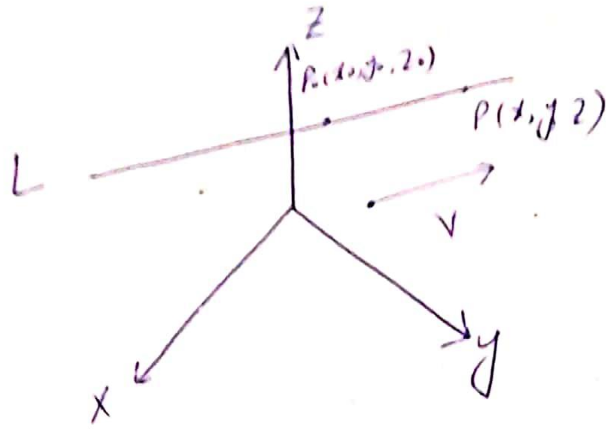
Lines and planes in space:-



Suppose that L is a line in a space passing through a point $P_0(x_0, y_0, z_0)$ parallel to vector $v = v_1 i + v_2 j + v_3 k$. The L is set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to v. Thus

$P_0P = tv$ for some scalar parameter

t.

$$(x - x_0)i + (y - y_0)j + (z - z_0)k = t(v_1 i + v_2 j + v_3 k)$$

$$xi - x_0 i + yj + y_0 j + zk - z_0 k = t(v_1 i + v_2 j + v_3 k)$$

$$xi + yj + zk = x_0 i + y_0 j + z_0 k + t(v_1 i + v_2 j + v_3 k)$$

Vector Equation for a line:-

A vector Equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to v is

$r(t) = r_0 + tv \quad -\infty < t < \infty$

where r is the position vector of a point $P(x, y, z)$ on L and r_0 is the position vector of $P_0(x_0, y_0, z_0)$

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3$$

Thes Equations give us the standard parametrization of the line for the parameter interval $-\infty < t < \infty$

Parametric Equations for a line

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to

$V = v_1 i + v_2 j + v_3 k$

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3 \quad -\infty < t < \infty$$

Examples:-

① Find parametric Equations for the line through $(-2, 0, 4)$ parallel to $v = 2i + 4j - 2k$.

$P_0(x_0, y_0, z_0) = (-2, 0, 4) \quad v_1 i + v_2 j + v_3 k = 2i + 4j - 2k$

$x = x_0 + tv_1 \qquad y = y_0 + tv_2, \qquad z = z_0 + tv_3$

$= -2 + 2t, \quad y = 0 + 4t \qquad = 4 - 2t$

Find the parametric Equations for the line through $P(-3,2,-3)$ and $Q(1,-1,4)$

$$r(t) = r_0 + tv$$
$$= r_0 + t|v|\frac{v}{|v|}$$

$PQ = 4i - 3j + 7k$

$(x_0, y_0, z_0) = (-3, 2, -3)$

$x = x_0 + tv_1$, $y = y_0 + tv_2$, $z = z_0 + tv_3$

$= -3 + 4t$ $\quad = 2 - 3t$ $\quad = -3 + 7t$

we can choose $Q(1,-1,4)$ as base point

$x = x_0 + tv_1$, $y = -1 - 3t$, $z = 4 + 7t$

$= 1 + 4t$

Distance from a point to a line :-

$$d = \frac{|PS \times v|}{|v|}$$

Example:- Find the distance from the point $S(1,1,5)$ to the line

$L: x = 1+t$, $y = 3-t$, $z = 2t$

Sol. $P(1,3,0)$ parallel to $v = i - j + 2k$

$PS = (1-1)i + (1-3)j + (5-0)k$

$= -2j + 5k$

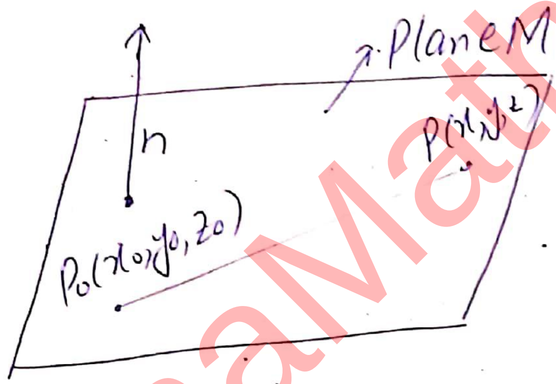
$$PS \times V = \begin{vmatrix} i & j & k \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= i(-4+5) - j(0-5) + k(0+2)$$

$$= \hat{i} + 5\hat{j} + 2k$$

$$d = \frac{|PS \times V|}{|V|} = \frac{\sqrt{1+25+4}}{\sqrt{1+1+1}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

An Equation for a plane in space.-



$P_0(x_0, y_0, z_0)$ normal to the non zero vector $n = Ai + Bj + Ck$

$$n \cdot P_0P = 0$$

$$(Ai + Bj + Ck) \cdot [(x-x_0)i + (y-y_0)j + (z-z_0)k] = 0$$

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

Equation for a plane

Vector Equation $n \cdot \overrightarrow{P_0P} = 0$

Component Equation $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$

Component Equation simplified

$$Ax + By + Cz = D, \text{ where}$$

$$D = Ax_0 + By_0 + Cz_0$$

Examples:-

① Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $n = 5i + 2j - k$.

Sol:

$$5(x-(-3)) + 2(y-0) + (-1)(z-7) = 0$$

$$5(x+3) + 2y - z + 7 = 0$$

$$5x + 15 + 2y - z + 7 = 0$$

$$5x + 2y - z = -22$$

② Find the Equation for the plane through $A(0,0,1)$, $B(2,0,0)$ and $C(0,3,0)$

$$\overrightarrow{AB} = ((2-0), 0, -1)$$

$$= (2, 0, -1)$$

$$\vec{AC} = (0, 3, -1)$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix}$$

$$= 3i + 2j + 6k$$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$3(x - 0) + 2(y - 0) + 6(z - 1) = 0$$

$$3x + 2y + 6z - 6 = 0$$

$$3x + 2y + 6z = 6.$$

Line of Intersection:-

Find a vector parallel to the line of intersection of the plane $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

$$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= 14i + 2j + 15k.$$

Ex Find the point where the line

$x = 8/3 + 2t, \quad y = -2t, \quad z = 1 + t$

intersects the plane $3x + 2y + 6z = 6$

Sol The Point

$$\left(8/3 + 2t, -2t, 1+t\right)$$

$$3\left(8/3 + 2t\right) + 2(-2t) + 6(1+t) = 6.$$

$$8 + 6t - 4t + 6 + 6t = 6$$

$$8t = -8$$

$$\boxed{t = -1}$$

The point of intersection

$$(x, y, z)\big|_{t=-1} = \left(8/3 - 2, 2, 1-1\right) = \left(2/3, 2, 0\right)$$

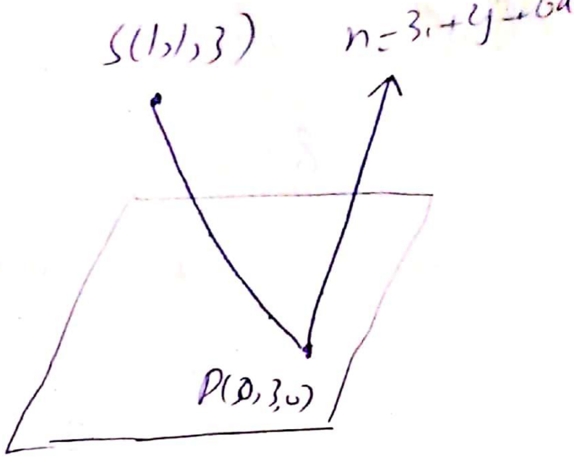
The Distence from a point to plane

$$d = \left| PS \cdot \frac{n}{|n|} \right|, \quad n = Ai + Bj + Ck$$

Ex Find the distence from $S(1,1,3)$ to the plane $3x + 2y + 6z = 6$

$$n = 3i + 2j + 6k$$

$PS = i - 2j + 3k$



$|n| = \sqrt{9+4+36} = 7$

$d = \left| PS \cdot \frac{n}{|n|} \right|$

$= (i - 2j + 3k) \cdot \left(\frac{3}{7}i + \frac{2}{7}j + \frac{6}{7}k\right)$

$= \frac{3}{7} - \frac{4}{7} + \frac{18}{7} = \frac{3-4+18}{7} = \frac{17}{7}$

Angles Between planes:-

Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Sol: The vectors

$n_1 = 3i - 6j - 2k$, $n_2 = 2i + j - 2k$

$\theta = \cos^{-1}\left(\frac{n_1 \cdot n_2}{|n_1||n_2|}\right)$

$n_1 \cdot n_2 = 6 - 6 + 4 = 4$

$|n_1| = \sqrt{9+36+4}$
$= \sqrt{49} = 7$

$|n_2| = \sqrt{4+1+4} = 3$

$= \cos^{-1}\left|\frac{4}{3 \times 7}\right|$

≈ 1.38 radian.

MV Calculus

Lecture 01:-

Functions of several variables:-

$$z = f(x, y)$$

dependent variable

independent variables

Domain and Range:-

Examples:- Find domain and sketch

(a) $f(x, y) = \dfrac{\sqrt{x+y+1}}{x-1}$

$$D = \{(x,y) \mid x+y+1 \geq 0,\ x \neq 1\}$$

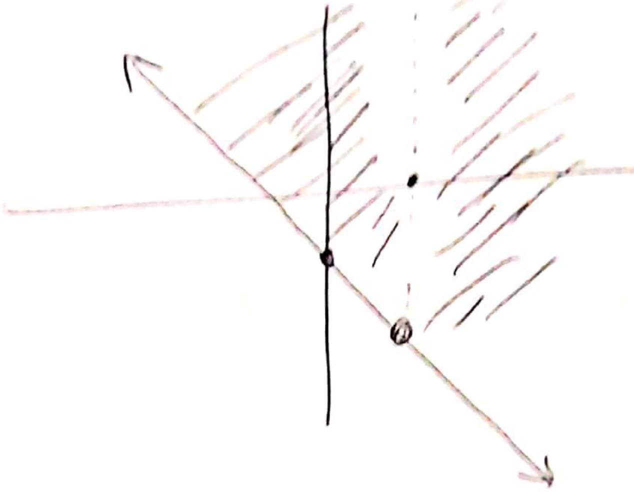
(b) $f(x, y) = x \ln(y^2 - x)$

$y^2 - x > 0 \qquad y^2 > x$ or $x < y^2$

$$D = \{(x, y) \mid x < y^2\}$$

(a) $x + y + 1 \geq 0$

$y \geq -x - 1$



(b) $y^2 - x > 0$

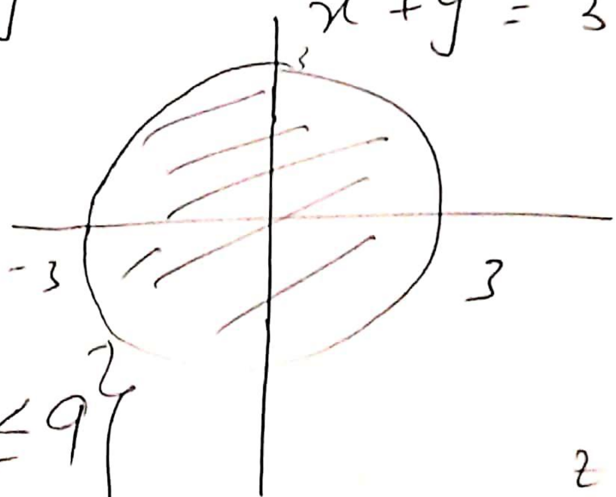
$y^2 > x$

(c) Find the domain and Range

$$f(x,y) = \sqrt{9 - x^2 - y^2}$$

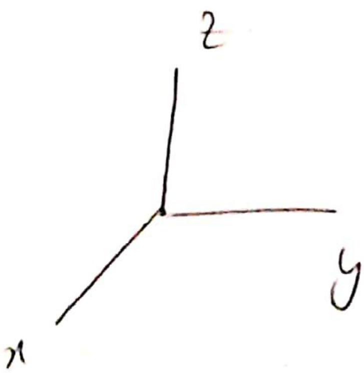
$9 \geq x^2 + y^2$

$x^2 + y^2 = 3^2$



$D = \{(x,y) \mid x^2 + y^2 \leq 9\}$

$R = [0,3]$



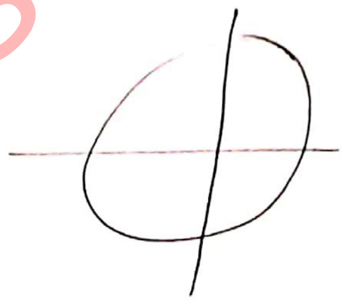
$$f(x,y,z) = \sqrt{9-x^2-y^2-z^2}$$

$$9-x^2-y^2-z^2 \geq 0$$

$$x^2+y^2+z^2 \leq 3^2$$

$$D = \{(x,y,z) : x^2+y^2+z^2 \leq 9\}.$$

$$R = [0,3].$$

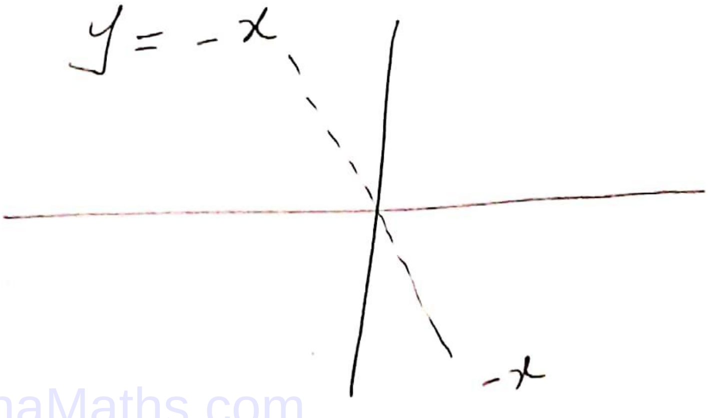


(e) $$f(x,y) = \frac{x-y}{x+y}.$$

$$x+y \neq 0.$$

$$y \neq -x.$$

avoid the line $y = -x$



$$D = \{(x,y)\ y \neq -x\}$$

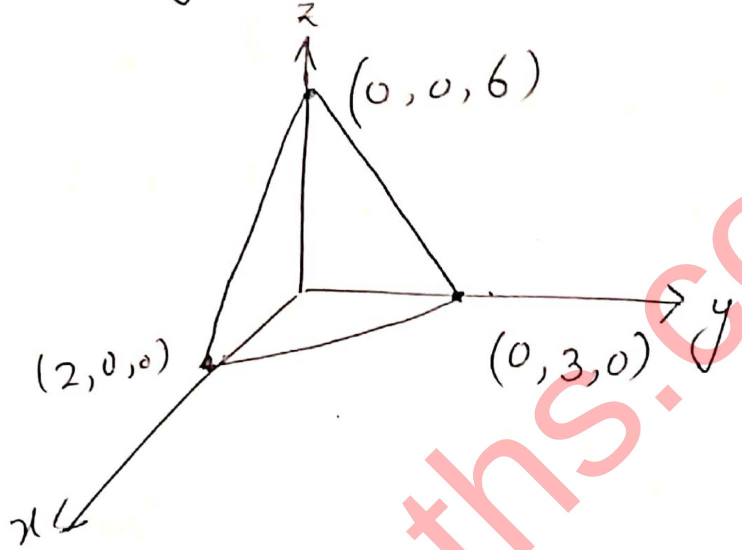
$$R = -\infty < z < \infty$$

Sketch the graph of function (3)

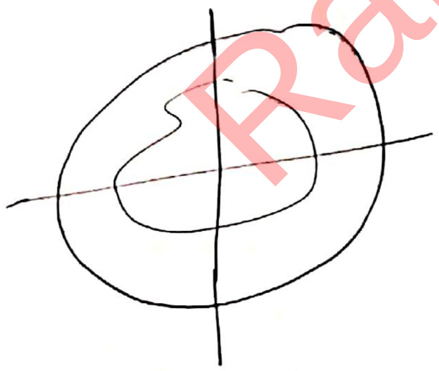
$$f(x,y) = 6 - 3x - 2y$$

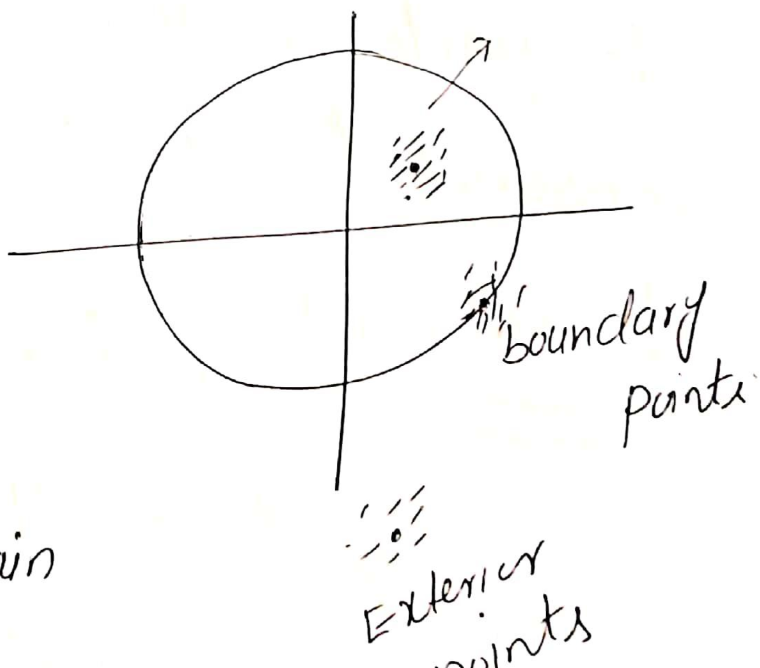
$$z = 6 - 3x - 2y$$

$$3x + 2y + z = 6$$



open, closed, bounded and unbounded regions:-





enclosed in a circle bounded domain

otherwise unbounded

Definitions:-

The interior points of a region as set, make up the interior of region. A region is open if it consists entirely of interior points. A region is closed if it contains all its boundary.

Bounded/unbounded:-

A region in the plane is bounded if it lies inside a disk of finite radious. A region is unbounded if it is not bounded.

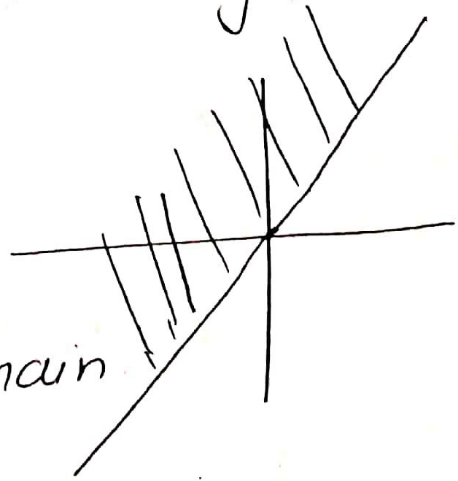
Examples:-

(i) $f(x,y) = \sqrt{y-x}$ $\quad y \geq x$.

$y - x \geq 0, \quad y \geq x$

unbounded

closed domain



$$f(x,y)=\frac{1}{\sqrt{y-x}}$$

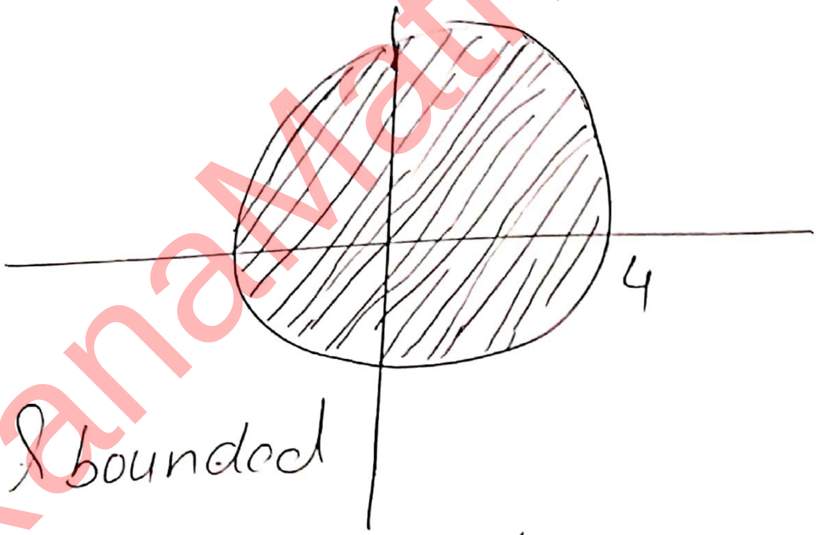
unbounded open

(iii) $$f(x,y)=\sqrt{16-x^2-y^2}$$

$$16-x^2-y^2 \geq 0$$

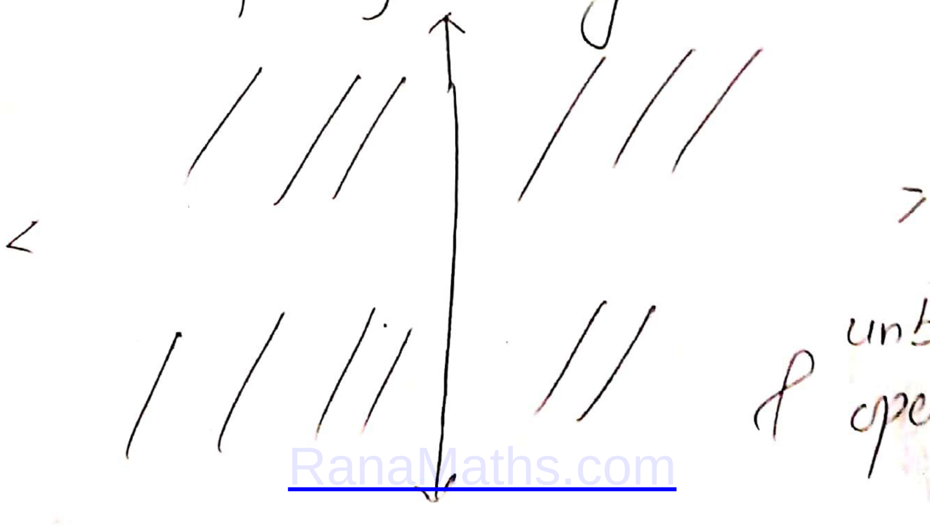
$$16 \geq x^2+y^2$$

$$x^2+y^2 \leq 4^2$$



closed & bounded

(iv) $$f(x,y)=\frac{x}{y}$$



unbounded & open

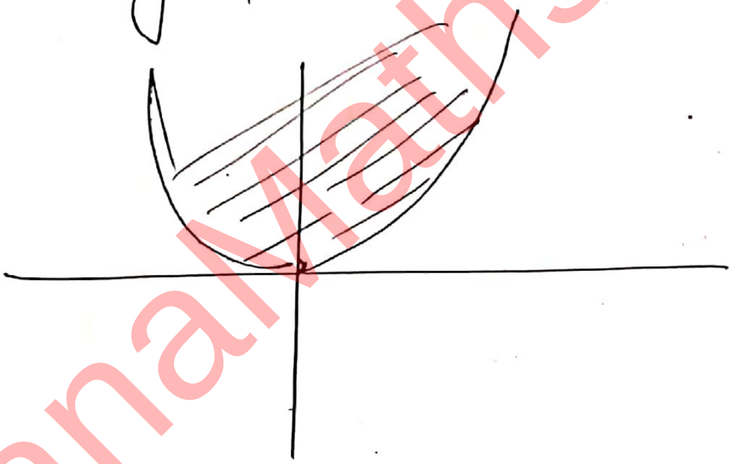
(v) $f(x,y,z) = \sqrt{16 - x^2 - y^2 - z^2}$

closed & bounded

(vi) Describe the domain of the function $f(x,y) = \sqrt{y - x^2}$

$y - x^2 \geq 0$

$y \geq x^2$



closed & unbounded.

14.3

Graphs, Level curves, and contours of functions of two variables:-

Definitions:-

The set of points in the plane where a function $f(x,y)$ has a constant value $f(x,y)=c$ is called level curve of f.

The set of points (x,y,z) in space (incl) where a function of three variables has a constant value $f(x,y,z)=c$ is called a level surface.

Examples:-

(a) Sketch the curves of function $f(x,y) = 6-3x-2y$ for the values $K = -6, 0, 6, 12$.

Sol:- $f(x,y) = k$.

$6-3x-2y=-6$.

$-3x-2y+12=0$

$3x+2y-12=0$

$x=0$

$y=6$, $(0,6)$

& $y=0$.

$x=4$

$(4,0)$

for $k=0$.

$6-3x-2y=0$.

$3x+2y-6=0$.

$(0,3)$

and $(2,0)$.

$k=6$

$6-3x-2y=6$:

$3x+2y=0$.

$x=0$

$(0,0)$

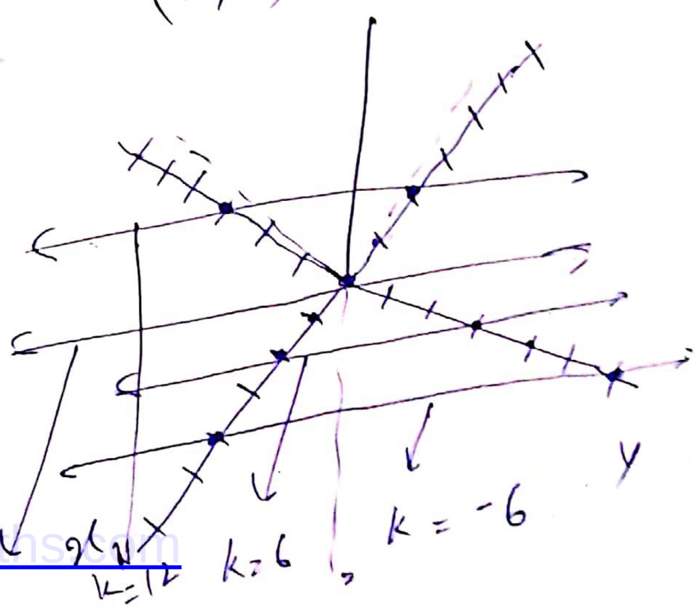
$(0,0)$

$6-3x-2y=12$.

$3x+2y+6=0$

$(0,-3)$

$(-2,0)$



(b) Sketch the level curves of the function $g(x,y) = \sqrt{9-x^2-y^2}$ for $k = 0, 1, 2, 3$.

Sol:-

$g(x,y) = k$:

$k=0$

$\sqrt{9-x^2-y^2} = 0$

$9-x^2-y^2 = 0$

$x^2+y^2 = 3^2$

$k=3$

$\sqrt{9-x^2-y^2} = 3$

$9-x^2-y^2 = 3^2$

$9-x^2-y^2 = 9$

$x^2+y^2 = 0$

$x^2+y^2 = 0^2$

$k=1$

$\sqrt{9-x^2-y^2} = 1$

$9-x^2-y^2 = 1$

$x^2+y^2 = 8$

$x^2+y^2 = 2.82$

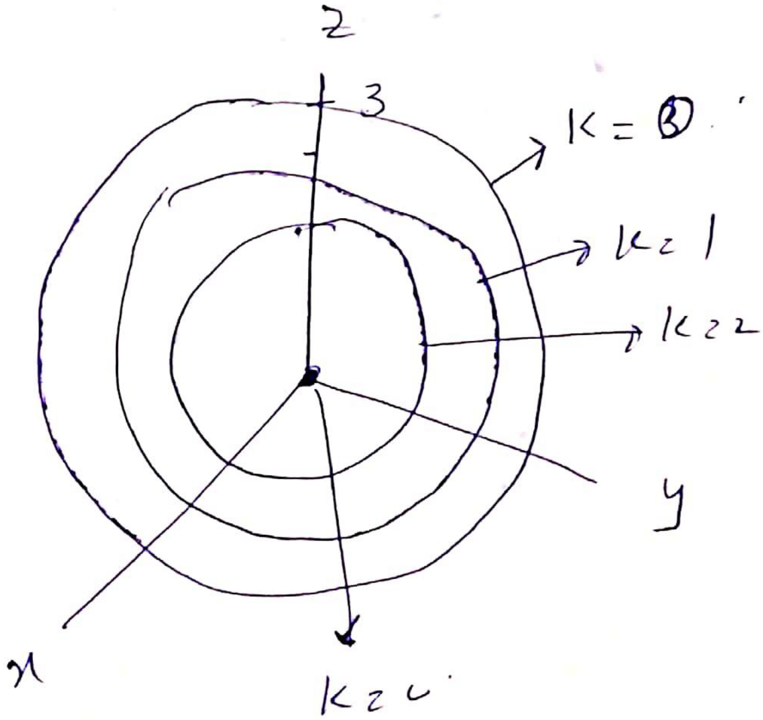
$k=2$

$\sqrt{9-x^2-y^2} = 2$

$9-x^2-y^2 = 4$

$x^2+y^2 = 5$

$x^2+y^2 = (2.23)^2$



(c) Find the level surface of the function $f(x,y,z) = x^2+y^2+z^2$ for $k = 0,1,2$.

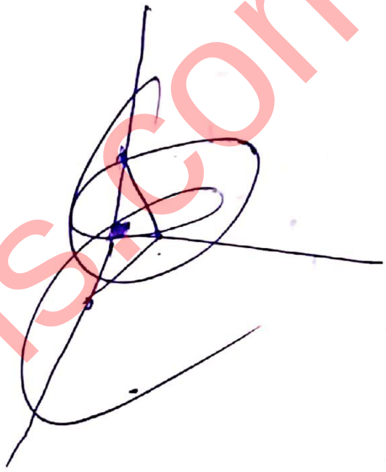
Sol:

$$f(x,y,z) = 0$$

$$x^2+y^2+z^2 = 0$$

x	0	0	0
y	0	0	0
z	0	0	0

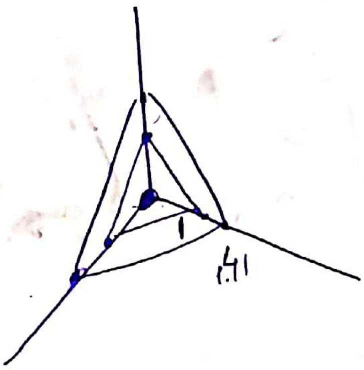
$$x^2+y^2+z^2 = 1$$



x	0	1 ~~1.41~~	0
y	0	0	~~1.41~~ 1
z	~~1.41~~ 1	0	0

$$x^2+y^2+z^2 = 2$$

$$k = 2$$



$$x^2+y^2+z^2 = 2$$

x	0	0	1.41
y	0	1.41	0
z	1.41	0	0

Partial Derivatives:-

$$\frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h,\, y_0) - f(x_0,y_0)}{h}$$

$$\frac{\partial f}{\partial y}\bigg|_{(x_0,y_0)} = \frac{d}{dy}(x_0,y)\bigg|_{y=y_0} = \lim_{h\to 0} \frac{f(x_0,\, y_0+h) - f(x_0,y_0)}{h}$$

Examples:- 1

$$f(x,y) = 7x^2 - x^3y^4 + 5x^4y^3$$

$$f_x = 14x - 3x^2y^4 + 20x^3y^3$$

$$f_y = 0 - 4y^3x^3 + 5x^4 3y^2$$

$$= -4x^3y^3 + 15x^4y^2$$

2 Find f_x and f_y, as function if

$$f(x,y) = \frac{2y}{y+\cos x}$$

$$f_x = \frac{\partial}{\partial x}\left(\frac{2y}{y+\cos x}\right)$$

$$= \frac{(y+\cos x)\frac{\partial}{\partial x}(2y) - 2y\frac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2}$$

$$= \frac{(y+\cos x)(0) - 2y(0-\sin x)}{(y+\cos x)^2}$$

$$= \frac{2y\sin x}{(y+\cos x)^2}$$

with x held constant, we get

$$f_y = \frac{\partial}{\partial y}\left(\frac{2y}{y+\cos x}\right)$$

$$= \frac{(y+\cos x)\frac{\partial}{\partial y}(2y) - 2y\frac{\partial}{\partial y}(y+\cos x)}{(y+\cos x)^2}$$

$$= \frac{(y+\cos x)(2) - 2y(1+0)}{(y+\cos x)^2}$$

$$f_y = \frac{\cancel{2y} + 2\cos x - \cancel{2y}}{(y+\cos x)^2} = \frac{2\cos x}{(y+\cos x)^2}$$

Find $\frac{\partial z}{\partial x}$ if the Equation

$$yz - \ln z = x + y$$

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}(\ln z) = \frac{\partial}{\partial x}(x+y)$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1+0$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\left(\frac{yz-1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x} = \frac{z}{yz-1}$$

Second-order Partial Derivatives:-

$\frac{\partial^2 f}{\partial x^2}$ or f_{xx}, $\frac{\partial^2 f}{\partial y^2}$ or f_{yy}

$\frac{\partial^2 f}{\partial x \partial y}$ or f_{yx} and $\frac{\partial^2 f}{\partial y \partial x}$ or f_{xy}

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Examples:

If $f(x,y) = x\cos y + ye^{x}$, find the 2nd-order derivatives

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos y + ye^{x})$$

$$\frac{\partial f}{\partial x} = \cos y + ye^{x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(\cos y + ye^{x})$$

$$\frac{\partial^2 f}{\partial x^2} = ye^{x}$$

$$\frac{\partial^2 f}{\partial y \partial x} = ? \Rightarrow \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}(\cos y + ye^{x})$$

$$= -\sin y + e^{x}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(-x\sin y + e^{x})$$

$$= -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial y}(x\cos y + ye^x)\right.$$

$$= \frac{\partial}{\partial y}\left(x(-\sin y) + e^x\right).$$

$$= -\cos y\, x + 0$$

$$= -x\cos y$$

Theorem:-

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Clairaut's theorem.

Partial Derivatives of Higher order:-

$$\frac{\partial^3 f}{\partial x\, \partial y^2} = f_{yyx}$$

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx},$$

①

$$f(x,y,z) = 1 - 2xy^2z + x^2y$$

$f_{yxyz} = ?$

$$f_y = -4xyz + x^2$$

$$f_{yx} = -4yz + 2x.$$

$$f_{yxy} = -4z + 0$$

$$f_{yxyz} = -4.$$

Max-Min Tests:-

The extreme value of $f(x,y)$ can occure only at

(i) boundary points of domain of f.

(ii) Critical points (Interior points where $f_x = f_y = 0$ or points where f_x or f_y fails to exist.

If the first and second order partial derivatives of f are continuous throughout a disk centred at a point (a,b) and $f_x(a,b) = f_y(a,b) = 0$, the nature of $f(a,b)$ can be test with the 2nd derivative test

(i) $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b) local maximum.

(ii) $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b) local minimum.

(iii) $f_{xx} f_{yy} - f_{xy}^2 < 0$ at (a,b) saddle point

(iv) $f_{xx} f_{yy} - f_{xy}^2 = 0$ (a,b) $\Rightarrow$ test is inconclusive.

Saddle point:-

has domain points (x,y) where $f(x,y) > f(a,b)$ and domain points (x,y) $f(x,y) < f(a,b)$ The corresponding point $((a,b),(f(a,b)))$ on the surface $z = f(x,y)$ is called a saddle point.

Examples:

Given $f(x,y) = 10 - 3x^2 - 2y^2 + 12x + 8y$, identify any critical point, saddle points and local extrema

1. $C(a,b)$, $f_x = 0$, $f_y = 0$

2. $D = f_{xx} f_{yy} - f_{xy}^2$

3. (a) $D > 0$, $f_{xx} > 0$, $f(a,b) \rightarrow$ local min

(b) $D > 0$, $f_{xx} < 0$, $f(a,b) \rightarrow$ local max

(c) $D < 0$ $f(a,b) \rightarrow$ Neither

$P(a,b) \rightarrow$ saddle point.

$f_x = -6x + 12$

$-6x + 12 = 0$, $\boxed{x = 2}$

$f_y = -4y + 8$

$\boxed{y = 2}$

$P(2,2)$

$f_{xx} = -6$, $f_{yy} = -4$, $f_{xy} = 0$

$D = f_{xx} \cdot f_{yy} - f_{xy}^2$

$= (-6)(-4) - 0^2 = 24$

$D > 0$

$f_{xx} < 0$

$f(2,2) = 30$

(2) $f(x,y) = 2x^4 + 2y^4 - 8x'y + 12$

$f_x = 8x^3 - 8y, \quad f_y = 8y^3 - 8x$

$f_x = 8(x^3 - y) \quad f_y = 8(y^3 - x)$

$f_{xx} = 24x^2$

$, f_{yy} = 24y^2$

$(0,0)$
$(1,1)$
$(-1,-1)$

$f_{xy} = -8$

$D = f_{xx} \cdot f_{yy} - f_{xy}^2$

$D(0,0) = 0 - (-8)^2 = 64. \quad D < 0$

$(0,0)$ Saddle point

$D(1,1) = 24 \cdot 24 - (-8)^2 = 512 \quad D > 0$

local min $(1,1)$

$f(1,1) = 2 + 2 - 8 + 12 = 8$

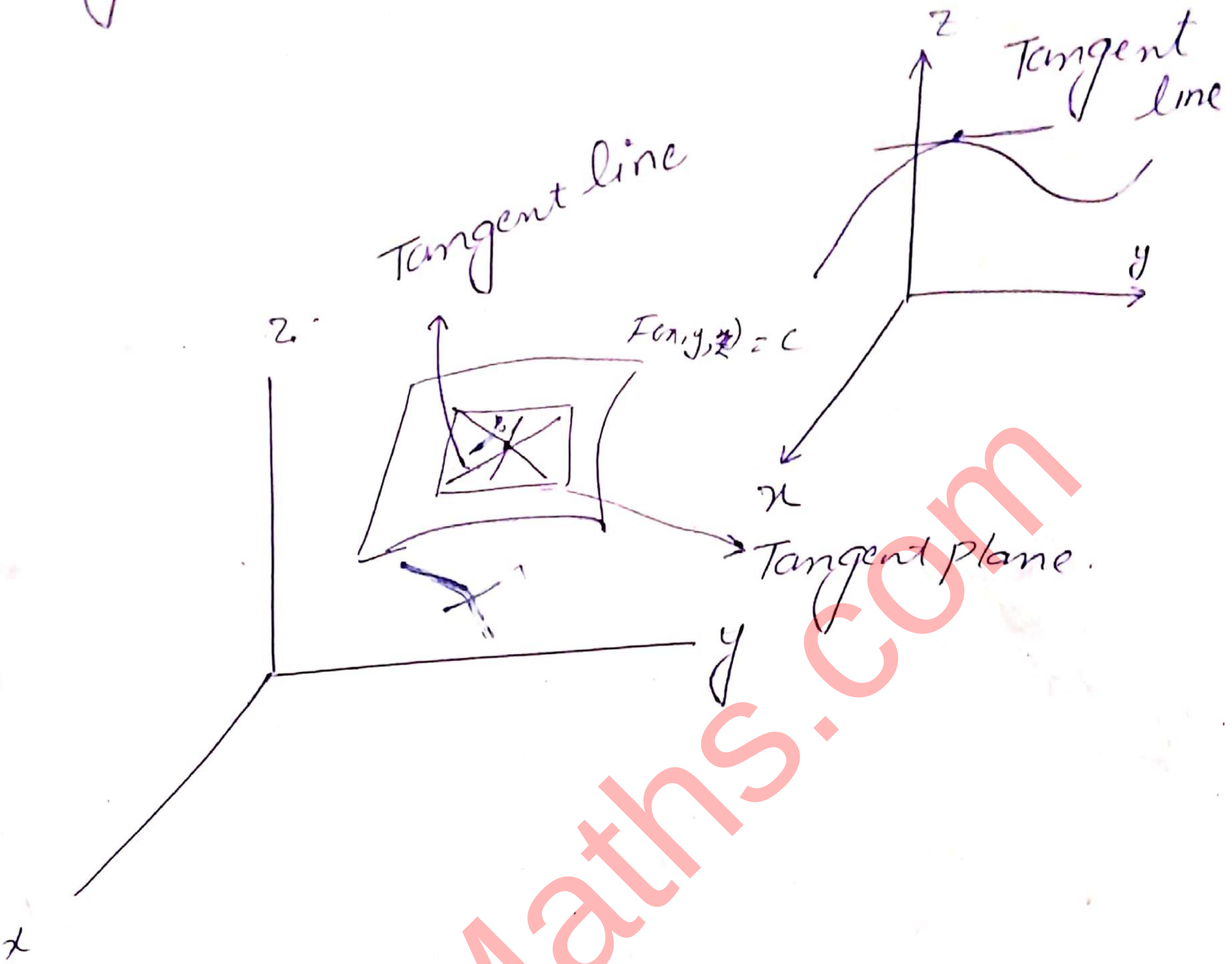
$D(-1,-1) = 24 \cdot 24 - (-8)^2 = +512 \quad D > 0 \quad f_{xx} > 0$

local min

$f(-1,-1) = 2(-1)^4 + 2(-1)^4 - 8(-1)(-1) + 12$

$= 8$

14.6 Tangent planes and Normal lines:- ①



Definition:- Assume that $f(x, y, z)$ has continuous first order partial derivatives and that point $P_0(x_0, y_0, z_0)$ is a point on the level surface $S: f(x, y, z) = c$.

If $\nabla f(x_0, y_0, z_0) \neq 0$, then $n = \nabla f(x_0, y_0, z_0)$ is a normal vector to S ~~at~~ P_0 and the tangent plane to S at P_0 is the plane with equation

$$f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)+f_z(x_0,y_0,z_0)(z-z_0)=0$$

and normal to the surface $f(x,y,z)=c$ at P_0 is given by

$$x = x_0 + f_x(x_0,y_0,z_0)t$$
$$y = y_0 + f_y(x_0,y_0,z_0)\,t$$
$$z = z_0 + f_z(x_0,y_0,z_0)\cdot t$$

Examples:- consider the ellipsoid

$$x^2+4y^2+z^2=18$$

(a) Find an equation of the tangent plane to the ellipsoid at the point $(1,2,1)$

(b) Find the parametric equations of the line that is normal to the ellipsoid at the point $(1,2,1)$

Sol:-

$$\nabla F = \langle 2x, 8y, 2z\rangle$$

$$f_x = 2x$$
$$f_y = 8y$$
$$f_z = 2z$$

$$n = \nabla F(1,2,1)$$
$$= \langle 2,16,2\rangle$$

$$f_x(x-x_0)+f_y(y-y_0)+f_z(z-z_0)=0$$ ②

$$2(x-1)+16(y-2)+2(z-1)=0.$$

$$2x-2+16y-32+2z-2=0.$$

$$2x+16y+2z=36$$

$$x+8y+z=18$$

(b)

$$x = x_0 + f_x \cdot t$$

$$x = 1+2t.$$

$$y = 2+16t.$$

$$z = 1+2t.$$

<u>2</u> plane Tangent to surface $z=f(x,y)$ at $(x_0, y_0, f(x_0, y_0))$

$$f_x(x-x_0)+f_y(y-y_0)-(z-z_0)=0.$$

<u>Example:-</u> Find the plane tangent to the surface $z = x\cos y - ye^x$ at $(0,0,0)$

$$f_x(0,0) = 1$$

$$f_y(0,0) = -2$$

$$1(x-0)-1(y-0)-(z-0)=0$$

$$x-y-z=0$$

Example:-

The surfaces

$$f(x,y,z)=x^2+y^2-2=0$$

$g(x,y,z)=x+z-4$. Find parametric equations for the line tangent to point $P_0(1,1,3)$

$$\nabla f = 2xi+2yj$$

$$\nabla g = i+k$$

parallel to $v=\nabla f\times\nabla g$

$$V=(2i+2j)\times i+k$$

$$= \begin{vmatrix} i & j & k \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\hat{i}-2\hat{j}-2\hat{k}$$

$$x = x_0+f_1 t = 1+2t$$

$$y = 1-2t$$

$$z = 3-2t$$

Solved

Estimating the change in f in a Direction u: ①

To estimate the change in the value of a differentiable function f when we move a small distance ds from a point P_0 in a particular direction u, use formula.

$$df = (\nabla f|_{P_0} \cdot u)\, ds.$$

Example:

Estimate how much the value

$$f(x,y,z) = y\sin x + 2yz.$$

will change if the point $P(x,y,z)$ moves 1 unit from $P_0\ (0,1,0)$ to $P_1(2,2,-2)$

Sol.

$$P_0P_1 = 2\hat{i} + \hat{j} - 2\hat{k}$$

$$u = \frac{P_0P_1}{|P_0P_1|} = \frac{2}{3}i + \frac{1}{3}j - \frac{2}{3}k$$

$$\nabla f: \quad y\cos x\, i + (\sin x + 2z)j + 2yk$$

$$\nabla f|_{(0,1,0)} = i + 2k$$

$$\nabla f \cdot u = (i + 2k)\left(\frac{2}{3}i + \frac{1}{3}j + \frac{2}{3}k\right)$$

$$= \frac{2}{3} - \frac{4}{3} = -2/3.$$

$$df = \left(\nabla f|_{P_0} \cdot u\right) ds$$

$$= -2/3(.1)$$

$$= -0.067 \text{ units}$$

Linearization:-

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0).$$

The approximation

$$f(x,y) = L(x,y)$$

Example: Find the linearization of

$$f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$$

at point $(3,2)$

$$f(3,2) = 9 - 3(2) + \frac{1}{2}(4) + 3$$

$$= 9 - 6 + 2 + 3 = 8$$

$$f_x = 2x - y$$

$$f_x(3,2) = 6 - 2 = 4$$

$$f_y = -x + y$$

$$f_y(3,2) = -3 + 2 = -1$$

$$L(x,y) = 8 + 4(x-3) - 1(y-2)$$

$$= 8 + 4x - 12 - y + 2$$

$$= 4x - y - 2$$

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Consider the function
$f(x,y) = x^2 + y^2 + 2xy - x - y + 1$ over
Square $0 \le x \le 1$ and $0 \le y \le 1$
Find extreme values.

(i) on $x = 0$.

$f(x,y) = y^2 - y + 1$ for $0 \le y \le 1$

$f_y = 2y - 1 \Rightarrow y = \frac{1}{2}$

$f(0,\frac{1}{2}) = \frac{1}{4} - \frac{1}{2} + 1 = \frac{1 - 2 + 4}{4} = \frac{3}{4}.$

$f(0,0) = 1$, $f(0,1) = 1$

(ii) for $y = 0$

$f(x,0) = x^2 - x + 1$

$f'(x,0) = 2x - 1 = 0$

$x = \frac{1}{2}$

$f(\frac{1}{2},0) = \frac{1}{4} - \frac{1}{2} + 1$

$= \frac{3}{4}$

$f(0,0) = 1$

$f(1,0) = 1 - 1 + 1 = 1$

(iii) $x = 1$

$f(1,y) = y^2 + y + 1$

$f'(1,y) = 2y + 1 = 0, \quad y = -\frac{1}{2}, \quad f(1, -\frac{1}{2}) = ?$

$f(1,0) = 1$

$f(1,1) = \cancel{1} + 1 + 2 \cancel{-1} \cancel{-1} \cancel{+1} = 3$

$f(1, -\frac{1}{2}) = 1^2 + \frac{1}{4} + \cancel{2}(1)(-\frac{1}{\cancel{2}}) - 1 + \frac{1}{2} + 1$

$= \cancel{1} + \frac{1}{4} \cancel{-1} \cancel{-1} + \frac{1}{2} \cancel{+1}$

$= \frac{1+2}{4} = \frac{3}{4}.$

(iv) $y = 1$

$f(x,1) = x^2 + x + 1$

$\boxed{x = -\frac{1}{2}}$

$f(-\frac{1}{2}, 1) = (-\frac{1}{2})^2 + 1^2 + \cancel{2}(-\frac{1}{\cancel{2}})(1) + \frac{1}{2} \cancel{-1} \cancel{+1}$

$= \frac{1}{4} + \cancel{1} \cancel{-1} + \frac{1}{2}$

$= \frac{3}{4}$

Lagrange Multipliers:-

①

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq 0$ when $g(x, y, z) = 0$. To find the local maximum and minimum values of f subject to constraint $g(x, y, z) = 0$, find the value x, y, z and λ that simultaneously satisfy the equations:

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = 0$$

For two independent variables, the condition is similar.

Examples:-

①

$f(x, y, z)$ $\qquad$ $g(x, y, z) = K$.

Given $f(x, y, z) = 3x^2 + y^2 - 2z^2$ and $3x + 2y - 8z = 50$, use lagrange Multipliers to find any maximum or minimum valus

Sol: $f(x, y, z)$, $g(x, y, z) = k$

x, y, z, λ

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad f_z = \lambda g_z$$

$$f_x = \lambda g_x \qquad\qquad f_y = \lambda g_y$$

$$6x = \lambda 3 \qquad\qquad 2y = \lambda 2$$

$$x = \frac{\lambda}{2} \qquad\qquad y = \lambda$$

$$-4z = \lambda(-8)$$

$$z = 2\lambda$$

$$3\left(\tfrac{1}{2}\lambda\right) + 2\lambda - 8(2\lambda) = -50$$

$$\frac{3}{2}\lambda + 2\lambda - 16\lambda = -50$$

$$\frac{3}{2}\lambda - 14\lambda = -50$$

$$3\lambda - 28\lambda = -100$$

$$-25\lambda = -100$$

$$\boxed{\lambda = 4}$$

$$x = \frac{1}{2}(4) = 2, \quad y = 4, \quad z = 8$$

$P(2,4,8)$ min

$$f(x,y,z) = 3(4) + 16 - 2(64)$$

$$= -100$$

$$P(4,-3,7) = \cancel{3(4)^2 + 9 - 2(49)} = -41$$

$$\cancel{= 3 + 9 - 50} = -38$$

$f(x,y,z) = 4x + 2y + 6z$ and $x^2 + y^2 + z^2 = 14$ use lagrange Multipliers to find any maximum or minimum values

Sol:-

$f_x = \lambda g_x$

$4 = \lambda 2x$

$x = \frac{2}{\lambda}$

$f_y = \lambda g_y$

$2 = 2\lambda y$

$y = \frac{1}{\lambda}$

$f_z = \lambda g_z$

$6 = \lambda 2z$

$z = \frac{3}{\lambda}$

$$\left(\frac{2}{\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 + \left(\frac{3}{\lambda}\right)^2 = 14$$

$$\frac{4}{\lambda^2} + \frac{1}{\lambda^2} + \frac{9}{\lambda^2} = 14$$

$$\frac{14}{\lambda^2} = 14 \quad , \boxed{\lambda = \pm 1}$$

if $\lambda = 1$

$x = 2, \quad y = 1, \quad z = 3$

$\lambda = -1$

$x = -2, \quad y = -1, \quad z = -3$

$f(2,1,3) = 4(2) + 2(1) + 6(3) = 28$ max

$f(-2,-1,-3) = 4(-3) + 2(-1) + 6(-3) = -28$ min

(3) Find the greatest and smallest values of the function

$f(x,y) = xy$, takes on ellipse

$$\frac{x^2}{8} + \frac{y^2}{2} = 1$$

$$g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1$$

$f_x = \lambda g_x$ | $x = \lambda y$

$$y = \lambda \frac{2x}{8}$$

$$y = \frac{\lambda}{4} x$$

$$y = \frac{\lambda}{4}(\lambda y)$$

$$y = \frac{\lambda^2}{4} y$$

$y = 0$ or $\lambda = \pm 2$

Case 1: $y = 0$ then $x = y = 0$ But $(0,0)$ not ellipse, Hence $y \neq 0$

So case 2 $\lambda = \pm 2$

when $\lambda = 2$

$x = 2y$ | $\frac{(2y)^2}{8} + \frac{y^2}{2} = 1$

$$\frac{4y^2}{8} + \frac{y^2}{2} = 1$$

$$4y^2 + 2y^2 = 8$$

$$8y^2 = 8$$

$$y = \pm 1$$

$$x = \pm 2$$

$(\pm 2, 1), (\pm 2, -1)$

the extreme values are $xy = 2$ and -2.

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BS(CS) - D

Assignment # 1

QUESTION: 01

Volume of cylinder = $\pi r^2 h$

Volume of hemisphere = $2/3 \pi r^3$

Total Volume (V)

$$V = \pi r^2 h + \left(\frac{2}{3}\pi r^3\right) \times 2$$

↳ ∵ Two Hemispheres

$$V = \pi r^2 h + \frac{4}{3}\pi r^3$$

$$V(r,h) = \pi\left(r^2 h + \frac{4}{3} r^3\right)$$

QUESTION: 02

Volume (V) = 15cm^3

Surface Area (A) = ?

let length, width as hight as l, w and h.

Surface Area (A) = $lw + 2(hw) + 2(hl)$

$$\text{Cost}(C) = 6lw + 3(2hw + 2hl)$$

$$C = 6lw + 6hw + 6hl \quad \text{— (A)}$$

$$V = lwh$$

$$l \times w = \frac{15 \times (10^{-2})^3 \; m^3}{h}$$

$$h = \frac{1.5 \times 10^{-5} \; m}{lw}$$

Substitute in (1)

$$C = 6\left(lw + (w + l)\frac{1.5 \times 10^{-5}}{lw}\right)$$

$$= 6\left(lw + \left(\frac{1}{l} + \frac{1}{w}\right) 1.5 \times 10^{-5}\right)$$

$$= 6lw + 9 \times 10^{-5}\left(1/l + 1/w\right)$$

QUESTION: 03

$$f(x, y) = 100\, x^{0.6} y^{0.4} \quad \text{——— (A)}$$

Let $x' = 2x$, $y' = 2y$

$$f(x', y') = 100\, x'^{0.6} \cdot y'^{0.4}$$

$$= 100 . (2x)^{0.6} . (2y)^{0.4}$$

$$= 100 . 2^{0.6} . 2^{0.4} x^{0.6} y^{0.4}$$

$$= 2^{0.6} . 2^{0.4} \left(100\, x^{0.6} y^{0.4}\right)$$

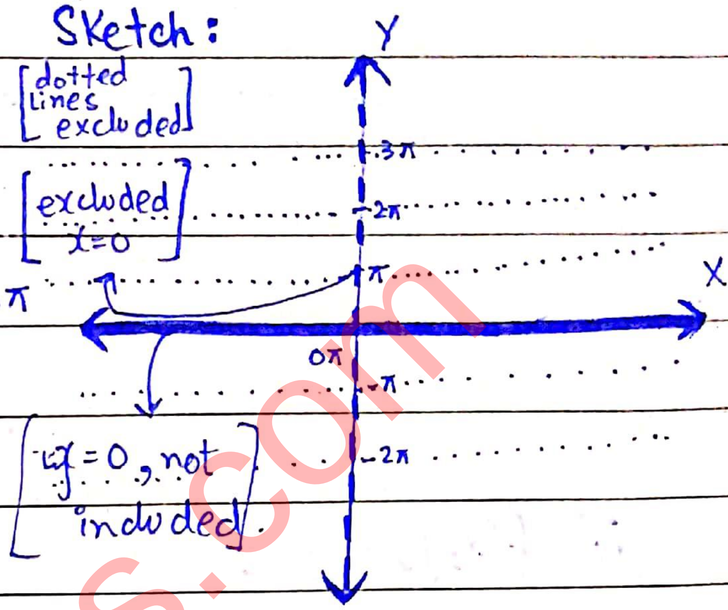
$$= 2^{0.6+0.4} f(x, y) \quad (\text{From (A)})$$

Thus, Proved $f(x', y') = 2 f(x, y) \Rightarrow f(2x, 2y) = 2 f(x, y)$

QUESTION: 04

i) $f(x,y) = \dfrac{e^{1/x}}{\sin y}$

Sketch:



Domain = ?

$x \neq 0$, $\sin y \neq 0 \Rightarrow y \neq n\pi$

$D: \{(x,y) \mid x \neq 0, y \neq n\pi\}$

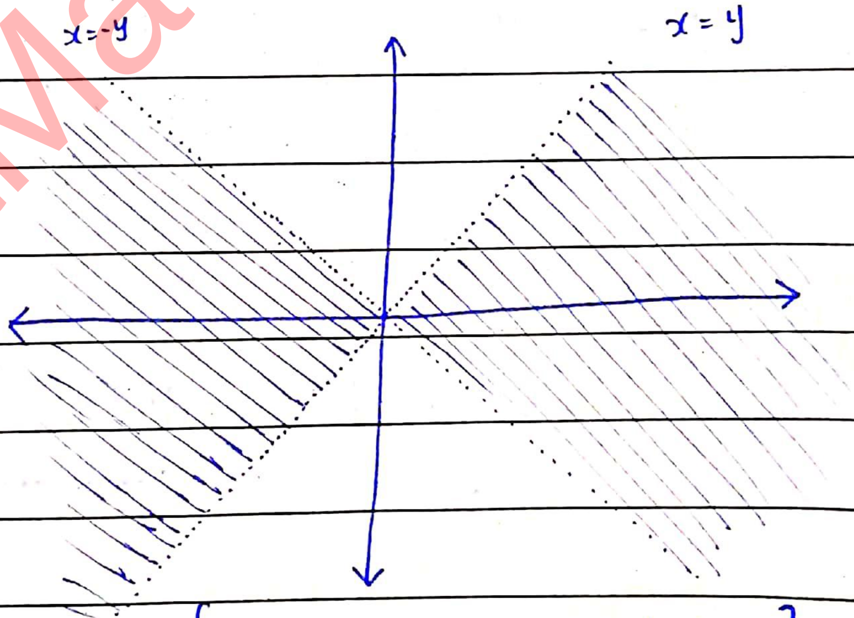
ii) $f(x,y) = \sqrt{\dfrac{x^2+y^2}{x^2-y^2}}$

Domain:

$x^2 - y^2 > 0$

$|x| > |y|$

$D: \{(x,y) \mid |x| > |y|\}$



[dotted linex are excluded]

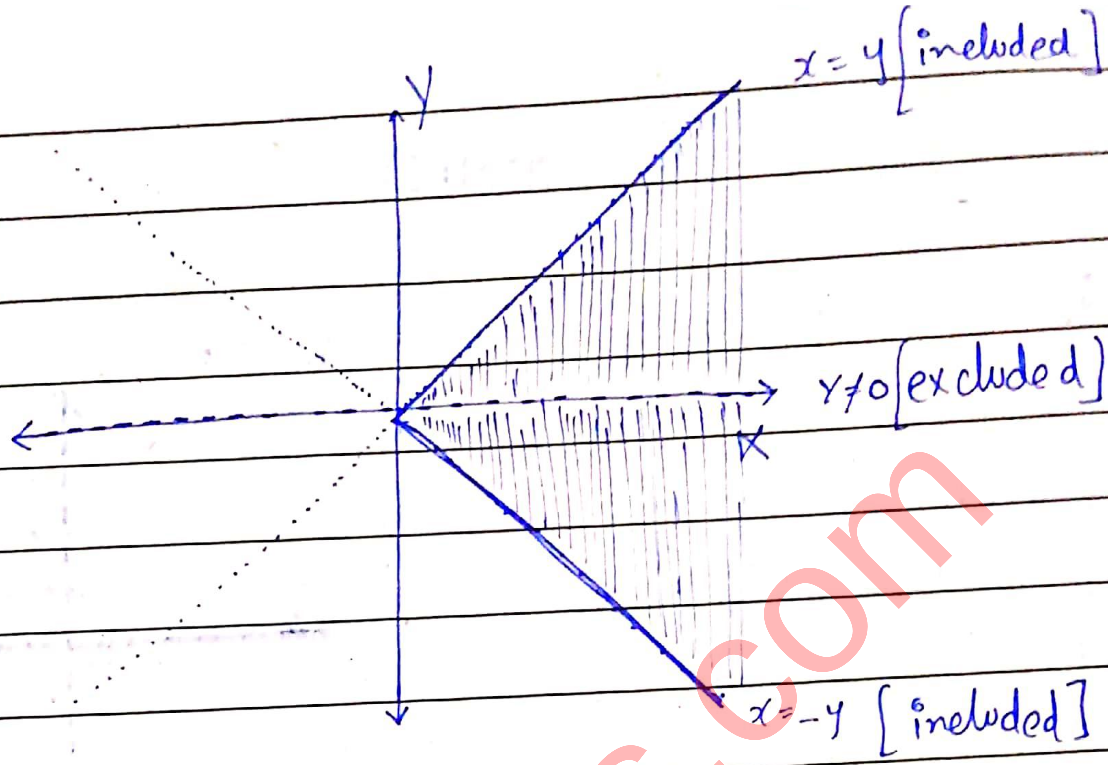
iii) $f(x,y) = \sin^{-1}(x^2/y^2)$

$\rightarrow$ $-1 \leq x^2/y^2 \leq 1$ domain of $\sin^{-1}$

$\rightarrow$ $x^2/y^2 \geq 0$ alway positive

$\rightarrow$ $y \neq 0$ $\quad 0 \leq x^2/y^2 \Rightarrow x > 0$, $x^2/y^2 \leq 1 \Rightarrow |x| \leq |y|$

Thus $D: \{x, y \mid x > 0, y \neq 0, |x| \leq |y|$

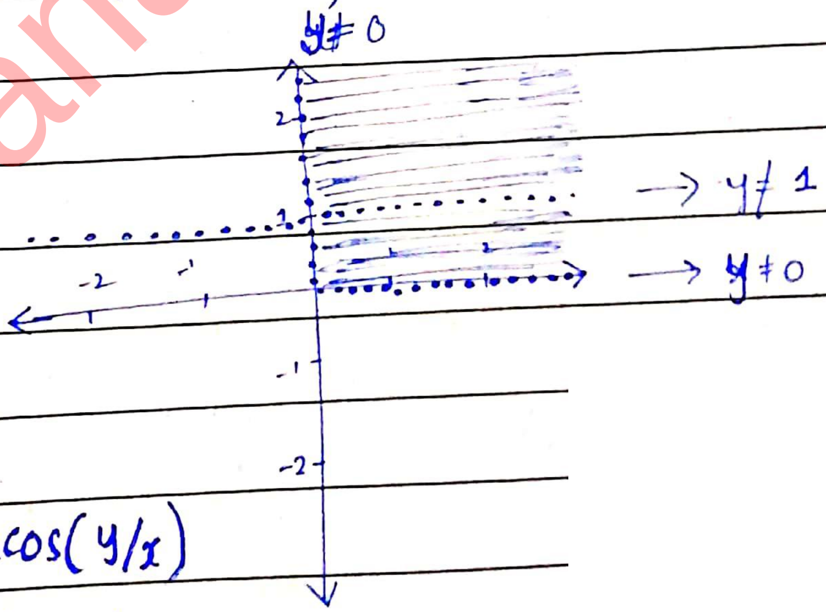


(iv) $f(x,y) = \dfrac{\ln x}{\ln y}$

→ $x > 0,\ y > 0$

→ $\ln y \neq 0 \Rightarrow y \neq 1$

$D : \{(x,y) \mid x > 0, y > 0, y \neq 1\}$



(v) $f(x,y) = \arccos(y/x)$

$-1 \leq y/x \leq 1$

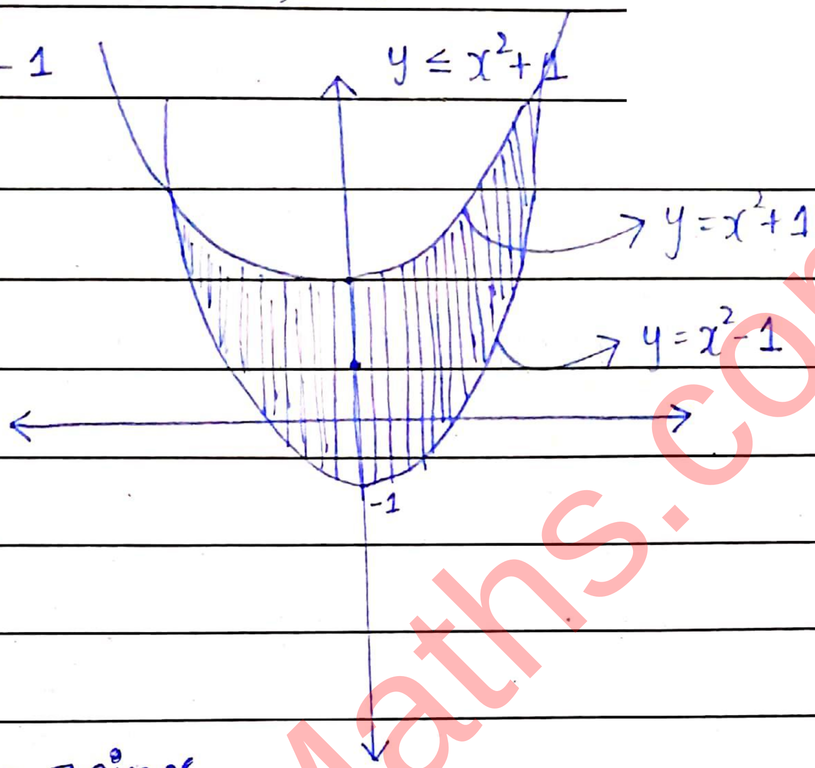
$D : \{(x,y) \mid -x \leq y \leq x\}$

(vi) $f(x,y) = \arccos(y - x^2)$

$$-1 \le y - x^2 \le 1$$

$y - x^2 \ge -1$, $y - x^2 \le 1$

$y \ge x^2 - 1$, $y \le x^2 + 1$



(vii) $f(x,y) = \dfrac{z \sin x}{\cos y}$

$\cos y \neq 0 \Rightarrow y \neq (2n+1)\pi/2$

$D : \{(x,y,z) \mid y \neq (2n+1)\pi/2\}$

(viii) $f(x,y) = \dfrac{\sqrt{4-x^2} + \sqrt{1-y^2}}{1 - \sqrt{9-z^2}}$

$4 - x^2 \ge 0 \Rightarrow |x| \le 4$, $1 - y^2 \ge 0 \Rightarrow |y| \le 1$

$9 - z^2 \ge 0 \Rightarrow |z| \le 3$, $1 - \sqrt{9-z^2} \neq 0$

$(\sqrt{9-z^2})^2 \neq 1^2$

$D : \{(x,y,z) \mid |x| \le 4, |z| \le 3, |y| \le 1$; $9 - 1 = z^2 \Rightarrow |z| \neq 2\sqrt{2}$

$|z| \neq 2\sqrt{2}\}$

(ix) $f(x,y,z) = e^{\sqrt{9-(x^2+y^2+z^2)}}$

$9-(x^2+y^2+z^2) \geq 0$

$x^2+y^2+z^2 \leq 9$

$D : \{(x,y,z) \mid x^2+y^2+z^2 \leq 9\}$

Question: 05

QUESTION : 06

$c = 9/\sqrt{4-(x^2+y^2)}$

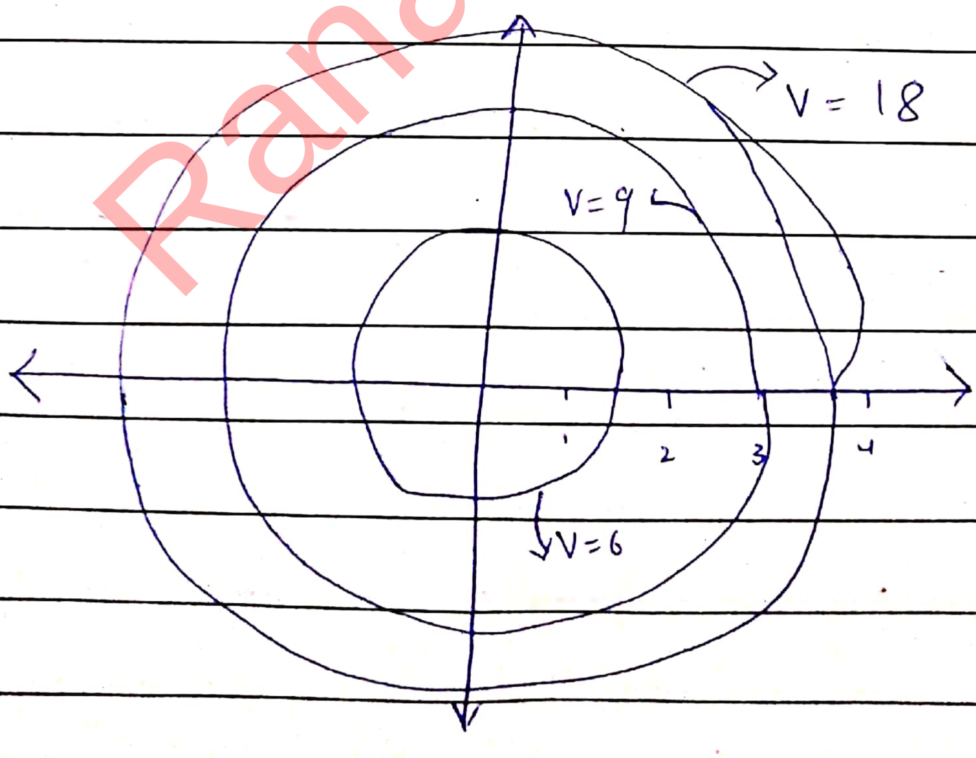
$\left(\frac{9}{c}\right)^2 = 4-(x^2+y^2)$

$x^2+y^2 = 4 - \frac{81}{c^2}$

for $c=18$: $x^2+y^2 = 15/4$

$c=9$: $x^2+y^2 = 3$

$c=6$: $x^2+y^2 = 7/4$

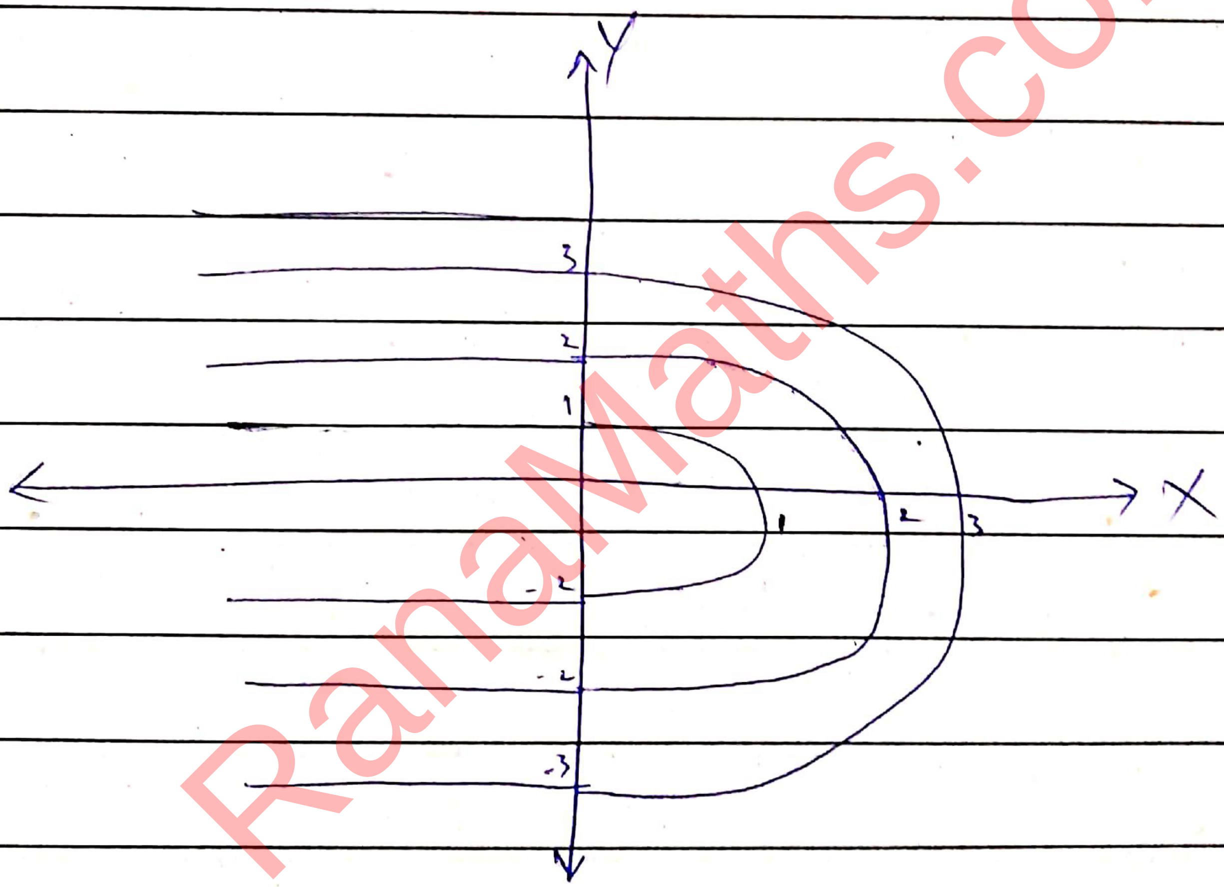


Question : 07

$c = \sqrt{x^2+y^2}$ for

$x^2+y^2 = c^2$

$(y) = c$



QUESTION:08

a) $[(x,y) : 4 < x^2 \le 9]$

$= \sqrt{2^2} < \sqrt{x^2} \le \sqrt{3^2}$

$= 2 < |x| \le 3$

$\Rightarrow |x| > 2 \qquad |x| \le 3$

$-2 > x > 2$ — (A) $\qquad -3 \le x \le 3$ — (B)

$[-3, -2) \cup (+2, 3]$

$\longrightarrow$ Open intarval

$\longrightarrow$ boundry points lie on $x = \pm 2$, $x = \pm 3$

b) $y \le x^2$

* boundry point lie on $y = x^2$

~~interval~~

* open interval

a) $\lim\limits_{(x,y)\to(2,e)} x^2 y \ln y$

$= 2^2 e \ln e$

$= 4e$

b) $\lim\limits_{(x,y)\to(3,2)} \dfrac{y^2-4}{xy-2x} = \dfrac{(\cancel{y-2})(y+2)}{x(\cancel{y-2})}$

$= \dfrac{2+2}{3} = 4/3$

c) $\lim\limits_{(x,y)\to(0,0)} \left(\dfrac{x^2-y^2}{x^2+y^2}\right)^2$

Path 1 :

$\to x=0 \quad \lim\limits_{(0,y)\to(0,0)} \left(\dfrac{0-y^2}{0+y^2}\right)^2 = +1$

Path 2 :

$\to y=0 \quad \lim\limits_{(x,0)\to(0,0)} \left(\dfrac{x^2-0}{x^2+0}\right)^2 = -1$

~~it not exist~~

$\lim\limits_{(x,y)\to(0,0)} \left(\dfrac{x^2-y^2}{x^2+y^2}\right)^2 \Rightarrow$ doesn't exist because x^2+y^2 resonates back, i.e: depends on θ in polar coordinates.

d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y^2}{x^4+y^4}$

Path 1 $\quad \lim\limits_{(0,y)\to(0,0)} \dfrac{3 \cdot 0^2 y^2}{0^4+y^4} = 0$

$x=0$

Path 2 $\quad \lim\limits_{(1,y)\to(1,0)} \dfrac{3y^2}{1+y^2} = 3/2$

$x=1$

Limit not exist.

[Question 10]

i) $x = r\cos\theta,\ y = r\sin\theta$

$$= \frac{r\cos\theta\, r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \frac{r^3\cos\theta\sin^2\theta}{r^2}$$

$$= r\cos\theta\sin^2\theta$$

$$\lim_{r\to} f(x) = 0$$

$$f(0) = 0$$

limiting value,

$\boxed{\text{Zero continous}}$

ii)

let

$x = r\cos\theta,\ y = r\sin\theta$

$$\frac{r^2\cos^2\theta \cdot r\sin\theta}{r^2(\cos^2\theta + \sin^2\theta)}$$

$$= r\cos\sin\theta.$$

$r \to$

$\lim = 0$

limit exist

[Question 11]

ii) $x = r\cos\theta,\ y = r\sin\theta$

$$= \frac{\cos\theta(r^2\cos^2\theta)^2\sin^2\theta)}{r^2(\cos^2\theta + \sin^2\theta}$$

$$= \frac{\cos(r^2)}{r^2}$$

$r \to 0$

limit undefine

$\boxed{\text{Not continous}}$

iii)

let $x = r\cos\theta,\ y = r\sin\theta$

$$= \frac{3(r\cos\theta)(r\sin\theta)}{r^2(\cos^2\theta + \sin^2\theta)}$$

$$= 3\cos\theta\sin\theta.$$

$\Rightarrow \theta$ ranges from

-1 to $+1$

Limit not exist and function not continous.

Question No: 12

a) Continous on whole domain (Trig)

b) Denominat alway greater than zero, so continous for all (x,y).

c) $x-y>0 \rightarrow y<x$

$x^2+y^2>0 \Rightarrow y^2>-x^2$

continous whe

$y<x$ ad $y^2>-x^2$

d

Question No: 14

$T = 100(x^2+y^2)^{-1/2}$

$T_x = \frac{d}{dx} 100(x^2+y^2)^{-1/2}$

$T_x = \frac{-100x}{\sqrt{(x^2+y^2)^3}}$

$\rightarrow (1,0) = -100$

$\rightarrow (0,1) = 0$.

Question 15

$T_y = \frac{d}{dy} \frac{100}{\ln 2} \ln(x^2+y^2)$

$= \frac{100}{\ln 2} \cdot \frac{2y}{x^2+y^2}$

sub points:

$\rightarrow (2,0) \Rightarrow \frac{100}{\ln 2} \cdot \frac{2(0)}{2^2+0^2} = 0$

$\rightarrow (0,2) = \frac{100}{\ln 2} \frac{2(2)}{0^2+2^2} = 144.3$

Question 16

$f_x(0,0) = \frac{f(h,0)-f(0,0)}{h}$

$h \to 0 \quad \frac{0 \cdot 0}{h} = 0$

$\cancel{f(y)}$ $f_y(0,0) = 0$

$f_x, f_y = 0$.

$\Rightarrow \cos\theta \sin\theta$ [After Substitu $x = r\cos\theta$ $y = r\sin\theta$]

θ range from $\{ 1$, So $\}$

Limit not exist.

Question 17

Surface A: $lw + 2lh + 2wh$

Volume : $lwh = 500 \Rightarrow h = \frac{500}{lw}$ — (1)

$$A(l,w) = lw + 2l \cdot \frac{500}{lw} + 2 \cdot w \frac{500}{lw}$$

$$= lw + \frac{1000}{w} + \frac{1000}{l}$$

$A_l = w - \frac{1000}{l^2}$, $A_w = l - \frac{1000}{w^2}$

$A_l = 0 \Rightarrow w = \frac{1000}{l^2}$ $A_w = 0 \Rightarrow l = \frac{1000}{w^2}$

$w = \dfrac{1000}{\frac{1000 \times 1000}{w^2}}$ $\Rightarrow w = 10m$

$\Rightarrow l = 10m$

$\Rightarrow h = \frac{500}{l \times w} = 5m$

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CS - D

ASSIGNMENT - 02

14.8

Langrage multipliers

Q1)

$$f(x,y) = x^2 + 4y^2$$

$$x^2+y^2=1 \Rightarrow g(x,y) = x^2+y^2-1$$

According to Langrange multipliers,

$$\nabla f = \lambda \nabla g$$

$f_x = \lambda g_x$ $\qquad$ $f_y = \lambda g_y$ $\qquad$ $x^2+y^2=1 \rightarrow (iii)$

$2x = 2x\lambda \rightarrow (i)$ $\qquad$ $8y = \lambda 2y \rightarrow (ii)$

from (i) either $\lambda = 1$, or $x = 0$

if $\lambda = 1$ then

$y=0$ (from (ii))

Put in (iii)

$x = \pm 1$

$(1,0)$, $(-1,0)$

$y = \pm 1$

$(0,1)$

$(0,-1)$

Possible extreme values are

$(1,0)$, $(-1,0)$, $(0,1)$, $(0,-1)$

$f(1,0)=1$, $f(-1,0)=1$, $f(0,1)=4$, $f(0,-1)=4$

So, Highest points $= (0, \pm 1)$

Lowest points $= (\pm 1, 0)$

Q3)

$$f(x,y) = \sin \pi x \cos \pi y \qquad R: \left[0, \tfrac{1}{4}\right] \times \left[\tfrac{1}{4}, \tfrac{1}{2}\right]$$

$$\iint_R f(x,y)\, dA = \int_{1/4}^{1/2}\int_{0}^{1/4} (\sin \pi x)(\cos \pi y)\, dx\, dy = \int_0^{1/4}\int_{1/4}^{1/2} \sin \pi x \cos \pi y \, dy\, dx$$

$$= \int_0^{1/4} \left(\frac{\sin \pi y}{\pi}\right)\Bigg|_{1/4}^{1/2} \sin \pi x\, dx = \frac{1}{\pi}\left[\sin\frac{\pi}{2} - \sin\frac{\pi}{4}\right] \int_0^{1/4} \sin \pi x\, dx$$

$$= \frac{1}{\pi^2}\left[1-\frac{1}{\sqrt{2}}\right]\left[-\cos\pi x\right]_0^{\frac{1}{4}} = \frac{-1}{\pi^2}\left[\frac{\sqrt{2}-1}{\sqrt{2}}\right]\left[\cos\frac{\pi}{4} - \cos 0\right]$$

$$= \frac{-1}{\pi^2}\left[\frac{\sqrt{2}-1}{\sqrt{2}}\right]\left[\frac{1}{\sqrt{2}}-1\right] = \frac{-1}{\pi^2}\left[\frac{\sqrt{2}-1}{\sqrt{2}}\right]\left[\frac{1-\sqrt{2}}{\sqrt{2}}\right] = \frac{(\sqrt{2}-1)^2}{2\pi^2}$$

$$= \frac{2-2\sqrt{2}+1}{2\pi^2} = \frac{3-2\sqrt{2}}{2\pi^2} = 0.0086$$

$$0 \le 0.0086 \le 0.3125$$

double Integral

15.1 Double and iterate over Rectangles:-

Theorem 1- Fubini's Theorem (First form)

If $f(x,y)$ is continuous throughout the rectangular region $R: a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

$$= \int_a^b\int_c^d f(x,y)\,dy\,dx.$$

Examples:-

$$\int_0^2 \underbrace{\int_1^3 xy^2\,dy}\,dx$$

$$= \int_1^3 xy^2\,dy$$

$$= x\frac{y^3}{3}\Big|_{y=1}^{y=3}$$

$$= \frac{x}{3}\left[3^3 - 1^3\right] = \frac{x}{3}(27-1)$$

$$= \frac{26}{3}x$$

$$= \int_0^2 \frac{26}{3} x \, dx$$

$$= \frac{26}{3} \frac{x^2}{2} \Big|_0^2$$

$$= \frac{26}{6} (4-0)$$

$$= \frac{13}{3}(4) = \frac{26 \times 2}{3} = \frac{52}{3}$$

$$\int_0^2 \int_1^3 xy^2 \, dy \, dx = \frac{52}{3}$$

Now By changing order of integration

$$\int_1^3 \int_0^2 xy^2 \, dx \, dy.$$

$$\int_0^2 xy^2 \, dx = \frac{x^2 y^2}{2} \Big|_{x=0}^{x=2}$$

$$= \frac{y^2}{2}(2^2 - 0^2)$$

$$= \frac{y^2}{2}(4)$$

$$= 2y^2$$

$$\int_1^3 2y^2\,dy = \left.\frac{2y^3}{3}\right|_1^3$$

$$= \frac{2}{3}(27-1)$$

$$= \frac{52}{3}$$

Example 2

$$\iint_R (2y-3x^2y^2)\,dA$$

$$R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 2\}$$

$$\int_0^1\int_0^2 (2y-3x^2y^2)\,dy\,dx.$$

$$= \int_0^2\int_0^1 (2y-3x^2y^2)\,dx\,dy$$

$$\int_0^1\int_0^2 (2y-3x^2y^2)\,dy\,dx = ?$$

$$\int_0^2 (2y-3x^2y^2)\,dy = \left.\frac{2y^2}{2} - \frac{3x^2y^3}{3}\right|_{y=0}^{y=2}$$

$$= (2^2-0^2) - x^2(2^3-0^3)$$

$$= 4 - x^2(8)$$

$$= 4 - 8x^2$$

$$\int_0^1 (4-8x^2)\,dx = 4x - \frac{8x^3}{3}\Big|_0^1$$

$$= 4(1-0) - \frac{8}{3}(1-0)$$

$$= 4 - 8/3$$

$$= \frac{12-8}{3} = 4/3$$

$$\int_0^2 \int_0^1 (2y-3x^2y^2)\,dx\,dy = ?$$

$$\int_0^1 (2y-3x^2y^2)\,dx = 2xy\Big|_{x=0}^{x=1} - \frac{\cancel{3}x^3y^2}{\cancel{3}}\Big|_0^1$$

$$= 2y(1-0) - y^2(1-0)$$

$$= 2y - y^2$$

$$\int_0^2 (2y-y^2)\,dy = \frac{2y^2}{2} - \frac{y^3}{3}\Big|_0^2$$

$$= (4-0) - \frac{1}{3}(8-0)$$

$$= 4 - \frac{8}{3}$$

$$= \frac{12-8}{3}$$

$$= 4/3$$

Double Integrals over General Regions:-

Theorem 2.

Let $f(x,y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a,b]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

② If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c,d]$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

$$\int_0^1 \int_0^{2y} (4+2x-y^2)\,dx\,dy$$

$$\int_0^{2y} (4+2x-y^2)\,dx = 4x + \frac{\cancel{2}x^2}{\cancel{2}} - y^2x \Big|_0^{2y}$$

$$= 4(2y-0) + (2y)^2 - y^2(2y)$$

$$= 8y + 4y^2 - 2y^3$$

$$= \int_0^1 (8y + 4y^2 - 2y^3)\,dy$$

$$= \frac{8y^2}{\cancel{2}} + \frac{4y^3}{3} - \frac{2y^4}{4} \Big|_0^1$$

$$= 4y^2 + \frac{4}{3}y^3 - \frac{1}{2}y^4 \Big|_0^1$$

$$= 4 + \frac{4}{3} - \frac{1}{2}$$

$$= \frac{24+8-3}{6} = \frac{29}{6}$$

---

use double Integral find the volume of tetrahedron bounded by plane $z = 4 - 4x - 2y$

$$V = \iint_R (4 - 4x - 2y)\,dA \qquad 0 = 4 - 4x - 2y$$

$$y = 2 - 2x.$$

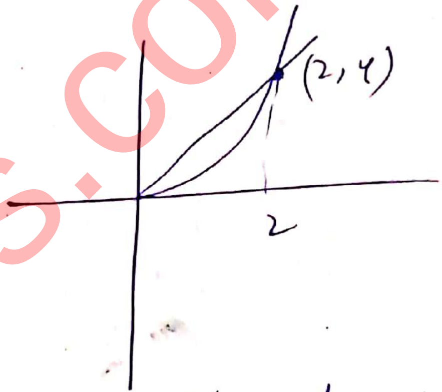
Finding Limits of Integration:-

Example:-

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\, dy\, dx.$$

write an equivalent integral with order of integration reversed

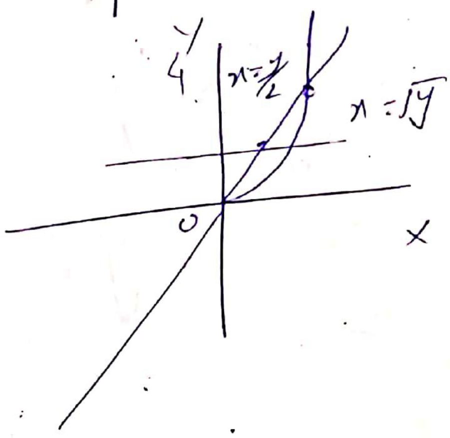
$x^2 \le y \le 2x$



$y = x^2$, $y = 2x$

$x = \sqrt{y}$

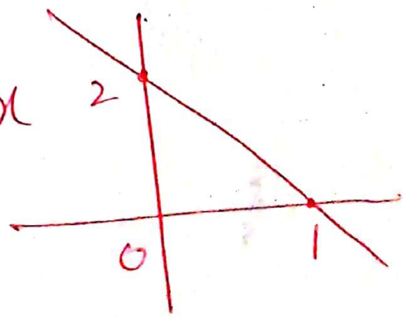
$x = \frac{y}{2}$



$x=0$, $x=2$

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\, dx\, dy$$

$$= \int_0^1 \int_0^{2-2x} (4-4x-2y)\, dy\, dx$$



$$= \frac{4}{3}$$

15.5 Triple Integrals in Rectangular coordinates:-

$$\iiint_D F(x,y,z)\,dV$$

$$\iiint_B x^3yz^2\,dV \quad \{0 \le x \le 2,\ -2 \le y \le 3,\ 0 \le z \le 1\}$$

$$\int_0^2 \int_{-2}^3 \underbrace{\int_0^1 x^3yz^2\,dz}\,dy\,dx$$

$$\int_0^1 x^3yz^2\,dz$$

$$= x^3y\frac{z^3}{3}\Big|_0^1$$

$$= \frac{x^3y}{3}(1-0)$$

$$= \frac{x^3y}{3}$$

$$= \int_0^2 \underbrace{\int_{-2}^3 \frac{x^3y}{3}\,dy}\,dx$$

(5)

$$= \int_0^2 \underbrace{\int_{-2}^3 x^3 y/3 \, dy}\, dx$$

$$\int_{-2}^{3} \frac{x^3 y}{3} dy$$

$$= \frac{x^3 y^2}{6} \Big|_{-2}^{3}$$

$$= \frac{x^3}{6}(9-4)$$

$$= \frac{5}{6} x^3$$

$$= \int_0^2 \frac{5}{6} x^3 dx$$

$$= \frac{5}{6} \frac{x^4}{4} \Big|_0^2$$

$$= \frac{5}{\cancel{24}_{3}} (\cancel{16}^{2} - 0)$$

$$= 10/3$$

(2) $\int_0^3 \int_0^x \int_0^{x-y} 4xy \, dz \, dy \, dx$.

$$\int_0^{x-y} 4xy\,dz = 4xy\, z\Big|_{z=0}^{z=x-y}$$

$$= 4xy(x-y)$$

$$= 4x^2y - 4xy^2$$

$$= \int_0^3 \underbrace{\int_0^x 4x^2y - 4xy^2\, dy}\, dx.$$

$$\int_0^x (4x^2y - 4xy^2)\,dy$$

$$= \frac{4x^2y^2}{2} - \frac{4xy^3}{3}\Big|_{y=0}^{y=x}$$

$$= \frac{4x^2x^2}{\cancel{2}} - \frac{4x\,x^3}{3}$$

$$= 2x^4 - \frac{4}{3}x^4$$

$$= \frac{6x^4 - 4x^4}{3} = \frac{2}{3}x^4$$

$$\int_0^3 \frac{2}{3}x^4\,dx.$$

$$= \frac{2}{3}\frac{x^5}{5}\Big|_0^3$$

$$= \frac{2}{3}\cdot\frac{3^5}{5} = \frac{2}{\cancel{3}}\cdot\frac{\overset{81}{\cancel{243}}}{5} = \frac{162}{5}$$

Average value of function in space:-

Average value of F over D

$$= \frac{1}{\text{volume of } D} \iiint_D F\,dV$$

<u>Example:-</u>

Find the average value of $F(x,y,z) = xyz$ throughout the cubic region D bounded by the coordinate planes and planes $x=2$, $y=2$ and $z=2$ in the octant

Volume of region $= xyz$
$= 2(2)(2)$
$= 8$

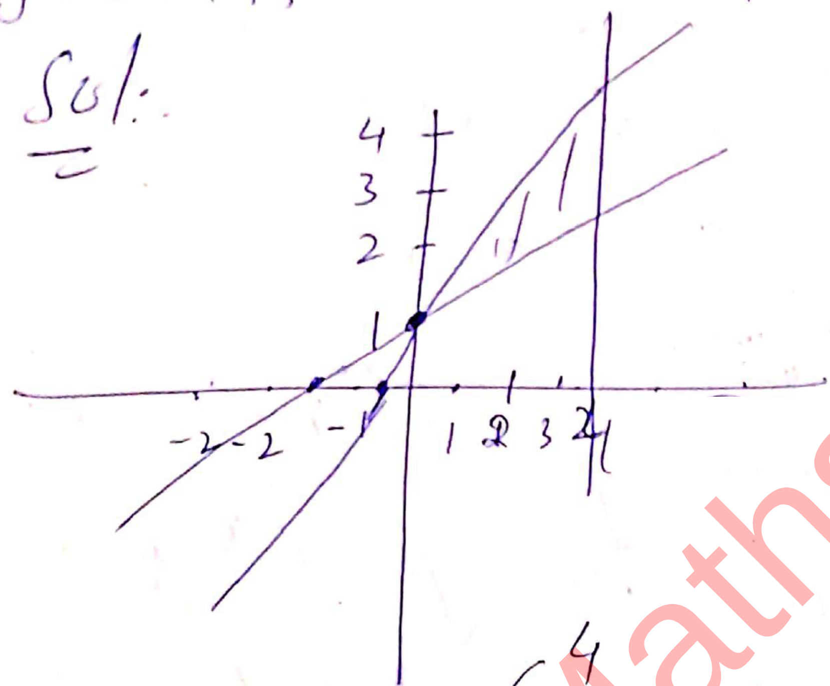
$$\int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz = 8$$

$$\text{Avg.} = \frac{1}{\text{vol}} \iiint_{\text{cube}} xyz\,dV$$

$$= \frac{1}{8}(8) = 1$$

① Using integration find the area of the triangular region whose sides have equations $y = 2x+1$, $y = 3x+1$, and $x = 4$.

Sol:



$f(x) \geq g(x)$

$a \leq x \leq b$

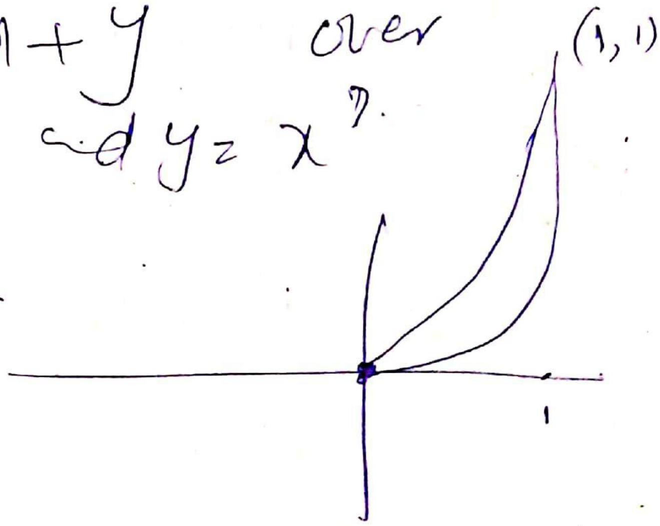
$\int_a^b (f(x) - g(x)) dx$

$$\text{Area} = \int_0^4 (3x+1) - (2x+1)\, dx$$

$$= 8 \text{ sq}$$

② $f(x,y) = x+y$ over the region $y = x^2$ and $y = x^3$.

$$= \int_0^1 \int_{x^3}^{x^2} (x+y)\, dy\, dx$$



15.4 Double Integral in polar coordinates:- (1)

$$\iint_R f(r,\theta)\,dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\, r\,dr\,d\theta$$

How the find limit of Integration:

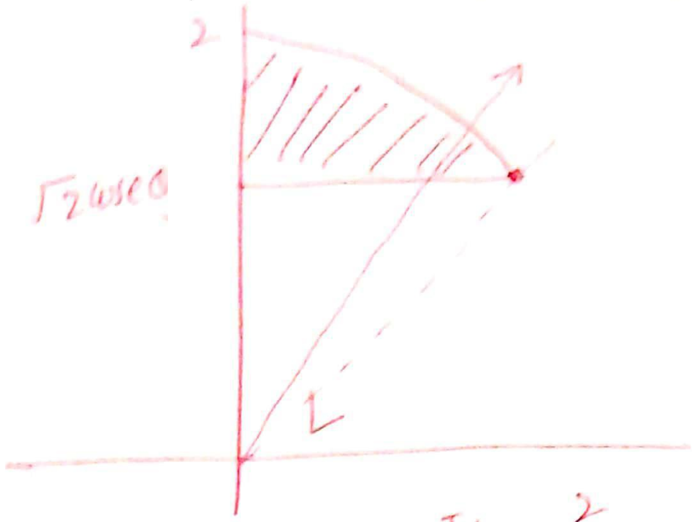
1. Sketch the region and label the bounding curves.

2. Find r limit of integration:-

Imagine a ray L from the origin cutting through R in the direction of increasing r. Mark the r-values where L enters and leaves.

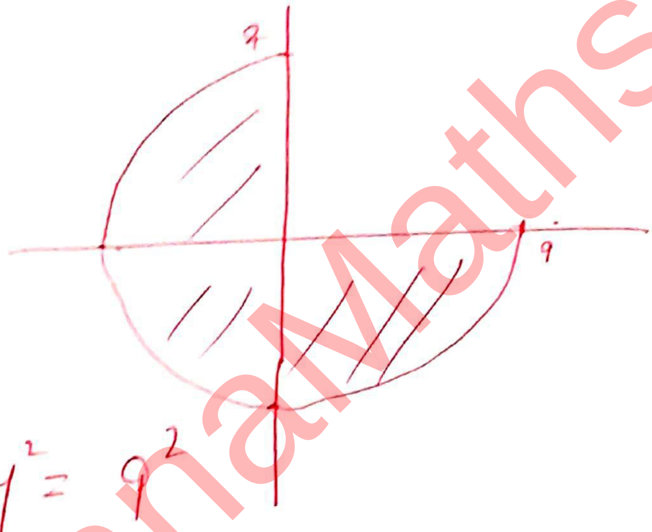
3. Find the θ-limits of Integration.

Find the largest and smallest θ-values.



$$\iint_R f(r,\theta)\,dA = \int_{\frac{\pi}{4}}^{\pi/2}\int_{\sqrt{2}\,\mathrm{cosec}\,\theta}^{2} f(r,\theta)\,r\,dr\,d\theta$$

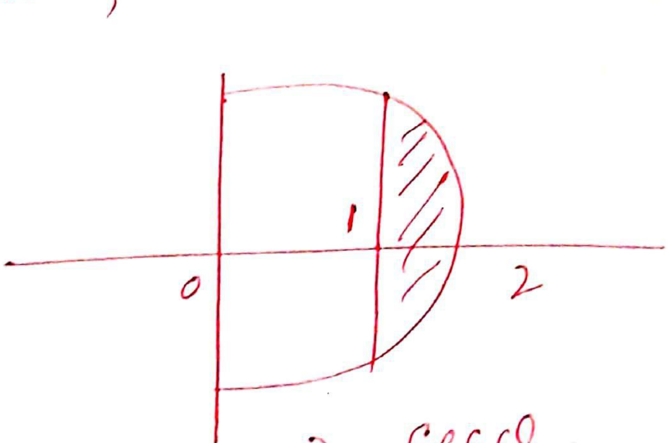
(1)



$x^2+y^2=9^2$

$r=9$ $\quad \frac{\pi}{2} \le \theta \le 2\pi \quad , \quad 0 \le r \le 9.$

(2)



$x^2+y^2=2^2$ | $x=1$ | $2=\sec\theta$

$r=2$ | $r\cos\theta=1$ | $\theta=\frac{\pi}{3}$ or $-\frac{\pi}{3}$.

$\sec\theta \le r \le 2$ | $r=\sec\theta$ | $-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$

Find the limit of Integration ②

for integrating $f(r,\theta)$ over the region R that lies inside the cardioid $\boxed{r=2(1+\cos\theta)}$ and outside the circle $\boxed{r=2}$ in the 1st quadrant.

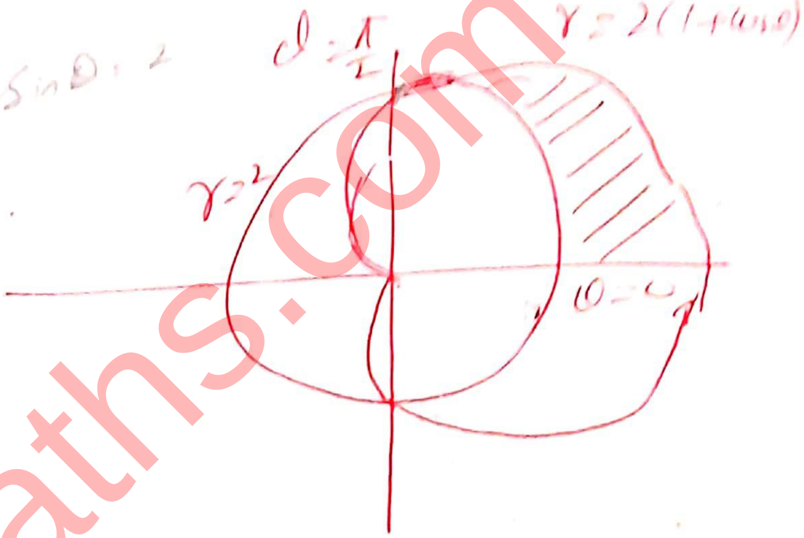
$2 = 2(1+\cos\theta)$

$1 = 1+\cos\theta$

$\cos\theta = 0$

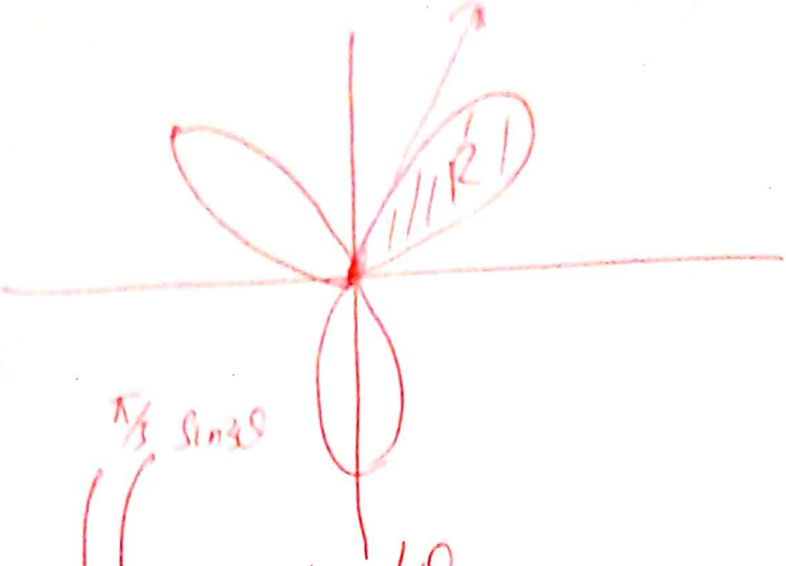
$\theta = \cos^{-1}(0)$

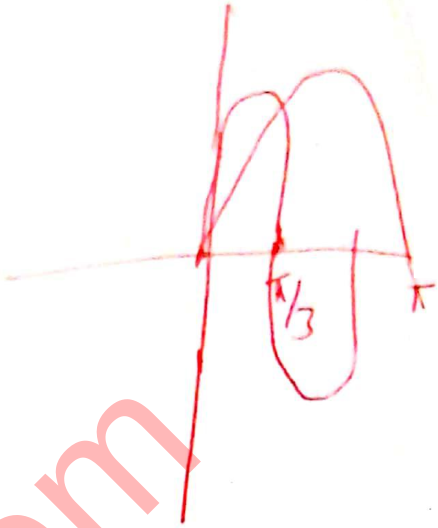
$\theta = \pi/2$



$$= \int_0^{\pi/2}\int_2^{2(1+\cos\theta)} f(r,\theta)\, r\,dr\,d\theta$$

④ use polar double integral to find the area enclosed by the three-petaled rose $r=\sin 3\theta$





$$A = \int_0^{\pi/3}\int_0^{\sin 3\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/3} \frac{r^2}{2}\Big|_0^{\sin 3\theta} d\theta$$

$$= \int_0^{\pi/3} \frac{(\sin 3\theta)^2}{2}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/3} \sin^2 3\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/3} \frac{1-\cos 6\theta}{2}\, d\theta$$

$$= \frac{1}{4}\left(\pi/3\right) = \frac{1}{12}\pi$$

$$\text{Area} = 3\,\frac{1}{12}\pi = \frac{\pi}{4}$$

15.4 Double Integrals in Polar form:-

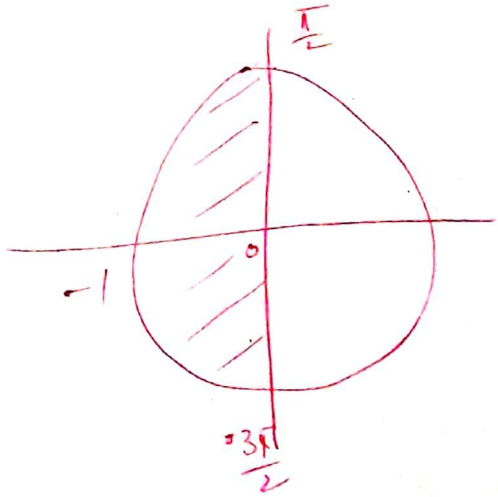
(1)

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$dx\,dy = r\,dr\,d\theta$$

(1) $$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\,dy\,dx$$

$x = r\cos\theta \qquad y = r\sin\theta$

$$x^2+y^2 = 1$$
$$r^2\cos^2\theta + r^2\sin^2\theta = 1$$
$$r = 1$$



$r = 0, \quad 1$

$\theta = \frac{\pi}{2}, \; 3\pi/2$

$$I = \int_{\pi/2}^{3\pi/2}\int_{0}^{1} r\cos\theta \, r\,dr\,d\theta.$$

$$= \int_{\pi/2}^{3\pi/2} \int_0^1 r^2 \cos\theta \, dr \, d\theta$$

$$= \int_{\pi/2}^{3\pi/2} \cos\theta \left. \frac{r^3}{3} \right|_0^1 d\theta$$

$$= \frac{1}{3} \int_{\pi/2}^{3\pi/2} \cos\theta \, d\theta$$

$$= \frac{1}{3} \left[\sin\frac{3\pi}{2} - \sin\frac{\pi}{2} \right]$$

$$= \frac{1}{3}(-1-1)$$

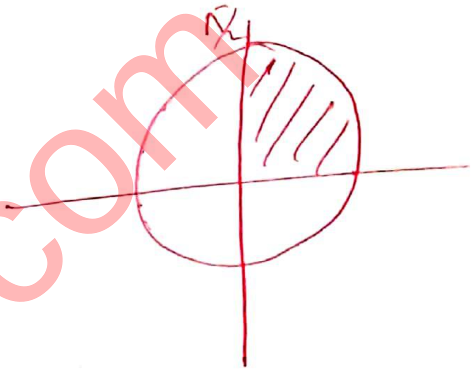
$$= -2/3$$

Convert polar into Cartesian:-

① $$\int_0^{\pi/2}\int_0^1 r^3 \sin\theta\cos\theta \, dr\, d\theta$$

$r=0,\ r=1,\ \theta=0,\ \theta=\pi/2$

$x^2+y^2=1^2$



$\Rightarrow$

$x = r\cos\theta$

$y = r\sin\theta$

$\cos\theta = \frac{x}{r}$

$\sin\theta = \frac{y}{r}$

$$= \iint x^2+y^2 \sin\theta\cos\theta \quad r\,dr\,d\theta$$

$$= \iint (x^2+y^2)\frac{x}{r}\left(\frac{y}{r}\right) r\,dr\,d\theta$$

$$= \iint \cancel{(x^2+y^2)} \frac{xy}{\cancel{x^2+y^2}} \quad r\,dr\,d\theta$$

$$= \int_0^1\int_0^{1-x^2} xy\, dy\, dx$$

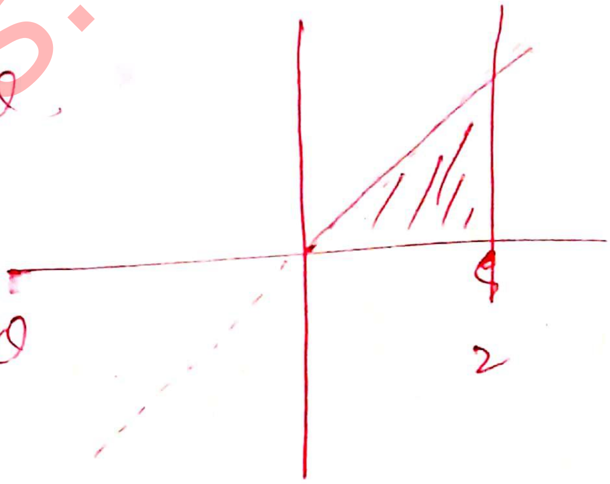
(2) $\int_0^{\frac{\pi}{4}} \int_0^{2\sec\theta} r^5 \sin^2\theta \, dr \, d\theta$.

$r=0$, $r = 2\sec\theta$, $r = \frac{2}{\cos\theta} \Rightarrow x \cancel{r} = \frac{2}{\frac{x}{\cancel{r}}}$

$\underline{x=2}$

$\theta = 0$, $\theta = \frac{\pi}{4}$

$\tan\theta = \frac{y}{x} \Rightarrow \frac{y}{x} = 1$, $y = x$.



$= \iint r^4 \sin^2\theta \; r \, dr \, d\theta$.

$= \iint r^{\cancel{4}\,2} \frac{y^2}{\cancel{r^2}} r \, dr \, d\theta$

$= \iint r^2 \; y^2 \; r \, dr \, d\theta$.

$= \int_0^x \int_0^2 (x^2+y^2) \, y^2 \, dx \, dy$

5.5

Triple Integrals in Rectangular coordinates :- ①

$$\iiint$$

Examples

1.

$$\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\,dz\,dy\,dx.$$

$$= \int_0^1\int_0^1 x^2z+y^2z+\frac{z^3}{3}\Big|_0^1 \,dy\,dx$$

$$= \int_0^1\int_0^1 x^2+y^2+\frac{1}{3}\,dy\,dx$$

$$= \int_0^1 \left(x^2y+\frac{y^3}{3}+\frac{1}{3}y\right)\Big|_0^1 dx$$

$$= \int_0^1 x^2+\frac{1}{3}+\frac{1}{3}\,dx.$$

$$= \int_0^1 x^2+\frac{1}{3}+\frac{1}{3}\,dx$$

$$= \frac{x^3}{3}+\frac{1}{3}x+\frac{1}{3}x\Big|_0^1$$

$$= \frac{1}{3}+\frac{1}{3}+\frac{1}{3}$$

$$= 1$$

(3) Evaluate

$$\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$$

$$= \int_0^{\log 2}\int_0^{x} \frac{e^{x+y+z}}{1}\Big|_0^{x+y} dy\,dx$$

$$= \int_0^{\log 2}\int_0^{x} e^{x+y+x+y} - e^{x+y}\,dy\,dx.$$

$$= \int_0^{\log 2}\int_0^{x} \left(e^{2x+2y} - e^{x+y}\right) dy\,dx.$$

$$= \int_0^{\log 2} \frac{e^{2x+2y}}{2} - \frac{e^{x+y}}{1}\Big|_0^{x} dx$$

$$= \int_0^{\log 2} \left(\frac{e^{2x+2x}}{2} - \frac{e^{2x+0}}{2}\right) - \left(e^{2x} - e^{x}\right) dx$$

$$= \int_0^{\log 2} \left(\frac{e^{4x}}{2} - \frac{e^{2x}}{2} - e^{2x} + e^{x}\right) dx$$

$$= \int_0^{\log 2} \left(\frac{e^{4x}}{2} - \frac{3e^{2x}}{2} + e^{x}\right) dx.$$

(2)

$$\frac{e^{4x}}{2\times 4}\Big|_0^{\log 2} - \frac{3}{4}e^{2x}\Big|_0^{\log 2} + e^{x}\Big|_0^{\log 2}$$

$$= \frac{1}{8}\left(e^{4\log 2} - 1\right) - \frac{3}{4}\left(e^{2\log 2} - 1\right) + e^{\log 2} - 1$$

$$= \frac{e^{\log 2^4}}{8} - \frac{1}{8} - \frac{3}{4}e^{\log 2^2} + \frac{3}{4} + e^{\log 2} - 1$$

$$= \frac{2^4}{8} - \frac{1}{8} - \frac{3}{4}\cdot 4 + \frac{3}{4} + 2 - 1$$

$$= \frac{16}{8} - \frac{1}{8} - 3 + \frac{3}{4} + 1$$

$$= \frac{16 - 1 - 24 + 6 + 8}{8}$$

$$= 5/8 \quad \text{Ans}$$

(3) Evaluate $\iiint \dfrac{dx\,dy\,dz}{(1+x+y+z)^3}$ over t[he]

volume of tetrahedron $x=0, y=0,$ [$z=0$]

$x+y+z=1$

$z=0, \quad z=1-x-y$

$y=0, \quad y=1-x$

$x=0, \quad x=1$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1+x+y+z)^{-3}\, dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left| \frac{(1+x+y+z)^{-2}}{-2} \right|_0^{1-x-y} dy\,dx.$$

$$= \int_0^1 \int_0^{1-x} \frac{(1+x+y+1-x-y)^{-2}}{-2} - \frac{(1+x+y+0)^{-2}}{-2}\, dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \frac{(2)^{-2}}{-2} - \frac{(1+x+y)^{-2}}{-2}\, dy\,dx.$$

$$\frac{1}{2}\int_0^1\int_0^{1-x} (1+x+y)^{-2} - \frac{1}{4}\, dy\, dx$$

$$= \frac{1}{2}\left[\int_0^1 \frac{(1+x+y)^{-1}}{-1}\Big|_0^{1-x} - \frac{1}{4}y\Big|_0^{1-x} dx\right.$$

$$= \frac{1}{2}\left[\int_0^1 \frac{1+\cancel{x}+1-\cancel{x}}{-1} - \frac{(1+x)^{-1}}{-1} - \frac{1}{4}(1-x)\right) dx$$

$$= \frac{1}{2}\int_0^1 \left(\frac{2}{-1} + (1+x)^{-1} - \frac{1}{4} + \frac{1}{4}x\right) dx$$

$$= \frac{1}{2}\int_0^1 \left((1+x)^{-1} + \frac{1}{4}x - \frac{1}{4} - 2\right) dx$$

$$= \frac{1}{2}\left[\ln|1+x|\Big|_0^1 + \frac{x^2}{8}\Big|_0^1 - \frac{1}{4}x\Big|_0^1 - 2x\Big|_0^1\right]$$

$$= \frac{1}{2}\left[(\ln 2 - \ln 1) + \frac{1}{8} - \frac{1}{4} - 2\right]$$

$$= \frac{1}{2}\left[\ln 2 + \frac{1}{8} - \frac{1}{4} - 2\right] = \frac{1}{2}\left[\ln 2 + \frac{1}{8} - \frac{3}{4}\right]$$

$$= \frac{1}{2}\left[\frac{8\ln 2 + 1 - 6}{8}\right]$$

$$= \frac{1}{2}\left[\frac{8\ln 2 - 5}{8}\right]$$

$$= \frac{1}{2}\left[\ln 2 - 5/8\right] \text{ Ans}$$

(4) Find the volume of D enclosed by surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$

Sol:

$$V = \iiint_D dz\,dy\,dx$$

$$z = x^2 + 3y^2, \quad z = 8 - x^2 - y^2$$

$$x^2 + 3y^2 = 8 - x^2 - y^2$$

$$4y^2 = 8 - 2x^2$$

$$y^2 = \frac{4 - x^2}{2}$$

$$y = \pm\sqrt{\frac{4 - x^2}{2}}$$

$$4 - x^2 = 0$$

$$x^2 = 4$$

$$\boxed{x = \pm 2}$$

$$V = \int_{-2}^{2}\int_{-\sqrt{4-x^2/2}}^{\sqrt{4-x^2/2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2/2}}^{\sqrt{4-x^2/2}} (8-x^2-y^2-x^2-3y^2)\,dy\,dx.$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2/2}}^{\sqrt{4-x^2/2}} (8-2x^2-4y^2)\,dy\,dx.$$

$$= \int_{-2}^{2} (8-2x^2)y - \frac{4}{3}y^3 \Big|_{-\sqrt{4-x^2/2}}^{+\sqrt{4-x^2/2}} dx.$$

$$= \int_{-2}^{2} (8-2x^2)\sqrt{\frac{4-x^2}{2}} - \frac{4}{3}\left(\sqrt{\frac{4-x^2}{2}}\right)^3 + (8-2x^2)\sqrt{\frac{4-x^2}{2}} - \frac{4}{3}\left(\sqrt{\frac{4-x^2}{2}}\right)^3 dx$$

$$= \int_{-2}^{2} 2(8-2x^2)\sqrt{\frac{4-x^2}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2} dx$$

$$= \int_{-2}^{2} 2.2(4-x^2)\sqrt{\frac{4-x^2}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2} dx$$

$$= \int_{-2}^{2} 8 \; \frac{4-x^2}{2}\sqrt{\frac{4-x^2}{2}} - 8/3 \left(\frac{4-x^2}{2}\right)^{3/2} dx.$$

$$= \int_{-2}^{2} 8 \left(\frac{4-x^2}{2}\right)^{3/2} - 8/3 \left(\frac{4-x^2}{2}\right)^{3/2} dx.$$

$$= \int_{-2}^{2} \left(\frac{4-x^2}{2}\right)^{3/2} (8 - 8/3) dx$$

$$= \int_{-2}^{2} \left(\frac{4-x^2}{2}\right)^{3/2} \left(\frac{24-8}{3}\right) dx$$

$$= \frac{16}{3}\int_{-2}^{2} \left(\frac{4-x^2}{2}\right)^{3/2} dx.$$

$$= \frac{16}{3 \; 2^{3/2}} \int_{-2}^{2} (4-x^2)^{3/2} dx.$$

$$= \frac{16}{3 . 2\sqrt{2}} \int_{-2}^{2} (4-x^2)^{3/2} dx.$$

⑤

$$\frac{8}{3\sqrt{2}}\int_{-2}^{2}(4-x^2)^{3/2}dx$$

$$=\frac{4\sqrt{2}\cancel{\sqrt{2}}}{3\cancel{\sqrt{2}}}\int_{-2}^{2}(4-x^2)^{3/2}dx$$

$$=\frac{4\sqrt{2}}{3}\int_{-2}^{2}(4-x^2)^{3/2}dx.$$

$x = 2\sin u$

$dx = 2\cos u\,du$

$\sin u = \frac{x}{2}$

when $x = 2$

$\sin u = 1$ $\boxed{u = \frac{\pi}{2}}$

$x = -2$

$u = -\pi/2$.

$$=\frac{4\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}(4-4\sin^2 u)^{3/2}\,2\cos u\,du$$

$$=\frac{4\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}(4(1-\sin^2 u))^{3/2}\,2\cos u\,du.$$

$$= \frac{4\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2} (4\cos^2 u)^{3/2}\, 2\cos u\, du$$

$$= \frac{4\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2} (2\cos u)^3\, 2\cos u\, du$$

$$= \frac{4\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2} 8\times 2\cos^4 u\, du.$$

By reduction order

$$\cos^4 u = \frac{3}{8}u + \frac{1}{4}\sin(2u) + \frac{1}{32}\sin 4u$$

$$= \frac{64\sqrt{2}}{3}\left(\frac{3}{8}u + \frac{1}{4}\sin 2u + \frac{1}{32}\sin 4u\right\Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{64\sqrt{2}}{3}\left(\frac{3}{8}\cdot\frac{\pi}{2} - \frac{3}{8}\left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{64\sqrt{2}}{3}\cdot\left(\frac{3\pi}{16} + \frac{3\pi}{16}\right)$$

$$= \frac{\cancel{64}^{4}\sqrt{2}}{\cancel{3}}\cdot\frac{\cancel{6}^{2}\pi}{\cancel{16}}$$

$$= 8\sqrt{2}\cdot\pi$$

$$= 8\pi\sqrt{2}$$

$\cos^4 x$

$$= \int (\cos^2 x)^2 \, dx$$

$$= \int \left(\frac{1+\cos 2x}{2} \right)^2 dx$$

$$= \frac{1}{4} \int (1+\cos 2x)^2 \, dx$$

$$= \frac{1}{4} \int (1+\cos^2 2x + 2\cos 2x) \, dx$$

$$= \frac{1}{4} \int 1 + 2\cos 2x + \left(\frac{1+\cos 4x}{2} \right) dx$$

$$= \frac{1}{4} \int 1 + 2\cos 2x + \frac{1}{2} + \frac{\cos 4x}{2} \, dx$$

$$= \frac{1}{4} \int \frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x \, dx$$

$$= \frac{1}{4} \left(\frac{3}{2}x + \frac{2\sin 2x}{2} + \frac{1}{2} \frac{\sin 4x}{4} + C \right.$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

Avg value of a function in space:-

$$\text{Average value of } F \text{ over } D = \frac{1}{\text{volume of } D} \iiint_D F\,dV.$$

Example:-

Find the avg value of $F(x,y,z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the planes $x=2$, $y=2$ and $z=2$ in the first octant.

$$\int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz = \int_0^2\int_0^2 \frac{x^2}{2}\Big|_0^2 yz\,dy\,dz$$

$$= \int_0^2\int_0^2 2yz\,dy\,dz$$

$$= \int_0^2 \frac{2y^2}{2} z\Big|_0^2 dz$$

$= 8$

$$\text{Avg value} = \frac{1}{\text{Volume}} \iiint_{\text{cube}} xyz \, dV$$

$$= \frac{1}{\cancel{8}}(\cancel{8})$$

$$= 1$$

15.6

Application of multiple Integrals:

(1)

1) Masses and first moments:-

If $\delta(x, y, z)$ is the density (mass per unit volume) of an object occupying a region D in space, the integral of δ over D gives the mass of an object.

1.1 Three dimensional solid:

: mass $\quad M = \iiint_D \delta \, dV \qquad \delta = \delta(x, y, z)$.

1.1.1 First moments about coordinate planes:-

$$M_{yz} = \iiint_D x \, \delta \, dV, \quad M_{xz} = \iiint_D y \, \delta \, dV,$$

$$M_{xy} = \iiint_D z \, \delta \, dV$$

1.1.2 Centre of mass

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}.$$

Two-dimensional plates

Mass $M = \iint_R \delta \, dv \qquad \delta(x,y)$

First moments

$$M_y = \iint_R x \delta \, dA$$

$$M_x = \iint_R y \delta \, dv$$

centre of mass

$$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$$

centre of mass $(\bar{x}, \bar{y}, \bar{z})$

$\therefore$ When density is contant then centre of mass is called centroid.

Examples: (1)

Find the centre of mass of a two-dimensional plate that occupies the quarter ~~cycle~~ circle $x^2+y^2 \le 1$ in the first quadrant and has density $k(x^2+y^2)$.

$$M = \int_0^1 \int_0^{\sqrt{1-x^2}} k(x^2+y^2)\, dy\, dx.$$

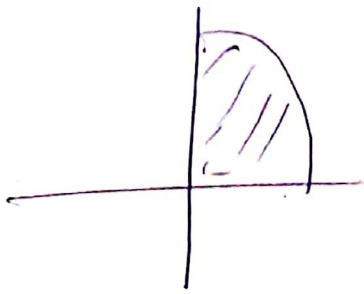
$$= k\int_0^1 x^2 y + \frac{y^3}{3}\Big|_0^{\sqrt{1-x^2}} dx$$

$$= k\int_0^1 x^2\sqrt{1-x^2} + \frac{(1-x^2)^{3/2}}{3}\, dx$$

Now, we change into polar.

$$M = \int_0^1 \int_0^{\sqrt{1-x^2}} k(x^2+y^2)\, dy\, dx$$

$$dx\,dy = r\,dr\,d\theta.$$



$$M = \int_0^{\pi/2}\int_0^1 k\, r^2\, r\, dr\, d\theta.$$

$$= \int_0^{\pi/2} k\frac{r^4}{4}\Big|_0^1 d\theta = k\int_0^{\pi/2} \frac{1}{4}\, d\theta.$$

$$= k\,\frac{1}{4}\theta\Big|_0^{\pi/2}$$

$$= \frac{k\pi}{8}$$

$$M_x = K\int_0^{\pi/2}\int_0^1 r^3\, r\sin\theta\, dr\, d\theta$$

$$= K\int_0^{\pi/2}\int_0^1 r^4 \sin\theta\, dr\, d\theta$$

$$= K\int_0^{\pi/2} \frac{r^5}{5}\sin\theta \Big|_0^1 d\theta$$

$$= K\int_0^{\pi/2} \frac{1}{5}\sin\theta\, d\theta$$

$$= K -\frac{1}{5}\cos\theta \Big|_0^{\pi/2}$$

$$= -K\frac{1}{5}\left(\cos\pi/2 - \cos 0\right)$$

$$= -K\frac{1}{5}(0-1)$$

$$= K/5$$

Similarly, $M_y = K/5$

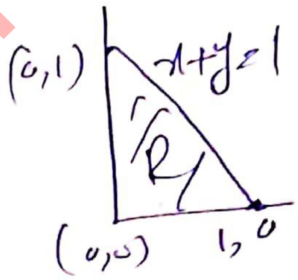
$$\bar{x} = \frac{M_y}{M} = \frac{K/5}{\frac{K\pi}{8}} = \frac{\cancel{K}}{5}\times\frac{8}{\cancel{K}\pi} = 8/5\pi$$

$$\bar{x} = \bar{y} = \frac{8}{5\pi}.$$

Example 02

A triangular domain with vertices $(0,0), (0,1)$ and $(1,0)$ has density function $\delta(x,y) = xy$. Find it total mass.

Sol



$$\int_0^1\int_0^{-x+1} \delta(x,y)\, dy\, dx$$

$$= \frac{1}{24}$$

Moments of Inertia:-

Three-dimensional Solid

About x-axis $I_x = \iiint (y^2+z^2)\,\delta\, dv$

y-axis $I_y = \iiint (x^2+z^2)\,\delta\, dv$

z-axis $I_z = \iiint (x^2+y^2)\,\delta\, dv.$

About the line $I_L = \iiint r^2(x,y,z)\,\delta\, dv$

Two-dimensional plate:-

About x-axis
$I_x = \iint y^2 \delta\, dA$

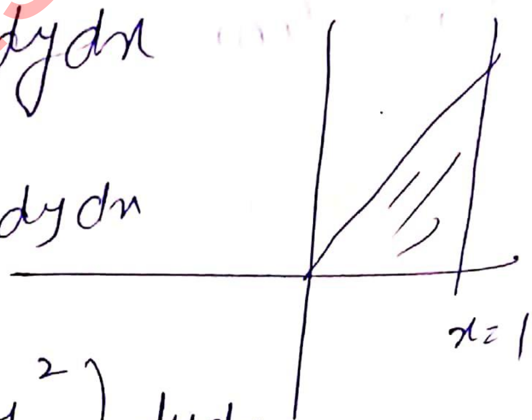
About y-axis $I_y = \iint x^2 \delta\, dA.$

About a line $I_L = \iint r^2(x,y)\,\delta\, dA$

Example: (2)

A thin plate covers the triangular region bounded by the x-axis and the lines $x=1$ and $y=2x$ in the first quadrant. The plate density at a point (x,y) is $\delta(x,y)=6x+6y+6$. Find the plate moments of inertia about the coordinate axes and origin.

$$I_x = \int_0^1 \int_0^{2x} y^2\, \delta(x,y)\, dy\, dx$$



$$= \int_0^1 \int_0^{2x} y^2(6x+6y+6)\, dy\, dx$$

$$= \int_0^1 \int_0^{2x} (6xy^2+6y^3+6y^2)\, dy\, dx$$

$$= \int_0^1 \left. \frac{6xy^3}{\cancel{3}} + \frac{6y^4}{4} + \frac{6y^3}{\cancel{3}} \right|_0^{2x} dx$$

$$= \int_0^1 2x(8x^3) + \frac{3}{2}16x^4 + 2\cdot 8x^3\, dx$$

$$= \int_0^1 (16x^4 + 24x^4 + 16x^3)\, dx = \int_0^1 40x^4 + 16x^3\, dx$$

$$= 12$$

(4)

$$I_y = \int_0^1 \int_0^{2x} x^2 \delta(x,y)\,dy\,dx$$

$$= 39/5$$

$$I_0 = 12 + 39/5 = \frac{60+39}{5} = \frac{99}{5}$$

Lecture

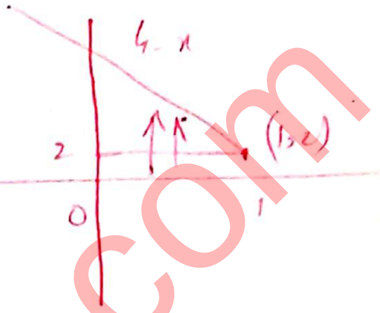
$y = 4 - x$

$y = 2$

$x = 0, \; x = 1$

① $$\int_0^1 \int_2^{4-2x} f \, dy\, dx$$

$$\int_2^4 \int_0^{\frac{4-y}{2}} f(x,y)\, dx\, dy$$



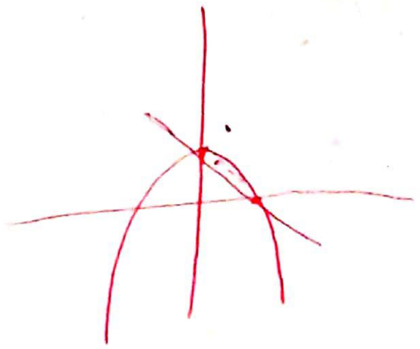
② $$\int_0^1 \int_{1-x}^{1-x^2} f(x,y)\, dy\, dx$$

$x = 0$
$x = 1$

$y = 1 - x^2$
$\sqrt{1-y} = x$

$y = 1 - x$
$x = 1 - y$

$$\int_0^1 \int_{1-y}^{\sqrt{1-y}} f(x,y)\, dx\, dy$$



③ $\int_0^{\frac{\pi}{6}} \int_{\sin x}^{1/2} xy^2 \, dy \, dx$

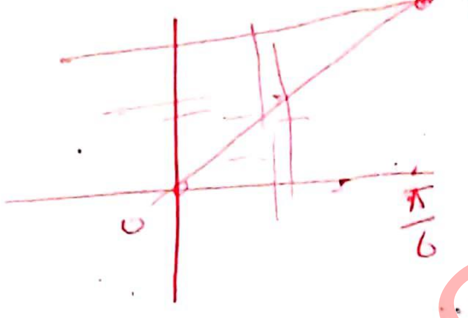
$x=0$, $x=\frac{\pi}{6}$ | $y=\frac{1}{2}$, $y=\sin x$

$x=\sin^{-1}y$

$y=\sin x$

$y=\frac{1}{2}$

$\int_0^{\frac{1}{2}} \int_0^{\sin^{-1}y} xy^2 \, dx \, dy$



15.3

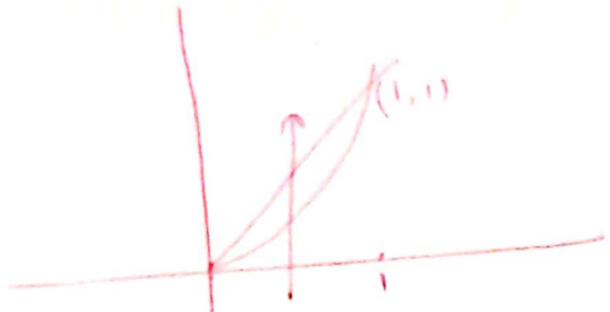
Area By double Integration:-

The area of a closed, bounded region R is

$$A = \iint_R dA.$$

Examples:- ①

Find the area of region R bounded by $y=x^3$ and $y=x^2$ in the first quadrant



$$A = \int_0^1 \int_{x^2}^{x} dy\,dx$$

$$= \int_0^1 y\Big|_{x^2}^{x} dx$$

$$= \int_0^1 (x - x^2)\,dx$$

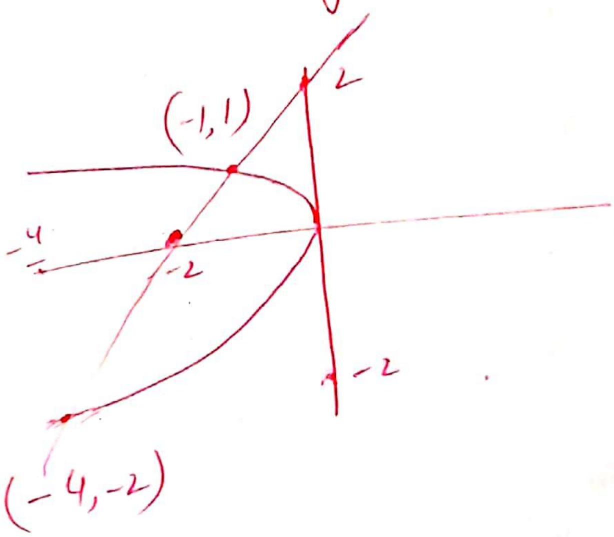
$$= \frac{x^2}{2} - \frac{x^2}{3}\Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}.$$

② Find the area

The parabola $x = -y^2$ and the line $y = x + 2$

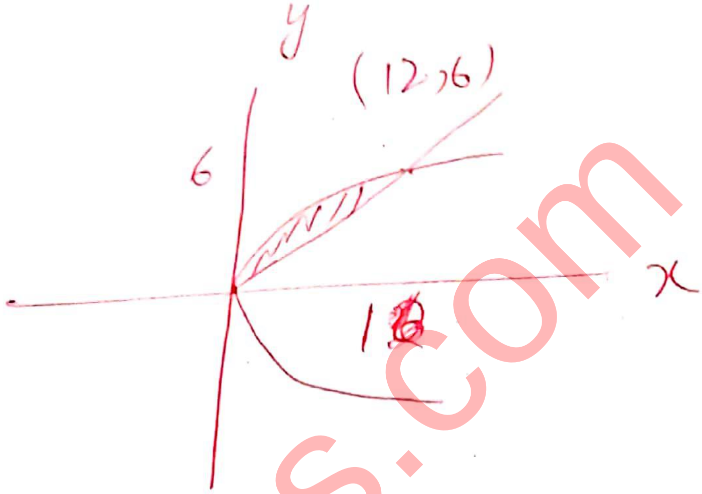
$$= \int_{-2}^{1} \int_{y-2}^{-y^2} dx\,dy$$



$$= \frac{9}{2}$$

③ Sketch and find the area

$$\int_0^6 \int_{y^2/3}^{2y} dx\,dy$$



$= 12$

Average value:-

$$\text{Average value of } f \text{ over } R = \frac{1}{\text{area of } R} \iint_R f\,dA$$

Examples:-

① Find the averge value of $f(x,y) = \sin(x+y)$ over the rectangle $0 \le x \le \pi$, $0 \le y \le \pi/2$

③

$$\text{avg} = \frac{1}{\text{area of }R}\int_0^{\pi}\int_0^{\pi/2}\sin(x+y)\,dy\,dx$$

$$= \frac{1}{\frac{\pi^2}{2}}\int_0^{\pi}\int_0^{\pi/2}\sin(x+y)\,dy\,dx$$

$$= \frac{2}{\pi^2}\int_0^{\pi}-\cos(x+y)\Big|_0^{\pi/2}dx$$

$$= \frac{2}{\pi^2}\int_0^{\pi}-\left[\cos\left(x+\frac{\pi}{2}\right)-\cos x\right]dx$$

$$= \frac{2}{\pi^2}\ -\sin\left(x+\frac{\pi}{2}\right)+\sin x\Big|_0^{\pi}$$

$$= \frac{2}{\pi^2}\left[-\left(\sin\frac{3\pi}{2}-\sin\pi\right)+\left(\sin\frac{\pi}{2}-\sin 0\right)\right]$$

$$= \frac{2}{\pi^2}\left(1+1\right)$$

$$= \frac{4}{\pi^2}$$

340   180   31   46   275

(2) Find the avg height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2$, $0 \le y \le 2$.

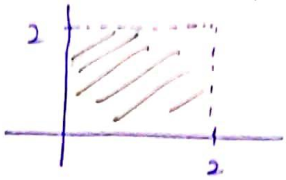
Sol:-

$$\text{avg height} = \frac{1}{4} \int_0^2 \int_0^2 (x^2 + y^2)\, dy\, dx$$

$$= 8/3$$

Assignment = 2

Q4) $z = 16 - x^2 - 2y^2 \qquad x = 2,\ y = 2$ and $z = 0,\ x = 0,\ y = 0$

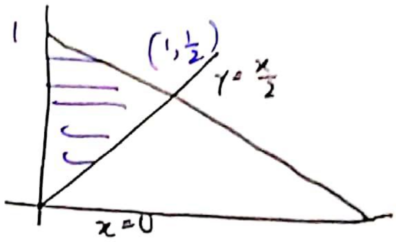


$$= \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\, dy\, dx = \int_0^2 \left((16y) - (x^2 y) - \frac{2y^3}{3} \right)\Bigg|_0^2 dx$$

$$= \int_0^2 \left(32 - 2x^2 - \frac{2(-8)}{3} \right) dx = \int_0^2 \left(32 - 2x^2 - \frac{16}{3} \right) dx = \left(32x - \frac{2x^3}{3} - \frac{16x}{3} \right)\Bigg|_0^2$$

$$= 32(2) - \frac{2(2)^3}{3} - \frac{16(2)}{3} = 64 - \frac{16}{3} - \frac{32}{3} = 64 - \frac{16+32}{3} = 48$$

Q5) $z = 0,\ x = 0,\ x + 2y + z = 2,\ x = 2y$



$x + 2y + z = 0$ assume $z = 0$

$y = 1 - \frac{x}{2}$

$$\iint_R (2 - x - 2y)\, dy\, dx = \int_0^1 \int_{x/2}^{1 - \frac{x}{2}} (2 - x - 2y)\, dy\, dx$$

$$= \int_0^1 \int_{x/2}^{1-\frac{x}{2}} (2 - x - 2y)\, dy\, dx = \int_0^1 \left[2y - xy - \frac{2y^2}{2} \right]_{x/2}^{1-\frac{x}{2}} dx$$

$$= \int_0^1 \left(\left[2\left(1 - \frac{x}{2}\right) - x\left(1 - \frac{x}{2}\right) - \left(1 - \frac{x}{2}\right)\right]^2 - \left(2\left(\frac{x}{2}\right) - x\left(\frac{x}{2}\right) - \frac{x^2}{4} \right) dx \right.$$

$$= \int_0^1 \left(2 - 2\left(\frac{x}{2}\right) - x + \frac{x^2}{2} - \left[1 + \frac{x^2}{4} - x \right] - x + \frac{x^2}{2} - \frac{x^2}{4} \right) dx$$

$$= \int_0 \left(2 - x - x + \frac{x^2}{2} - 1 - \frac{x^2}{4} + x - x + \frac{x^2}{2} + \frac{x^2}{4}\right.$$

$$= \int_0^1 (1 - 2x + x^2)\,dx = \left(\frac{x^3}{3} + x\right)\Big|_0^1 = \frac{1}{3} - 1 + 1 \Rightarrow \boxed{V = \frac{1}{3} \text{ cubic units}}$$

Q6)

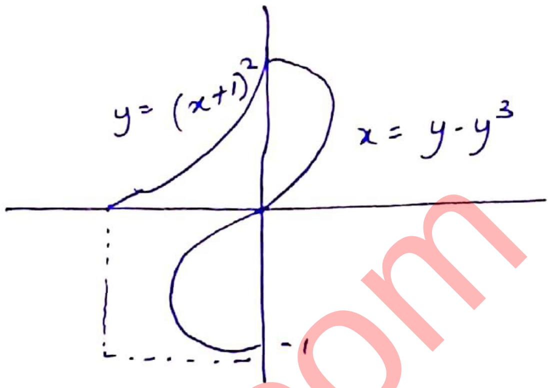
Using y-simple,

from $y=0$, $y=1$

$y = (x+1)^2$

$\sqrt{y} - 1 = x$

$y - y^3 = x$



for $y = (0,1)$

$$= \int_0^1 \int_{\sqrt{y}-1}^{y-y^3} dx\,dy = \int_0^1 (x)\Big|_{\sqrt{y}-1}^{y-y^3} dy = \int_0^1 (y - y^3 - \sqrt{y} + 1)\,dy$$

$$= \left(\frac{y^2}{2} - \frac{y^4}{4} - \frac{2}{3}(y^{3/2}) + y\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{4} - \frac{2}{3} + 1 = \frac{7}{12}$$

y-simple from $y=0$ to $y=-1$;

$x = -1 \qquad x = y - y^3$

$$\int_{-1}^{0}\int_{-1}^{y-y^3} dx\,dy = \int_{-1}^{0} (x)\Big|_{-1}^{y-y^3} = \int_{-1}^{0} (y - y^3 + 1)\,dy = \left(\frac{y^2}{2} - \frac{y^4}{4} + y\right)\Big|_{-1}^{0}$$

$$= \frac{1}{2} - \frac{1}{4} - 1 = \frac{2-1-4}{4} = -\frac{3}{4} \Rightarrow A = \frac{3}{4}$$

Total Area $= \frac{3}{4} + \frac{7}{12} = \frac{4}{3}$

Q7)

$z = x^2 + y^2 \qquad 3x + 2y^2 + z^2 = 9 \qquad P(1,1,2)$

$\nabla f = 2x\hat{i} + 2y\hat{j} - \hat{k} \qquad \nabla g = 6x\hat{i} + 4y\hat{j} + 2z\hat{k}$

Vector in direction of tangent $= \nabla f \times \nabla g$

$\nabla f\,\Big|_{(1,1,2)} = \langle 2, 2, -1\rangle \qquad \nabla g\,\Big|_{(1,1,2)} = \langle 6, 4, 4\rangle$

$\vec{v} = \nabla f \times \nabla g$

$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & -1 \\ 6 & 4 & 4 \end{vmatrix} = \hat{i}(8+4) - \hat{j}(8 \\ + \hat{k}(8-12)$$

$$\vec{v} = 12\hat{i} - 14\hat{j} - 4\hat{k}$$

Paremetric form ; $x(t) = 1+12t$, $y(t) = 1-14t$, $z(t) = 2-4t$

Q8 (b)

$$f(x,y) = xe^{xy}$$

$$\boxed{f_x(x,y) = xe^{xy} \cdot y + e^{xy}}$$

$$f_y(x,y) = xe^{xy} \cdot x + e^{xy}(0)$$

$$\boxed{f_y(x,y) = x^2e^{xy}}$$

Partial derivative exist and function is continuous, so it is differentiable.

$$z = z_0 + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$= 1 + [0+e^0][x-1] + 1[y-0]$$

$$= \cancel{1} + x - \cancel{1} + y$$

$$\Rightarrow \boxed{z = x+y}$$

$$(x,y) = (1.1, -0.1)$$

$$\boxed{z = 1.1 - 0.1 = 1}$$

Q9)

$$z = f(x,y) = x^2 + 3xy - y^2$$

$$dz = f_x(x,y)\,dx + f_y(x,y) \cdot dy$$

$$dz = (2x+3y)\,dx + (3x-2y)\,dy$$

$x=2$, $dx = 0.05$, $y=3$, $dy = -0.04$

$$dz = [2(2)+3(3)](0.05) + [3(2)-2(3)][-0.04]$$

$$= (4+9)(0.05) + 0 = 0.65$$

$$\Delta z = f(x,y) - f(x_0,y_0)$$

$$f(2,3) = (2)^3 + 3(2)(3) - (3)^2$$

$$\Rightarrow \boxed{f(2,3) = 13}$$

$f(2.05, 2.96) = (2.05)^2 + 8(2.05)(2.96) - (2.96)^2$

$= 4.2025 + 18.204 - 8.7616$

$\Rightarrow \boxed{f(2.05, 2.96) = 13.64}$

$\Rightarrow \boxed{\Delta z = 0.64}$

So, dz and Δz has difference of $\boxed{0.01}$

Q10)

$$= \iint_R 10{,}000 \frac{e^y}{1+\frac{|x|}{2}}\, dy\, dx$$

$$= \int_{-5}^{5}\int_{-2}^{0}\left[10{,}000 \frac{e^y}{1+\frac{|x|}{2}}\right] dy\, dx$$

$$= 10\,000 \int_{-5}^{0} \left[\frac{1-e^{-2}}{1+\frac{|x|}{2}}\right]_{-2}^{0} dy\, x$$

$$= 10{,}000\,(1-e^{-2})\left(\int_{-5}^{0} \frac{1}{1-\frac{x}{2}}\, dx + \int_{0}^{5} \frac{1}{1+\frac{x}{2}}\, dx\right)$$

$$= 10000\,(1-e^{-2})\left(\left[-2\ln\left(1-\frac{x}{2}\right)\right]_{-5}^{0} + \left[2\ln\left(1+\frac{x}{2}\right)\right]_{0}^{5}\right)$$

$$= 10000\,(1-e^{-2}) \cdot 4\ln\left(\frac{7}{2}\right)$$

Total Population of Bacteria $= 43328.7 \approx 43329$ Answer

Q11

$f(t) = b + at$ $\quad f(-20) = 2,\ f(20) = 7$

$2 = b - 20a \rightarrow (1)$ $\quad 7 = b + 20a \rightarrow (2)$

diameter $= 40\,m$ $\quad$ radius $= 20\,m$

volume $= ?$

If radius $= 20\,m$, then circle is

$x^2 + y^2 \le (20)^2$

$x^2 + y^2 \le 400$

Depth is constant at x-axis, but increases on y-axis

from $f(0, -20) = 2$ to $f(0, 20) = 7$

$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - 2}{20 + 20} = \frac{1}{8}$

$z = \frac{1}{8}(y + y_0) + 2 \qquad z = \frac{1}{8}y + \frac{9}{2}$

Volume $= \iint f(x,y)\, dA$

Using polar coordinates,

$x = r\cos\theta \qquad y = r\sin\theta \qquad 0 \le \theta \le 2\pi \qquad 0 \le r \le 20$

$x^2 + y^2 = r$

$$= \int_0^{2\pi}\int_0^{20} \left(\frac{1}{8} r\sin\theta + \frac{9}{2}\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{8}\left(\frac{r^3}{3}\right)\sin\theta + \left(\frac{r^2}{2}\right)\left(\frac{9}{2}\right)\right]_0^{20} d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{24}(8000)\sin\theta + 900\right) d\theta$$

$$= \left[\frac{-1000}{3}\cos\theta + 900\theta\right]_0^{2\pi}$$

$$= \left[\frac{-1000}{3} + 1800\pi\right] - \left[\frac{-1000(1)}{3} + 0\right]$$

$= 1800\pi \approx 5654.8$ Answer

Q2

The shortest distance is $OP \perp AB$

If we analyze from triangle AOB

$$\frac{OP}{OB} = \frac{OB}{AB} \Rightarrow OP \Rightarrow \frac{OA \cdot OB}{H \cdot B}$$

$$\Rightarrow \frac{5(3)}{\sqrt{(5)^2 + (3)^2}} \Rightarrow \frac{15}{\sqrt{34}} \approx 2.572 \text{ miles}$$

minimum cost $= 2.572 \times 250{,}000$

$= 643{,}120$ Answer