b \in F$ and $v \in V$

(4) $a(bv) = (ab)v \quad$ " " "

(5) $1v = v \quad \forall\, v \in V$

and 1 is the identity element of F w.r.t multiplication.

Notes:-

(1) If V is a vector space over field F then denote it as $V(F)$

(2) The elements of F are called scalars and the elements of V are called vectors.

(3) The scalars multiplication for a vector space V over the field F is the function

$$f : F \times V \longrightarrow V \quad \text{defined by}$$

$f(a, v) = av \quad \forall\, a \in F, v \in V$

(4) The additive identity of V is denoted by O_V or 0 and is called zero vector and additive identity of F is denoted by O_F and is called zero scalar.

Imp

Theorem 3 - V.V

Every field F is a vector space over itself.

Proof:- Let $(F, +, \cdot)$ is a field then

(1) $(F, +)$ is a group $\because F$ is field

(2) $a \cdot (l+m) = a \cdot l + a \cdot m \quad \forall\, a \in F$ and $l, m \in F$ (by distributive property of Field)

(3) $(a+b) \cdot l = a \cdot l + b \cdot l \quad \forall\, a, b \in F$, $l \in F$ (by distributive property of field)

(4) $a(bl) = (ab)l \quad \forall\, a, b \in F, l \in F$ (by associative property of multiplication of Field)

(5) $1l = l \quad \forall\, l \in F$

$\Rightarrow F$ is vector space over F i-e Every field is a vector space over itself.

Examples:-

The set $R^3 = \{(x, y, z) : x, y, z \in R\}$ is a vector space over field $(R, +, \cdot)$

proof:-

$\forall \; a \in R$ and $\forall \; (x, y, z) \in R^3$

$a(x, y, z) = (ax, ay, az) \in R^3$

i-e Scalar multiplication of elements of R^3 with element of R is defined and

I (1) $\forall \; (x_1, y_1, z_1), (x_2, y_2, z_2) \in R^3$

$\Rightarrow (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in$

i-e R^3 is closed under addition "+"

(2) $\forall \; (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$

$[(x_1, y_1, z_1) + (x_2, y_2, z_2)] + (x_3, y_3, z_3)$

$= (x_1 + x_2, y_1 + y_2, z_1 + z_2) + (x_3, y_3, z_3)$

$= [(x_1 + x_2) + x_3, (y_1 + y_2) + y_3, (z_1 + z_2) + z_3]$

$= (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3), z_1 + (z_2 + z_3))$

$\because$ associative law holds in R

$= (x_1, y_1, z_1) + (x_2 + x_3, y_2 + y_3, z_2 + z_3)$

$= (x_1, y_1, z_1) + [(x_2, y_2, z_2) + (x_3, y_3, z_3)]$

i-e "+" is associative in R^3

(3) $\forall (x, y, z) \in R^3 \ \exists \ (0,0,0) \in R^3$
such that

$(x, y, z) + (0,0,0) = (x+0, y+0, z+0)$
$= (x, y, z)$

$(0,0,0) + (x, y, z) = (0+x, 0+y, 0+z)$
$= (x, y, z)$

$\Rightarrow (0,0,0)$ is additive identity in R^3

(4) $\forall (x, y, z) \in R^3 \ \exists \ (-x, -y, -z) \in R^3$
such that

$(x, y, z) + (-x, -y, -z) = (-x, -y, -z) + (x, y, z)$
$= (0, 0, 0)$

$\Rightarrow$ Inverse of each element of R^3 is present in R^3

$\Rightarrow (R^3, +)$ is a group

Also

(5) $(x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$= (x_2 + x_1, y_2 + y_1, z_2 + z_1) \quad \because (R, +)$ is abelian

$= (x_2, y_2, z_2) + (x_1, y_1, z_1)$

$\Rightarrow (R^3, +)$ is an abelian group

II $\forall\ a \in R$ and $(x_1, y_1, z_1), (x_2, y_2, z_2) \in R^3$

$a[(x_1, y_1, z_1) + (x_2, y_2, z_2)]$

$= a(x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$= (a(x_1 + x_2), a(y_1 + y_2), a(z_1 + z_2))$

$= (ax_1 + ax_2, ay_1 + ay_2, az_1 + az_2)$

$\because$ distribution law hold in R

$= (ax_1, ay_1, az_1) + (ax_2, ay_2, az_2)$

$= a(x_1, y_1, z_1) + a(x_2, y_2, z_2)$

III $\forall\ a, b \in R$ and $(x, y, z) \in R^3$

$(a+b)(x, y, z)$

$= ((a+b)x, (a+b)y, (a+b)z)$

$= (ax + bx, ay + by, az + bz)$

$\because$ distribution law holds in R

$= (ax, ay, az) + (bx, by, bz)$

$= a(x, y, z) + b(x, y, z)$

IV $\forall\ a, b \in F$ and $(x, y, z) \in R^3$
we have

$a(b(x, y, z))$

$= a(bx, by, bz)$

$= (a(bx), a(by), a(bz))$

$= ((ab)x, (ab)y, (ab)z)$
$\because$ associative law holds in R

$= (ab)(x, y, z)$

V $1(x, y, z) = (1x, 1y, 1z) = (x, y, z)$
$\forall\ (x, y, z) \in R^3$
and $1 \in R$ is multiplicative identity in R.

$\Rightarrow R^3$ is a vector space over R.

Note:-
Similarly $R^n(R)$ is vector space $\forall\ n = 1, 2, 3, \ldots$

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Theorem 3 -

Let $V(F)$ be a vector space then

Imp (i) $aO_V = O_V \quad \forall\ a \in F$

Imp (ii) $O_F V = O_V \quad \forall\ v \in V$

(iii) $(-a)V = a(-V) \quad \forall\ a \in F$ and $v \in V$

(iv) If $aV = O_V$ then either $a = O_F$ or $V = O_V$

(v) $a(V_1 + V_2) = aV_1 - aV_2$

(i) $aO_V = O_V \quad \forall\ a \in F$

proof:-

$$aO_V = a(O_V + O_V)$$

$\Rightarrow \quad aO_V = aO_V + aO_V$ by second property of vector space

$\Rightarrow \quad aO_V + O_V = aO_V + aO_V$

$\Rightarrow \quad O_V = aO_V$ by cancellation law

OR

$$aO_V = O_V \quad \forall\ a \in F$$

(ii) $O_F V = O_V \quad \forall\ v \in V$

proof:-

$$O_F V = O_F V + O_V$$

$\Rightarrow \quad (O_F + O_F)V = O_F V + O_V$ by 3rd property of vector space

$\Rightarrow \quad O_F V + O_F V = O_F V + O_V$

$\Rightarrow \quad O_F V = O_V$ by cancellation law

(iii) $(-a)V = a(-V) \quad \forall\ a \in F$ and $v \in V$

proof:-

Since $0_F v = 0_V \quad \forall\ v \in V$

$\Rightarrow (a+(-a))v = 0_V$

$\Rightarrow av + (-a)v = 0_V$ by 3rd property of Vector space

$\Rightarrow av + (-a)v = (av) + (-av)$

$\Rightarrow (-a)v = -av$ by cancellation law — (1)

Now

$a 0_V = 0_V \quad \forall\ a \in F$

$\Rightarrow a(v+(-v)) = 0_V$

$\Rightarrow av + a(-v) = 0_V$ by 2nd property of V

$\Rightarrow av + a(-v) = av + (-av)$

$\Rightarrow a(-v) = -av$ by cancellation law — (2)

from (1) and (2)

$(-a)v = a(-v) = -av$

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(iv) V.V.Imp

If $av = 0_V$ then either $a = 0_F$ or $v = 0_V$

proof:-

Let $av = 0_V$

Let $a \neq 0_F$

$\Rightarrow \exists\ a^{-1} \in F$ such that

$a^{-1}a = aa^{-1} = 1$, the identity w.r.t multiplication of F

$\Rightarrow$ from (1) we can write
$$a^{-1}(av) = a^{-1}(O_v)$$
$\Rightarrow (a^{-1}a)v = a^{-1}O_v$ by 4th property of V

$\Rightarrow 1v = O_v$ from part (i)

$\Rightarrow v = O_v$ by 5th property of V

Now
Let $av = O_v$
and $v \neq O_v$
and since
$$v = v + O_v$$
$\Rightarrow v = v + av$
$\Rightarrow 1v = 1v + av$ by 5th property of V
$\Rightarrow 1v = (1+a)v$ by 3rd " " "
$\Rightarrow 1 = 1 + a$ by comparing co-efficients
since $v \neq O_v$

$\Rightarrow -1+1 = -1+1+a$
$\Rightarrow O_F = O_F + a$
$\Rightarrow O_F = a$
or $a = O_F$
$\Rightarrow$ if $av = O_v$
then either $a = O_F$ or $v = O_v$

(v) $a(v_1 - v_2) = av_1 - av_2$

Proof:-
$$a(v_1 - v_2)$$

$= a(v_1 + (-v_2))$

$= av_1 + a(-v_2)$ by 2nd property of V

$= av_1 - av_2$ from part (iii)

"Subspace" 2002

Let $V(F)$ be a vector space and W is a non-empty subset of V, then W is called a subspace of V if under the operation of V, W itself forms a vector space over field F.

Example :-
Let V be the vector space of R^3 then the set $W = \{(x, y, 0) : x, y \in R\}$ is a subspace of V.

Note :-
Every vector space $V \neq \{0v\}$ has two subspaces namely V itself and the singleton set $\{0v\}$.

These two subspaces are called improper subspaces of V any other subspace of V is called a proper subspace of V.

Theorem 3- V. V. Imp 2002
Let $V(F)$ be a vector space over

a field F then W is a subspace of V iff

(i) $w_1, w_2 \in W$, $w_1 + w_2 \in W$

(ii) $w \in W$ and $a \in F \Rightarrow aw \in W$

Proof:-

Suppose W is a non-empty subset of V satisfying (i) and (ii) we are to prove W is a vector space.

(1) For $w_1, w_2 \in W$ and $-1 \in F$

$\Rightarrow -1\, w_2 \in W$ by (ii) property (given)

$\Rightarrow -w_2 \in W$

and $w_1, -w_2 \in W$

$\Rightarrow w_1 + (-w_2) \in W$ by (i) property (given)

$\Rightarrow w_1 - w_2 \in W$

$\Rightarrow (W, +)$ is a subgroup of $(V, +)$

$\Rightarrow (W, +)$ is a group

and since subgroup of an abelian group is abelian.

$\Rightarrow (W, +)$ is an abelian group.

(2) $\forall\ a \in F$, $w_1, w_2 \in W$

$a(w_1 + w_2) = aw_1 + aw_2$

$\because W \subset V$

(3) $a, b \in F$ and $w \in W$

$(a+b)w = aw + bw \quad \because W \subset V$

(4) $\forall \; a, b \in F$ and $w \in W$

$a(bw) = (ab)w \quad \because W \subset V$

(5) $\forall \; w \in W$

$1w = w \quad \because W \subset V$

$\Rightarrow W$ is a vector space over field F

$\Rightarrow W(F)$ is a subspace of V

Conversely

Let W is a subspace of V then

(i) $\forall \; w_1, w_2 \in W, \; w_1 + w_2 \in W$

$\because (W, +)$ is group

(ii) $\forall \; a \in F, \; w \in W$

$\Rightarrow aw \in W \quad \because W$ is a V.S

expected V.V Imp

2000 2004 Theorem:- Let $V(F)$ be a vector space then a non-empty subset W of V is a subspace of $V(F)$ iff

$\forall \; a, b \in F$ and $w_1, w_2 \in W$

then $aw_1 + bw_2 \in \bar{W}$

proof:-

Let $\bar{W}$ is a non-empty subset of a vector space $\bar{V}(F)$ such that

$aw_1 + bw_2 \in \bar{W}$ $\forall$ $a, b \in F$ and $w_1, w_2 \in \bar{W}$ then

(1) $1 \in F \Rightarrow -1 \in F$

$\Rightarrow 1w_1 + (-1)w_2 \in \bar{W}$ $\forall$ $w_1, w_2 \in \bar{W}$ by given condition

$\Rightarrow w_1 - w_2 \in \bar{W}$

$\Rightarrow (\bar{W}, +)$ is a subgroup of $(\bar{V}, +)$

$\Rightarrow (\bar{W}, +)$ is a group.

Since subgroup of an abelian group is abelian group.

$\Rightarrow (\bar{W}, +)$ is abelian group.

(2) $\forall$ $a \in F$, $w_1, w_2 \in \bar{W}$

$a(w_1 + w_2) = aw_1 + aw_2$ $\because \bar{W} \subset \bar{V}$

(3) $a, b \in F$ and $w \in F$

$(a+b)w = aw + bw$ $\because \bar{W} \subset \bar{V}$

(4) $\forall$ $a, b \in F$ and $w \in \bar{W}$

$\Rightarrow a(bw) = (ab)w \quad \because W \subset V$

(8) $\forall \; w \in W \Rightarrow$

$1w = w \quad \because W \subset V$

$\Rightarrow W$ is a vector space over field F

$\Rightarrow W(F)$ is a subspace of V

Conversely

Let W(F) is a subspace of V(F) then

$\forall \; a, b \in F$ and $w_1, w_2 \in W$

$aw_1, bw_2 \in W \quad \because W$ is a V.S

$\Rightarrow aw_1 + bw_2 \in W \quad \because (W,+)$ is a group

Example:-

Let $R^3(R)$ be the vector space and consider the subset

$W = \{(x,y,z) : x+y+z=0\}$ of R^3

then

W is a subspace of R^3

Proof:-

Since $(0,0,0) \in R^3$ is such that $0+0+0 = 0$

$\Rightarrow (0,0,0) \in W \Rightarrow W \neq \phi$

Let $w_1 = (x_1, y_1, z_1)$ and $w_2 = (x_2, y_2, z_2)$ are any element of W.

then $x_1 + y_1 + z_1 = 0$ and $x_2 + y_2 + z_2 = 0$

Now let $a, b \in R$

then $aw_1 + bw_2$

$= a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$

$= (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)$

$= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$

and

$(ax_1 + bx_2) + (ay_1 + by_2) + (az_1 + bz_2)$

$= ax_1 + ay_1 + az_1 + bx_2 + by_2 + bz_2$

$= a(x_1 + y_1 + z_1) + b(x_2 + y_2 + z_2)$

$= a(0) + b(0)$

$= 0$

$\Rightarrow aw_1 + bw_2 \in W$

$\Rightarrow W$ is a Subgroup of R^3

Theorem:-
Let $V(F)$ be a vector space and W, U be two subspace of V then
$W \cap U$ is a subspace of V

Proof:-
Since W and U are subspace of V

$\Rightarrow \quad 0_v \in W$ and $0_v \in U$

$\Rightarrow \quad 0_v \in W \cap U$

$\Rightarrow \quad W \cap U \neq \phi$

Now let $a, b \in F$ and $s_1, s_2 \in W \cap U$

$\Rightarrow \quad s_1, s_2 \in W$ and $s_1, s_2 \in U$

$\Rightarrow \quad as_1 + bs_2 \in W$ and $as_1 + bs_2 \in U$

$\Rightarrow$ W and U are subspaces of V

$\Rightarrow \quad as_1 + bs_2 \in W \cap U$

$\Rightarrow \quad W \cap U$ is a subspace of V

Theorem:-
Intersection of any number of subspaces

of a vector space $V(F)$ is a subspace of $V(F)$

proof:- Let $\{W_i : i \in \Omega\}$ be a family of subspaces of $V(F)$ then we want to show that $\bigcap_{i\in\Omega} W_i$ is a subspace of V.

Since W_i is a subgroup of $(V,+)$ $\forall\, i \in \Omega$

$\Rightarrow 0_v \in W_i \quad \forall\, i \in \Omega$

$\Rightarrow 0_v \in \bigcap_{i\in\Omega} W_i$

$\Rightarrow \bigcap_{i\in\Omega} W_i \neq \phi$

Now let $a, b \in F$ and $s_1, s_2 \in \bigcap_{i\in\Omega} W_i$

$\Rightarrow s_1, s_2 \in W_i \quad \forall\, i \in \Omega$

$\Rightarrow as_1 + bs_2 \in W_i \quad \forall\, i \in \Omega$

$\Rightarrow as_1 + bs_2 \in W_i \quad \forall\, i \in \Omega$

$\Rightarrow W_i$ is a subspace of V

$\Rightarrow as_1 + bs_2 \in \bigcap_{i\in\Omega} W_i$

$\Rightarrow \bigcap_{\lambda \in \Omega} W_i$ is a subspace of V

Example:- Imp

Show that the union of two subspaces of a vector space $V(F)$ need not to be a subspace.

Proof:-

Let $V = R^2(R)$ be the vector space and

$$W = \{(x,0) : x \in R\}$$

and

$$U = \{(0,y) : y \in R\}$$

are subspaces of R^2.

Now consider

$$W \cup U = \{(x,y) : x,y \in R \text{ and } x=0 \text{ or } y=0\}$$

then $(1,0)$ and $(0,1) \in W \cup U$

and $(1,0) + (0,1) = (1+0, 0+1) = (1,1) \notin W \cup U$

$\Rightarrow W \cup U$ is not a group w.r.t "+"

$\Rightarrow W \cup U$ is not a vector space

$\Rightarrow W \cup U$ is not a subspace of $V(R)$

v.v imp

LINEAR COMBINATION

Let $V(F)$ be a vector space and Let $v_1, v_2, \ldots v_n \in V$ then any $v \in V$ is called a linear combination of $v_1, v_2, \ldots v_n$ if $\exists$ $a_1, a_2, \ldots a_n \in F$ such that

$$v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$$

Linear Span of a set

Let $S \neq \phi$ be a subset of a vector space V, then the set

$$L(S) = \{a_1v_1 + a_2v_2 + \cdots + a_nv_n \; ; \; a_i \in F \text{ and } v_i \in S\}$$

i-e the set of all linear combination of a finite number of element of S is called a linear span of S and it is denoted by $L(S)$ or $\langle S \rangle$ and the set S is called the Sanning or generating set of $L(S)$

Notes:-

If $S = \phi$ then we define

$$L(\phi) = \{0_V\}$$

Example:-

Consider the vector space $R^2(R)$ and $S = \{(1,3)(1,7)\}$ show that

$$(5,27) \in L(S)$$

Sol:-

Let $a, b \in R$ such that

$$a(1,3) + b(1,7) = (5,27)$$

$$\Rightarrow (a, 3a) + (b, b7) = (5,27)$$

$$\Rightarrow (a+b, 3a+7b) = (5,27)$$

$$\Rightarrow a+b = 5 \quad \text{---(i)}$$

$$3a+7b = 27 \quad \text{---(ii)}$$

by $3 \times$ (i) $-$ (ii)

$$3a + 3b = 15$$
$$\pm 3a \pm 7b = \pm 27$$
$$-4b = -12$$
$$b = 3$$

Use in (i)

$$a+3=5$$
$$a=2$$

i-e $a=2,\ b=3 \in R$

and

$$2(1,3)+3(1,7)=(5,27)$$

or $(5,27)=2(1,3)+3(1,7)$

$\Rightarrow (5,27) \in L(\{(1,2)(1,7)\})$

i-e $(5,27) \in L(S)$

Example:-

Write down the matrix $V=\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$ as a linear combination of the matrices $V_1=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $V_2=\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $V_3=\begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$ over R.

Sol:-

Let $a_1, a_2, a_3 \in R$ are such that $V = a_1V_1 + a_2V_2 + a_3V_3$

$$\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} = a_1\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + a_2\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + a_3\begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$$

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$$\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} a_1 & a_1 \\ a_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_2 & a_2 \end{bmatrix} + \begin{bmatrix} 0 & 2a_3 \\ 0 & -a_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1+0+0 & a_1+0+2a_3 \\ a_1+a_2+0 & 0+a_2-a_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & a_1+2a_3 \\ a_1+a_2 & a_2-a_3 \end{bmatrix}$$

Comparing

$$a_1 = 3 \quad \text{---} (1)$$

$$a_1 + 2a_3 = 1 \quad \text{---} (2)$$

$$a_1 + a_2 = 1 \quad \text{---} (3)$$

$$a_2 - a_3 = -1 \quad \text{---} (4)$$

Using $a_1 = 3$ in (2)

$$3 + 2a_3 = 1$$

$$\Rightarrow 2a_3 = 1-3$$

$$\Rightarrow 2a_3 = -2$$

$$\Rightarrow a_3 = -1$$

Using $a_1 = 3$ in (3)

$$3 + a_2 = 1$$

$$\Rightarrow a_2 = -2$$

$$\Rightarrow V = 3V_1 - 2V_2 - 1V_3$$

Example :-

For what value of k will the vector $(1,-2,k)$ in R^3 be a linear combination of the vectors $(3,0,-2)$ and $(2,-1,-5)$ over R.

Sol :-

Let $(1,-2,k) = a(3,0,-2)+b(2,-1,5)$

$\Rightarrow (1,-2,k) = (3a,0,-2a)+(2b,-b,5b)$

$\Rightarrow (1,-2,k) = (3a+2b, 0-b, -2a+5b)$

$\Rightarrow (1,-2,k) = (3a+2b, -b, -2a+5b)$

Comparing

$3a+2b = 1$ —— (1)

$-b = -2$ —— (2)

$-2a+5b = k$ —— (3)

(2) $\Rightarrow b = 2$

use in (1)

$3a+2(2) = 1$

$\Rightarrow 3a = 1-4$

$\Rightarrow a = -1$

Using "a" and "b" in (3)

$-2(-1)+5(2) = k$

$\Rightarrow 2 + 10 = k$

$\Rightarrow k = 12$ as required.

Imp

Theorem:-

Let $V(F)$ be a vector space and S be a subset of V then $L(S)$ is a subspace of V containing S and is the smallest subspace of V which contains S.

Proof:-

Let $S = \phi$

Then $L(S) = \{0_V\}$ is a subspace of V and $S \subset L(S)$ $\because \phi \subset L(S)$

If $S \neq \phi$

Now $L(S) = \{a_1v_1 + a_2v_2 + \cdots + a_nv_n : a_i \in F$ and $v_i \in S\}$

Now if $v \in S$ then

$1v \in L(S)$

$\Rightarrow v \in L(S)$

$\Rightarrow S \subset L(S)$

Now to prove $L(S)$ is a subspace of $V(F)$

Let $\omega_1, \omega_2 \in L(S)$

$\Rightarrow \omega_1 = a_1v_1 + a_2v_2 + \cdots + a_nv_n$
and $\omega_2 = b_1v_1 + b_2v_2 + \cdots + b_nv_n$ where $a_i \in F$, $b_i \in S$ $\forall i = 1, 2, \ldots n$

Now let $a, b \in F$ then

$aw_1 + bw_2$

$= a(a_1v_1 + a_2v_2 + \cdots + a_nv_n) + b(b_1v_1 + b_2v_2 + \cdots b_nv_n)$

$= aa_1v_1 + aa_2v_2 + \cdots aa_nv_n + bb_1v_1 + bb_2v_2 + \cdots + bb_nv_n$

$= (aa_1 + bb_1)v_1 + (aa_2 + bb_2)v_2 + \cdots + (aa_n + bb_n)v_n \in L(S)$

$\because aa_i + bb_i \in F,\ v_i \in S$

i-e $aw_1 + bw_2 \in L(S)$

$\Rightarrow L(S)$ is a subspace of $V(F)$

Now

Let W is a subspace of $V(F)$ containing S

i-e $S \subset W$

Now let

$a_1v_1 + a_2v_2 + \cdots + a_nv_n \in L(S)$

then $a_1, a_2, \cdots, a_n \in F$

and $\because v_1, v_2, \cdots, v_n \in S$

$\Rightarrow v_1, v_2, \cdots, v_n \in W \quad \because S \subset W$

$\Rightarrow a_1v_1, a_2v_2, \cdots, a_nv_n \in W$

$\because W$ is a subspace

$\Rightarrow a_1v_1 + a_2v_2 + \cdots + a_nv_n \in W$

$\because W$ is a subspace

$\Rightarrow L(S) \subset W$

i-e $L(S)$ is the smallest subspace of $V(F)$ containing S.

Notes:-

(1) $L(S)$ is said to be a subspace of $V(F)$ generated or spanned of S.

(2) $L(S)$ is also called the subspace of linear manifolds of S.

Theorem:- V.V.Imp

Let $V(F)$ be a vector space then

$$L(V) = V$$

proof:-

Since V is a subset of V

$L(V)$ is a subspace of V

$\Rightarrow \quad L(V) \subset V$ —— (1)

but $L(V)$ is the smallest subspace of V containing V.

i-e $\quad V \subset L(V)$ —— (2)

from (1) and (2)

$$L(V) = V$$

Note:-

Since $L(V) = V$ and V is a spanning set for $L(V)$

$\Rightarrow V$ is a spanning set for V

i-e every vector space V has a

spanning set V itself.

Theorem:-
If $V(F)$ be a vector space and S, T are two subsets of V then
(i) $S \subset T \Rightarrow L(S) \subset L(T)$
(ii) $L(L(S)) = L(S)$ V.V. Imp

proof:-
(i) $S \subset T \Rightarrow L(S) \subset L(T)$

proof:-
Let $a_1v_1 + a_2v_2 + \dots + a_nv_n \in L(S)$
then $a_1, a_2, \dots, a_n \in F$
and $v_1, v_2, \dots, v_n \in S$
$\Rightarrow v_1, v_2, \dots, v_n \in T \quad \because S \subset T$
$\Rightarrow v_1, v_2, \dots, v_n \in L(T) \quad \because T \subset L(T)$
$\Rightarrow a_1v_1, a_2v_2, \dots a_nv_n \in L(T)$
$\because L(T)$ is a subspace

$\Rightarrow L(S) \subset L(T)$
expected

(ii) $L(L(S)) = L(S)$ V.V. Imp
proofs-
Since $L(S)$ is a subspace of V
$\Rightarrow L(S)$ is a vector space over F
$\Rightarrow L(L(S)) = L(S) \quad \because L(W) = W$
$\forall$ vector space $W(F)$

Theorem:- 208

If S and T are two [redacted] subsets of a vector space $V(F)$ then

(i) $L(S) \cup L(T) \subset L(S \cup T)$

(ii) $L(S \cap T) \subset L(S) \cap L(T)$

proof:-

(i) $S \subset S \cup T \Rightarrow L(S) \subset L(S \cup T)$

Also $T \subset S \cup T \Rightarrow L(T) \subset L(S \cup T)$

$\Rightarrow L(S) \cup L(T) \subset L(S \cup T)$

(ii) proof:-

Since $S \cap T \subset S \Rightarrow L(S \cap T) \subset L(S)$

Also $S \cap T \subset T \Rightarrow L(S \cap T) \subset L(T)$

$\Rightarrow L(S \cap T) \subset L(S) \cap L(T)$

Example:-

Give an example of a vector space $V(F)$ in which for subsets S and T of V

$$L(S \cup T) \neq L(S) \cup L(T)$$

and $L(S \cap T) \neq L(S) \cap L(T)$

proof:-

Let $S = \{(1,0)\}$

$T = \{(0,1)\}$

then $S \cup T = \{(1,0), (0,1)\}$ are subset of $R^2(R)$

Now

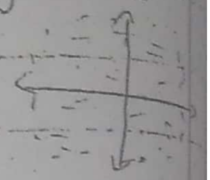
$$L(S) = \{x(1,0) : x \in R\}$$

$$= \{(x,0) : x \in R\} = x\text{-axis}$$

$L(T) = \{y(0,1) : y \in R\}$
$= \{(0,y) : y \in R\} = y\text{-axis}$

then
$L(S) \cup L(T) = (x\text{-axis}) \cup (y\text{-axis})$

and $L(S \cup T) = \{x(1,0) + y(0,1) : x, y \in R\}$
$= R^2$



Clearly
$L(S \cup T) \neq L(S) \cup L(T)$

Now if
$S = \{(0,0), (1,0)\}$
$T = \{(0,0), (-1,0)\}$

then
$S \cap T = \{(0,0)\}$

and
$L(S \cap T) = L(\{(0,0)\}) = \{(0,0)\}$

and
$L(S) = \{a(0,0) + b(1,0) : a, b \in R\}$
$= \{(b,0) : b \in R\} = x\text{-axis}$

Now
$L(T) = \{c(0,0) + d(-1,0) : c, d \in R\}$
$= \{(-d,0) : d \in R\} = x\text{-axis}$

Also $L(S) \cap L(T) = x\text{-axis}$
$\Rightarrow L(S) \cap L(T) \neq L(S \cap T)$

Finite dimensional vector space:
A vector space $V(F)$ is said to be a finite dimensional vector space if

$\exists$ a finite subset S of V such that

$$V = L(S)$$

i-e V can be spanned by a finite subset of V

A finite dimensional vector space is also called finitely generated vector space.

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Examples:-

Show that $R^2(R)$ is a finitely dimensional vector space.

proof:-

Consider the subset $S = \{(1,0), (0,1)\}$ of R^2 then

$$L(S) \subset R^2 \quad \text{---(1)}$$

and $\forall (x,y) \in R^2$

$$(x,y) = x(1,0) + y(0,1)$$

$$\Rightarrow (x,y) \in L(S)$$

$$\Rightarrow R^2 \subset L(S) \quad \text{---(2)}$$

from (1) and (2)

$$L(S) = R^2$$

Since S is finite

$\Rightarrow R^2$ is a finite dimensional vector space.

Note:-

Similarly $R^1, R^2, \ldots R^n$ are finite dimensional vector spaces

Infinite dimensional vector space:-

A vector space $V(F)$ is said to be infinite dimensional vector space if $\nexists$ (there does not exist) a finite subset $S(V)$ such that

$$L(S) = V$$

Example:-

Let $V = F[x]$ be the vector space of all polynomials with co-efficients from field F then V is infinitely generated or infinite dimensional vector space because $\nexists$ a finite subset $(S(V))$ S of V such that $L(S) = V$

Note that the subset $S = \{1, x, x^2, x^3, ---\}$ of V is such that $\forall$ $a_0(1) + a_1x + a_2x^2 + --- a_nx^n \in V$

$\because$ $a_0, a_1, a_2, --- a_n \in F$

and $1, x, x^2, --- x^n \in S$

$\Rightarrow$ $a_0(1) + a_1x^1 + a_2x^2 + --- + a_nx^n \in L(S)$

$\Rightarrow$ $V \subset L(S)$

but $L(S) \subset V$

$\Rightarrow$ $V = L(S)$

$\Rightarrow$ S is a spanning set of V

A subset $S \neq \phi$ of $V(F)$ is said to be linearly independent if its vectors are linearly independent.

Note:-
Empty set ϕ is defined to be linearly independent.

Example:-
The subset $S = \{(1,0),(0,1)\}$ of $R^2(R)$ is linearly independent.

Sol:-

Let $a(1,0) + b(0,1) = (0,0)$

$\Rightarrow (a,0) + (0,b) = (0,0)$

$\Rightarrow (a+0, 0+b) = (0,0)$

$\Rightarrow (a,b) = (0,0)$

$\Rightarrow a = 0, b = 0$

$\Rightarrow$ vectors are linearly independent

Alternative Method:-

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

$\Rightarrow$ vectors $(1,0)$ and $(0,1)$ are linearly independent.

Examples:-

Let V be the vector space of all functions defined from $\mathbb{R}$ to $\mathbb{R}$. Then show that

$V_1 = \cos^2 x$, $V_2 = \sin^2 x$, $V_3 = \cos 2x$

then V_1, V_2, V_3 are Linealy dependent

proof:-

Let $a_1 V_1 + a_2 V_2 + a_3 V_3 = O_V$

$$a_1 \cos^2 x + a_2 \sin^2 x + a_3 \cos 2x = O_V$$

$$\Rightarrow a_1 \cos^2 x + a_2 \sin^2 x + a_3 (\cos^2 x - \sin^2 x) = O_V$$

$$\Rightarrow a_1 \cos^2 x + a_2 \sin^2 x + a_3 \cos^2 x - a_3 \sin^2 x = O_V$$

$$\Rightarrow a_1 \cos^2 x + a_2 \sin^2 x + a_3 \cos^2 x - a_3 \sin^2 x = O_V$$

$$\Rightarrow (a_1 + a_3) \cos^2 x + (a_2 - a_3) \sin^2 x = O_V$$

Comparing co-efficients

$$a_1 + a_3 = 0$$
$$a_2 - a_3 = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

which is in echelon form and rank of the matrix is 2

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and $2 < 3$ (number of variables)

$\Rightarrow$ System has infinite solution

System can be rewritten as

$$a_1 + a_3 = 0 \quad \text{--- (1)}$$
$$a_2 - a_3 = 0 \quad \text{--- (2)}$$

and let $a_3 = c$ where c is constant

use in (2)

$\Rightarrow a_2 - c = 0$

$\Rightarrow a_2 = c$

use $a_3 = c$ in (1)

$a_1 + c = 0$

$\Rightarrow a_1 = -c$

$\Rightarrow (a_1, a_2, a_3) = (-c, c, c)$

$= c(-1, 1, 1)$

where c is any constant real number.

using $c = 1$

$(a_1, a_2, a_3) = (-1, 1, 1)$

using in

$a_1V_1 + a_2V_2 + a_3V_3 = 0$

$\Rightarrow -1V_1 + 1V_2 + 1V_3 = 0$

or $-1\cos^2 x + 1\sin^2 x + 1\cos^2 x = 0$

$\Rightarrow V_1, V_2, V_3$ are linearly dependent.

Example:-

Check whether the given vectors $(1,-2,1)$, $(1,2,-1)$, $(7,-4,1)$ in R^3 are linearly dependent or not.

Sol:-

Let $a(1,-2,1)+b(1,2,-1)+c(7,-4,1)=(0,0,0)$

$\Rightarrow (a,-2a,a)+(b,2b,-b)+(7c,-4c,c)=(0,0,0)$

$\Rightarrow (a+b+7c,\ -2a+2b-4c,\ a-b+c)=(0,0,0)$

Comparing

$$a+b+7c=0$$
$$-2a+2b-4c=0$$
$$a-b+c=0$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 7 \\ -2 & 2 & -4 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix of co-efficients is

$$A=\begin{bmatrix} 1 & 1 & 7 \\ -2 & 2 & -4 \\ 1 & -1 & 1 \end{bmatrix}$$

$a_1 + 2a_2 + ...$

$$\sim_R \begin{bmatrix} 1 & 2 & 7 \\ -2+2 & 1+4 & -4+14 \\ 1-1 & -1-2 & 1-7 \end{bmatrix} \begin{matrix} a_1 \\ R_2 + 2R_1 \\ R_3 - R_1 \end{matrix}$$

$$\sim_R \begin{bmatrix} 1 & 2 & 7 \\ 0 & 5 & 10 \\ 0 & -3 & -6 \end{bmatrix}$$

$$\sim_R \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{matrix} \\ \frac{1}{5} R_2 \\ \frac{1}{-3} R_3 \end{matrix}$$

$$\sim_R \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 1-1 & 2-2 \end{bmatrix} R_3 - R_2$$

$$\sim_R \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

matrix is in echelon form

and rank $A = 2 < 3$ muber of variable

$\Rightarrow$ system has infinite solution

and the system can be rewritten as

$$a + 2b + 7c = 0$$
$$1b + 2c = 0$$

Let $c =$ constant $= k$ use in (2)

$$b + 2k = 0$$
$$\Rightarrow b = -2k \quad \text{use in (1)}$$

$$a + 2(-2k) + 7k = 0$$
$$\Rightarrow a - 4k + 7k = 0$$
$$\Rightarrow a + 3k = 0$$
$$\Rightarrow a = -3k$$

$$\Rightarrow (a, b, c) = (-3k, -2k, k)$$
$$= k(-3, -2, 1)$$

where k is any constant

using $k = 1$

$$\Rightarrow (a, b, c) = (-3, -2, 1)$$

i-e

$$(-3)(1, -2, 1) + (-2)(2, 1, -1) + 1(7, -4, 1) = (0,$$

$\Rightarrow$ vectors $(1, -2, 1), (2, 1, -1), (7, -4, 1)$ are linearly dependent.

Alternative Method:-

For the set $\{(1, -2, 1), (2, 1, -1), (7, -4, 1)\}$ we can construct a square matrix of row vectors as

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 7 & -4 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 7 & -4 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & -1 \\ -4 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 2 & -1 \\ 7 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 7 & -4 \end{vmatrix}$$

$$= 1(1-4) + 2(2+7) + 1(-8-7)$$

$$= 1(-3) + 2(9) + 1(-15)$$

$$= -3 + 18 - 15$$

$$= -18 + 18$$

$$= 0$$

Given vectors are linarly dependent

2002 Example:-

Show that the vectors

$u_1 = (3, 1, 2)$, $u_2 = (0, -1, 5)$, $u_3 = (1, 2, -2)$

in $\mathbb{R}^3$ are L.I.D

Sol:-

For the set $\{(3, 1, 2), (0, -1, 5), (1, 2, -2)\}$ we can construct a square matrix of row vectors as

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 1 & 2 & -2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 1 & 2 & -2 \end{vmatrix}$$

$$= 3\begin{vmatrix} -1 & 5 \\ 2 & -2 \end{vmatrix} - 1\begin{vmatrix} 0 & 5 \\ 1 & -2 \end{vmatrix} + 2\begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix}$$

$$= 3(2-10) - 1(0-5) + 2(0+1)$$
$$= 3(-8) - 1(-5) + 2(1)$$
$$= -24 + 5 + 2$$
$$= -17$$

Given vectors are linearly independent.

Examples:- 198

Let R^4 be vector space over R and $u_1 = (1, 2, 0, -1)$, $u_2 = (-1, 3, 2, 0)$, $u_3 = (0, 0, 1, 0)$ be vectors of R^4. Examine the set $\{u_1, u_2, u_3\}$ for its linear independent or linearly dependent.

Sol:-

The matrix formed by given vectors as rows is

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ -1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\underset{R}{\sim} \begin{bmatrix} 1 & 2 & 0 & -1 \\ -1+1 & 3-2 & 2+0 & 0-1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad R_2+R_1$$

$$\underset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Since there is no zero row in the matrix

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$\Rightarrow$ vectors $(1,2,0,-1)$, $(-1,3,2,0)$ $(0,0,1,0)$ are linearly independent.

998 Example:-

Examine the set $\{(1,1,0,0), (0,1,-1,0), (0,0,0,3)\}$ for it linearly independent or linearly dependent.

Sol:-

The matrix formed by given vector as rows as

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Since matrix is already in echelon form and there is no zero row in the matrix
→ given vectors are L.I

2000 Theorem:- v.imp
The subset $\{0_v\}$ of vector space $V(F)$ linearly dependent.
proof:-
Since $\forall\ a \in F$
$$a\,0_v = 0_v$$
$$\Rightarrow a\,0_v = 0_v \quad \forall\ a \in F - \{0_F\}$$
$$\Rightarrow a\,0_v = 0_v \quad \forall\ a \neq 0_F$$
$\Rightarrow 0_v$ is linearly dependent
i-e $\{0_v\}$ is " "

Theorem:- Imp
The singleton subset of a vector space $V(F)$ has a non-zero vector
$\alpha \neq 0_v$

is linearly independent.

proof:-

Let $\{v\}$ be a subset of a vector space $V(F)$ and $v \neq 0_v$ then if

$$av = 0_v$$

then $a = 0_F$ $\because v \neq 0_v$

$\Rightarrow$ v is linearly independent

$\Rightarrow$ $\{v\}$ is linearly independent.

Theorem:- V. Imp

Let S be a non-empty subset of a vector space $V(F)$ and $0_v \in S$ then S is linearly dependent i-e a subset of V containing the zero vector is linearly dependent.

proof:-

Let $S = \{v, v_1, v_2, ---, v_n\}$ where $v = 0_v$ then we can write

$0_F v_1 + 0_F v_2 + --- + 0_F v_n + 1v$

$\Rightarrow$ $0_v + 0_v + --- + 0_v + 0_v$ $\quad \because v = 0_v \Rightarrow 1v = 0_v$

i-e $0_F v_1 + 0_F v_2 + --- + 0_F v_n + 1v = 0_v$

Since co-efficient of v is $1 \neq 0_F$

$\Rightarrow$ vectors $v_1, v_2, \ldots, v_n, v$ are linearly dependent.
i-e S is linearly dependent.

Theorem:-
Let $S = \{v_1, v_2, \ldots, v_m\}$ contains dependent set $T = \{u_1, u_2, \ldots u_r\}$ linearly dependent, then S is linearly dependent.

proof:-
Since T is linearly dependent
Therefore $a_1 u_1 + a_2 u_2 + \cdots + a_r u_r = O_V$

$\Rightarrow \exists$ non-zero $a_i \in F$ for $i = 1, 2, \ldots r$
and we can write

$$a_1 u_1 + a_2 u_2 + \cdots + a_r u_r + O_F u_{r+1} + \cdots + O_F u_m = O_V$$

where $u_{r+1}, u_{r+2}, \ldots u_m \in S - T$

$\Rightarrow u_1, u_2, \ldots u_r, u_{r+1}, \ldots u_m$ is linearly dependent because $\exists\, a_i \neq O_F$

$\Rightarrow U = \{u_1, u_2, \ldots u_m\}$ is linearly dependent but $U \subset S$ and

$$|U| = m = |S|$$

$\Rightarrow \quad U = S$

$\Rightarrow S$ is linearly dependent

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Theorem:-

If $\{v_1, v_2, ---, v_n\}$ is linearly Independent and $\{v_1, v_2, ---, v_n, v\}$ is linearly dependent Then v is a linear combination of $v_1, v_2, ---, v_n$

Proof:-

Since $v_1, v_2, ---, v_n$ are linearly independent

$\Rightarrow a_1v_1 + a_2v_2 + --- + a_nv_n = 0_V$

$\Rightarrow a_1 = 0_F, a_2 = 0_F, ---, a_n = 0_F$

If $v_1, v_2, ---, v_n, v$ are linearly dependent then

if $b_1v_1 + b_2v_2 + --- + b_nv_n + bv = 0_V$ — (1)

then at least one b_i is non-zero

we are to show $b \neq 0_F$

Suppose $b = 0_F$ then $bv = 0_V$

and $b_1v_1 + b_2v_2 + --- + b_nv_n = 0_V$

then $b_1 = 0_F, b_2 = 0_F, ---, b_n = 0_F$

$\because v_1, v_2, --- v_n$ are L.I

$\Rightarrow b_1v_1 + b_2v_2 + --- + b_nv_n = 0_V$

$\Rightarrow b_1 = 0_F, b_2 = 0_F, --- b_n = 0_F, b = 0_F$

$\Rightarrow v_1, v_2, ---, v_n, v$ are linearly independent.

which is a contradiction

$\Rightarrow b \neq 0_F$

from (1) we can write

$bv = -b_1v_1 - b_2v_2 - \cdots - b_nv_n$

since $b \neq 0_F \Rightarrow b^{-1} \in F$ and we can write

$b^{-1}bv = b^{-1}(-b_1v_1 - b_2v_2 - \cdots - b_nv_n)$

$\Rightarrow 1v = -b^{-1}b_1v_1 - b^{-1}b_2v_2 - \cdots - b^{-1}b_nv_n$

$\Rightarrow v = -b^{-1}b_1v_1 - b^{-1}b_2v_2 - \cdots - b^{-1}b_nv_n$ because $-b^{-1}b_i \in F$

$\Rightarrow v$ is a linear combination of $v_1, v_2, \ldots, v_n$

Theorem:- If $v_1, v_2, \ldots, v_n$ are in $V(F)$ then either they are linearly independent or some v_k is linear combination of the preceeding ones $v_1, v_2, \ldots, v_{k-1}$

proof:- Suppose $v_1, v_2, \ldots, v_n$ are not linearly independent then $v_1, v_2, \ldots, v_n$ linearly dependent

$\Rightarrow$ for $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0_v$

$\Rightarrow \exists$ non-zero $a_i \in F$ where $i = 1, 2, \ldots$

suppose k is the largest number such that $a_k \neq 0_F$

Then
$$\left.\begin{array}{l} a_{k+1} = 0_F \\ a_{k+2} = 0_F \\ \vdots \\ a_n = 0_F \end{array}\right\} \text{ if } k \neq n$$

$\Rightarrow a_1V_1 + a_2V_2 + \cdots + a_kV_k + 0_FV_{k+1} + \cdots + 0_FV_n = 0_V$

$\Rightarrow a_1V_1 + a_2V_2 + \cdots + a_kV_k = 0_V$ and $a_k \neq 0_F$

$\Rightarrow a_kV_k = -a_1V_1 - a_2V_2 - \cdots - a_{k-1}V_{k-1}$

$\Rightarrow a_k^{-1}a_kV_k = a_k^{-1}(-a_1V_1 - a_2V_2 - \cdots - a_{k-1}V_{k-1})$

$\Rightarrow (1)V_k = -a_k^{-1}a_1V_1 - a_k^{-1}a_2V_2 - \cdots - a_k^{-1}a_{k-1}V_{k-1}$

$\Rightarrow V_k = -a_k^{-1}a_1V_1 - a_k^{-1}a_2V_2 - \cdots - a_k^{-1}a_{k-1}V_{k-1}$

$\Rightarrow V_k$ is linearly ~~independent~~ combination of the preceeding vectors $V_1, V_2, \cdots, V_{k-1}$

Conversely suppose in vector $V_1, V_2, \cdots, V_n$ there is a vector V_k which is linear combination of the preceeding vectors $V_1, V_2, \cdots, V_{k-1}$

i-e $V_k = a_1V_1 + a_2V_2 + \cdots + a_{k-1}V_{k-1}$

$\Rightarrow 0_V = a_1V_1 + a_2V_2 + \cdots + a_{k-1}V_{k-1} - V_k$

OR

$a_1V_1 + a_2V_2 + \cdots + a_{k-1}V_{k-1} - 1V_k = O_V$

$\Rightarrow a_1V_1 + a_2V_2 + \cdots + a_{k-1}V_{k-1} - 1V_k + \underbrace{O_V + O_V + \cdots + O_V}_{n-k \text{ times}} = O_V$

$\Rightarrow a_1V_1 + a_2V_2 + \cdots + a_{k-1}V_{k-1} - 1V_k + O_F V_{k+1} + \cdots + O_F V_n = O_V$

$\Rightarrow V_1, V_2, \cdots, V_{k-1}, V_k, \cdots, V_n$ are linearly dependent $\because$ co-efficient of V_k is non-zero

expected V.V. Imp

Theorem:-

A set $S = \{V_1, V_2, \cdots, V_n\}$ of n vectors $(n \geq 2)$ in a vector space $V(F)$ is linearly dependent iff at least one of the vectors in S is a linear combination of the remaining vectors of S. also all the vectors are non-zero

Proof:-

Let $S = \{V_1, V_2, \cdots, V_n\}$ is linearly dependent Then for

$a_1V_1 + a_2V_2 + \cdots + a_nV_n = O_V$

at least one a_i is non-zero

Suppose $a_k \neq O_F$

$\Rightarrow a_k^{-1} \in O_F$

then $a_1V_1 + a_2V_2 + \cdots + a_kV_k + \cdots + a_nV_n =$

$\Rightarrow a_k v_k = -a_1 v_1 - a_2 v_2 - \cdots - a_{k-1} v_{k-1} - a_{k+1} v_{k+1} - \cdots - a_n v_n$

$\Rightarrow a_k^{-1} a_k v_k = a_k^{-1}[-a_1 v_1 - a_2 v_2 - \cdots - a_{k-1} v_{k-1} - a_{k+1} v_{k+1} - \cdots - a_n v_n]$

$\Rightarrow 1 v_k = -a_k^{-1} a_1 v_1 - a_k^{-1} a_2 v_2 - \cdots - a_k^{-1} a_{k-1} v_{k-1} - a_k^{-1} a_{k+1} v_{k+1} - \cdots - a_k^{-1} a_n v_n$

$\Rightarrow v_k = -a_k^{-1} a_1 v_1 - a_k^{-1} a_2 v_2 - \cdots - a_k^{-1} a_{k-1} v_{k-1} - a_k^{-1} a_{k+1} v_{k+1} - \cdots - a_k^{-1} a_n v_n$

$\Rightarrow v_k$ is a linear combination of the remaning vectors

Conversly let $v_k \in S$ is a linear combination of the remaning vectors $v_1, v_2, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n$ of S

then $v_k = b_1 v_1 + b_2 v_2 + \cdots + b_{k-1} v_{k-1} + b_{k+1} v_{k+1} + \cdots + b_n v_n$

$\Rightarrow 0_V = b_1 v_1 + b_2 v_2 + \cdots + b_{k-1} v_{k-1} - 1 v_k + b_{k+1} v_{k+1} + \cdots + b_n v_n$

or

$b_1 v_1 + b_2 v_2 + \cdots + b_{k-1} v_{k-1} - 1 v_k + b_{k+1} v_{k+1} + \cdots + b_n v_n = 0_V$

$\Rightarrow V_1, V_2, ---, V_{k-1}, V_k, V_{k+1}, ---, V_n$ are linearly dependent $\because$ co-efficie of V_k is non-zero

Theorem:-
Every subset of a linearly independent set is linearly independent.

proof:-
Let S be linearly independent set and T be its subset. Suppose T is linearly dependent then T is not linearly independent

$\Rightarrow \exists$ a vector V_k in T which is linear combination of remaning vectors of T

i-e $V_k = a_1V_1 + a_2V_2 + --- + a_{k-1}V_{k-1} + a_{k+1}V_{k+1} + --- + a_m V_m$

where $T = \{V_1, V_2, ---, V_m\}$

and suppose $S - T = \{V_{m+1}, V_{m+2}, --- V_n\}$

then we can write

$V_k = a_1V_1 + a_2V_2 + --- + a_{k-1}V_{k-1} + a_{k+1}V_{k+1} + 0_F V_{m+1} + --- + 0_F V_n$

$\Rightarrow V_k \in S$ is a linear combination of the remaning vector of S

$\Rightarrow$ S is linearly dependent which is a contraduction
$\Rightarrow$ T is not linearly dependent
$\Rightarrow$ T is linearly independent
$\Rightarrow$ Every subset of a linearly independent set is linearly independent.

Imp **Theorem:-**

A finite subset S of a vector space $V(F)$ is linearly independent iff no proper subset of S can generate $L(S)$.

Proof:-

Let $S = \{V_1, V_2, \ldots, V_n\}$ be a finite subset of $V(F)$.
Let S is linearly independent and T is a proper subset of S

Now $T \subset S$
$$L(T) \subset L(S)$$
Since T is proper subset of S
$\Rightarrow \exists$ $V_i \in S$ such that $V_i \notin T$
then V_i can be expressed as linear combination of element of T

but $T \subset S$

$\Rightarrow T - \{V_i\} \subset S - \{V_i\}$

$\Rightarrow T \subset S - \{V_i\}$

$\Rightarrow V_i$ can be expressed as a linear combination of element of $S - \{V_i\}$

$\Rightarrow V_i \in S$ can be expressed as linear combination of remaning vector of S

$\Rightarrow S$ is Linearly dependent.

$\Rightarrow V_i \in S$ is such that $V_i \notin L(T)$

$\Rightarrow V_i \in L(S)$ " " " $V_i \notin L(T)$

$\Rightarrow L(S) \not\subset L(T)$

$\Rightarrow L(S) \neq L(T)$

$\Rightarrow T$ cannot span $L(S)$

$\Rightarrow$ no proper subset of S can generate $L(S)$

Conversly

Suppose no proper subset of S can generate $L(S)$

and let S is linearly dependent then $\exists$ $V_k \in S$ such that V_k is a linear combination of the remaining vectors

$V_1, V_2, --- V_{k-1}, V_{k+1}, --- V_n$ of S

$\Rightarrow V_k = a_1V_1 + a_2V_2 + --- a_{k-1}V_{k-1} + a_{k+1}V_{k+1} + --- + --- a_nV_n$ ——— (1)

where $a_i \in F$

If $T = S - \{V_k\}$

then T is a proper subset of S

Suppose $V \in L(S)$

$\Rightarrow V = b_1V_1 + b_2V_2 + --- + b_{k-1}V_{k-1} + b_kV_k + b_{k+1}V_{k+1} + --- + b_nV_n$ where $b_i \in F$

$\Rightarrow V = b_1V_1 + b_2V_2 + --- + b_{k-1}V_{k-1} + b_k(a_1V_1 + a_2V_2 + --- + a_{k-1}V_{k-1} + a_{k+1}V_{k+1} --- + a_nV_n) + b_{k+1}V_{k+1} + --- + b_nV_n$

$\Rightarrow V = (b_1 + a_1b_k)V_1 + (b_2 + a_2b_k)V_2 + --- + (b_{k-1} + a_{k-1}b_k)V_{k-1} + (b_{k+1} + a_{k+1}b_k)V_{k+1} + --- + (b_n + a_nb_k)V_n$

$\Rightarrow V$ is linear combination of vector of T

$\Rightarrow V \in L(T)$

$\Rightarrow L(S) \subset L(T)$

but

$T \subset S$

$\Rightarrow L(T) \subset L(S)$

$\Rightarrow L(S) = L(T)$

$\Rightarrow$ a proper subset T of S generate $L(S)$

which is a contraduction

$\Rightarrow S$ is linearly independent.

Chapter :-

"BASIS AND DIMENSIONS"

Basis of a vector space:-
A set of a linearly independent vectors spanning a vector space V is called a basis for V

Example:-
The subset $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ of R^3 is a basis for R^3

proof:-
The matrix made by the vectors of S as a rows of matrix is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is in echelon form and there is no zero row in A
$\Rightarrow$ vectors of S are linearly independent

Since $S \subset R^3$
$\Rightarrow L(S) \subset R^3$
Now let $(x, y, z) \in R^3$

then $(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

$\Rightarrow$ (x,y,z) is a linear combination of vectors of S

$\Rightarrow$ $(x,y,z) \in L(S)$

$\Rightarrow$ $R^3 \subset L(S)$.

$\Rightarrow$ $R^3 = L(S)$

$\Rightarrow$ S spans R^3

$\Rightarrow$ S is a basis for R^3.

The set $\{(1,0,0), (0,1,0), (0,0,1)\}$ is called standard basis for R^3

Similarly standard basis for R is $\{1\}$

Standard basis for R^2 is $\{(1,0), (0,1)\}$

" " " $R^3 = \{(1,0,0), (0,1,0), (0,0,1)\}$

" " " $R^4 = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

Dimension of a vector space :-

The dimension of a vector space is the number of elements in a basis for V. It is denoted by dim V

Note:-
For the definition we conclude that the dimension of a vector space can not exceed the number of a element in spanning set.

Finite dimensional Vector space:-
If the basis of a vector space is finite then the vector space is called a finite dimensional vector space.

Infinite dimensional vector space:-
If there is no finite basis for a vector space then the vector space is called infinite dimensional vector space.

Examples:-

(1) R is finite dimensional vector space of dimension 1

(2) R^2 " " " " " " " 2

(3) R^3 " " " " " " " 3

(4) R^n " " " " " " " n

where $n \in Z^+$

(5) If $V = F[x]$ be the vector space of all polynomials with co-efficients in F then the set $\{1, x, x^2, \ldots\}$

of independent vectors of V forms a basis for V but it is not finite

$\Rightarrow$ V is infinite dimensional vector space.

Note:-

For vector space $V = \{0_V\}$

$$L(\phi) = \{0_V\}$$

$$\text{i.e} \quad L(\phi) = V$$

$\Rightarrow$ ϕ is a spanning set for $V = \{0_V\}$

$\Rightarrow$ ϕ is linearly independent by definition

$\Rightarrow$ ϕ is a basis for $V = \{0_V\}$

and $\dim(\{0_V\}) = 0$.

V.V. imp

Theorem:- V.V imp V.V imp

Any finite dimensional vector space contains a finite basis.

Proof:-

Let V be a finite dimensional vector space and

$S = \{v_1, v_2, \ldots, v_r\}$ be a finite spanning set for V

If $v_i = 0_V \ \forall\, i$ then

$$V = \{0_V\}$$

Then ϕ is basis for V
If at least one $V_i \neq O_V$ then
$V \neq \{O_V\}$
If $r = 1$
then $S = \{V_1\}$ and $V_1 \neq O_V$
$\Rightarrow$ S is linearly independent

$\Rightarrow$ S is a basis for V
Now
Suppose $S \neq$ singleton set
and $S = \{V_1, V_2, ---- V_r\}$

If S is linearly dependent then
S cannot be basis for V
and $\exists$ V_k which is linear
combination of the remaining vectors

Now let $T = \{V_1, V_2, ----, V_r\} - \{V_k\}$

and renumbring T as

$T = \{u_1, u_2, ----, u_{r-1}\}$.
If T is linearly dependent then
T cannot be a basis for V
and $\exists$ u_s which is linear
combination of the remaining
vectors.

and proceeding in the same way we get

$B = \{w_1, w_2, ----, w_n\}$

which is linearly independent and a spanning set for V

$\Rightarrow B$ is a basis for V

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Theorem :-

If $\{v_1, v_2, ----, v_n\}$ is basis for a vector space $V(F)$, then any other subset of linearly independent vector $\{w_1, w_2, -----, w_m\}$ contains less or equal elements than $\{v_1, v_2, ----, v_n\}$

i-e $m \leq n$.

Proof :-

Since $\{v_1, v_2, -----, v_n\}$ is a basis for V

$\Rightarrow \dim V = n$

Let $S = \{w_1, w_2, -----, w_m\}$ be a set of linearly independent vector in V.

Need to show that $m \leq n$

Let us suppose that $m > n$

i-e S contains more than n elements

i-e we can write

$S = \{w_1, w_2, -----, w_n, w_{n+1}, -----, w_m\}$

Since S is linearly independent so
$w_i \neq 0_V \quad \forall \quad i = 1, 2, ---, m$
because any set consisting of zero vector is linearly dependent.
Now
$w_1 \in V$ and $\{v_1, v_2, ---, v_n\}$ is a basis for V
$\Rightarrow$ w_1 can be expressed as a linear combination of vectors
$v_1, v_2, ----, v_n$
Thus $T = \{w_1, v_1, v_2, ---, v_n\}$ is linearly dependent.

So $\exists$ a vector v_i in T which can be expressed as a linear combination of the preceeding vectors $w_1, v_1, v_2, --- v_{i-1}$
$\exists$ $b_0, b_1, b_2, ---- b_{i-1} \in F$

Such that $v_i = b_0 w_1 + b_1 v_1 + b_2 v_2 + --- + b_{i-1} v_{i-1}$
Now for any $v \in V$ ~~$\exists \; a_1, a_2, ---- a_n \in F$~~
Since $\{v_1, v_2, ----, v_n\}$ is a basis for V
$\exists$ $a_1, a_2, ----, a_n \in F$
Such that $v = a_1 v_1 + a_2 v_2 + --- + a_i v_i + --- a_n v_n$

$\Rightarrow$ $v = a_1 v_1 + a_2 v_2 + --- + a_{i-1} v_{i-1} + a_i (b_0 w_1 + b_1 v_1 + --- + b_{i-1} v_{i-1})$
$+ a_{i+1} v_{i+1} + --- + a_n v_n$

$\Rightarrow V = (a_1 + a_i b_1)v_1 + (a_2 + a_i b_2)v_2 + \cdots (a_{i-1} + a_i b_{i-1})v_{i-1}$
$+ a_i b_0 w_1 + a_{i+1} v_{i+1} + \cdots + a_n v_n$

$\Rightarrow v \in L(\{w_1, v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\})$

$\Rightarrow V \subset L(\{w_1, v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\})$

but $L(\{w_1, v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}) \subset V$

$\Rightarrow V = L(\{w_1, v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\})$

Now we can assume $i = n$ i-e
we can arrange the set

as $T = \{w_1, u_1, u_2, \ldots, u_n\}$ where $u_n = v_i$

and $V = L(\{w_1, v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\})$

$= L(\{w_1, u_1, u_2, \ldots, u_{n-1}\})$

Proceeding on the same way we can write

$V = L(\{w_1, w_2, \ldots, w_n\})$

Now $w_{n+1} \in V$ and $V = L(\{w_1, w_2, \ldots, w_n\})$

$\Rightarrow w_{n+1} \in L(\{w_1, w_2, \ldots, w_n\})$

$\Rightarrow$ w_{m+1} is a linear combination of vectors $w_1, w_2, ----, w_n$

$\Rightarrow$ $\{w_1, w_2, -----, w_n, w_{m+1}\}$ is linearly dependent. but it is subset of a linearly independent set.

$\{w_1, w_2, ---, w_n, w_{n+1}, ---, w_m\}$

which is a contraduction because subset of a linearly independent set is linearly independent.

The contraduction is due to our wrong supposition that $m > n$

Hence $m \le n$

Note:-

This result shows if $\dim V = n$ then any set consisting of more than n elements is linearly dependent.

Theorem:-

Any two basis of a finite dimensional vector space have same number of elements.

Proof:-

Let a finite dimensional vector

space $V(F)$ has two basis S and T with m and n elements respectively. Consider S as basis and T as the set of linearly independent vectors

then $|T| \leq |S|$

$\Rightarrow n \leq m$ ——— (1)

Now consider T as basis and S as the set of linearly independent vectors then

$|S| \leq |T|$

$\Rightarrow m \leq n$ ——— (2)

from (1) and (2)

$m = n$

$\Rightarrow |S| = |T|$

i-e Any two basis of a finite dimensional vector space have same number of elements.

Imp

Theorem:-

Let V be a finite dimensional vector space of dimension n. Then a subset $S = \{v_1, v_2, \ldots, v_n\}$ of V is a basis for V iff every vector in V is uniquely expressible as L.C of the elements of S.

proof:-
Let the set $\{v_1, v_2, ---, v_n\}$ be a basis for V
Then every vector $v \in V$ can be expressed as a linear combination of vectors of $\{v_1, v_2, ---, v_n\}$

i-e $v = a_1v_1 + a_2v_2 + --- + a_nv_n$
where ~~and~~ $a_1, a_2, ----, a_n \in F$
Suppose another expression for v

$$v = b_1v_1 + b_2v_2 + ----- + b_nv_n$$
where $b_1, b_2, ----, b_n \in F$

$$v = v$$

$\Rightarrow a_1v_1 + a_2v_2 + ----+ a_nv_n = b_1v_1 + b_2v_2 + ---+ b_nv_n$

$\Rightarrow (a_1 - b_1)v_1 + (a_2 - b_2)v_2 + --- + (a_n - b_n)v_n = O_V$

$\Rightarrow a_1 - b_1 = O_F,\ a_2 - b_2 = O_F,\ ----\ a_n - b_n = O_F$
because $v_1, v_2, ----, v_n$ are linearly independent vectors of basis.

v.v imp

$\Rightarrow a_1 = b_1,\ a_2 = b_2, -----, a_n = b_n$

$\Rightarrow$ expression is Unique.

Conversly Let every vector of V can

be uniquely expressed as a linear combination of vectors of $\{v_1, v_2, ---, v_n\}$ then

$$V = L(\{v_1, v_2, -----, v_n\})$$

Now let

$$a_1 v_1 + a_2 v_2 + ----- + a_n v_n = 0_v$$

but expression for 0_v is

$$0_F v_1 + 0_F v_2 + ----- + 0_F v_n = 0_v$$

$\Rightarrow a_1 = 0_F, a_2 = 0_F, -----, a_n = 0_F$

as expression for 0_v is unique.

$\Rightarrow v_1, v_2, -----, v_n$ is linearly independent

$\Rightarrow \{v_1, v_2, -----, v_n\}$ is a basis for V

v.v. imp

Theorem:-

Let $V(F)$ be a vector space and

$$\dim V = n$$

Let S be a linearly independent subset of V containing n vectors then S is a basis.

proof:-

Let $\dim V = n$ and

$$S = \{v_1, v_2, -----, v_n\}$$

be a linearly independent subset of V

Now let $v \in V$

then the set $T = \{v_1, v_2, ---, v_n, v\}$ consisting of $n+1$ vectors and

$n+1 > n$

Since set consisting of $n+1$ elements can not be linearly independent

$\Rightarrow \{v_1, v_2, ---, v_n, v\}$ is linearly dependent.

$\Rightarrow a_1v_1 + a_2v_2 + --- + a_nv_n + av = 0_V$ ---(1)

$\Rightarrow$ at least one a_i or a is non-zero

we prove $a \neq 0_F$ because if

$a = 0_F$

$\Rightarrow a_1v_1 + a_2v_2 + --- + a_nv_n + 0_F v = 0_V$

$\Rightarrow a_1v_1 + a_2v_2 + --- + a_nv_n = 0_V$

$\Rightarrow a_1 = 0, a_2 = 0, -----, a_n = 0 = 0_V$

also $a = 0_F$

and then $T = \{v_1, v_2, ---, v_n, v\}$ is linearly independent.

which is a contradiction

$\Rightarrow a \neq 0_F$

Now from (1) we can write

$$aV = -a_1v_1 - a_2v_2 - \cdots - a_nv_n$$

$$a^{-1}(av) = a^{-1}(-a_1v_1 - a_2v_2 - \cdots - a_nv_n)$$

$$\Rightarrow v = -a^{-1}a_1v_1 - a^{-1}a_2v_2, \cdots - a^{-1}a_nv_n$$

$$\Rightarrow v \in L(S)$$

$$\Rightarrow V \subset L(S)$$

Also

$$L(S) \subset V$$

$\Rightarrow V = L(S)$ and S is linearly independent

$\Rightarrow S$ is basis for V

Theorem:- اہم

Statement یاد رکھیں

Let $V(F)$ be a finite dimensional vector space and let S be linearly independent subset of V then S is contain in some basis for V.

Proof:-

Let V be a finite dimensional vector space and

$$\dim V = n$$

$\Rightarrow V$ has basis consisting of n elements

$\Rightarrow$ Every linearly independent subset of of V can contain at most n elements

So if $S = \{v_1, v_2, ----, v_r\}$ is linearly independent set of V then
$r \leq n$

If $r = n$ then S is itself a basis for V and S is contained in basis S.

If $r < n$ then S can not be a basis for V

$\Rightarrow L(S) \neq V$

and let $v_{r+1} \in V - L(S)$

and consider

$$a_1 v_1 + a_2 v_2 + ---- + a_r v_r + a_{r+1} v_{r+1} = 0_V \quad \text{---} (1)$$

and if $a_{r+1} \neq 0_F$

$\Rightarrow a_{r+1}^{-1} \in F$

and from (1) we can write

$$a_{r+1} v_{r+1} = -a_1 v_1 - a_2 v_2 ---- - a_r v_r$$

$$\Rightarrow a_{r+1}^{-1} a_{r+1} v_{r+1} = a_{r+1}^{-1}(-a_1 v_1 - a_2 v_2 ---- - a_r v_r)$$

$$\Rightarrow v_{r+1} = -a_{r+1}^{-1} a_1 v_1 - a_{r+1}^{-1} a_2 v_2 ---- - a_{r+1}^{-1} a_r v_r$$

$\Rightarrow v_{r+1} \in L(S)$

which is a contradiction

$\Rightarrow a_{r+1} = 0_F$

and then eq (1) becomes

$$a_1v_1 + a_2v_2 + \cdots + a_rv_r = 0_V$$

$\Rightarrow a_1 = 0_F, a_2 = 0_F, \cdots, a_r = 0_F$

$\because S = \{v_1, v_2, \cdots v_r\}$ is L.I

and hence $\{v_1, v_2, \cdots, v_r, v_{r+1}\}$ is L.I

If $r+1 = n$ then $\{v_1, v_2, \cdots, v_{r+1}\}$ is a basis for V and S is contained in this set.

If $r+1 < n$, then proceeding on the same lines we reached after $n-r$ steps at the linearly independent set

$$\{v_1, v_2, \cdots, v_r, v_{r+1}, v_{r+2}, \cdots v_n\}$$

which becomes the basis for V and S is contained in this basis for V.

Theorem:-

If W is a subspace of a finite dimensional vector space $V(F)$ then

$$\dim W \leq \dim V$$

and if $\dim W = \dim V$ then $W = V$

Proof:-

Let $\dim V = n$ then any set

consisting of more than n elements is linearly dependent.

Now let S is ~~linearly independent~~ basis for W and $|S| = m$ i.e $\dim W = m$

$\Rightarrow$ S is linearly independent subset of V

$\Rightarrow$ $m \leq n$

$\Rightarrow$ $\dim W \leq \dim V$

and if $\dim W = \dim V$

$\Rightarrow$ Basis for W contains n linearly independent vectors

$\Rightarrow$ Basis for W is a basis for V

$\Rightarrow$ S is a basis for V

$\Rightarrow$ $L(S) = V$ —— (1)

but S is a basis for W.

$\Rightarrow$ $L(S) = W$ —— (2)

from (1) and (2)

$$V = W$$

Theorem:-
Any linearly independent set of vectors in a finite dimensional vector space can be extended to a basis for V.

proof:-
Let V be a finite dimensional vector space of dim n.

$\Rightarrow$ V has a basis consisting of n elements.
and if $S = \{v_1, v_2, ---, v_r\}$ is a linearly independent set then
$$r \leq n$$
If $r = n$ then S itself is a basis.

If $r < n$, then S can not be basis for V
$$\Rightarrow L(S) \neq V$$
and let $v_{r+1} \in V - L(S)$
and consider
$$a_1v_1 + a_2v_2 + --- + a_rv_r + a_{r+1}v_{r+1} = 0_V \quad \text{--- (1)}$$
and if $a_{r+1} \neq 0_F$
$$\Rightarrow a_{r+1}^{-1} \in F$$
and from (1) we can write
$$a_{r+1}v_{r+1} - a_1v_1 - a_2v_2 - --- - a_rv_r$$

$a_{r+1}^{-1} a_{r+1} v_{r+1} = a_{r+1}^{-1} (-a_1 v_1 - a_2 v_2 - \cdots - a_r v_r)$

$\Rightarrow v_{r+1} = -a_{r+1}^{-1} a_1 v_1 - a_{r+1}^{-1} a_2 v_2 - \cdots - a_{r+1}^{-1} a_r v_r$

$\Rightarrow v_{r+1} \in L(S)$

which is a contradiction

$\Rightarrow a_{r+1} = 0_F$

and then eq (1) becomes

$a_1 v_1 + a_2 v_2 + \cdots + a_r v_r = 0_V$

$\Rightarrow a_1 = 0_F,\ a_2 = 0_F, \ldots, a_r = 0_F$

$\because S = \{v_1, v_2, \ldots v_r\}$ is L.I

and hence $\{v_1, v_2, \ldots, v_r, v_{r+1}\}$ is L.I

If $r+1 = n$ then the set $\{v_1, v_2, \ldots, v_r, v_{r+1}\}$ be the basis for V and

if $r+1 < n$, then proceeding on the same lines after $n-r$ steps we reached at the linearly independent set $\{v_1, v_2, \ldots, v_r, v_{r+1}, \ldots v_n\}$

which is a basis for V

Theorem:-

Let $V(F)$ be a finitely generated

vector space and let W be a subspace of V then any basis for W can be extended to a basis for V.

proof:-

Let S is a basis for W then S is linearly independent subset of V and since every linearly independent subset of V can be extended to a basis for V

$\Rightarrow$ S can be extended for a basis for V

Q:- Determine whether or not the following from a basis for the vectors space R^3 or not $(1,1,1),(1,2,3),(2,-1,1)$

Sol:-

Since $\dim R^3 = 3$

$\Rightarrow$ Any subset of R^3 consisting of three linearly independent vectors forms a basis for R^3

Therefore given vectors form a basis for R^3 if they are linearly independent.

Now the matrix formed by the given vectors as the rows of matrix is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 1 & 1 \\ 1-1 & 2-1 & 3-1 \\ 2-2 & 1-2 & 1-2 \end{bmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\underset{\sim}{R} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0+0 & -3+3 & 2+6 \end{bmatrix} \quad R_3 + 3R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 8 \end{bmatrix}$$

Since there is no zero row in echelon form
$\Rightarrow$ Given vector are linearly independent
Hence form a basis for R^3

2020
Q:- Check the following set
$\{(1,1,0,0), (0,2,-1,0), (0,0,0\,3), (0,1,2,3)\}$
for the basis of R^3.

Sol:-

Since $\dim R^4 = 4$

$\Rightarrow$ Any subset of R^4 consisting of four linearly independent vectors form a basis for R^4.

Since given vectors set consist of four vectors of R^4, therefore given set form a basis for R^4, if given set is linearly independent.

Now the matrix formed by the given vectors as the rows of matrix is

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\underset{R}{\sim} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & -1 & 0 \end{bmatrix} \quad \text{by } R_{24}$$

$$\underset{R}{\sim} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad \text{by } R_{34}$$

$$\sim_R \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0-0 & 2-2 & -1-4 & 6-6 \\ 0 & 0 & 0 & 3 \end{bmatrix} \text{ by } R_3 - 2R_1$$

$$\sim_R \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -6 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

There is no zero row in echelon form of matrix A

$\Rightarrow$ Given vectors are linearly independent

Q:- Check the following vectors form the basis of R^3 over R.
2001

(i) $\{(0,1,1), (0,2,1), (1,5,3)\}$

(ii) $\{(1,1,0), (3,1,3), (5,3,3)\}$

Q: Let $S = \left\{ \begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}, \begin{bmatrix} 2 & -4 \\ -1 & 7 \end{bmatrix}, \begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix} \right\}$ be a subset of vector space $V(R)$ of all 2×2 matrices. Find dimension of $(L(S))$ and basis for $L(S)$

proof:-

Since $|S| = 4$

$\Rightarrow L(S)$ has $\dim \leq 4$

Now to check S is linearly independent or not.

Consider

$$\begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix} = a\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix} + b\begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix} + c\begin{bmatrix} 2 & -4 \\ -1 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} a & -5a \\ -4a & 2a \end{bmatrix} + \begin{bmatrix} b & b \\ -b & 5b \end{bmatrix} + \begin{bmatrix} 2c & -4c \\ -c & 7c \end{bmatrix}$$

$$\begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} a+b+2c & -5a+b-4c \\ -4a-b-c & 2a+5b+7c \end{bmatrix}$$

$$\Rightarrow \begin{cases} a+b+2c = 1 & \text{—— (1)} \\ -5a+b-4c = -7 & \text{—— (2)} \\ -4a-b-c = -5 & \text{—— (3)} \\ 2a+5b+7c = 1 & \text{—— (4)} \end{cases}$$

adding (1) and (3) we have

$-3a + c = -4$ —— (5)

adding (2) and (3)

$$\Rightarrow -9a - 5c = -12 \quad \text{---} (6)$$

Now by $-3 \times (5) + (6)$

$$\begin{array}{rcl} \cancel{9a} - 3c &=& \cancel{12} \\ -\cancel{9a} - 5c &=& -\cancel{12} \\ \hline -8c &=& 0 \end{array}$$

$$\Rightarrow c = 0$$

Using $c = 0$ in (5)

$$\Rightarrow -3a + 0 = -4$$

$$\Rightarrow a = \frac{4}{3}$$

Using $a = \frac{4}{3}$, $c = 0$ in (1)

$$\frac{4}{3} + b + 2(0) = 1$$

$$\Rightarrow b = 1 - \frac{4}{3}$$

$$\Rightarrow b = -\frac{1}{3}$$

and $a = +\frac{4}{3}$, $b = -\frac{1}{3}$, $c = 0$

Satisfies all the equation (1), (2), (3), (4) i-e

$$\begin{bmatrix}1&-7\\-5&1\end{bmatrix} = \frac{4}{3}\begin{bmatrix}1&-5\\-4&2\end{bmatrix} + \left(\frac{-1}{3}\right)\begin{bmatrix}1&1\\-1&5\end{bmatrix} + 0\begin{bmatrix}2&-4\\-1&7\end{bmatrix}$$

$\Rightarrow$ Set S is linearly dependent

Now consider the set

$$T = S - \left\{\begin{bmatrix}1&-7\\-5&1\end{bmatrix}\right\} \quad \text{then } L(T) = L(S)$$

and since

$$T = \left\{\begin{bmatrix}1&-5\\-4&2\end{bmatrix}, \begin{bmatrix}1&1\\-1&5\end{bmatrix}, \begin{bmatrix}2&-4\\-1&7\end{bmatrix}\right\}$$

Now consider.

$$a_1\begin{bmatrix}1&-5\\-4&2\end{bmatrix} + b_1\begin{bmatrix}1&1\\-1&5\end{bmatrix} + c_1\begin{bmatrix}2&-4\\-1&7\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}a_1+b_1+2c_1 & -5a+b_1+4c_1\\ -4a-b_1-c_1 & 2a_1+5b_1+7c_1\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$a_1+b_1+2c_1 = 0 \quad \text{— (i)}$$
$$-5a_1+b_1+4c_1 = 0 \quad \text{— (ii)}$$
$$-4a_1-b_1-c_1 = 0 \quad \text{— (iii)}$$
$$2a_1+5b_1+7c_1 = 0 \quad \text{— (iv)}$$

adding (i) and (iii)

$$-3a_1+c_1 = 0 \quad \text{— (v)}$$

adding (ii) and (iii)

$-9a_1 - 5c_1 = 0$ ——— (vi)

by $-3 \times (v) + (vi)$

$$\cancel{9a_1} - 3c_1 = 0$$
$$-\cancel{9a_1} - 5c_1 = 0$$
$$-8c_1 = 0$$
$$\Rightarrow c_1 = 0$$

using $c_1 = 0$ in (v)

$$-3a_1 + 0 = 0$$
$$\Rightarrow a_1 = 0$$

using $a_1 = 0$, $c_1 = 0$ in eq (1)

$$0 + b_1 + 2(0) = 0$$
$$\Rightarrow b_1 = 0$$

and $a_1 = 0$, $b_1 = 0$, $c_1 = 0$ satisfies all the (i), (ii), (iii), (iv) equations.

$\Rightarrow$ set T is linearly independent and hence form a basis for $L(S)$ and

$$\dim(L(S)) = 3.$$

Sum of Subspaces:-

Let U and W be two subspaces of vector space $V(F)$ then the sum of U and W is defined as

$U+W = \{u+w : u \in U \text{ and } w \in W\}$

Note:-

Since $\forall$ $u \in U$ and $w \in W$

$u, w \in V$

$\Rightarrow$ $u+w \in V$

$\Rightarrow \{u+w : u \in U, w \in W\} \subset V$

$\Rightarrow$ $U+W \subset V$

Theorem:

If U and W are subspaces of a vector space $V(F)$, then $U+W$ is the smallest subspace of V containing both U and W

proof:-

If U and W are subspace of V then

$0_V \in U$ and $0_V \in W$

$\Rightarrow$ $0_V + 0_V \in U+W$

$\Rightarrow$ $0_V \in U+W$

$\Rightarrow$ $U+W \neq \phi$

Now let $u_1+w_1, u_2+w_2 \in U+W$

then for $a, b \in F$

$a(u_1+w_1) + b(u_2+w_2)$

$= au_1 + aw_1 + bu_2 + bw_2$ $\because u_1, u_2, w_1, w_2 \in V$

$= (au_1 + bu_2) + (aw_1 + bw_2)$

$\in \bar{U} + \bar{W}$ $\because \bar{U}$ and $\bar{W}$ are subspace of V

$\Rightarrow \bar{U}+\bar{W}$ is a subspace of V

Also $\forall\ u \in \bar{U}$ since $O_V \in \bar{W}$

$\Rightarrow u + O_V \in \bar{U}+\bar{W}$

$\Rightarrow u \in \bar{U}+\bar{W}$

$\Rightarrow \bar{U} \subset \bar{U}+\bar{W}$

Similarly

$\bar{W} \subset \bar{U}+\bar{W}$

Now consider L be any other subspace containing both $\bar{U}$ and $\bar{W}$ then

$\forall\ u \in \bar{U}, u \in L$ } $u+w \in L$

and $\forall\ w \in \bar{W}, w \in L$ } $\because (L,+)$ is a group

$\Rightarrow \bar{U}+\bar{W} \subset L$

$\Rightarrow \bar{U}+\bar{W}$ is the smallest subspace of $V(F)$ containing both $\bar{U}$ and $\bar{W}$.

Theorem :- Result یاد رکھیں

If S_1 and S_2 are two finite subsets of vector space $V(F)$ then

$$L(S_1 \cup S_2) = L(S_1) + L(S_2)$$

Proof:- let $S_1 = \{u_1, u_2, ----, u_n\}$
$S_2 = \{w_1, w_2, ----, w_m\}$

Now let $v \in L(S_1 \cup S_2)$
then v is linearly combination of vectors of $S_1 \cup S_2$

$\Rightarrow v = a_1u_1 + a_2u_2 + --- + a_nu_n + b_1w_1 + ---- + b_mw_m$

$\Rightarrow v = u + w$ where $u \in L(S_1)$ and $w \in L(S_2)$

$\Rightarrow v \in L(S_1) + L(S_2)$

$\Rightarrow L(S_1 \cup S_2) \subset L(S_1) + L(S_2)$ —— (1)

Conversly let
$v \in L(S_1) + L(S_2)$

$\Rightarrow v = u + w$ where $u \in L(S_1)$ and $w \in L(S_2)$

$\Rightarrow v = (a_1u_1 + a_2u_2 + ---- + a_nu_n) + (b_1w_1 + b_2w_2 + --- + b_mw_m)$

$\Rightarrow v = a_1u_1 + a_2u_2 + ---- + a_nu_n + b_1w_1 + b_2w_2 + ---- b_mw_m$

$\Rightarrow v$ is linearly Combination of vectors of $S_1 \cup S_2$

$\Rightarrow v \in L(S_1 \cup S_2)$

$\Rightarrow L(S_1) + L(S_2) \subset L(S_1 \cup S_2)$ —— (2)

from (1) and (2)
$L(S_1 \cup S_2) = L(S_1) + L(S_2)$

Consider $R^3 = \{(x,y,z) : x,y,z \in R\}$ then R^3 is the direct sum of its subspaces
$U = \{(a,b,0) : a,b \in R\}$ and
$W = \{(0,0,c) : c \in R\}$
because $U + W \subset R^3$
and $\forall\ (a,b,c) \in R^3$ 2005
$(a,b,c) = (a,b,0) + (0,0,c) \in U+W$ Imp
$\Rightarrow R^3 \subset U + W$
$\Rightarrow R^3 = U + W$ (i)
if (ii)
$(x,y,z) \in U \cap W$ then
$(x,y,z) \in U$ and $(x,y,z) \in W$
$\Rightarrow z = 0,\ x = 0$ and $y = 0$
$\Rightarrow (x,y,z) = (0,0,0)$
$\Rightarrow U \cap W = \{(0,0,0)\}$
$\Rightarrow R^3 = U \oplus W$

Example:-
Show that R^3 is direct sum of its subspaces $U = \{(x,y,z) : x = y = z\}$ and $W = \{(0,y,z) : y,z \in R\}$
Sol:-
Since $(x,y,z) \in R^3$

Notes:-
If S_1 and S_2 are subspaces of a vec[tor] space $V(F)$ then $L(S_1 \cup S_2) = L(S_1) + L[(S_2)]$
$= S_1 + S_2$

Direct sum of subspaces:-
Let U and W be subspaces of a vector space $V(F)$. Then we say V a direct sum of U and W if

(1) $V = U + W$ and
(2) $U \cap W = \{0_V\}$
If V is a direct sum of U and W then we write
$$V = U \oplus W$$

Direct sum of sub spaces:-
Let $V(F)$ be a vector space, then [V] is said to be a direct sum of subs[paces] U and W if every $v \in V$ can b[e] uniquely written as $v = u + w$ where $u \in U$ and $w \in$ [W]
and we write V is a direct sum of U and W as
$$V = U \oplus W$$

Example:-

can be uniquely expressed as
$(x,y,z) = (x,x,x) + (0,-x+y,-x+z)$
where $(x,x,x) \in U$ and $(0,-x+y,-x+z) \in W$
$\Rightarrow R^3$ is direct sum of U and W
i-e $R^3 = U + W$

2005 Theorem :- Imp 2005

Imp The vector space $V(F)$ is the direct sum of its subspaces U and W iff

(i) $U + W = V$

(ii) $U \cap W = \{0_V\}$

proof:-

Let $V = U \oplus W$, then for any $v \in V$ v can be uniquely expressed as :

$v = u + w$ where $u \in U$ and $w \in W$

$\Rightarrow V \subset U + W$

also $U + W \subset V$ $\because$ sum of two subspace is a subspace.

$\Rightarrow V = U + W$

Now let $v \in U \cap W$

$\Rightarrow v \in U$ and $v \in W$

Now $v \in U \Rightarrow v + 0_V \in U + W$

and $v \in W \Rightarrow 0_V + v \in U + W$

but $V = U \oplus W$

$\Rightarrow$ expression for v is unique

i-e $u_1 + w_1 = u_2 + w_2$

$\Rightarrow u_1 = u_2$, $w_1 = w_2$

So $v + O_v = O_v + v$

$\Rightarrow v = O_v$ and $O_v = v$

$\Rightarrow v = O_v$

$\Rightarrow U \cap W = \{O_v\}$

Conversely let U and W be subspaces of V satisfying (i) $V = U + W$

(ii) $U \cap W = \{O_v\}$

Now let $v \in V$ then $v \in U + W$

then $v = u + w$ where $u \in U$ and $w \in W$

Let also $v = u' + w'$ " $u' \in U$ and $w' \in W$

$\Rightarrow u + w = u' + w'$

$\Rightarrow u - u' = w' - w$

Let $s = u - u' = w' - w$

and $s = u - u' \in U$

$s = w' - w \in W$

$\Rightarrow s \in U \cap W$

$\Rightarrow s \in \{O_v\}$

$\Rightarrow s = O_v$

$\Rightarrow u - u' = O_v$ and $w' - w = O_v$

$\Rightarrow u = u'$ and $w' = w$

$\Rightarrow$ expression for v is unique

$\Rightarrow$ Every $v \in V$ can be uniquely expressed as $v = u + w$ where $u \in U$ and $w \in W$

$\Rightarrow$ V is direct sum of U and W
i-e $V = U \oplus W$

2002 Theorem:-
If U and W are finite dimensional subspaces of a vector space $V(F)$, then
$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$$
proof:-
If U and W are finite dimensional subspaces of $V(F)$ then, there are two cases
Case I:-
If $U \cap W \neq \{0_v\}$
Since U and W are finite dimensional subspaces of V
$\Rightarrow$ $U \cap W$ is finite dimensional being a subspace of finite dimensional subspaces U and W.
Let $\{v_1, v_2, ---, v_r\} = A$ is a basis of $U \cap W$
Since $U \cap W \subset U$
and $U \cap W \subset W$
So for the basis of U and W we have to extend A
So Let
$\{v_1, v_2, --- v_r, u_1, u_2, --- u_s\} = B$ is

a basis for $\bar{U}$ and $\{v_1, v_2, \ldots, v_r, w_1, w_2, \ldots, w_t\} = C$ is a basis for $\bar{W}$

then $\dim \bar{U} = r+s$, $\dim \bar{W} = r+t$

$\dim(\bar{U} \cap \bar{W}) = r$

Now consider the set

$D = A \cup B \cup C = \{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots u_s, w_1, \ldots w_t\}$

Now

$L(D) = L(A \cup B \cup C) = L(B \cup C) \quad \because A \subset B$

$= L(B) + L(C)$

$= \bar{U} + \bar{W}$

$\Rightarrow D$ is a spanning set for $\bar{U} + \bar{W}$

$\Rightarrow \dim(\bar{U} + \bar{W}) \leq r+s+t$

Now we show that the vectors of D are linearly independent

for this consider

$a_1v_1 + a_2v_2 + \cdots + a_rv_r + b_1u_1 + \cdots + b_su_s + c_1w_1 + c_2w_2 + \cdots + c_tw_t = 0$ — (1)

$\Rightarrow a_1v_1 + \cdots + a_rv_1 + b_1u_1 + \cdots b_su_s = -c_1w_1 \cdots - c_tw_t$

Let

$x = a_1v_1 + \cdots + a_rv_r + b_1u_1 + \cdots + b_su_s = -c_1w_1 \cdots - c_tw_t$

Now

$x = a_1v_1 + a_2v_2 + \cdots + a_rv_r + b_1u_1 + \cdots + b_su_s \in L(B) = \bar{U}$ — (2)

and

$x = -c_1w_1 - c_2w_2, \ldots - c_tw_t \in L(C) = \bar{W}$ — (3)

$\Rightarrow x \in U \cap W$

and since basis for $U \cap W$ is

$A = \{v_1, v_2, ---, v_r\}$

$\Rightarrow x \in L(A)$ i-e

$x = d_1 v_1 + d_2 v_2 + --- + d_r v_r$ —— (4)

where $d_i \in F$

from (3) and (4)

$d_1 v_1 + d_2 v_2 + --- + d_r v_r = -c_1 w_1 - c_2 w_2 --- - c_t w_t$

$d_1 v_1 + d_2 v_2 + --- + d_r v_r + c_1 w_1 + c_2 w_2 + --- + c_t w_t = 0_V$

$\Rightarrow d_1 = 0, d_2 = 0, --- d_r = 0, c_1 = 0, c_2 = 0, --- c_t = 0$

$\because v_1, v_2, --- v_r, w_1, w_2, --- w_r$ are L.I.

using $c_1 = 0, c_2 = 0, --- c_t = 0$ in (1)

$a_1 v_1 + a_2 v_2 + --- + a_r v_r + b_1 u_1 + b_2 u_2 + --- + b_s u_s = 0_V$

$\Rightarrow a_1 = 0, a_2 = 0, --- a_r = 0, b_1 = 0, b_2 = 0, --- b_s = 0$

$\because v_1, v_2, --- v_r, u_1, u_2, --- u_s$ are L.I

Therefore from (1)

$a_1 v_1 + a_2 v_2 + --- + a_r v_r + b_1 u_1 + b_2 u_2 + --- + b_s u_s$
$+ c_1 w_1 + c_2 w_2 + --- + c_t w_t = 0_V$

$\Rightarrow a_1 = 0, --- a_r = 0, b_1 = 0, --- b_s = 0, c_1 = 0, --- c_t = 0$

$\Rightarrow \{v_1, v_2, --- v_r, u_1, u_2, --- u_s, w_1, --- w_t\} = D$

is linearly independent

$\Rightarrow D$ is a basis for $\bar{U} + \bar{W}$

$\Rightarrow \dim(\bar{U} + \bar{W}) = r + s + t$

$\Rightarrow \dim(\bar{U} + \bar{W}) = (r + s) + (r + t) - r$

$\Rightarrow \dim(\bar{U} + \bar{W}) = \dim \bar{U} + \dim \bar{W} - \dim(\bar{U} \cap$

Case II :-

If $\bar{U} \cap \bar{W} = \{0_v\}$ then $\dim(U \cap W) = 0$

and Let

$A = \{u_1, u_2, --- u_s\}$ is a basis for $\bar{U}$

and

$B = \{w_1, w_2, --- w_t\}$ " " " $\bar{W}$

$\Rightarrow \dim \bar{U} = s, \quad \dim \bar{W} = t$

Now Consider the set

$C = A \cup B = \{u_1, u_2, --- u_s, w_1, --- w_t\}$

then

$L(C) = L(A \cup B) = L(A) + L(B) = \bar{U} + \bar{W}$

$\Rightarrow C$ is a spanning set for $\bar{U} + \bar{W}$

$\Rightarrow \dim(\bar{U} + \bar{W}) \leq s + t$

Now we prove C is linearly independent

for this Let

$a_1u_1 + a_2u_2 + --- + a_su_s + b_1w_1 + --- + b_tw_t = 0_v \quad \text{--- (1)}$

$\Rightarrow a_1u_1 + a_2u_2 + --- + a_su_s = -b_1w_1 - b_2w_2 --- - b_tw_t$

and Let

$x = a_1u_1 + a_2u_2 + --- + a_su_s = -b_1w_1 - b_2w_2 --- - b_tw_t$

$\Rightarrow x = a_1u_1 + a_2u_2 + --- + a_su_s \in L(A) = \bar{U}$

and

$x = -b_1w_1 - b_2w_2 --- - b_tw_t \in L(B) = \bar{W}$

$\Rightarrow x \in \bar{U} \cap \bar{W}$

$\Rightarrow x \in \{0_v\}$

$\Rightarrow x = 0_v$

$\Rightarrow a_1u_1 + a_2u_2 + \cdots + a_su_s = 0_V$

and $-b_1w_1 - b_2w_2 - \cdots - b_tw_t = 0_V$

$\Rightarrow a_1 = 0, a_2 = 0, \cdots a_s = 0$ and $-b_1 = 0, \cdots -b_t = 0$

$\because u_1, u_2, \cdots u_s$ and $w_1, w_2, \cdots w_t$ are L.I

$\Rightarrow a_1 = 0, a_2 = 0, \cdots a_s = 0, b_1 = 0, \cdots b_t = 0$

from (1)

$u_1, u_2, \cdots u_s, w_1, w_2, \cdots w_t$ are L.I

$\Rightarrow C$ is a basis for $U + W$

$\Rightarrow \dim(U + W) = s + t$

$\Rightarrow \dim(U + W) = s + t - 0$

$\Rightarrow \dim(U + W) = \dim U + \dim W - \dim(U \cap W)$

Note:-

If $\dim(U \oplus W) = \dim(U + W)$

$= \dim U + \dim W - \dim(U \cap W)$

$= \dim U + \dim W - \dim(\{0_V\})$

$= \dim U + \dim W - 0 \quad \because U \cap W =$

$= \dim U + \dim W$

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